Thermodynamics and Phase Structure of Dyonic Taub-NUT-AdS

GeomGravX, University of Tartu.

Adel M. Awad

Ain Shams U. & British U. in Egypt

(E. Elkhateeb, S. Eissa, A. Barqawy and M. Tharwat)

Phys. Rev. D 109 (2024), 084026,
Phys. Rev. D 108 (2023), 064022,

Phys. Rev. D 105 (2022), 124034.

Content

- *Taub-NUT (TN) Metric*
- *Taub-NUT Charges*
- *Path Integral & Regularity Conditions*

Minkowski

- *TN-AdS Dyons*
- *Extended Thermodynamics*
- *TN-Phases-Spherical Case*
- *TN-Phases-Topological Cases*

Anti de Sitter

Taub-NUT Metric

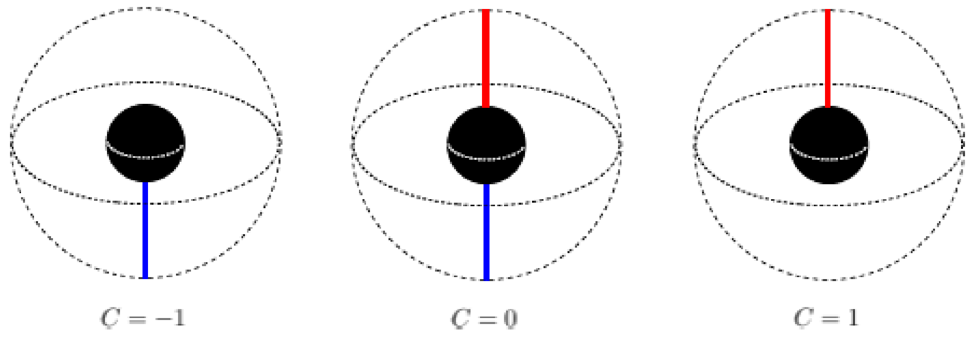
- *It is a vacuum soln. of GR with two parameters, m and n.*

$$ds^2 = -f(dt + (2n\cos\theta + C)\,d\phi)^2 + f^{-1}dr^2 + (r^2 + n^2)(d\theta^2 + \sin\theta^2\, d\phi^2),$$

where, $f = \frac{r^2 - n^2 - 2m\,r}{r^2 + n^2}$, *and* C *is some parameter.*

- *As n → 0, we get Schwarzschild Soln.*

- *As m → 0, we do not get Minkowski space!*



Location of Misner String*

- From 60's that it was known as a gravitational dyon. Misner showed that it has a string-like singularity similar to Dirac' String (i.e., it is a gauge artifact!), which can be removed upon imposing $\beta = 8\pi n$.

*R. Durka, Int. J. Mod. Phys. D 31 (2022) 04, 2250021

Taub-NUT Charges

- *The electric and magnetic charges of a soln. are given by*

$$Q_e = \frac{1}{4\pi}\int_{S^2} * F \qquad\qquad Q_m = -\frac{1}{4\pi}\int_{S^2} F$$

- *Also, one can calculate the total mass of a soln. using Kumar' integral*

$$M = -\frac{1}{4\pi}\int_{S^2} * d\chi$$ *(where, χ is a time-like Killing1-form).*

- *It is intriguing to know that the nut parameter is*

$$n = \frac{1}{4\pi}\int_{S^2} d\chi$$

- *The nut parameter is seen as a magnetic-type mass*!

TN unconstrained Thermodynamics:

- *The price for removing Misner string is to impose the relation*

$$\beta(r_0) = 8\pi n. \quad \left(\text{since } \beta = \frac{4\pi}{\acute{f}(r_0)}, \text{this impose a relation betn. } n \text{ and } r_0\right)$$

- *This leads to $M=3n/4$, and $S \neq A_h/4$!*
- *Temperature, $\text{limit}_{(n\to 0)}$ $T \neq T_{Sch}$ (diverges in this limit!)*
- *Notice that for Anti-de-Sitter cases with flat and hyperbolic horizon, TN solutions have no Misner strings and we don't have to impose this condition!*
- *If n and r_0 are not related, we must have a pair of thermodynamic quantities; (X,Y) such that*

$$dU = TdS + \cdots + XdY$$

$\phi_e dQ = e\phi_e d\mathcal{N} = \mu d\mathcal{N}$ → Number of Particles

$\beta_s = 8\pi n$ ← $T_s dS_s$ *or* $\phi_N dN$ → N=n, $\phi_N = -n/2r_0$

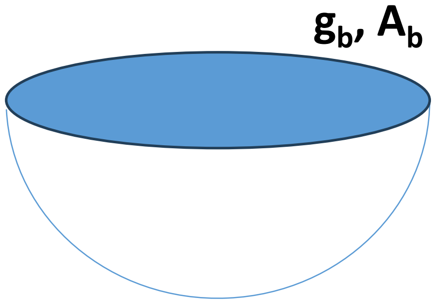
Path Integral & Regularity Conditions

- *BH thermodynamics is defined through Euclidean Path Integral,*

$$Z = \int D[g, A] e^{-I(g,A)} \sim e^{-I_{on}} = e^{-\beta F},$$

- *Here we fix the values of the fields at the boundary, i.e., g_b and A_b.*
- *Also, **g** and **A** must be regular.*
- *For example, Regularity of the metric, **g,** at the horizon leads to*

$$\beta = \frac{4\pi}{f'(r_0)}.$$



Path Integral & Regularity Conditions

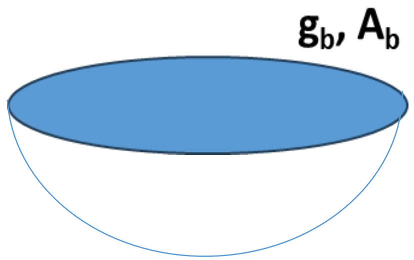
In short:

- *For fixed charges we have*

  $Z_{Can} = exp(-I_{on}) = exp(-\beta F)$, *(Can. Ens.)*, $F = U\text{-}TS$ (*Helmholtz pot.*),

- *For fixed chem. Pot. We have*

  $Z_{grnd} = exp(-I_{on}) = exp(-\beta G)$, *(Grand Can.)*, $G = U\text{-}TS\text{-} \mu N$ (*Gibbs pot.*).



- *But we could have more than one solution with the same boundary values, in this case*

$$Z= exp(-I_1) + exp(-I_2) = = exp(-\beta F_1) + exp(-\beta F_2)$$

- *In these cases, one soln. might dominate in certain range of temperature, then the other could dominate in another range. If this happens, we have phase transition at which $F_1 = F_2$, at certain transition temperature!*
- *In this work we adopt, the standard definition of internal energy and entropy;*

$$U = -\partial_\beta \ln Z\,, \quad S = \beta\, \partial_\beta I - I.$$

TN-AdS Dyons*

- Metric:

$$ds^2 = -f(r)\,(\,dt - 2n(\cos\theta + k)\,d\phi\,)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)\,(\,d\theta^2 + \sin^2\theta\,d\phi_e^2)$$

$$f(r) = \frac{r^2 + q_e^2 + q_m^2 - n^2 - 2\,m\,r}{r^2 + n^2} + \frac{r^4 + 6\,n^2\,r^2 - 3\,n^4}{(r^2 + n^2)\;L^2}$$

$$A = \left(\frac{n\,q_m - q_e r}{r^2 + n^2} + \phi_e\right) dt + \left(\left[\frac{2\,n\,q_e r + q_m\,(r^2 - n^2)}{(r^2 + n^2)}\right] cos\theta + C\right) d\phi$$

- Regularity of the 1-form leads to:

$$C_{\pm} = \mp(q_m + 2n\phi_e),$$

$$A_{\pm}^{\phi} = \frac{(q_m + 2n\phi_e)(\cos\theta \mp 1)}{(r^2 + n^2)\sin^2\theta}.$$

$$q_e = \frac{n\,q_m + \phi_e\,(n^2 + r_0^2)}{r_0}.$$

Charges:

$$Q_e^{\infty} = q_e,$$

$$Q_m^h = (q_m + 2\,n\phi_e).$$

* Phys. Rev. D 108 (2023), 064022.

Extended Thermodynamics*

$$P = \frac{3}{8\pi L^2}$$

- Temperature: $$T = \frac{(1-\phi_e^2)\, r_0^2 - (q_m + n\,\phi_e)^2}{4\pi\, r_0^3} + \frac{3\, r_0^2\,(n^2 + r_0^2)}{4\pi\, L^2\, r_0^3}.$$
- I is calculated using counter-terms: $I(\beta, n, Q_m, \phi_e, P) = \beta G$ (G is grand Pot,)

$$G = \frac{\left[\phi_e\,(n^2 + r_0^2) + n q_m\right]^2}{4 r_0^3} - \frac{r_0^4 + 3n^4}{4L^2 r_0} + \frac{(n^2 - r_0^2)\,(2\phi_e^2 - 1) + 3q_m^2 + 4nq_m\phi_e}{4r_0}.$$

$$dG = -\,S\,dT + \phi_n\,dn + \phi_m\,dQ_m - Q_e\,d\phi_e + V\,dP,$$

where,

$$\left(\frac{\partial G}{\partial T}\right)_{n,Q_m,\phi_e,P} = -S, \qquad \left(\frac{\partial G}{\partial n}\right)_{T,Q_m,\phi_e,P} = \phi_n,$$

$$\left(\frac{\partial G}{\partial P}\right)_{T,n,Q_m,\phi_e} = V, \qquad \left(\frac{\partial G}{\partial Q_m}\right)_{T,n,\phi_e,P} = \phi_m,$$

$$Q_e = -\left(\frac{\partial G}{\partial \phi_e}\right)_{T,n,Q_m,P} = Q_e^{\infty}.$$

* Phys. Rev. D 109 (2024), 084026.

Extended Thermodynamics

- 1st Law is satisfied: $$dU = T\,dS + \phi_n\,dn + \phi_m\,dQ_m + \phi_e\,dQ_e - P\,dV.$$

With, $U = M - n\,\phi_n - PV$ $\quad S = \pi\left(r_0^2 + n^2\right)$ $\quad M = -\int_\Sigma (\star d\xi + 2\Lambda\omega) = \mathrm{m}$

$$P = \frac{3}{8\pi L^2}, \qquad V = \frac{4\pi r_0^3}{3}\left(1 + \frac{3n^3}{r_0^2}\right)$$

$$\phi_n = \frac{n\,(Q_m - n\,\phi_e)^2 + r_0^2\,(3\,n\,\phi_e^2 - n - 2\,Q_m\,\phi_e)}{2r_0^3} + \frac{3\,n\,(r_0^2 - n^2)}{2\,r_0\,L^2},$$

$$\phi_e = \Phi_e|_\infty - \Phi_e|_h, \qquad \phi_m = \Phi_m|_\infty - \Phi_m|_h = \frac{(Q_m - n\,\phi_e)}{r_0},$$

- Smarr and Gibbs-Duhem relations are satisfied.

$$G = M - T\,S - n\,\phi_n - Q_e\,\phi_e.$$

$$M = 2\,T\,S + 2\,n\,\phi_n + Q_e\,\phi_e + Q_m\,\phi_m - 2\,P\,V.$$

TN-Phases-Spherical Solution*

- Temperature:

$$T = \frac{3(r_h^2+n^2)^2 - l^2(p^2+q^2-k(r_h^2+n^2))}{4\pi r_h l^2 (r_h^2+n^2)}.$$

- Pressure: $$P = \frac{3}{8\pi L^2},$$

- Rewrite the above eqn. as *P=P(v), and search for critical points;*

$$\frac{\partial P}{\partial v} = 0\,, \qquad \frac{\partial^2 P}{\partial v^2} = 0\,,$$

* Phys. Rev. D 109 (2024), 084026; Phys. Rev. D 108 (2023), 064022.

TN-Phases-Spherical Solution

- Critical Points (Canon. Case $\phi_e=0$, and q_m and n are fixed):

$$\bar{P}_c^{(1,2)} = \frac{1}{n^4}\left(6q_m^2 - n^2 + 2\sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}\right),$$

$$\bar{T}_c^{(1,2)} = \pm\frac{1}{3n^4}\left[16\left(3q_m^2 - \sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}\right)^{3/2} + \left(8n^2-96q_m^2\right)\sqrt{3q_m^2-\sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}}\right],$$

$$r_{0(c)}^{(1,2)} = \mp\sqrt{3q_m^2 - \sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}},$$

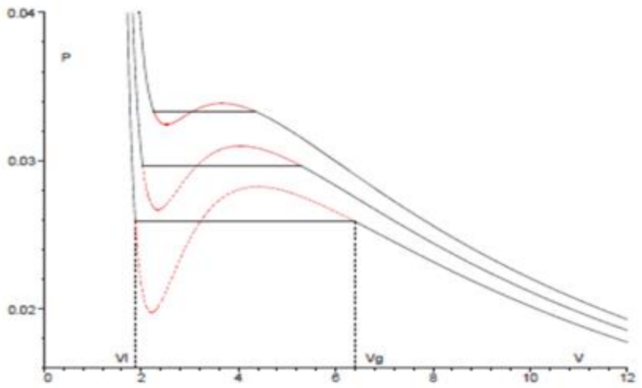
$$\bar{P}_c^{(3,4)} = \frac{1}{n^4}\left(6q_m^2 - n^2 - 2\sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}\right),$$

$$\bar{T}_c^{(3,4)} = \pm\frac{1}{3n^4}\left[16\left(3q_m^2 + \sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}\right)^{3/2} + \left(8n^2-96q_m^2\right)\sqrt{3q_m^2+\sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}}\right],$$

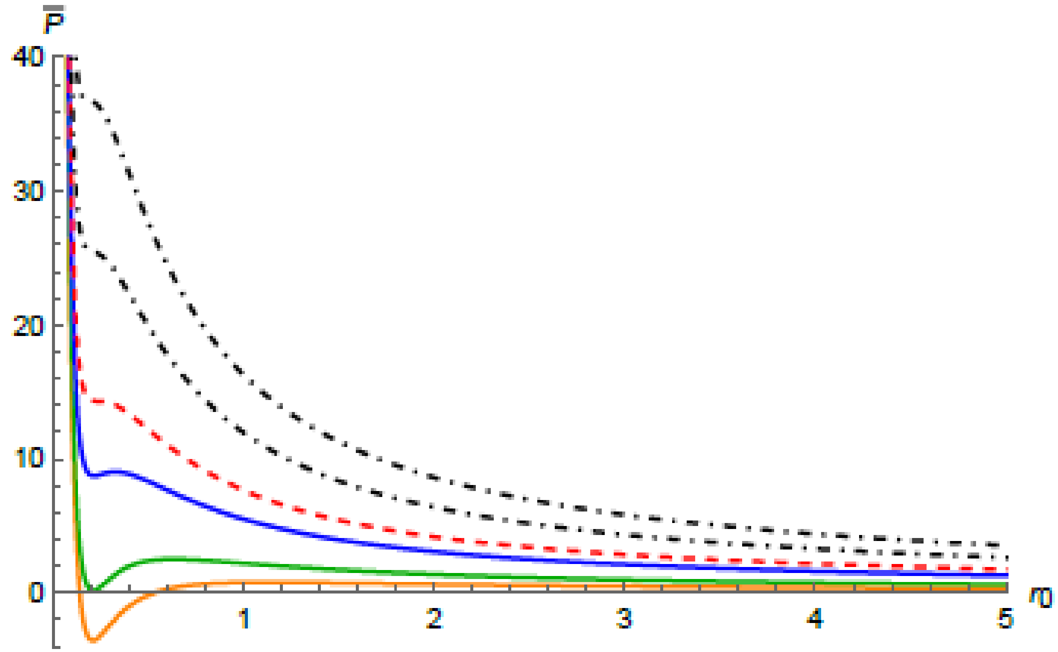
$$r_{0(c)}^{(3,4)} = \mp\sqrt{3q_m^2 + \sqrt{3}\sqrt{q_m^2\left(3q_m^2-n^2\right)}}.$$

TN-Phases-Spherical Solution

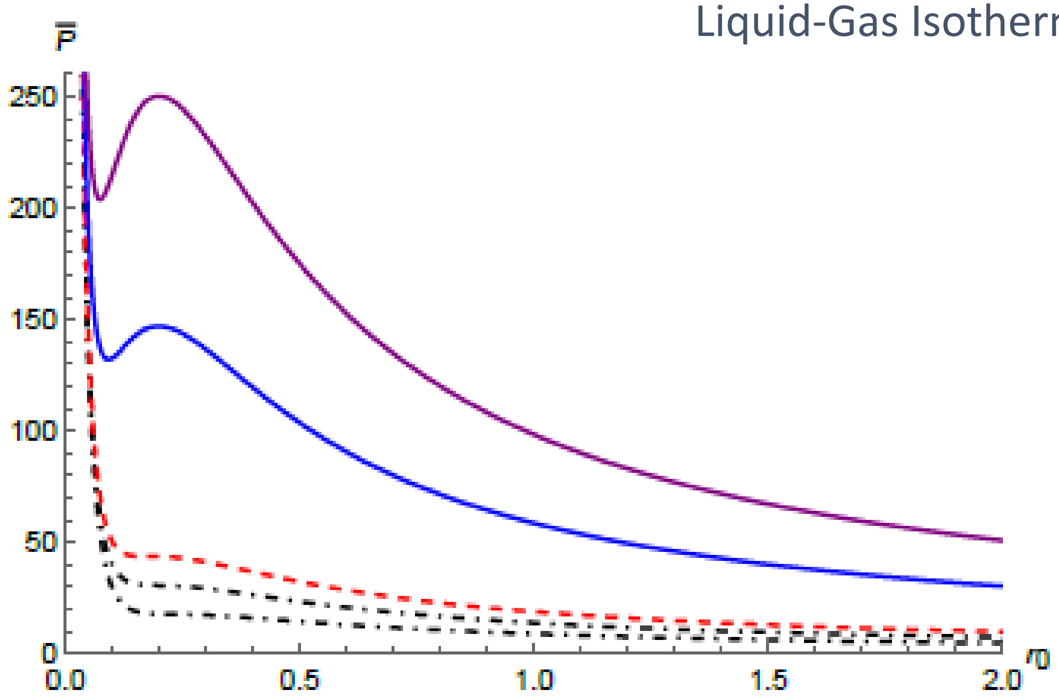
- We have two C. P. 's (Canon. Case):



Liquid-Gas Isotherms



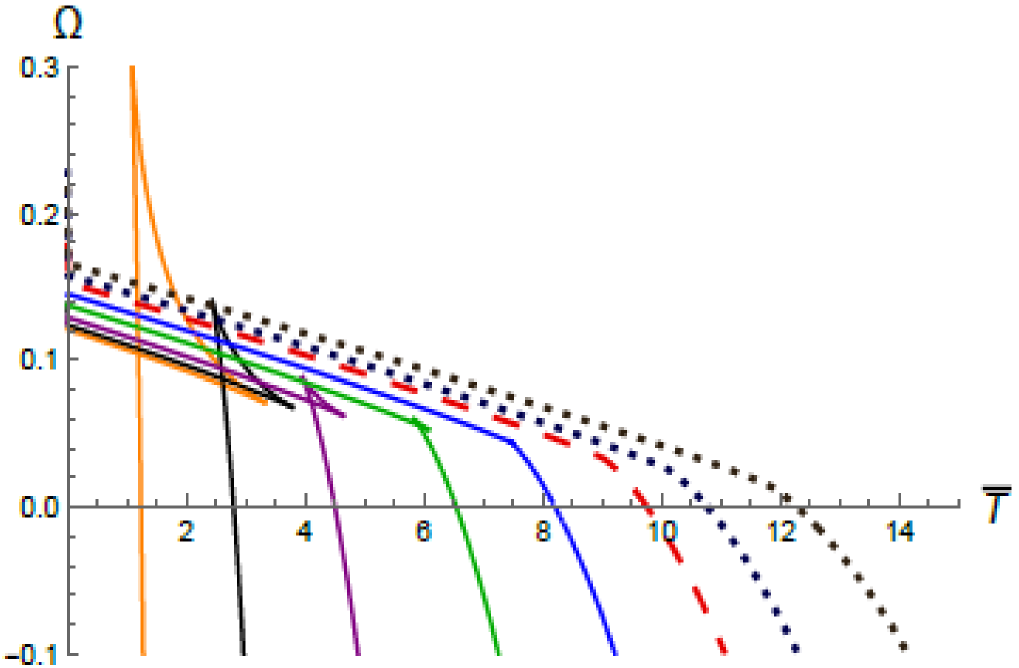
1st point



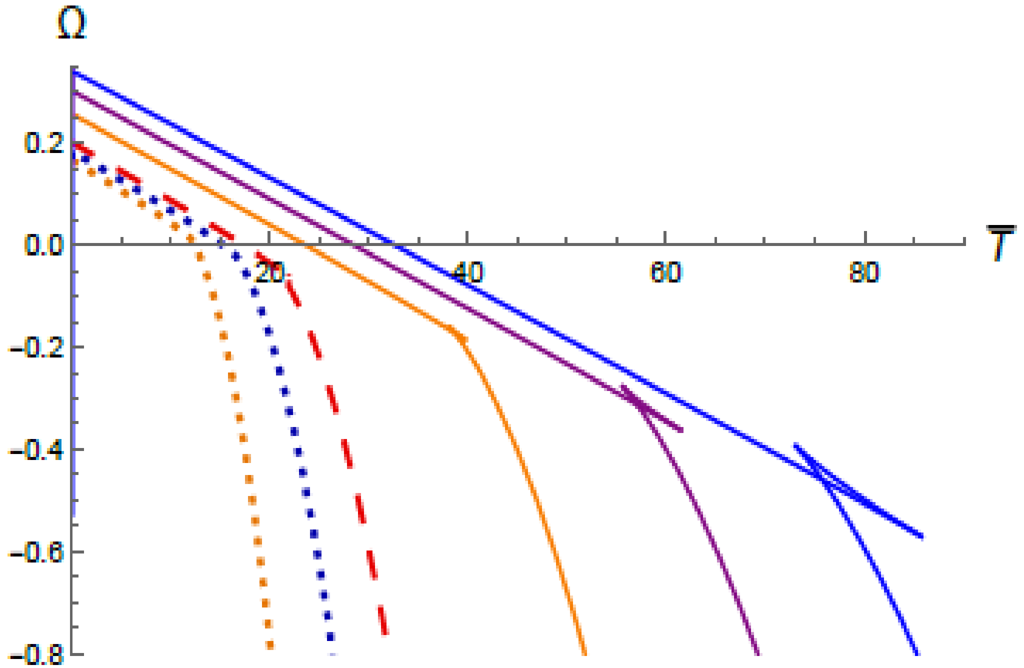
2nd point

TN-Phases-Spherical Solution

- Behavior of Free Energy. (Canon. Case):



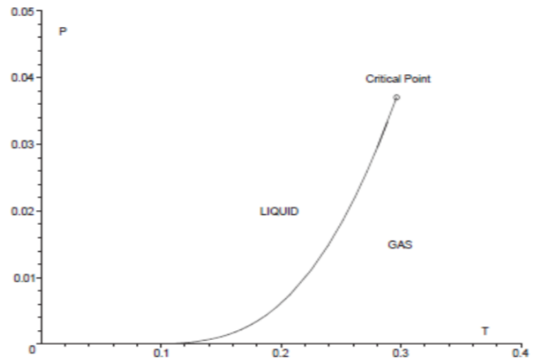
1st point



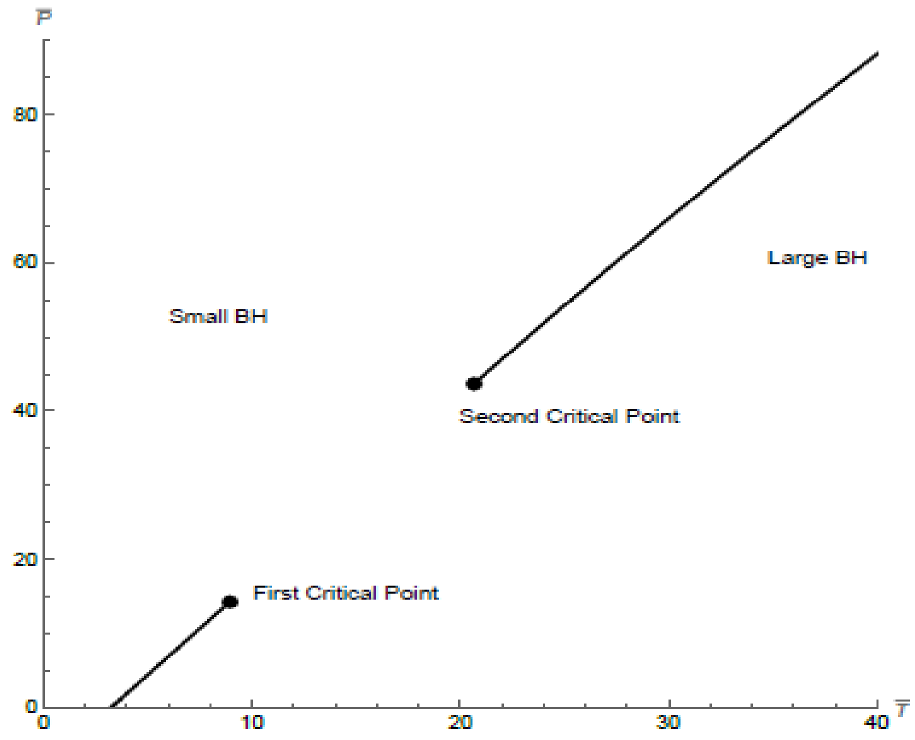
2nd point

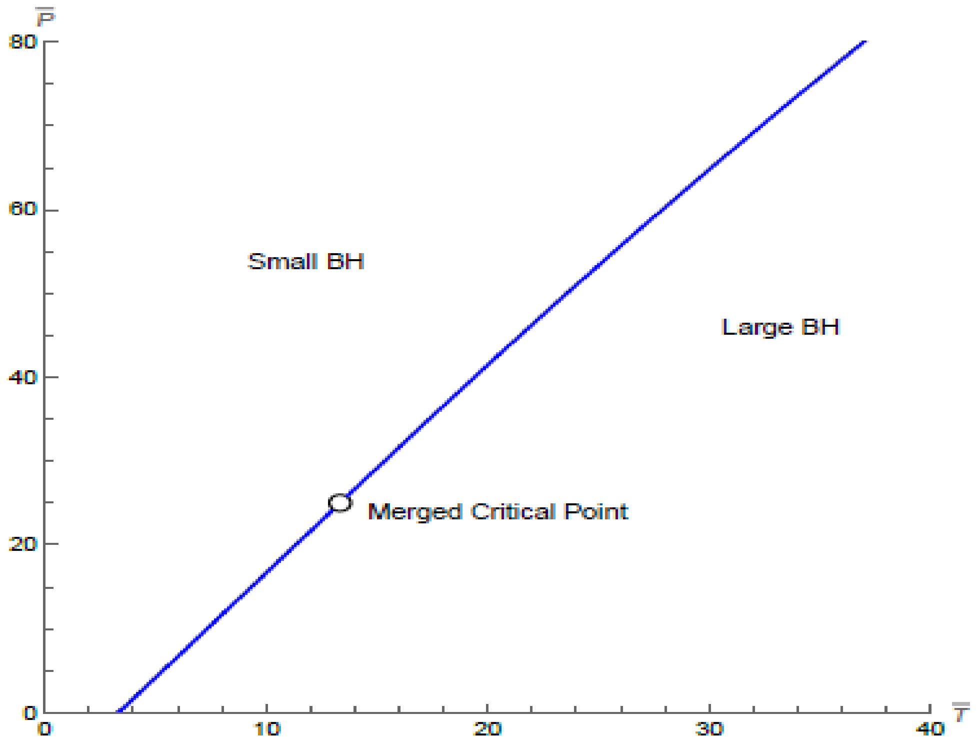
TN-Phases-Spherical Solution

- P-T diagram (Canon. Case):



$q_m = \frac{n}{\sqrt{3}}$





TN-Phases-Spherical Solution

- Critical Points (mixed Case, q_m=0):

$$\bar{P}_c^{(1,2)} = \left(\frac{1-\phi_e^2}{n^4}\right)\left(\frac{6n^2\phi_e^2}{\phi_e^2-1} + \frac{2\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1}\right) - \frac{\left(1-13\phi_e^2\right)}{n^2},$$

$$\bar{T}_c^{(1,2)} = \pm\frac{1}{3n^4}\Bigg[16\left(1-\phi_e^2\right)\left(-\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}\right)^{3/2} + n^2\left(8-104\phi_e^2\right)\sqrt{-\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}}\Bigg],$$

$$r_{0(c)}^{(1,2)} = \pm\sqrt{-\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}}.$$

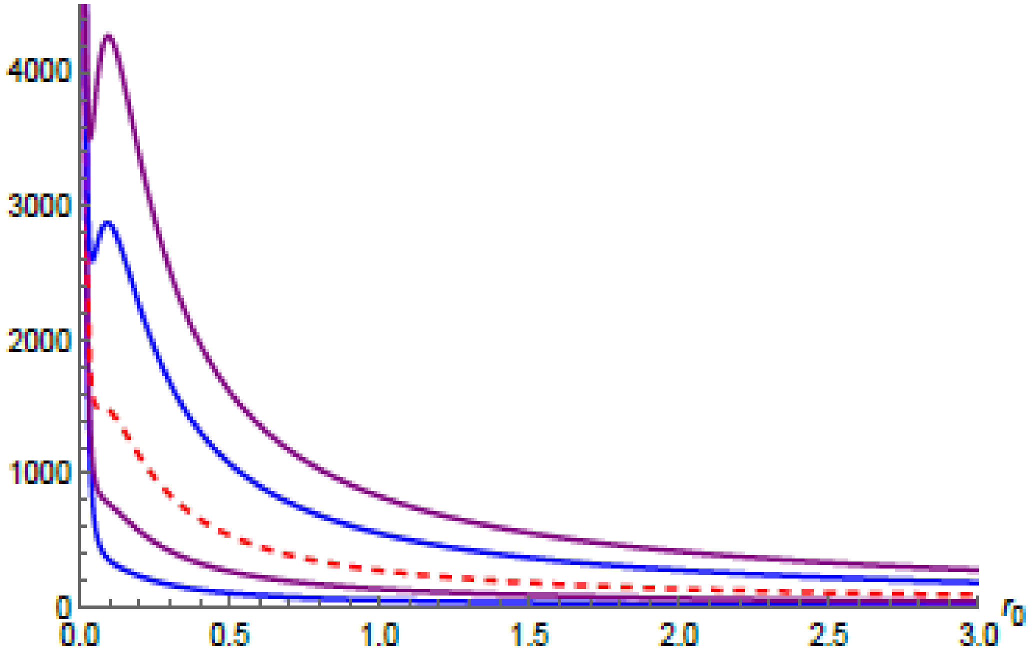
$$\bar{P}_c^{(3,4)} = \left(\frac{1-\phi_e^2}{n^4}\right)\left(\frac{6n^2\phi_e^2}{\phi_e^2-1} - \frac{2\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1}\right) - \frac{\left(1-13\phi_e^2\right)}{n^2},$$

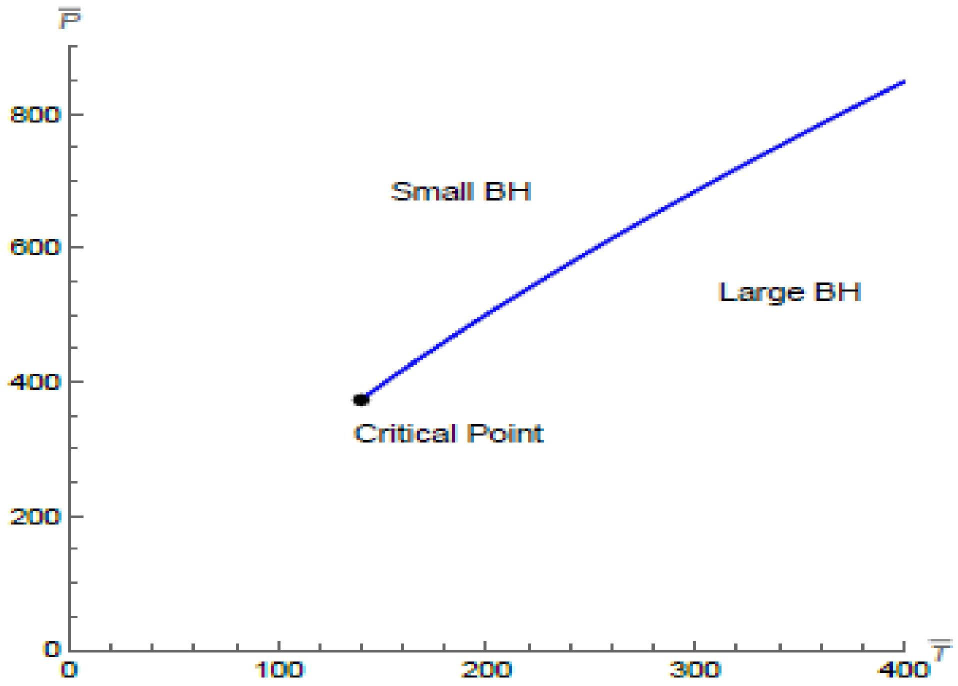
$$\bar{T}_c^{(3,4)} = \pm\frac{1}{3n^4}\Bigg[16\left(1-\phi_e^2\right)\left(\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}\right)^{3/2} + n^2\left(8-104\phi_e^2\right)\sqrt{\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}}\Bigg],$$

$$r_{0(c)}^{(3,4)} = \pm\sqrt{\frac{\sqrt{3}\sqrt{n^4\left(4\phi_e^4-\phi_e^2\right)}}{\phi_e^2-1} - \frac{3n^2\phi_e^2}{\phi_e^2-1}}.$$

TN-Phases-Spherical Solution

- We have similar results to the Canon. Case for mixed Case (q_m=0, 1/2 < ϕ_e<1):
- But for the mixed Case (q_m=0, $\phi_e \geq$1) we have one crit. point:





TN-Phases-Topological Solution*

- Metric:

$$ds^2 = -f(r)\left(dt + 2n(\frac{x^2}{2} + C_0)d\phi\right)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\left(dx^2 + x^2 d\phi^2\right),$$

$$ds^2 = -f(r)\left(dt + 2n(\cosh\eta + C_{-1})d\phi\right)^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)\left(d\eta^2 + \sinh^2\eta\, d\phi^2\right)$$

- Thermodynamics are consistent, i.e., 1^{st} law, Gibbs-Duhem and Smarr.

- We get critical points:

$$\frac{\partial P}{\partial v} = 0\,, \quad \frac{\partial^2 P}{\partial v^2} = 0\,,$$

- In these cases, we obtain only one critical point.

* Phys. Rev. D 109 (2024), 084026

TN-Phases-Topological Solution (k=0)

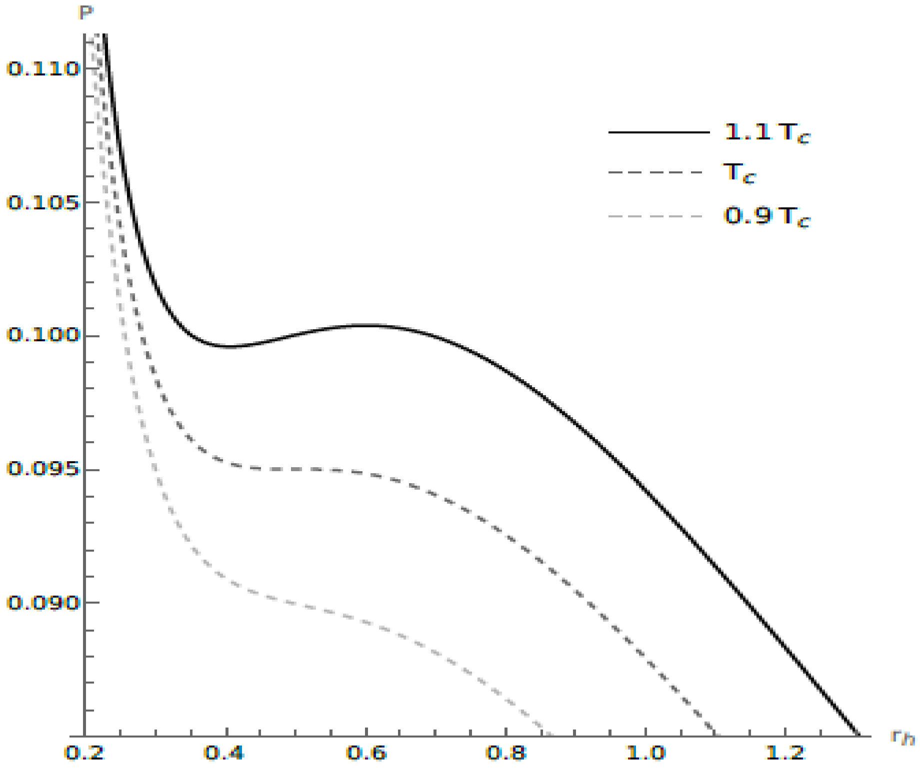


Figure 6: Pressure as a function of r_h as T changes around the critical point. T goes from $1.1T_c$ to T_c to $0.9T_c$. The critical radius occurs at $r_c = 0.496$. The thermodynamic parameters take the following values: $\Phi_e = 1.1$, $Q_m^h = 1.32$, $n = 1$, $T_c = 0.251$, and $k = 0$.

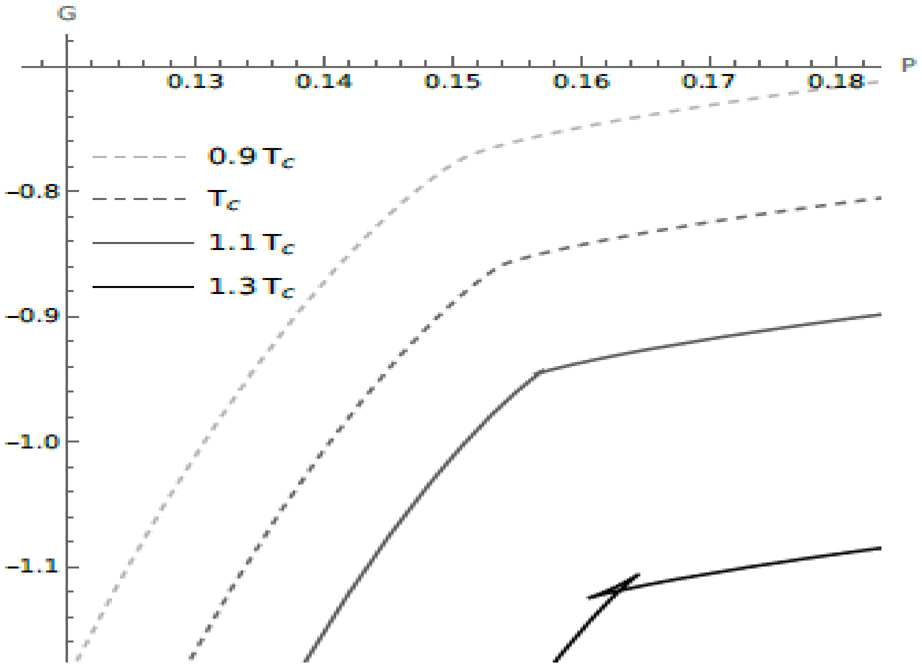


Figure 7: Gibbs energy vs pressure for $r \in [0.01, 0.3]$, where $\Phi_e = 2.00$, $Q_m^h = 2.30$, $n = 1.1$, $k = 0$, $T_c = 0.244$.

TN-Phases-Topological Solution (k=0)

- P-T diagram:

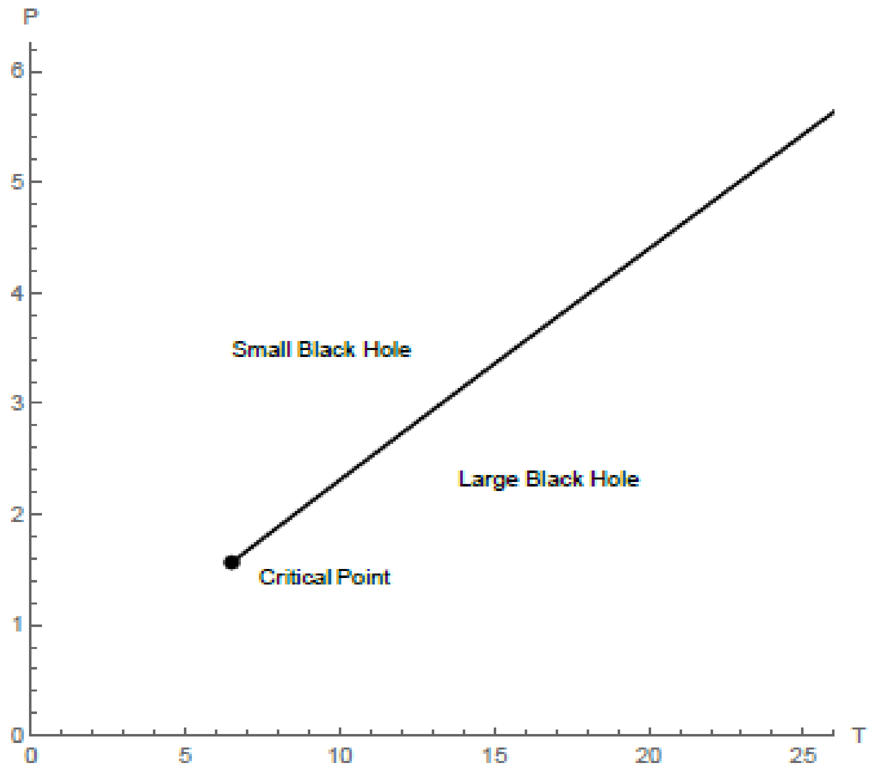


Figure 8: P-T phase diagram for $\Phi_e = 0.10$, $Q_m^h = 2.30$, $n = 1.1$, $k = 0$, $T_c = 6.49$, $P_c = 1.56$.

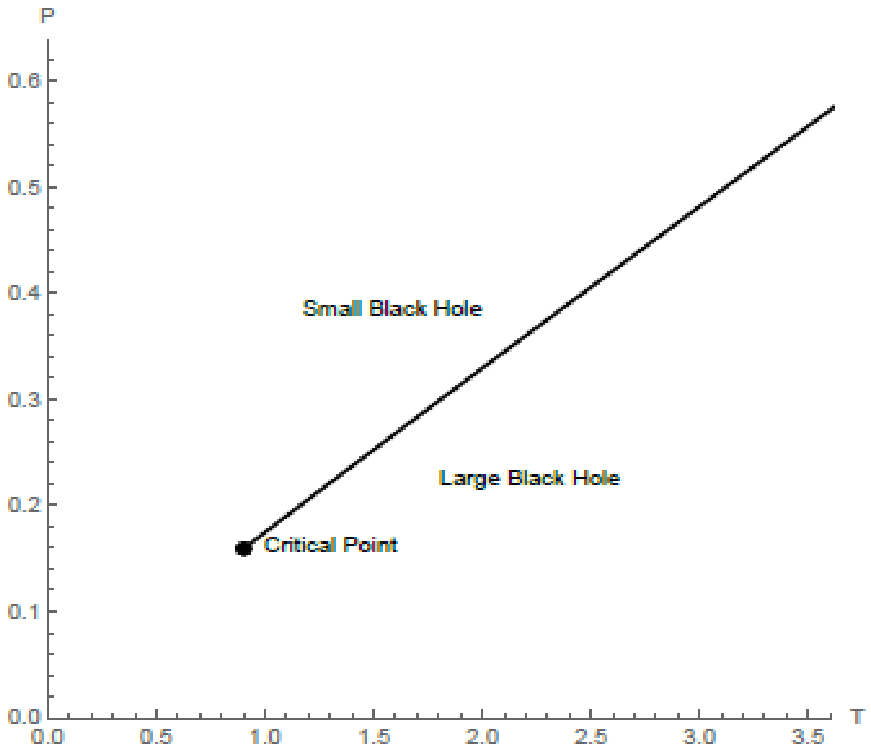


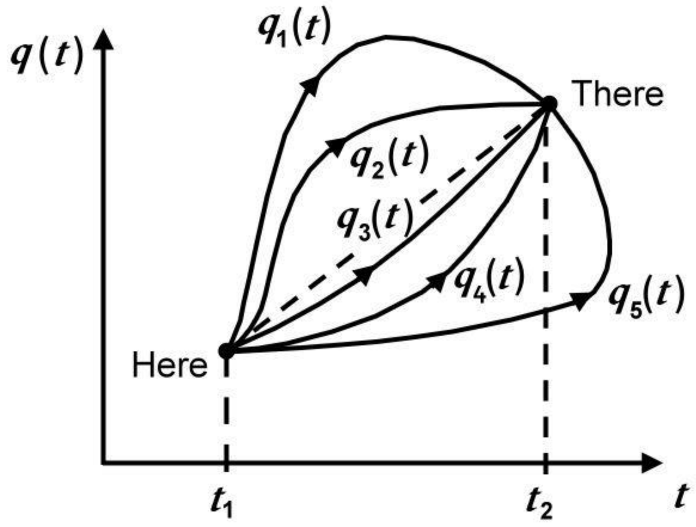
Figure 9: P-T phase diagram for $\Phi_e = 0$, $Q_m^h = 1.30$, $n = 1.50$, $k = 0$, $T_c = 0.902$, $P_c = 0.159$.

Conclusion

- *We have introduced a consistent thermodynamics for the Taub NUT which satisfy, 1st law, Smarr's and Gibbs-Duhem relations without imposing $\beta = 8\pi n$!*
- *Upon removing the above constraint, it is evident that the nut parameter n and its chemical potential ϕ_n plays an important role in fixing TN thermodynamics leading to an area law for entropy and a sensible limit for the temperature as $n \to 0$.*
- *In this thermodynamics we have small and large black hole phases as in charged AdS black holes but with additional critical point associated with n.*
- *For the flat dyonic case we obtain a first order phase transition for large enough temperatures and pressures ending at one critical point that depend on n.*
- *It would be interesting to understand what these phase-transition correspond to in the boundry field theory.*

Thank You

Euclidean Path Integral & Thermal Field Theory



- In QM PI is defined as

$$\langle q_f,t_f|\, q_i,t_i\rangle = \int D[q]\, e^{\,i\, I[q]/h} = \langle q_f|\, e^{-i\,H\,\Delta t}\,|q_i\rangle$$

$q(t_i) = q_i$ & $q(t_f) = q_f$.

- In QFT PI is given by

$$Z = \langle \phi_f,t_f|\, \phi_i,t_i\,\rangle = \int D[\phi]\, e^{\,i\, I[\phi]/h} = \langle \phi_f|\, e^{-i\,H\,\Delta t}\,|\phi_i\rangle$$

$\phi(t_i) = \phi_i$ & $\phi\,(t_f) = \phi_{f.}$

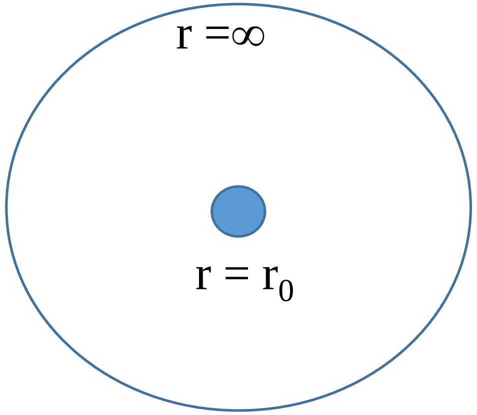
- QFT at finite temp., $\phi_i = \phi_f$ and taking $\Delta t \rightarrow -i\beta$.

$Z = \sum_\phi \langle\phi|e^{-\beta H}\,|\phi\rangle = \mathrm{Tr}\, e^{-\beta H}$. (in FT without gravity β is arbitrary!)

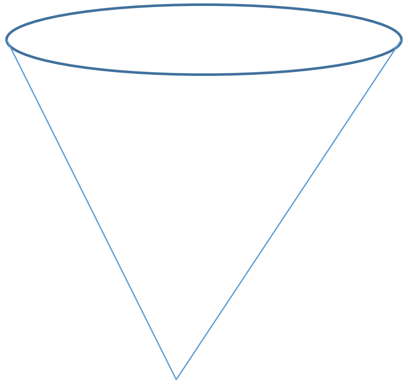
Euclidean Path Integral & Regularity Conditions

$Z = \sum_{(g,\phi)} \langle g,\phi|\, e^{-\beta H}\, |g,\phi\rangle = \mathrm{Tr}\, e^{-\beta H}$. (with gravity β is not arbitrary!)

- g and ϕ *must be regular!*



BH boundaries



2-dim conical space
(R, *θ*) *coordinates*

- *Expand Eucl. metric ($t \to i\tau$) around* $r = r_0$,
- $ds^2 = f(r)\, d\tau^2 + 1/f(r)\, dr^2 + \ldots$

Where, $f = f_0 + f'\,(r - r_0) + \ldots$

$ds^2 = f'\,(r-r_0)\, d\tau^2 + 1/(f'\,(r-r_0))\, dr^2 + \ldots$

$dR = (f'\,(r-r_0))^{-1/2}\, dr \to R = 2\,[\,(r-r_0)/f'\,]^{1/2}$

$ds^2 = R^2\, d(f'\tau/2)^2 + dR^2 + \ldots$

- *Compare with 2-dim conical space*

$ds^2 = R^2\, d\theta^2 + dR^2\,, \to \beta = [\tau] = 2[\theta]/f' = 4\pi/f'$

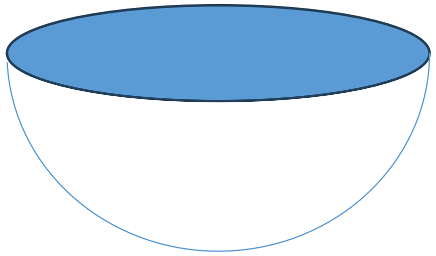
Euclidean Path Integral & Regularity Conditions

- Gauge pot. regularity conditions;
- A_{elec} =(-q/r+c) dt, as r → r_0, | A_{elec} |^2 =(-q/r+c)^2 /f (reg. only if) → c =q/r_0.
- A_{elec} =(-q/r+c) dt, as r → ∞, A_{elec} =c dt → c = ϕ, is fixed! (grand. Canon. Ens.) since $I=I(g_b, \phi)$.
- A_{mag} =(p cos(θ)+c') dϕ, as r → r_0 is regular.
- A_{mag} =(p cos(θ)+c') dϕ , as r → ∞, p is fixed! (canon. Ens.). I=I(g_b,p).
- A_{mag} =(p cos(θ)+c') dϕ , at , θ=0, π, | A_{mag} |^2 =(p cos(θ)+c')^2 /sin(θ)^2, (reg. only if) → c' = $\pm$ p.
- Work term in 1st law, ϕ dQ = ϕ d(eN) = μdN

Euclidean Path Integral & Phase-Transitions

- Our approach requires that we specify the boundary values of g_b and A_b to construct thermodynamics.

$$I=I(g_b,A_b)=I(\beta,p,\phi)$$

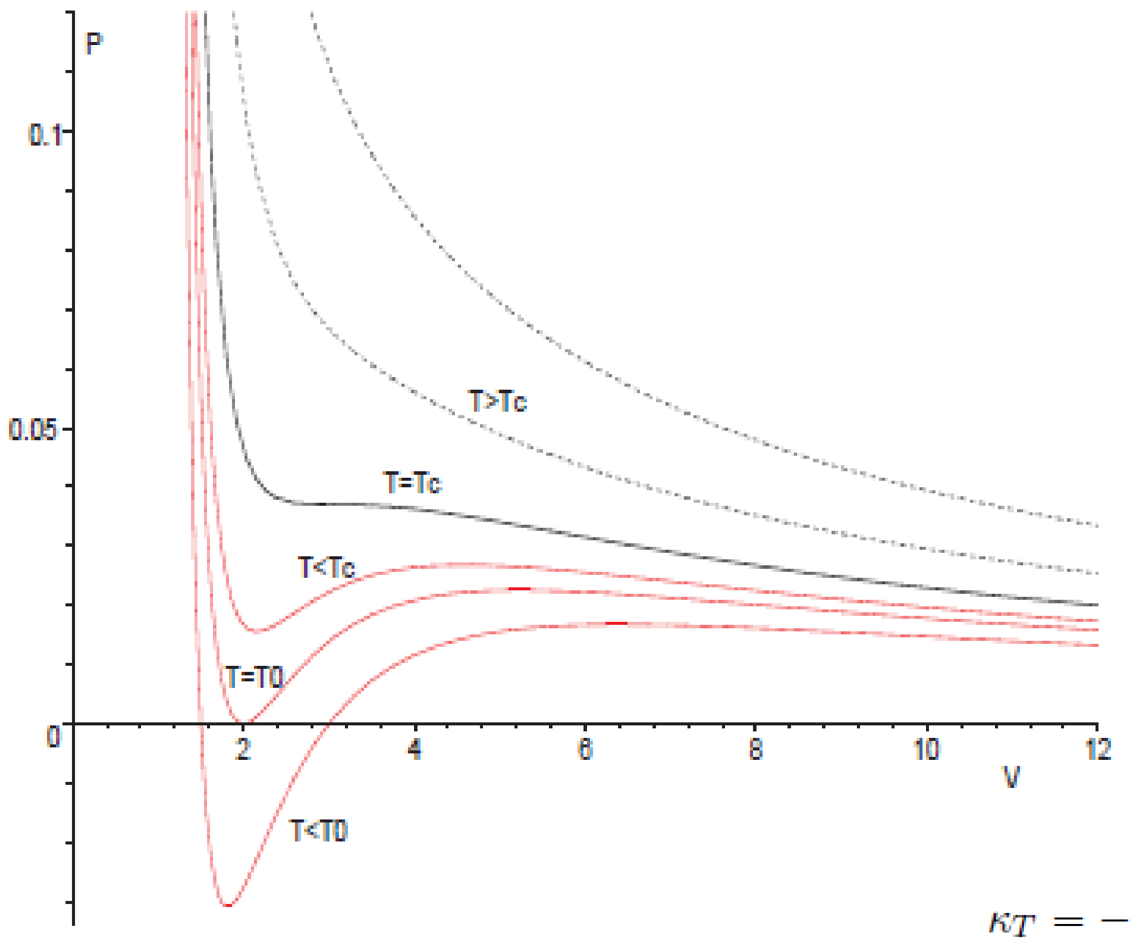


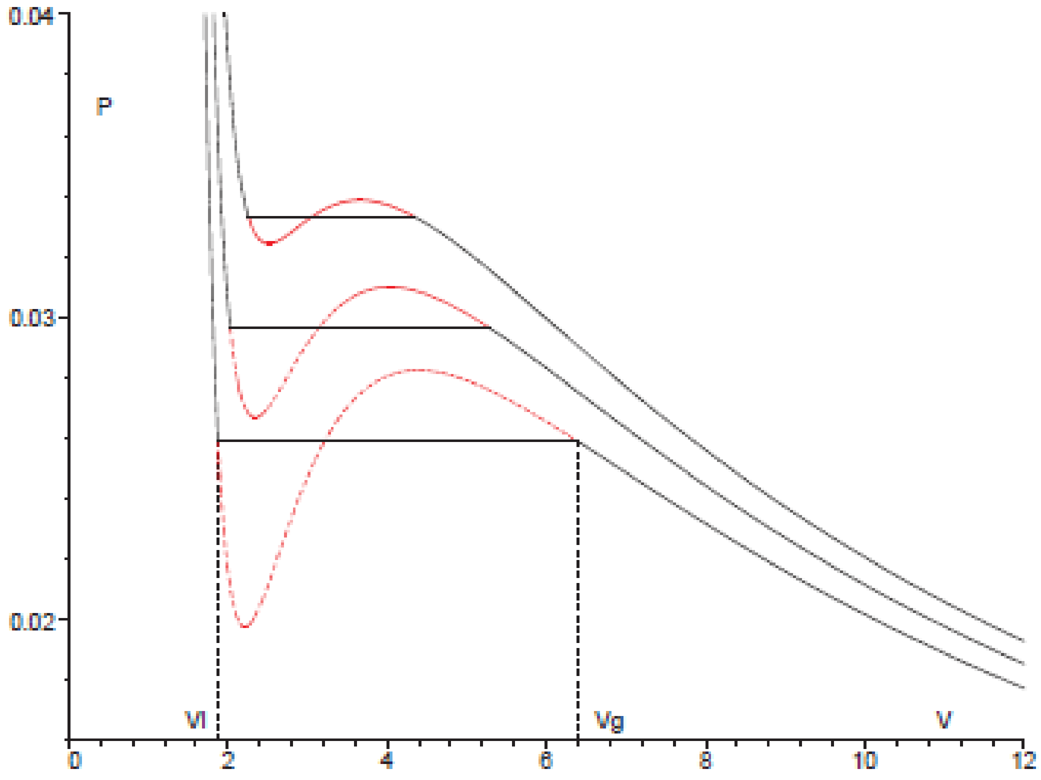
- But what if we have more than one solution that have the same boundary values.

$$Z= \exp(-I_1) + \exp(-I_2) = = \exp(-\beta F_1) + \exp(-\beta F_2)$$

- We might have certain range of temp. in which $F_1 < F_2$, then this could be reversed in another range, $F_2 < F_1$. In this case we have phase transition!
- The temp. at which $F_1 = F_2$, is the transition temp.!

Fluid' Behavior: P-V diagram





$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T$$

Behavior around Critical Points:

- Critical behavior is characterized by power law behavior for certain quantities (critical exponents):
- Specific heat,

$$C_v = T\frac{\partial S}{\partial T}\Big|_v \propto |t|^{-\alpha} . \qquad \alpha = 0.$$

- Ordered parameter,

$$\eta = v_g - v_l \propto |t|^{\beta} . \qquad \beta = 1/2 .$$

- Compressibility,

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T \propto |t|^{-\gamma} . \qquad \gamma = 1 .$$

- Pressure at $T=T_c$,

$$|P - P_c| \propto |v - v_c|^{\delta} . \qquad \delta = 3 .$$

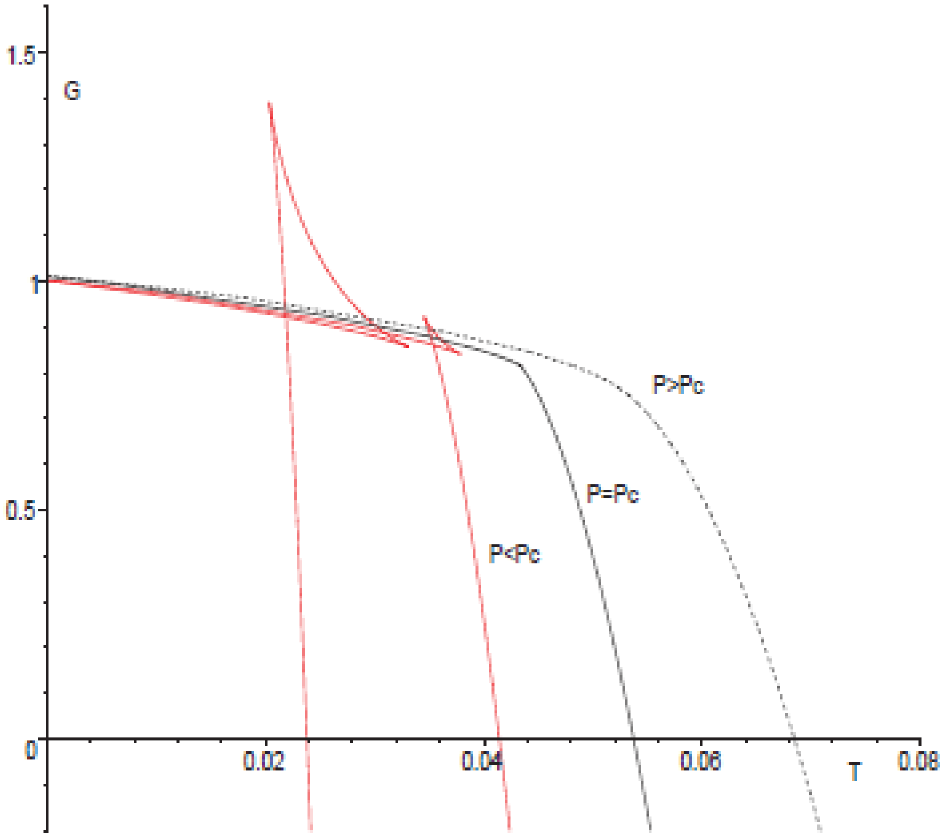
Critical Behavior of Ch-AdS:

$$C_v = T\frac{\partial S}{\partial T}\Big|_v \propto |t|^{-\alpha}. \qquad \alpha = 0.$$

$$\eta = v_g - v_l \propto |t|^{\beta}. \qquad \beta = 1/2.$$

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T \propto |t|^{-\gamma}. \qquad \gamma = 1.$$

$$|P - P_c| \propto |v - v_c|^{\delta}. \qquad \delta = 3.$$



Gibbs Energy:

- Gibbs energy is

$G = \mu_1 N_1 + \mu_2 N_2, \quad dN_1 = -dN_2$

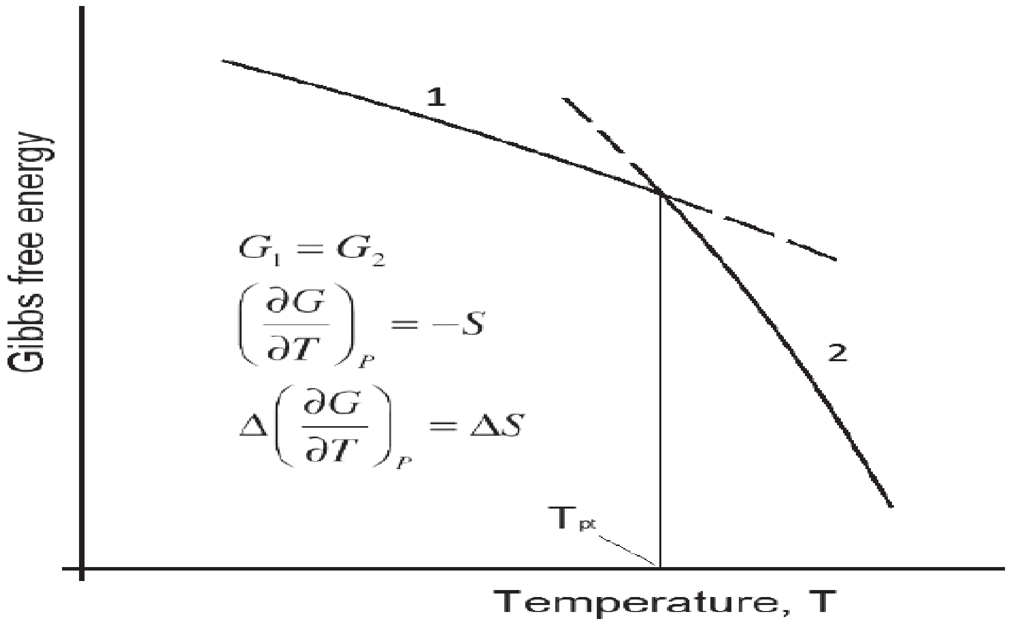
$dG = (\mu_1 - \mu_2)\, dN_1$

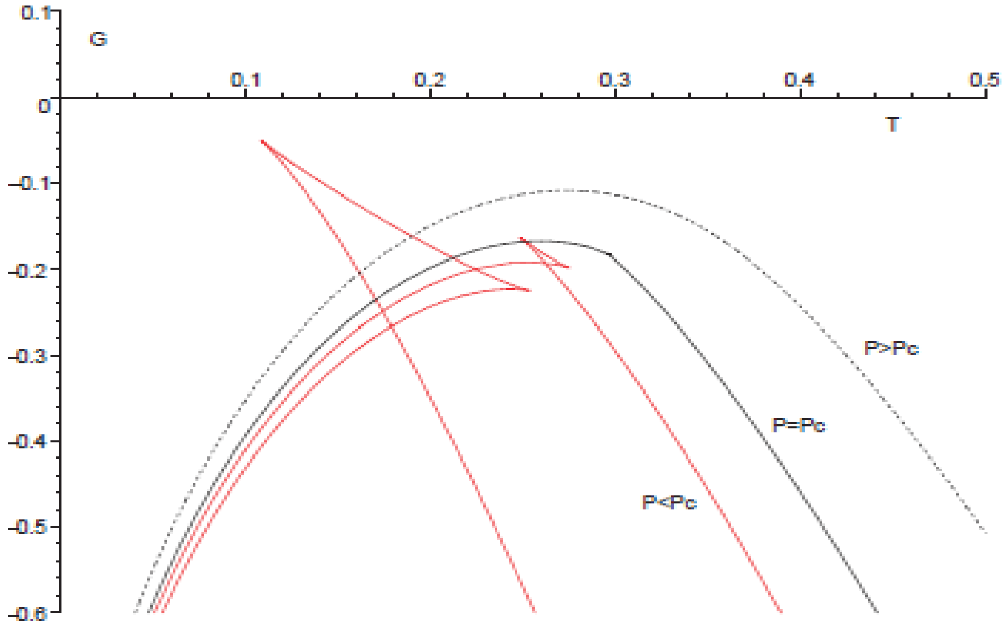
At transition:

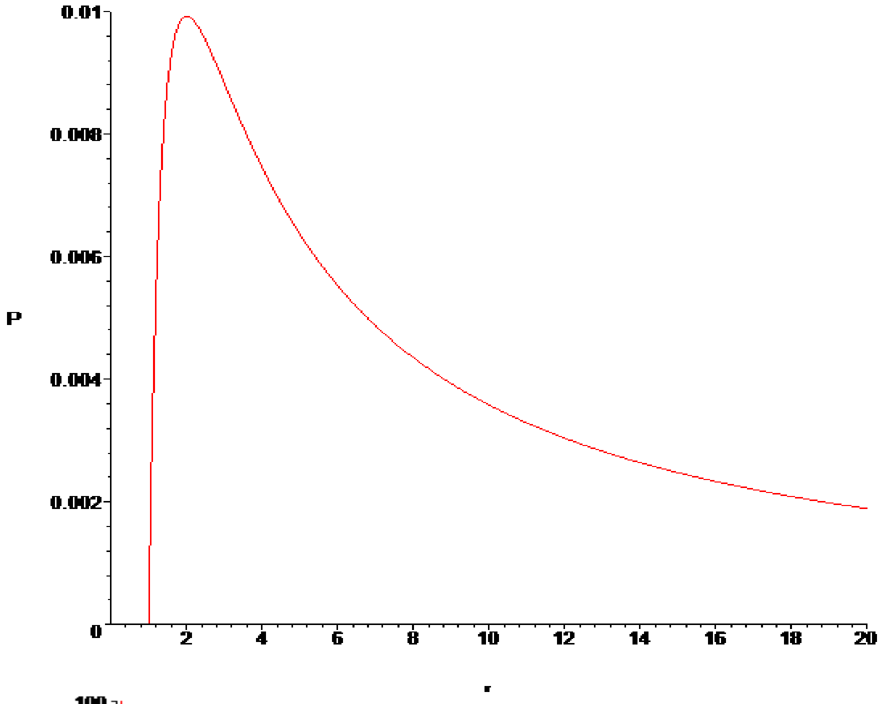
$T_1 = T_2,\ P_1 = P_2,\ \text{and } G_1 = G_2,$

$\mu_1 = \mu_2$

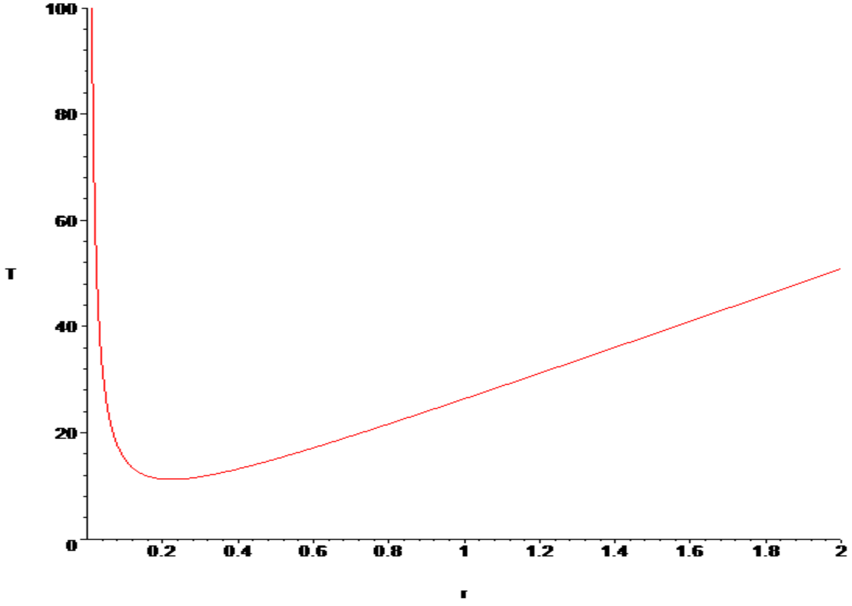
- *For VdW fluids we get* *swallowtail behaviors*



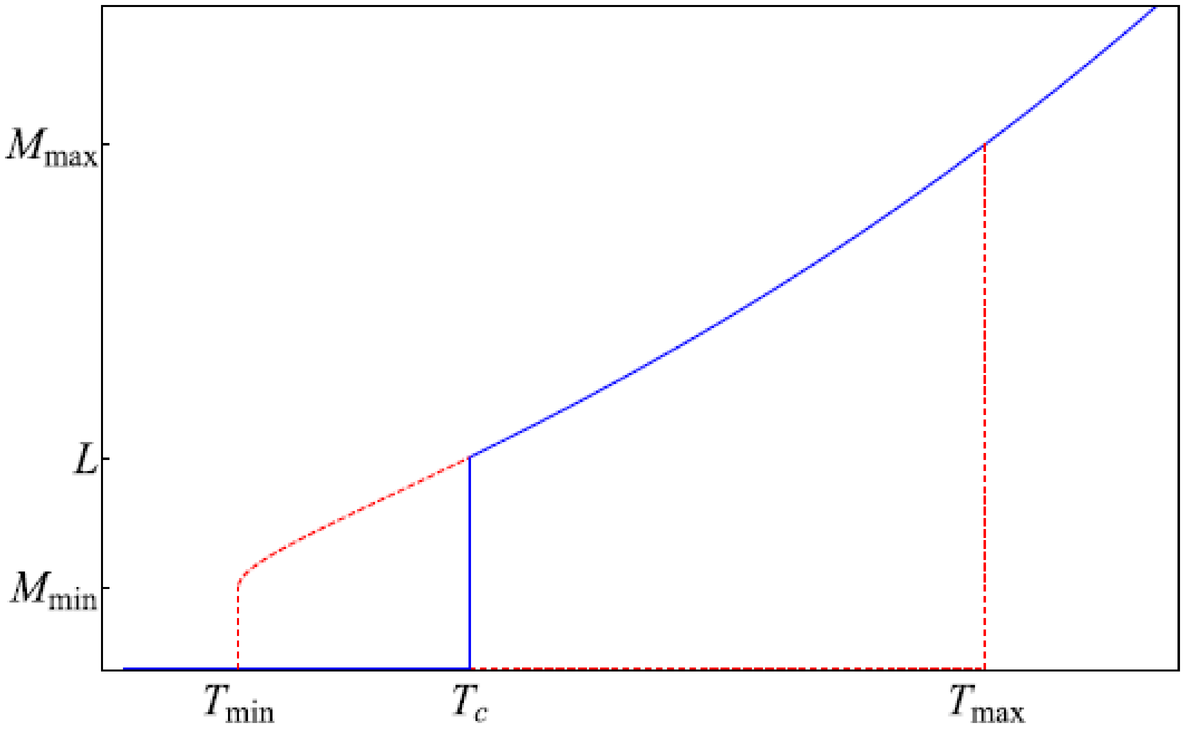




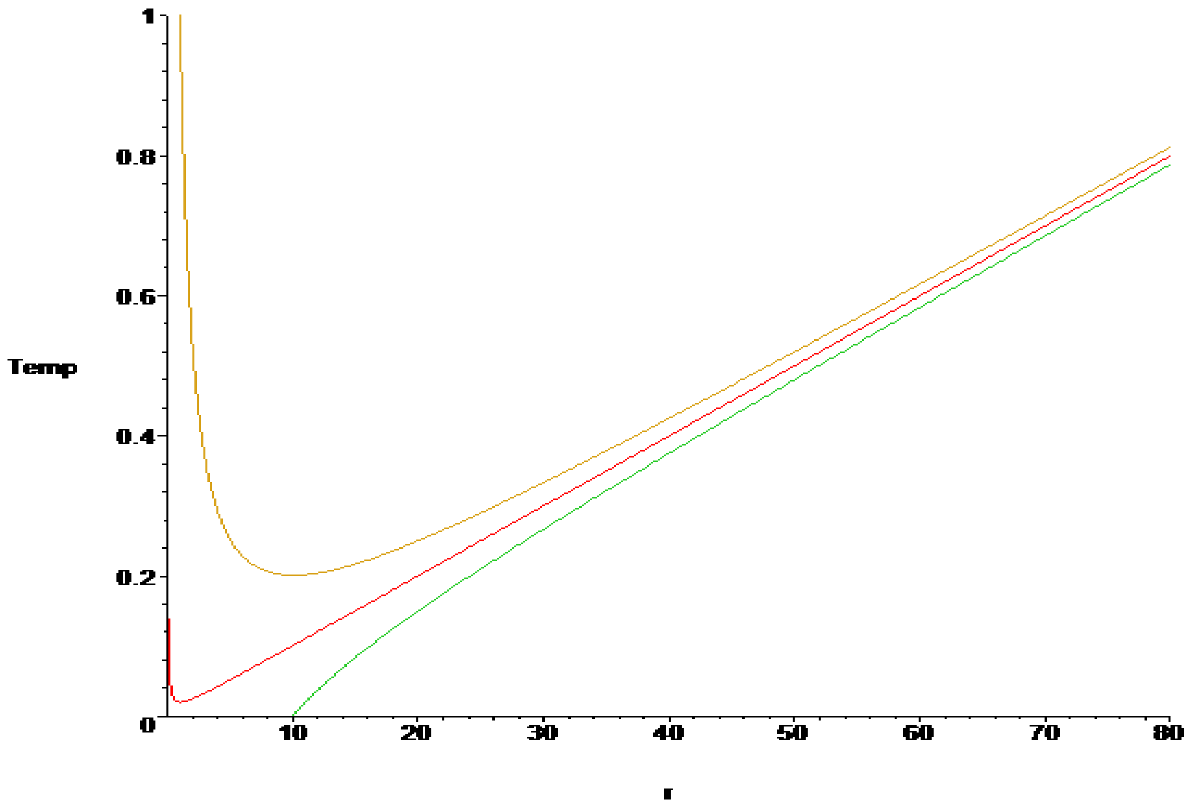
0.01
0.008
0.006
P
0.004
0.002
0
2
4
6
8
10
12
14
16
18
20
r

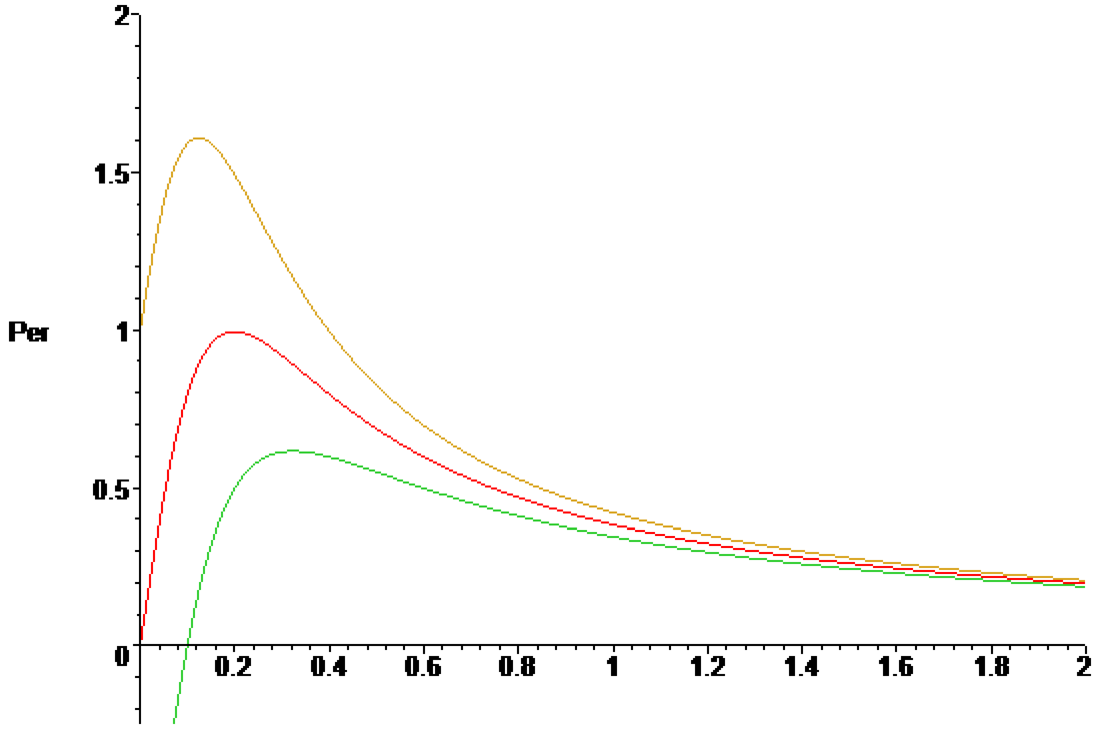


100
80
60
T
40
20
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
r



$M_{\max}$
L
$M_{\min}$
$T_{\min}$
T_c
$T_{\max}$


Temp
1
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
r


Per
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

A Brief History of Taub-NUT

- Taub discovered it in 1950's, then Newman, Unti, and Tamburinoin 60's wrote it in its present form.
- It was known in the 60's that it's a gravitational dyon. This lead Misner to treat its string-like singularity similar to Dirac' String(i.e., it is a gauge artifact!), imposing $\beta = 8\pi n$.
- Bonner in late 60's thought of the string as a source for angular momentum and didn't impose $\beta = 8\pi n$.
- With $\beta = 8\pi n$ *Lorentzian TN contains closed-time like curves.*

Taub-NUT Metric

- *We have following perturbations away from η_{ab} give;*

$$h_{00} = \frac{2m}{r} = -2\phi_N \qquad h_{0\phi} = 2n\cos\theta = A_\phi \text{ (monopole-like poten.)}.$$

- *Test mass eqns. give*

$$m_t \frac{d^2x^i}{dt^2} = -\, m_t \left[\partial_i \phi_N + \vec{v} \times (\vec{\nabla} \times \vec{A})\right].$$

- *The action of this test particle reads*

$$S_p \sim \int dt \left(\frac{1}{2} m_t\, v^2 - m_t \phi_N + m_t\, \vec{A} \cdot \vec{v}\right).$$

- *This is analogous to the action of a test charge "q" under the Lorentz force or a dyon!*

$$S_q \sim \int dt \left(\frac{1}{2} m\, v^2 - q\phi_e + q\vec{A}_e \cdot \vec{v}\right).$$

ϕ_e=Q/r

P $\cos\theta\ \hat{e}_\phi$

Misner String

- *TN metric has a string-like singularity along z-axis. To see that when C=0,*

let us consider the norm vector $\nabla_a t$,

$$|\nabla t|^2 = \frac{(2n\cos\theta + C)^2}{(r^2+n^2)\,(\sin\theta)^2} - \frac{1}{f(r)}.$$

- *This shows singular behaviors at f(r)=0 (horizon), and at* $\theta=0$, *and* π, *which can not be both removed by choosing C.*
- *If we try to remove divergence at* $\theta=0$, *by setting C=-2n,*

$$ds^2{}_{(+)} = -f(dt^+ + 2n\,(\cos\theta - 1)\,d\phi)^2 + f^{-1}dr^2 + (r^2+n^2)(d\theta^2 + \sin\theta^2\,d\phi^2)$$

- *Also, for doing the same at* $\theta=\pi$.

$$ds^2{}_{(-)} = -f(dt^- + 2n\,(\cos\theta + 1)\,d\phi)^2 + f^{-1}dr^2 + (r^2+n^2)(d\theta^2 + \sin\theta^2\,d\phi^2)$$

- *These two patches can cover the manifold smoothly only if*

$$\Delta t^{\pm} = 4n\Delta\phi \quad \Rightarrow \quad \beta = 8\pi n$$

Path Integral & Regularity Conditions*

- *For elec. solution, fixing A at boundary, $r\to\infty$, fixes ϕ_e (electric pot.)*

  $A_{ele} = \left(-\frac{q}{r} + C'\right) dt \,|_B = C' dt$, *or* C' *is fixed. This is a grand Canonical Ensemble.*

- *Reg. of* $\boldsymbol{A}$ *at the horizon* → $|A_{ele}|^2 = \frac{\left(C' - \frac{q}{r}\right)^2}{f(r)}\Big|_{r=r0}$ → $C' = q/r_0 = \phi_e$
- *In this case, $I_{on} = \beta G$, where G is the Gibbs potential.*
- *For magnetic solution, fixing,* $A_{mag} = (p \cos\theta + C'')d\phi$, *as* $r\to\infty$, *fixes, "p" i.e., this is a Canonical Ensemble.*
- *Also,* $\left|A_{mag}\right|^2$ *has to be regular at* r_0, *this leads to*

$$\left|A_{mag}\right|^2 = \frac{(p \cos\theta + C'')^2}{(\sin\theta)^2} \to C'' = \pm p.$$

- *In this case, $I_{on} = \beta F$, where F is Helmholtz free energy.*

* Phys. Rev. D 109 (2024), 084026; Phys. Rev. D 108 (2023), 064022.

Van der Waals Fluids in a Nutshell

- Equation of State of real gas:

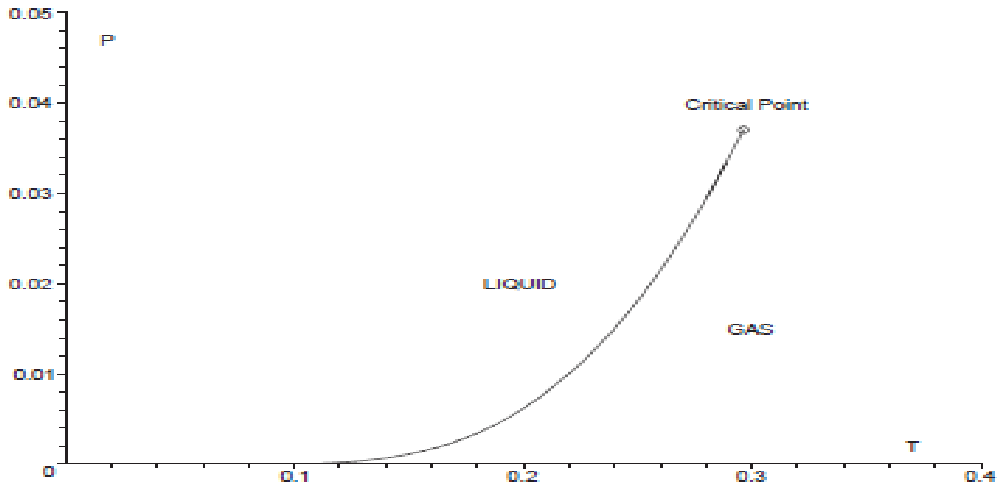
$$\left(P + \frac{a}{v^2}\right)(v - b) = kT\,.$$

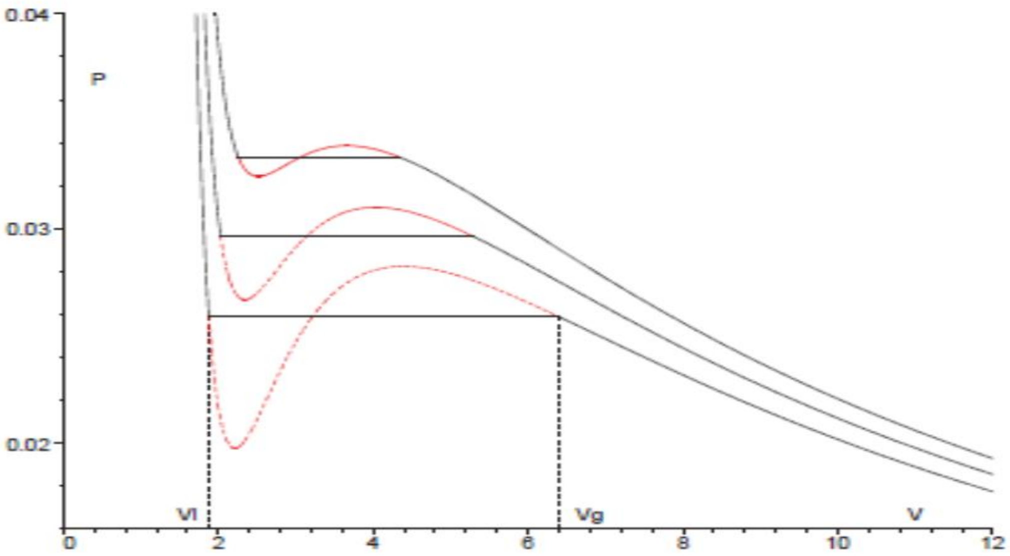
- Critical points are solutions of

$$\frac{\partial P}{\partial v} = 0\,, \qquad \frac{\partial^2 P}{\partial v^2} = 0\,,$$

$\Longrightarrow$ one critical point;

$$kT_c = \frac{8a}{27b}\,, \quad v_c = 3b\,, \quad P_c = \frac{a}{27b^2}\,.$$





Charged-AdS solution*

- Solution:

$$ds^2 = -V\,dt^2 + \frac{dr^2}{V} + r^2 d\Omega_2^2\,, \qquad V = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}\,.$$

$$F = dA\,, \quad A = -\frac{Q}{r}dt\,.$$

- On-shell gravitational action:

$$I_{EM} = \frac{\beta}{4l^2}\left(l^2 r_+ - r_+^3 + \frac{3l^2Q^2}{r_+}\right)$$

- In the original work of Chamblin-Emparan-Johnson-Myers, they found an analogue betn. Charge AdS BH and Van der Waals fluid, where,

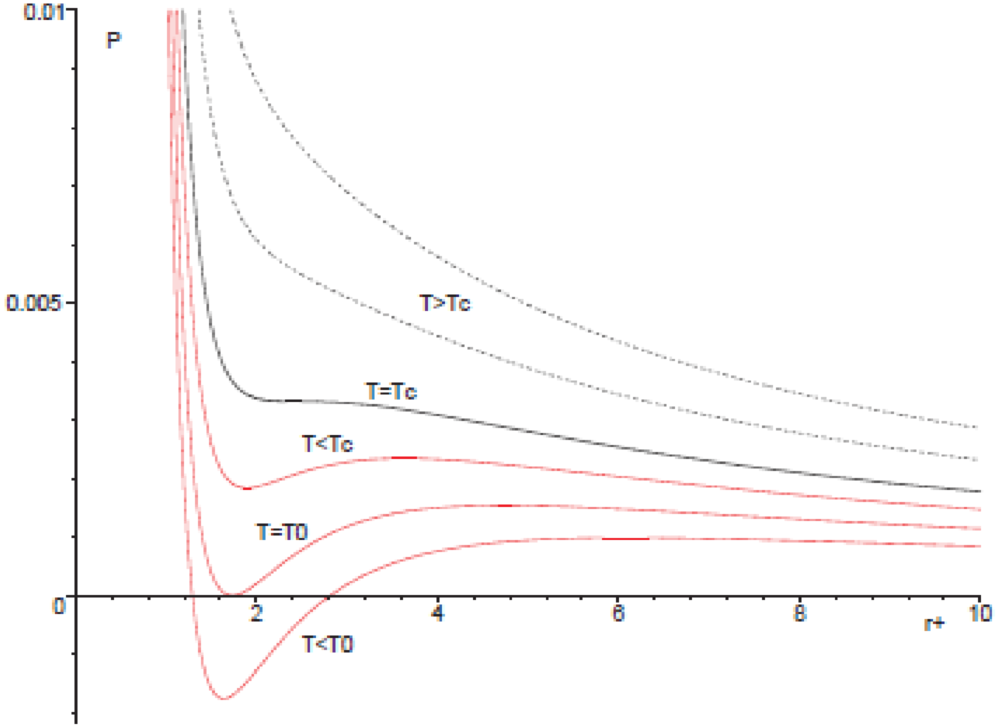
Temperature	$\Longleftrightarrow$	Q
Pressure	$\Longleftrightarrow$	β
volume	$\Longleftrightarrow$	r_+

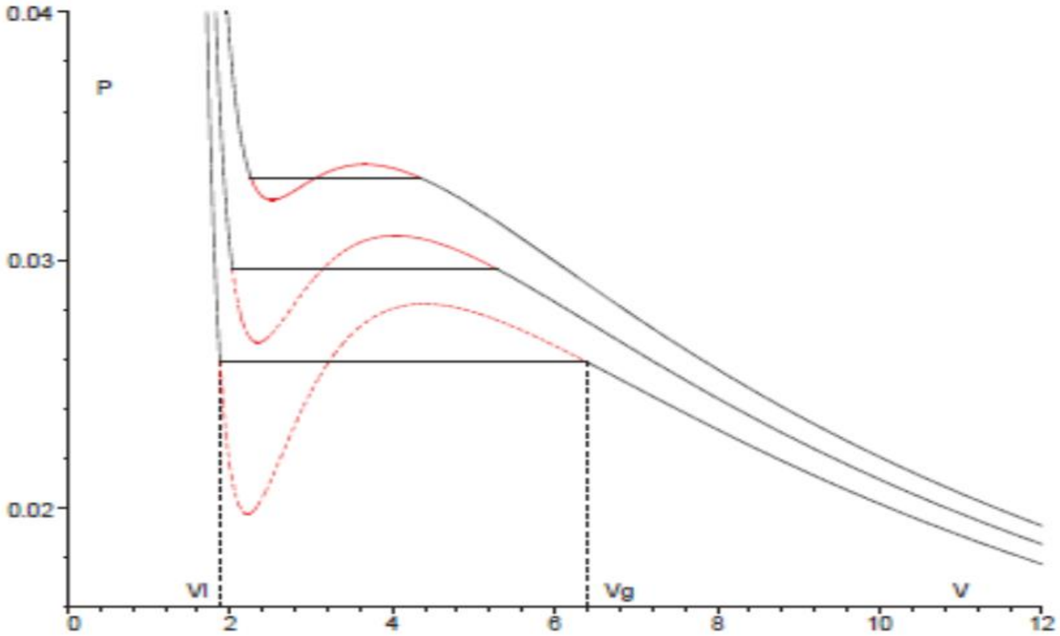
Charged-AdS solution

- In extended thermodynamics:

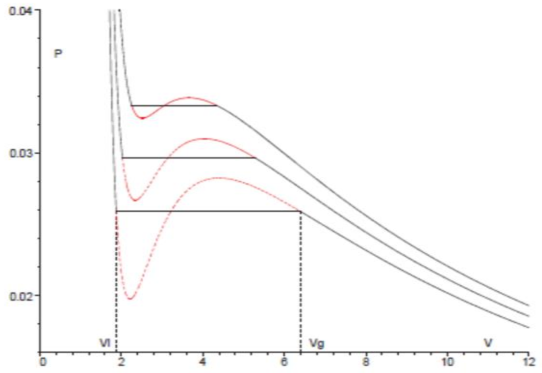
$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}\,, \qquad v = 2l_P^2 r_+\,.$$

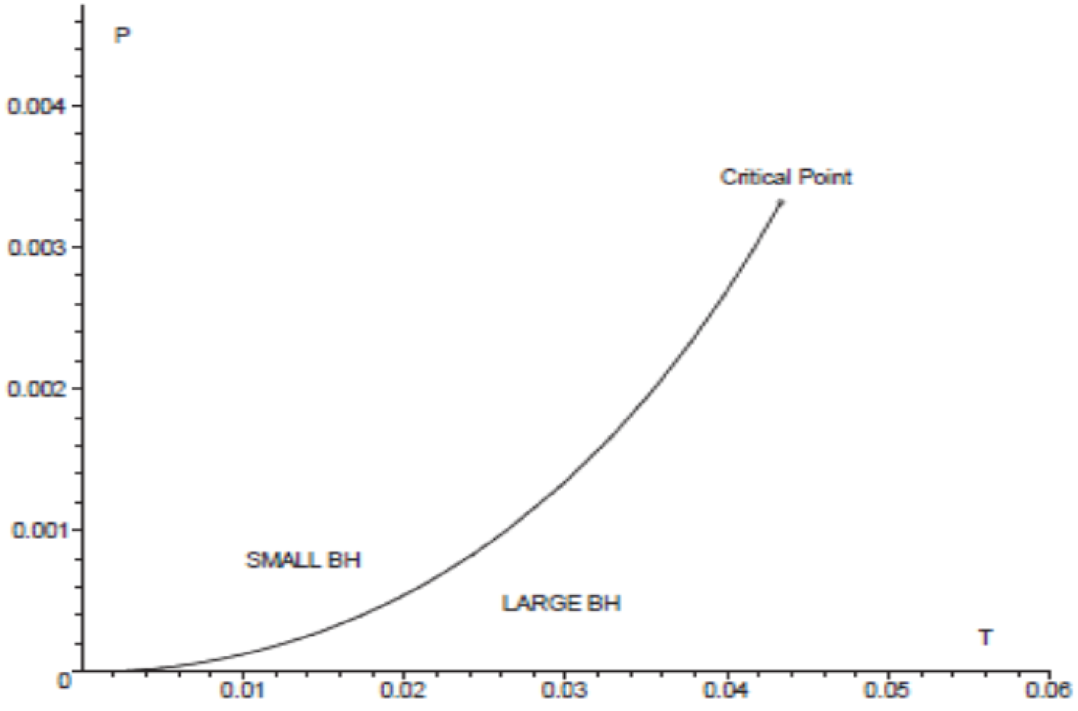
*



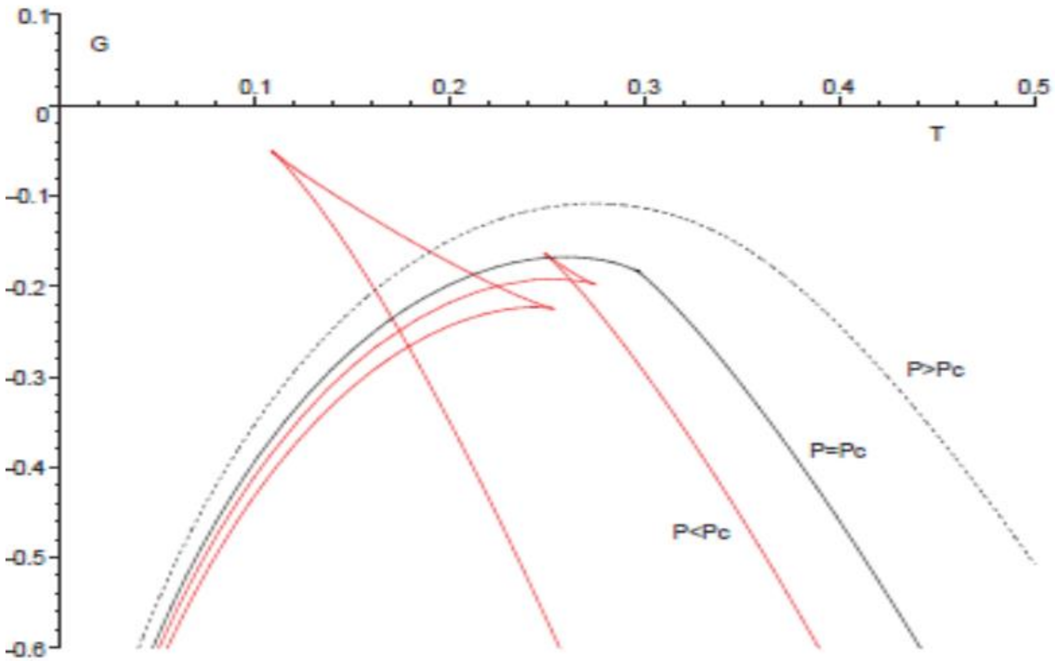


Charged-AdS solution





P-T Daigram



Free Energy Swallow tail behavior

Surfaces and Fluxes*

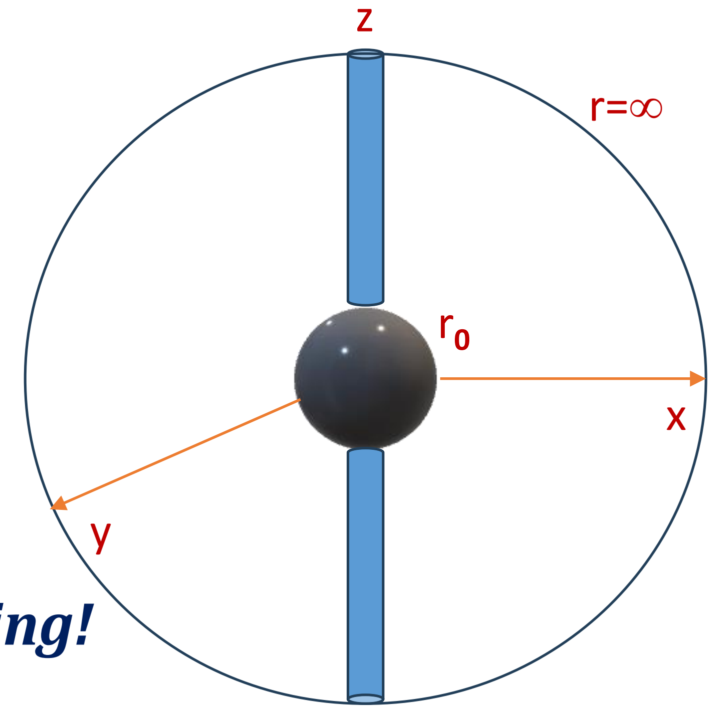
- *We calculate charges fluxes for C=0.*

$$M = -\frac{1}{4\pi}\int_{S^2} * \, d\chi = -\frac{1}{4\pi}\int \left(\frac{4n^2 f}{r^2+n^2}\cos\theta dr\wedge d\phi + f'\,(r^2+n^2)\sin\theta\, d\theta\wedge d\phi\right)$$

- *Mass: $M^{\infty}=m$, $M^{+}= M^{-}=-n\phi_n$, $M^{h}=m -2n\phi_n$.*

$\phi_n = -n/2r_0$

- *xx*
- *Nut Charge: $N^{\infty}=n$, $N^{+}= N^{-}=n/2$, $N^{h}=0$.*
- *Ang. Mom.: $J^{\infty}=0$, $J^{+}= nm$, $J^{-}=-nm$, $J^{h}=0$.*
- *Elec. Charge: $Q_e^{\infty}=q$, $Q_e^{+}= Q_e^{-}=-n\phi_m$, $Q_e^{h}=q -2n\phi_m$.*
- *Magn. Charge: $Q_m^{\infty}=p$, $Q_m^{+}= Q_e^{-}=-n\phi_e$, $Q_e^{h}=p+2n\phi_e$.*
- ***There are mass and charge distributions along Misner String!***



* Phys. Rev. D 109 (2024), 084026; Phys. Rev. D 108 (2023), 064022.