Properties of generalized Schwarzschild spacetimes with extra dimensions

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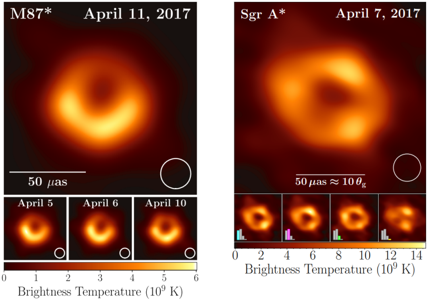
  - ADM mass

Black holes

Event Horizon Telescope Collaboration: *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. Lett. **875**, 17 (2019).

Event Horizon Telescope Collaboration: *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, Astrophys. J. Lett. **930**, 21 (2022).



Extra dimensions

Kaluza–Klein theory

Th. Kaluza: *On the Unification Problem in Physics*, Int. J. Mod. Phys. D **27**, No. 14 (2018) 1870001 (translation); Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966-972 (original).

L. Randall, R. Sundrum: *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83**, 3370-3373 (1999).

string theory

E. Witten: *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. B **471**, 135-158 (1996).

Vacuum solution

ansatz with n extra dimensions $\zeta^A = \zeta^1, ..., \zeta^n$:

$$ds^2 = -f(r)^\alpha dt^2 + f(r)^\beta dr^2 + r^2 d\Omega^2_{(2)} + f(r)^\gamma \delta_{AB} \underbrace{d\zeta^A d\zeta^B}_{\text{extra}}$$

Ricci tensor $R_{\mu\nu}$:

$$R_{00} = \alpha f^{\alpha-\beta}\ F_1 \quad R_{rr} = \ F_2 \quad R_{\vartheta\vartheta} = \ F_3$$

$$R_{\varphi\varphi} = \ F_3\ \sin^2\vartheta \quad R_{AB} = -\gamma f^{\gamma-\beta}\ F_1\ \delta_{AB}$$

$$F_1 = \frac{1}{r}\frac{f'}{f} + \frac{1}{4}(\alpha - \beta + n\gamma - 2)\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\frac{f''}{f}$$

$$F_2 = \beta\frac{1}{r}\frac{f'}{f} + \frac{1}{4}\left[\alpha(-\alpha + \beta + 2) + n\gamma(\beta - \gamma + 2)\right]\left(\frac{f'}{f}\right)^2 - \frac{1}{2}(\alpha + n\gamma)\frac{f''}{f}$$

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}(\alpha - \beta + n\gamma)\, r\frac{f'}{f}\right]$$

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Vacuum solution

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}(\alpha - \beta + n\gamma)\, r\frac{f'}{f}\right] = 0$$

$$\Downarrow$$

$$\boxed{f = (1 + ar^q)^{-1/\beta}} \qquad q = \frac{2\beta}{\alpha - \beta + n\gamma}$$

exception: $\alpha - \beta + n\gamma = 0 \Rightarrow f = 1 \Rightarrow$ Minkowski spacetime

the rest is then

$$F_1 = C_1\,\Phi \qquad F_2 = (C_2 + C_3\, ar^q)\Phi \quad \text{where:}$$

$$\Phi = \frac{1}{(\alpha - \beta + n\gamma)^2}\frac{ar^{q-2}}{(1 + ar^q)^2}$$

$$C_1 = -\alpha - \beta - n\gamma$$

$$C_2 = 2\beta^2 + \beta(\alpha + n\gamma) - (\alpha + n\gamma)^2$$

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Vacuum solution(s)

algebraic equations

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C_1 &= -\alpha - \beta - n\gamma \\
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can be solved by $\alpha = -\beta - n\gamma \Rightarrow C_1 = 0,\ C_2 = 0 \Rightarrow$

$$C_3 = -n\gamma\ \left[2\beta + (n+1)\gamma\right]$$

possible solutions are:

- $\gamma = 0 \to \alpha = -\beta$:

$f^\alpha = 1 + \frac{a}{r} \quad f^\beta = \left(1 + \frac{a}{r}\right)^{-1} \quad f^\gamma = 1$

trivial extension of the Schwarzschild spacetime

- $2\beta + (n+1)\gamma = 0 \to \alpha/\beta = (n-1)/(n+1),\ \gamma/\beta = -2/(n+1)$:

$f^\alpha = \left(1 + \frac{a}{r}\right)^{-\frac{n-1}{n+1}} \quad f^\beta = \left(1 + \frac{a}{r}\right)^{-1} \quad f^\gamma = \left(1 + \frac{a}{r}\right)^{\frac{2}{n+1}}$

nontrivial extension

Vacuum solution(s)

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nontrivial extension

Vacuum solutions

trivial extension

$$ds^2 = - \left(1+\frac{a}{r}\right) dt^2 + \left(1+\frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \\ + \delta_{AB}\, d\zeta^A d\zeta^B$$

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existence of horizon for $a < 0$ at $r = |a|$!!!

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existence of horizon for $a < 0$ at $r = |a|$!!!

Generalized Weyl solutions

ansatz

$$ds^2 = -e^{2U_1(r,z)}dt^2 + \sum_{a=2}^{D-2} e^{2U_a(r,z)}d\phi_a^2 + e^{2\nu(r,z)}(dr^2 + dz^2),$$

vacuum solutions satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) U_a = 0, \qquad \sum_{a=1}^{D-2} U_a = \ln r,$$

$\Rightarrow$ Laplace's equation with rod source $\longrightarrow$ "rod diagrams"

$$\frac{\partial \nu}{\partial r} = -\frac{1}{2r} + \frac{r}{2}\sum_{a=1}^{D-2}\left[\left(\frac{\partial U_a}{\partial r}\right)^2 - \left(\frac{\partial U_a}{\partial z}\right)^2\right], \quad \frac{\partial \nu}{\partial z} = r\sum_{a=1}^{D-2}\frac{\partial U_a}{\partial r}\frac{\partial U_a}{\partial z}.$$

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⇒ **Laplace's equation with rod source ⟶ "rod diagrams"**

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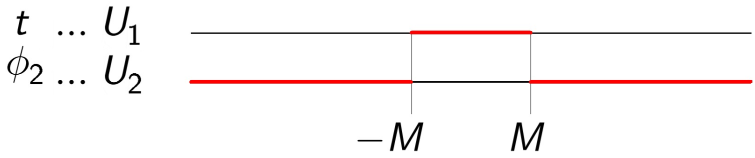
Generalized Weyl solutions

$D = 4$ Minkowski spacetime

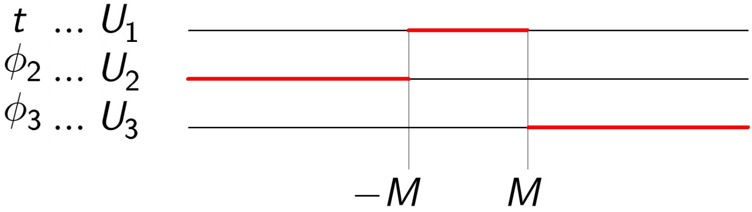
Generalized Weyl solutions

$D = 4$ Schwarzschild spacetime



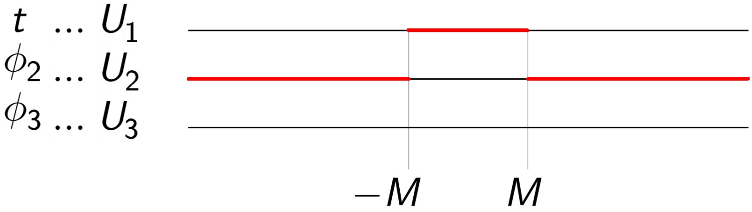
Generalized Weyl solutions

$D = 5$ Schwarzschild–Tangherlini spacetime



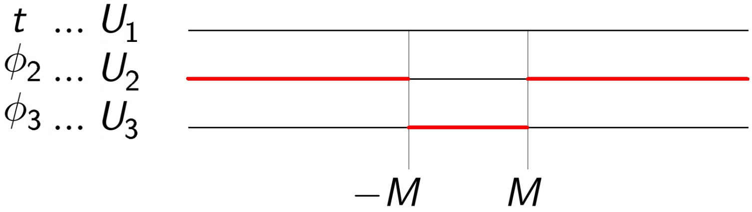
Generalized Weyl solutions

$D = 5$ black string



Generalized Weyl solutions

$D = 5$ Kaluza–Klein bubble



Weyl form of the nontrivial extension

$$ds^2 = -e^{2A}dt^2 + e^{2B}d\varphi^2 + e^{2C}\delta_{AB}d\zeta^A d\zeta^B + e^{2D}\left(d\rho^2 + dz^2\right)$$

$$A = \frac{1}{2}\frac{1-n}{1+n}\ln\frac{Q+a}{Q-a}, \quad B = \ln\rho - \frac{1}{2}\ln\frac{Q+a}{Q-a},$$

$$C = \frac{1}{1+n}\ln\frac{Q+a}{Q-a}, \quad D = \frac{1}{2}\ln\frac{(L-a)^2}{4Q_+Q_-},$$

$$Q = Q_+ + Q_-$$

$$Q_\pm = \sqrt{\left(z \pm \frac{a}{2}\right)^2 + \rho^2}$$

$$z = r\left(1 + \frac{a}{2r}\right)\cos\vartheta, \; \rho = r\sqrt{1 + \frac{a}{r}}\sin\vartheta$$

$$ds^2 = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} dt^2 + \left(1+\frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \left(1+\frac{a}{r}\right)^{\frac{2}{n+1}} \delta_{AB}d\zeta^A d\zeta^B$$

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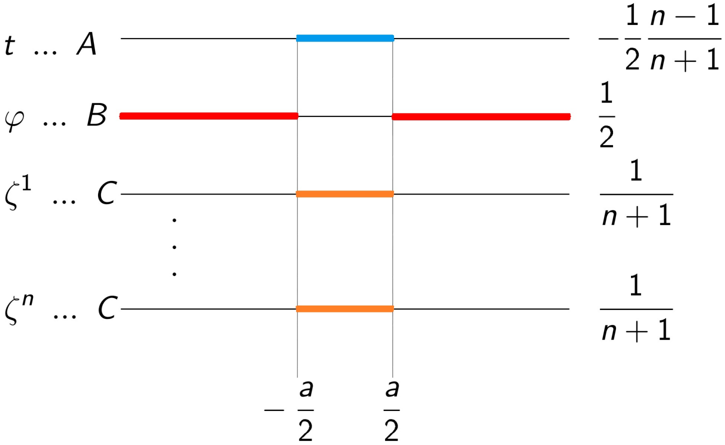
Weyl form of the nontrivial extension



Two types of solutions

trivial extension

$$
\begin{aligned}
ds^2 = & - \left(1 + \frac{a}{r} \right) dt^2 + \left(1 + \frac{a}{r} \right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \\
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$$
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existence of horizon for $a < 0$ at $r = |a|$!!!

Two types of solutions

trivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \delta_{AB}\, d\zeta^A d\zeta^B$$

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Kretschmann scalar

trivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \delta_{AB}d\zeta^A d\zeta^B$$

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12a^2}{r^6} \quad \text{the same as original Schwarzschild}$$

nontrivial extension

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$$K = \frac{12a^2}{r^6}\left[1 - \frac{1}{3}\frac{n(n-1)}{(n+1)^3}\left(1+\frac{r}{a}\right)^{-2}\left(\frac{3}{4}n + 1 + (n+1)\frac{r}{a}\right)\right]$$

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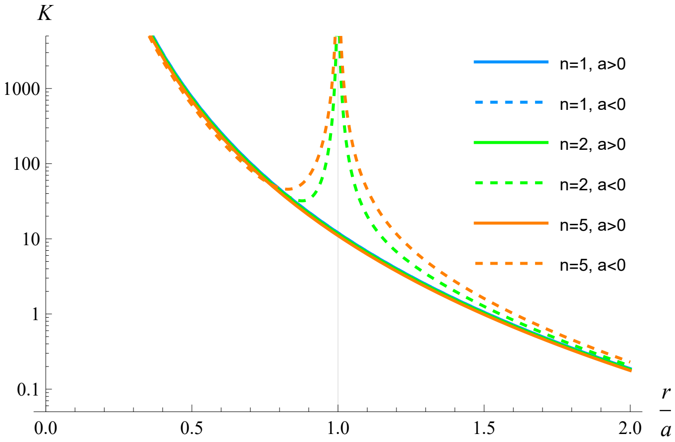

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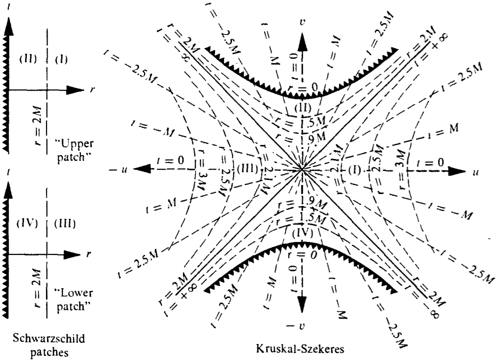
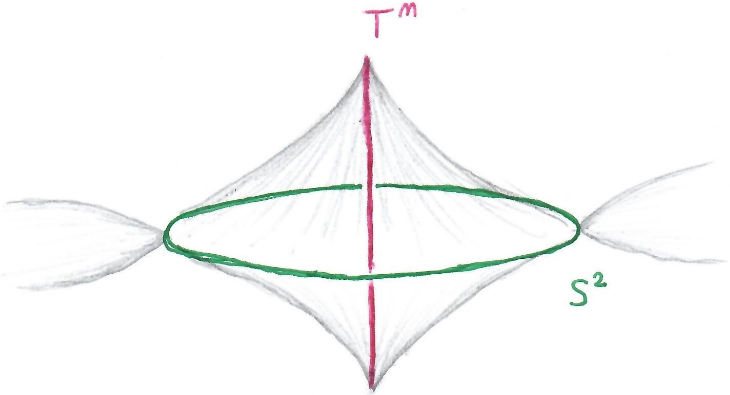


Kaluza–Klein bubble - "bubble of nothing"



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Two types of solutions

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Size of extra dimensions

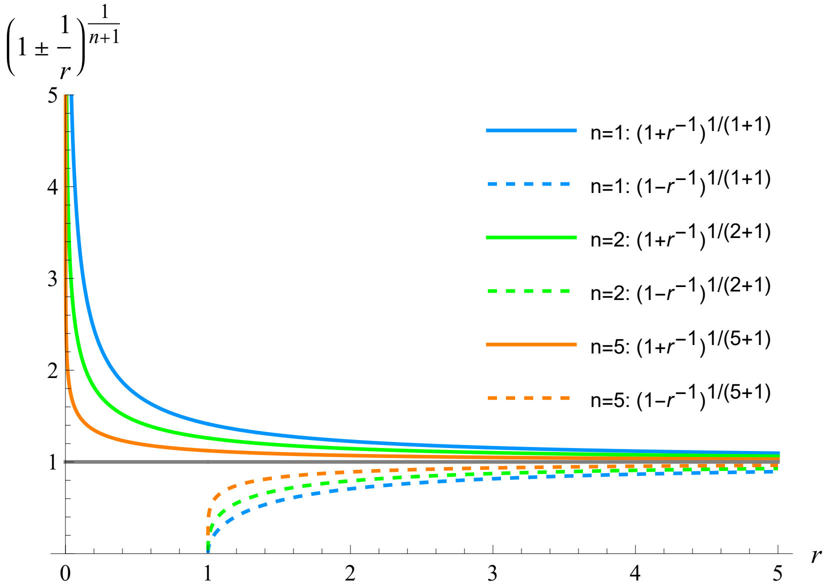
trivial extension

$$g_{AB} = \delta_{AB}$$

nontrivial extension

$$g_{AB} = \left(1 + \frac{a}{r}\right)^{\frac{2}{n+1}} \delta_{AB}$$

Size of extra dimensions



Size of extra dimensions

size at Planck length from a naked singularity

$$\frac{p_{\mathrm{phys.}}(r_{\mathrm{phys.}} = l_{\mathrm{Pl}})}{p_{\mathrm{phys.}}(r_{\mathrm{phys.}} \to \infty)} \approx \left[1.22 \cdot 10^{38} \frac{n+1}{n-1} \frac{M_{\mathrm{New.}}}{M_{\mathrm{sol.}}}\right]^{\frac{2}{3}\frac{1}{n+1}}$$

for only two extra dimension and supermassive black hole mass

$$n = 2 \;\; \& \;\; M_{\mathrm{New.}} = 10^{10} M_{\mathrm{sol.}} \Longrightarrow \frac{p_{\mathrm{phys.}}(r_{\mathrm{phys.}} = l_{\mathrm{Pl}})}{p_{\mathrm{phys.}}(r_{\mathrm{phys.}} \to \infty)} \approx 6 \cdot 10^{6}$$

smaller for smaller mass and larger number of extra dimensions

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Newtonian limit

$$ds^2 \approx -(1+2\phi)dt^2 + (1-2\psi)\delta_{ij}dx^i dx^j$$

$$g_{00} \approx -(1+2\phi) \quad \boxed{\phi = -\frac{\kappa M_{\mathrm{New.}}}{r}}$$

trivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right) \Rightarrow \phi = \frac{1}{2}\frac{a}{r} \Rightarrow \boxed{M_{\mathrm{New.}} = -\frac{a}{2\kappa}}$$

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Invariants

Komar mass

definition:

$$M_{\mathrm{Kom.}} = -\frac{1}{8\pi\kappa}\int_{\partial\Omega} \star d\sigma$$

Killing one-form $\sigma = g_{00}dt$
domain $\partial\Omega = S^2(R) \times S^1(L) \times \ldots \times S^1(L) \quad \longrightarrow \quad 4\pi R^2 L^n$
dual of two-form $\star d\sigma = g_{00,r} d\vartheta \wedge d\varphi \wedge d\zeta^1 \wedge \ldots \wedge d\zeta^n$

result:

$$M_{\mathrm{Kom.}}(R) = -\frac{1}{2\kappa} R^2 g_{00,r}(R) L^n \xrightarrow{R\to\infty} \boxed{M_{\mathrm{New.}} L^n}$$

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Invariants

Einstein and Landau–Lifshitz

Einstein mass

$$M_{\mathrm{Ein.}} = \int\limits_{\partial\Omega} E_{0}{}^{0i} dS_i$$

$$E_{\mu}{}^{\nu\rho} = \frac{1}{16\pi\kappa}\frac{g_{\mu\sigma}}{\sqrt{-g}}\left[(-g)\left(g^{\nu\sigma}g^{\rho\lambda} - g^{\rho\sigma}g^{\nu\lambda}\right)\right]_{,\lambda}$$

Landau–Lifshitz mass

$$M_{\mathrm{L.,L.}} = \int\limits_{\partial\Omega} L^{00i} dS_i$$

$$L^{\mu\nu\rho} = \frac{1}{16\pi\kappa}\Big[(-g)\Big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\Big)\Big]_{,\sigma}$$

results:

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Invariants

ADM

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$$E = \frac{1}{16\pi\kappa} \oint_{\partial\Omega} \sqrt{\gamma}\gamma^{\mu\nu}\gamma^{\alpha i}\left(\gamma_{\mu\alpha,\nu} - \gamma_{\mu\nu,\alpha}\right) dS_i$$

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$$\begin{aligned} M_{\mathrm{ADM}} &= \lim_{R\to\infty}\left(E(R) - E_0(R)\right) \\ &= \boxed{-\frac{2n+1}{n+1}\frac{a}{2\kappa}L^n} \end{aligned}$$

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Summary

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Physical properties

	horizons	singularities	$M_{\text{New.;Kom.}}$	$M_{\text{Ein.;L.,L.;ADM}}$
1. Mink. $(a = 0)$	none	none	0	0
2. trivial $(a > 0)$	none	$r = 0$	$-$	$-$
3. trivial $(a < 0)$	$r = -a$	$r = 0$	$+$	$+$
4. nontriv. $(a > 0)$	none	$r = 0$	$+$	$-$
5. nontriv. $(a < 0)$	$r = -a$	$r = \{0, -a\}$	$-$	$+$

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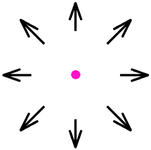
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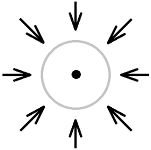
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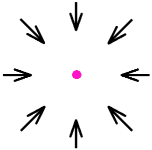
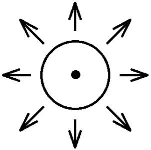

5.


Work in progress

- more than $1+3$ standard dimensions (straightforward)

$$ds^2 = -\left(1+\frac{a}{r^{q-1}}\right)^{-\frac{n-1}{n+1}} dt^2 + \left(1+\frac{a}{r^{q-1}}\right)^{-1} dr^2 + \\ + r^2 d\Omega_{(q)}{}^2 + \left(1+\frac{a}{r^{q-1}}\right)^{\frac{2}{n+1}} \delta_{AB} d\zeta^A d\zeta^B$$

- nonzero cosmological constant (complicated)

$$R_{\mu\nu} = \frac{2\Lambda}{n+q} g_{\mu\nu}$$

$$\underbrace{\left\{1 - f^{-\beta}\left[1+\frac{1}{2}\left(\alpha-\beta+n\gamma\right) r\frac{f'}{f}\right]\right\}\sin^2\vartheta}_{R_{\varphi\varphi}=R_{\vartheta\vartheta}\sin^2\vartheta} = \frac{2\Lambda}{n+q}\underbrace{r^2\sin^2\vartheta}_{g_{\varphi\varphi}=g_{\vartheta\vartheta}\sin^2\vartheta}$$

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Summary

- Eur. Phys. J. Plus **141** (2026) 1, 40, [arXiv:**2409.14349**]
- Schwarzschild: $ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega^2_{(2)}$
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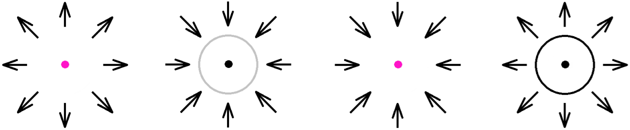

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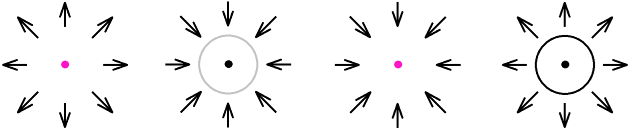

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Thank You for listening!