

Non–Minimally Coupled Weyl Connection Gravity

Cosmological Perturbations

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- 1** **Modified Gravity Theories**

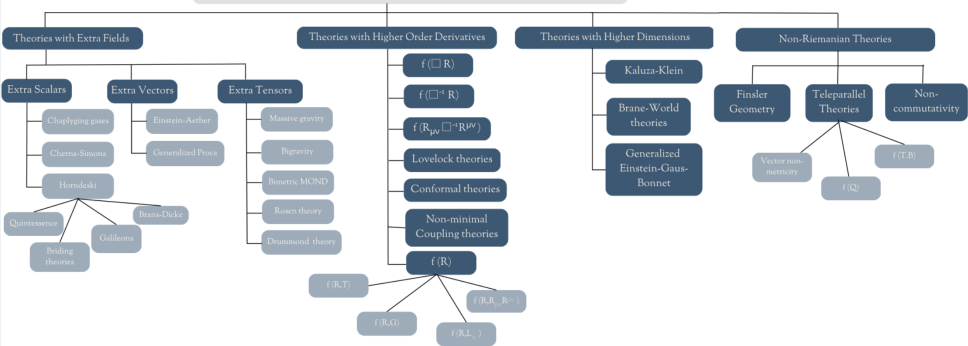
- 2** **Non-Minimally Coupled Weyl Connection Gravity**

- 3** **Modified Friedmann Equations**

- 4** **Cosmological Perturbations**

- 5** **Conclusion**

Modified Gravity Theories



The Weyl connection introduces a vector field that encodes non-metricity:

$$D_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu},$$

where A_λ is the Weyl vector field and $D_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} - \bar{\Gamma}_{\mu\lambda}^\rho g_{\rho\nu} - \bar{\Gamma}_{\nu\lambda}^\rho g_{\rho\mu}$.

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The **generalized Ricci tensor** is given by:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \underbrace{\frac{1}{2}A_\mu A_\nu + \frac{1}{2}g_{\mu\nu} (\nabla_\lambda - A_\lambda) A^\lambda + \tilde{F}_{\mu\nu}}_{\bar{R}_{\mu\nu}} + \frac{1}{2} (\nabla_\mu A_\nu + \nabla_\nu A_\mu),$$

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The **scalar curvature** is given by:

$$\bar{R} = R + \underbrace{3\nabla_\lambda A^\lambda - \frac{3}{2}A_\lambda A^\lambda}_{\bar{R}}.$$

Non-minimal matter–curvature coupling model, with Weyl connection, defined by the action:

$$S = \int (f_1(\bar{R}) + f_2(\bar{R})\mathcal{L}) \sqrt{-g}d^4x,$$

where $f_1(\bar{R})$ and $f_2(\bar{R})$ are generic functions of the scalar curvature.

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Varying the action with respect to the vector field, we obtain the **constraint-like equations**:

$$\nabla_\lambda \Theta(\bar{R}) = -A_\lambda \Theta(\bar{R}),$$

where $\Theta(\bar{R}) = F_1(\bar{R}) + F_2(\bar{R})\mathcal{L}$ and $F_i(\bar{R}) = \frac{df_i(\bar{R})}{d\bar{R}}$, $i \in \{1, 2\}$.

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Varying the action with respect to the metric, we obtain the **field equations**:

$$\bar{R}_{(\mu\nu)}\Theta(\bar{R}) - \frac{1}{2}g_{\mu\nu}f_1(\bar{R}) = \frac{f_2(\bar{R})}{2}T_{\mu\nu}.$$

It is possible to derive a **non-conservation law for the energy–momentum tensor**:

$$\nabla_{\mu} T^{\mu\nu} = \frac{2}{f_2(\bar{R})} \left[\frac{F_2(\bar{R})}{2} (g^{\mu\nu} \mathcal{L} - T^{\mu\nu}) \nabla_{\mu} R + \nabla_{\mu} (\Theta(\bar{R}) B^{\mu\nu}) - \frac{1}{2} (F_1(\bar{R}) g^{\mu\nu} + F_2(\bar{R}) T^{\mu\nu}) \nabla_{\mu} \bar{R} \right],$$

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- A generalization of the coupling can result in an extra force in the geodesic equation;
- Non-metricity also plays a significant role, introducing further contributions to the exchange between geometry and matter sectors.

Modified Friedmann Equations: the Minimal Case

We restrict our analysis to the minimal coupling case,

$$f_2(\bar{R}) = 1, \quad \Theta(\bar{R}) = F_1(\bar{R}) \equiv F(\bar{R}).$$

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For a spatially flat FLRW background in conformal time, we consider

$$d\tilde{s}^2 = a^2(\eta) (-d\eta^2 + \delta_{ij} dx^i dx^j) \quad | \quad \tilde{A}_\mu = (\tilde{A}(\eta), 0, 0, 0)$$

where $a(\eta)$ is the scale factor and \tilde{A}_μ is the background Weyl vector.

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Background field equations (perfect fluid):

$$\begin{aligned} (3\tilde{A}' - 6\mathcal{H}') F(\bar{R}) + a^2(\eta)f(\bar{R}) &= a^2(\eta)\tilde{\rho} \\ (2\mathcal{H}' - \tilde{A}' + 4\mathcal{H}^2 - 4\mathcal{H}\tilde{A} + \tilde{A}^2) F(\bar{R}) + a^2(\eta)f(\bar{R}) &= a^2(\eta)\tilde{P} \\ \tilde{\rho}' + 3\mathcal{H}(\tilde{\rho} + \tilde{P}) &= 0 \end{aligned}$$

At the background level, the generalized non-conservation equation reduces to the standard conservation law.

Using the gauge-invariant Bardeen scalars, Φ and Ψ , the perturbed metric is

$$ds^2 = a^2(\eta) \left(-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j \right).$$

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The Weyl vector is decomposed as

$$A_\mu = \left(\tilde{A}(\eta) + \delta A_0(\eta, \mathbf{x}), \delta A_1(\eta, \mathbf{x}), \delta A_2(\eta, \mathbf{x}), \delta A_3(\eta, \mathbf{x}) \right).$$

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Perturbed constraint:

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Using $\Theta(\bar{R}) = \tilde{\Theta} + \delta\bar{\Theta}$,
the constraint gives:

$$\delta\bar{\Theta}' = -\delta A_0 \tilde{\Theta} - \tilde{A} \delta\bar{\Theta},$$

$$\partial_i(\delta\bar{\Theta}) = -\delta A_i \tilde{\Theta}.$$

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SVT decomposition



$$\delta A_i = \partial_i \Sigma + \delta \hat{A}_i$$

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SVT decomposition

$$\xrightarrow{\delta A_i = \partial_j \Sigma + \delta \hat{A}_i}$$

which implies

$$\delta\bar{\Theta} = -\Sigma \tilde{\Theta},$$

$$\Sigma' = \delta A_0.$$

The vector component vanishes: $\hat{A}_i = 0$.

Linearized field equations (Fourier space):

$$\left(2\kappa^2(2\Phi + \Sigma) + 6(2\mathcal{H} - \tilde{A})\Phi' - 3(2\mathcal{H}' - \tilde{A}')\Sigma + 3(4\mathcal{H}^2 - \tilde{A}(4\mathcal{H} - \tilde{A}))\Psi + 3(2\mathcal{H} - \tilde{A})\delta A_0 \right) F(\bar{R}) = -a^2(\eta)\delta\rho,$$

$$(4\Phi' + 2\delta A_0 + (\tilde{A} - 2\mathcal{H})(\Sigma - 2\Psi)) F(\bar{R}) = -a^2(\eta)(\tilde{\rho} + \tilde{P})v,$$

$$2(\Phi - \Psi + \Sigma)F(\bar{R}) = a^2(\eta)\Pi,$$

$$\left(\frac{4}{3}\kappa^2(\Phi - \Psi + \Sigma) + 4\Phi'' + 2(2\mathcal{H} - \tilde{A})(2\Phi' + \Psi') + 2\delta A_0' + (2(\mathcal{H}' + 2\mathcal{H}^2) - \tilde{A}' - \tilde{A}(4\mathcal{H} - \tilde{A}))(\Psi - \Sigma) + 3(2\mathcal{H}' - \tilde{A}')\Psi + (2\mathcal{H} - \tilde{A})\delta A_0 \right) F(\bar{R}) = a^2(\eta)\delta P,$$

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Perturbed conservation equations (Fourier space):

$$\delta\rho' - 3(\tilde{\rho} + \tilde{P})\Phi' - \kappa^2(\tilde{\rho} + \tilde{P})v + 3\mathcal{H}(\delta\rho + \delta P) = 0,$$

$$(\tilde{\rho} + \tilde{P})v' + (\tilde{P}' + \mathcal{H}(\tilde{\rho} + \tilde{P}))v + \delta P + (\tilde{\rho} + \tilde{P})\Psi - \frac{2}{3}\kappa^2\Pi = 6a^{-2}(\eta)F(\bar{R})\left(\tilde{A}\delta A_0 + (\tilde{A}' + \tilde{A}(2\mathcal{H} - \tilde{A}))\Sigma\right).$$

Linearized field equations (Fourier space):

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Perturbed conservation equations (Fourier space):

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$$(\tilde{\rho} + \tilde{P})v' + (\tilde{P}' + \mathcal{H}(\tilde{\rho} + \tilde{P}))v + \delta P + (\tilde{\rho} + \tilde{P})\Psi - \frac{2}{3}\kappa^2\Pi = 6a^{-2}(\eta)F(\bar{R})\left(\tilde{A}\delta A_0 + (\tilde{A}' + \tilde{A}(2\mathcal{H} - \tilde{A}))\Sigma\right).$$

- For the chosen background, vector and tensor modes vanish due to the **absence of Weyl vector-induced vector modes**.
- δA_0 has the same dimension as \tilde{A} and \mathcal{H} .
- Σ has the same dimension as Φ and Ψ .

- The non-metricity in Non-Minimally Coupled Weyl Connection Gravity **plays a dynamical role**, modifying the gravitational sector.
- The non-conservation of the energy–momentum tensor leads **to extra force terms**, which may lead to observable signatures.
- Although the cosmological background satisfies the standard conservation law, the Weyl vector induces **non-trivial modifications** at the perturbative level.
- Scalar perturbations acquire **new degrees of freedom** through the Weyl vector perturbations, leading to modified field and conservation equations.

Future work:

- Compare the scalar perturbations with cosmological observations.
- Derive the generalized Layzer–Irvine equation and study its implications for structure formation and dark matter.
- Extend the analysis beyond scalar perturbations to investigate whether the Weyl vector can source vector modes.

Thank You for Your Attention!

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