

# CANONICAL AND NON-CANONICAL SYMMETRIES IN $f(R)$ COSMOLOGY

FRANCESCA SPINNATO, PhD student  
Scuola Superiore Meridionale, Naples, Italy  
*email: f.spinnato@ssmeridionale.it*



GeomGravX

*10 years of Geometric Foundations  
of Gravity and eXtensions*

29th June 2026



# THE STATE-OF-THE-ART



## GENERAL RELATIVITY

$$S_{HE} = \frac{1}{2k} \int \sqrt{-g} R d^4x$$

### SUCCESSSES

#### WEAK FIELD TESTS

LIGHT DEFLECTION

SAPHIRO DELAY

THE PRECESSION OF  
MERCURY'S PERIHELION

#### STRONG FIELD TESTS

BLACK HOLES

GRAVITATIONAL  
WAVES

#### IR SCALES

GALAXY ROTATION CURVES  
PROBLEM

THE LATE-TIME ACCELERATED  
EXPANSION OF THE UNIVERSE

### SHORTCOMINGS

#### UV SCALES

QUANTUM GRAVITY  
PROBLEM

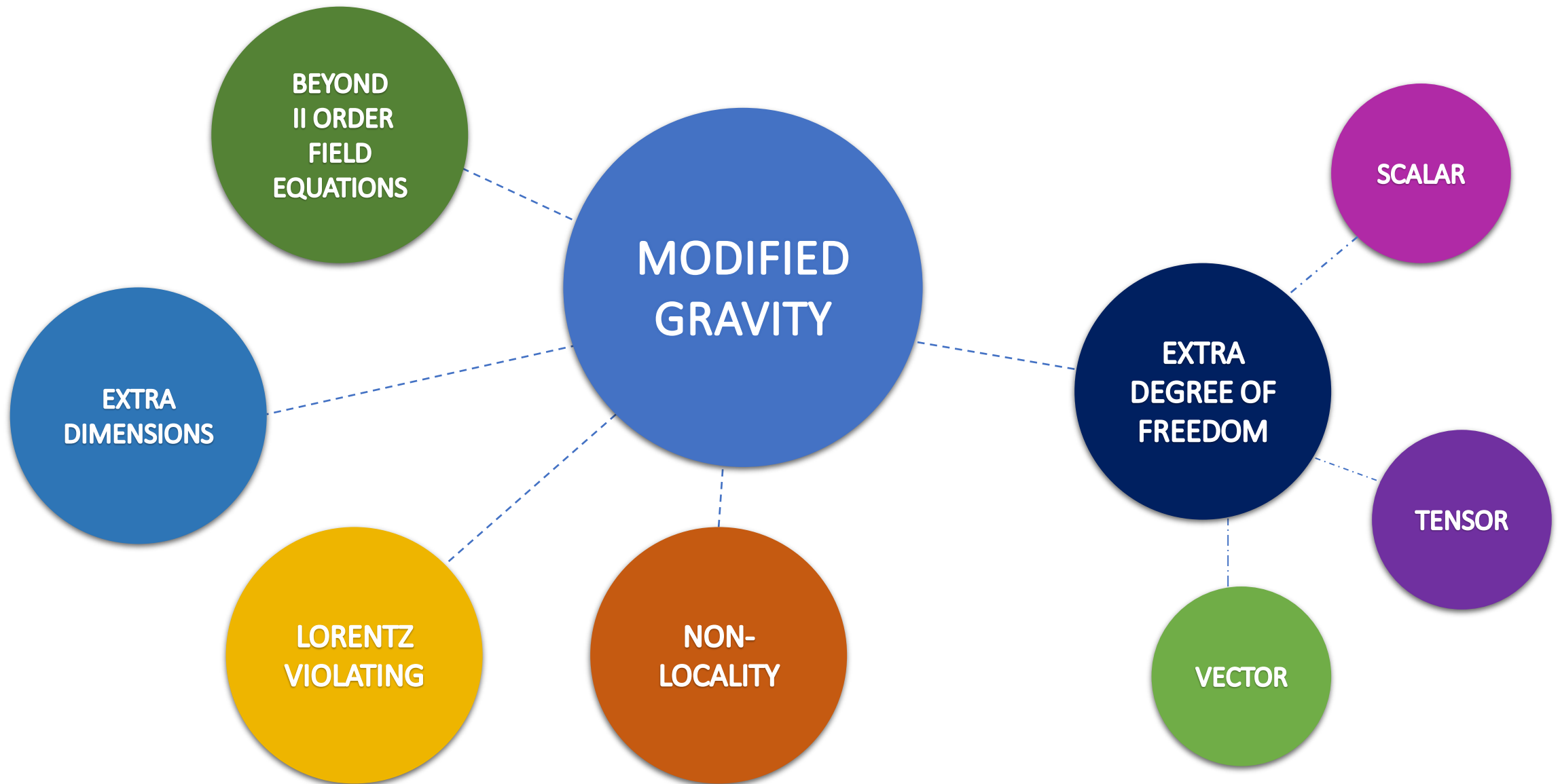
# MODIFIED GRAVITY

GEOMETRIC PART

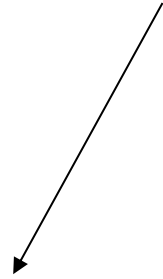
MATTER CONTENT

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

# MODIFIED GRAVITY

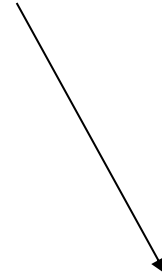


# MODIFIED GRAVITY



## ALTERNATIVE THEORIES

RELAXING BASIC ASSUMPTIONS



## EXTENDED THEORIES

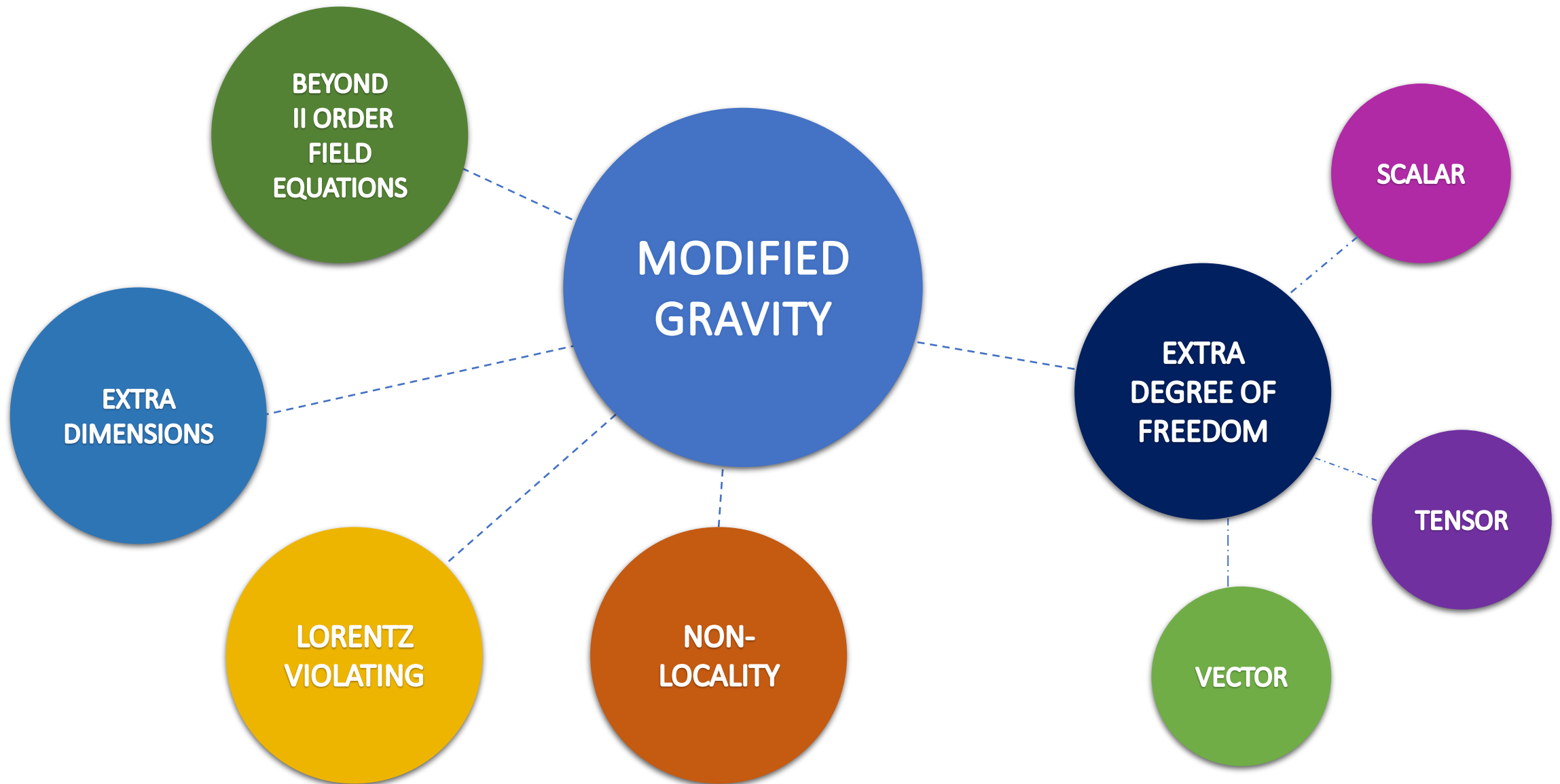
SUBSTITUING THE LINEAR DEPENDENCE OF THE ACTION  
ON THE RICCI SCALAR

# BE CAREFUL

A new relativistic theory must satisfy few requirements from a phenomenological point of view:

- To recover Newtonian dynamics in weak-field, slow motion limit
- To pass Classical Solar System tests
- To address Large-Scale Structure problem
- To reproduce in a self-consistent way the main cosmological parameters

# MODIFIED GRAVITY



# NOETHER SYMMETRY APPROACH



## NOETHER'S FIRST THEOREM

*To every differentiable symmetry generated by local actions, there corresponds a conserved current (Noether current)*

WHY?

*EXTEND/MODIFY GRAVITY* → HIGHLY COMPLEX FIELD EQUATIONS

*NOETHER SYMMETRY APPROACH* → SYMMETRIES → CONSERVED QUANTITIES

# ATTENTION!

NOT EVERY MODEL IDENTIFIED BY THE NOETHER SYMMETRY APPROACH WILL BE NECESSARILY VIABLE!

- THEORETICAL LIMITATIONS
- RULED OUT BY EXPERIMENTAL EVIDENCE

# NOETHER SYMMETRY APPROACH



## CANONICAL CASE

$$\left\{ \begin{array}{l} \mathcal{L}(x^a, \phi^i, \partial_a \phi^i) \rightarrow \mathcal{L}(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i) \\ \tilde{x}^a = x^a + \delta x^a \\ \tilde{\phi}^i = \phi^i + \delta \phi^i \\ \partial_a \tilde{\phi}^i = \partial_a \phi^i + [\partial_a(\delta \phi^i) - \partial_a \phi^i \partial_b(\delta x^b)] \end{array} \right.$$

### THE FIRST PROLONGATION OF NOETHER VECTOR

$$X^{[1]} = \delta x^a \frac{\partial}{\partial x^a} + \delta \phi^i \frac{\partial}{\partial \phi^i} + \delta(\partial_a \phi^i) \frac{\partial}{\partial(\partial_a \phi^i)}$$

### THE NOETHER IDENTITY

$$\boxed{X^{[1]} \mathcal{L} + \partial_a(\delta x^a) \mathcal{L} = \partial_a g^a}$$

# NOETHER SYMMETRY APPROACH



## NON-CANONICAL CASE

$$\left\{ \begin{array}{l} \mathcal{L}(x^a, \phi^i, \partial_a \phi^i, \partial_b \partial_a \phi^i) \rightarrow \mathcal{L}(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i, \partial_b \partial_a \tilde{\phi}^i) \\ \tilde{x}^a = x^a + \delta x^a \\ \tilde{\phi}^i = \phi^i + \delta \phi^i \\ \partial_a \tilde{\phi}^i = \partial_a \phi^i + [\partial_a (\delta \phi^i) - \partial_a \phi^i \partial_b (\delta x^b)] \\ \partial_b \partial_a \tilde{\phi}^i = \partial_b \partial_a \phi^i + [\partial_b \partial_a (\delta \phi^i) - \partial_a \phi^i \partial_b \partial_c (\delta x^c) - 2 \partial_b \partial_a \phi^i \partial_c (\delta x^c)] \end{array} \right.$$

### THE SECOND PROLONGATION OF NOETHER VECTOR

$$X^{[2]} = \delta x^a \frac{\partial}{\partial x^a} + \delta \phi^i \frac{\partial}{\partial \phi^i} + \delta (\partial_a \phi^i) \frac{\partial}{\partial (\partial_a \phi^i)} + \delta (\partial_b \partial_a \phi^i) \frac{\partial}{\partial (\partial_b \partial_a \phi^i)}$$

### THE NOETHER IDENTITY

$$X^{[2]} \mathcal{L} + \partial_a (\delta x^a) \mathcal{L} = \partial_a g^a$$

# NOETHER SYMMETRY APPROACH



WHAT ABOUT CONSERVED QUANTITIES?

## CANONICAL CASE

$$\mathcal{L}(x^a, \phi^i, \partial_a \phi^i) \rightarrow \mathcal{L}(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i)$$

PROPERLY REWRITING

$$\left[ \delta x^a \frac{\partial}{\partial x^a} + \delta \phi^i \frac{\partial}{\partial \phi^i} + \delta(\partial_a \phi^i) \frac{\partial}{\partial(\partial_a \phi^i)} \right] \mathcal{L} + \partial_a(\delta x^a) \mathcal{L} = \partial_a g^a$$

$$\frac{d}{dx^a} \left( \mathcal{L} \delta x^a - \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} \partial_b \phi^i \delta x^a + \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \delta \phi^i - g^a \right) = 0$$

$$\rightarrow j^a = \delta x^a \left[ \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} \partial_b \phi^i - \mathcal{L} \right] - \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \delta \phi^i + g^a$$

# NOETHER SYMMETRY APPROACH



WHAT ABOUT CONSERVED QUANTITIES?

## NON-CANONICAL CASE

$$\mathcal{L}(x^a, \phi^i, \partial_a \phi^i, \partial_b \partial_a \phi^i) \rightarrow \mathcal{L}(\tilde{x}^a, \tilde{\phi}^i, \partial_a \tilde{\phi}^i, \partial_b \partial_a \tilde{\phi}^i)$$

PROPERLY  
REWRITING

$$\left[ \delta x^a \frac{\partial}{\partial x^a} + \delta \phi^i \frac{\partial}{\partial \phi^i} + \delta(\partial_a \phi^i) \frac{\partial}{\partial(\partial_a \phi^i)} + \delta(\partial_b \partial_a \phi^i) \frac{\partial}{\partial(\partial_b \partial_a \phi^i)} \right] \mathcal{L} + \partial_a(\delta x^a) \mathcal{L} = \partial_a g^a$$

$$\frac{d}{dx^a} \left( \begin{aligned} & \delta x^a \left[ \partial_b \phi^i \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} - \partial_a \phi^i \partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} \right) + \partial_a \partial_b \phi^i \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} - \mathcal{L} \right] + \\ & + \left[ \partial_a \left( \frac{\partial \mathcal{L}}{\partial_b \partial_a \phi^i} \right) - \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \right] \delta \phi^i - [\partial_a(\delta \phi^i) - \partial_a(\delta x^a) \partial \phi^i] \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} + g^a \end{aligned} \right) = 0$$

$$\rightarrow j^a = \begin{aligned} & \delta x^a \left[ \partial_b \phi^i \frac{\partial \mathcal{L}}{\partial(\partial_b \phi^i)} - \partial_a \phi^i \partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} \right) + \partial_a \partial_b \phi^i \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} - \mathcal{L} \right] + \\ & + \left[ \partial_a \left( \frac{\partial \mathcal{L}}{\partial_b \partial_a \phi^i} \right) - \frac{\partial \mathcal{L}}{\partial(\partial_a \phi^i)} \right] \delta \phi^i - [\partial_a(\delta \phi^i) - \partial_a(\delta x^a) \partial \phi^i] \frac{\partial \mathcal{L}}{\partial(\partial_b \partial_a \phi^i)} + g^a \end{aligned}$$

# WHAT ARE WE GOING TO DO?



1. CHOOSE A MODIFIED THEORY;
2. ESTABLISH A METRIC;
3. WRITE THE GRAVITATIONAL ACTION;
4. FIND CANONICAL AND NON-CANONICAL LAGRANGIAN;
5. SELECT VIABLE MODELS APPLYING THE NOETHER SYMMETRY APPROACH;
6. COMPARE THE TWO CASES.

# $f(R)$ COSMOLOGY

- FOUR DIMENSIONS;
- NO KINETIC AND POTENTIAL TERMS;
- NO MATTER TERM;
- SPATIALLY FLAT FRIEDMAN-LEMAITRE-ROBERTSON-WALKER (FLRW) METRIC

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

$f(R)$  GRAVITY

$$S = \int \sqrt{-g} f(R) d^4x$$

THE RICCI SCALAR IN SPATIALLY  
FLAT FLRW METRIC

$$R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)$$

INTEGRATING OVER THE THREE-  
DIMENSIONAL HYPERSURFACE

$$\int \sqrt{-g} d^4x \rightarrow \int a^3 dt$$

$$S = \int a^3 f(R) dt$$

# $f(R)$ COSMOLOGY

CANONICAL POINT-LIKE LAGRANGIAN

$$S = \int a^3 \left\{ f(R) - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \right\} dt$$

$$\frac{\delta S}{\delta R} = a^3 f_R - \lambda = 0 \rightarrow \lambda = a^3 f_R$$

$$S = \int a^3 \left\{ f(R) - f_R \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \right\} dt$$

$$S = \int \left[ a^3 f - a^3 f_R R - 6 f_R a^2 \frac{d(\dot{a})}{dt} - 6 f_R a \dot{a}^2 \right] dt$$

$$-6 \int f_R a^2 \frac{d(\dot{a})}{dt} dt = -6 f_R a^2 \dot{a} + 6 \int [f_{RR} \dot{a} \dot{R} a^2 + 2 f_R a \dot{a}^2] dt$$

$$S = \int [a^3 (f - f_R R) + 6 f_{RR} \dot{a} \dot{R} a^2 + 6 f_R a \dot{a}^2] dt$$

$$\rightarrow \mathcal{L} = a^3 (f - f_R R) + 6 f_{RR} \dot{a} \dot{R} a^2 + 6 f_R a \dot{a}^2$$

LAGRANGIAN  
MULTIPLIERS  
METHOD

INTEGRATING BY  
PARTS

# $f(R)$ COSMOLOGY

NON CANONICAL POINT-LIKE LAGRANGIAN

$$S = \int a^3 \left\{ f(R) - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \right\} dt$$

$$\frac{\delta S}{\delta R} = a^3 f_R - \lambda = 0 \rightarrow \lambda = a^3 f_R$$

$$S = \int a^3 \left\{ f(R) - f_R \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] \right\} dt$$

$$S = \int [a^3(f - f_R R) - 6f_R a^2 \ddot{a} - 6f_R a \dot{a}^2] dt$$

$$\rightarrow \mathcal{L} = a^3(f - f_R R) - 6f_R a^2 \ddot{a} - 6f_R a \dot{a}^2$$

LAGRANGIAN  
MULTIPLIERS  
METHOD

# NOETHER SYMMETRY APPROACH

MINISUPERSPACE:

$$\mathcal{S} = \{a(t), R(t)\} = q^i$$

APPLICATION

SET OF INFINITESIMAL TRANSFORMATIONS:

CANONICAL CASE

$$\left\{ \begin{array}{l} \mathcal{L}(t, q^i, \dot{q}^i) \rightarrow \mathcal{L}(\tilde{t}, \tilde{q}^i, \tilde{\dot{q}}^i) \\ \tilde{t} = t + \epsilon \xi + \mathcal{O}(\epsilon^2) \\ \tilde{q}^i = q^i + \epsilon \eta^i + \mathcal{O}(\epsilon^2) \\ \tilde{\dot{q}}^i = \dot{q}^i + \epsilon \eta^{[1]} = \dot{q}^i + \epsilon(\dot{\eta}^i - \dot{q}^i \dot{\xi}) \end{array} \right.$$

NOETHER IDENTITY:

$$X^{[1]} \mathcal{L} + \dot{\xi} \mathcal{L} = \dot{g}$$

where

FIRST PROLONGATION OF  
NOETHER VECTOR

$$\begin{aligned} X^{[1]} &= \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i \dot{\xi}) \frac{\partial}{\partial \dot{q}^i} \\ &= \xi \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + (\dot{\alpha} - \dot{a} \dot{\xi}) \frac{\partial}{\partial \dot{a}} + (\dot{\beta} - \dot{R} \dot{\xi}) \frac{\partial}{\partial \dot{R}} \end{aligned}$$

GENERATORS OF INFINITESIMAL  
TRANSFORMATIONS

$$\eta^i = \{\alpha(t, a, R), \beta(t, a, R)\}$$

# NOETHER SYMMETRY APPROACH

MINISUPERSPACE:

$$\mathcal{S} = \{a(t), R(t)\} = q^i$$

APPLICATION

SET OF INFINITESIMAL TRANSFORMATIONS:

NON-CANONICAL  
CASE

$$\left\{ \begin{array}{l} \mathcal{L}(t, q^i, \dot{q}^i, \ddot{q}^i) \rightarrow \mathcal{L}(\tilde{t}, \tilde{q}^i, \tilde{\dot{q}}^i, \tilde{\ddot{q}}^i) \\ \tilde{t} = t + \epsilon \xi + \mathcal{O}(\epsilon^2) \\ \tilde{q}^i = q^i + \epsilon \eta^i + \mathcal{O}(\epsilon^2) \\ \tilde{\dot{q}}^i = \dot{q}^i + \epsilon \eta^{[1]} = \dot{q}^i + \epsilon(\dot{\eta}^i - \dot{q}^i \dot{\xi}) \\ \tilde{\ddot{q}}^i = \ddot{q}^i + \epsilon \eta^{[2]} = \ddot{q}^i + \epsilon(\ddot{\eta}^i - \ddot{\xi} \dot{q}^i - 2\dot{\xi} \ddot{q}^i) \end{array} \right.$$

NOETHER IDENTITY:

$$X^{[2]} \mathcal{L} + \dot{\xi} \mathcal{L} = \dot{g}$$

where

FIRST PROLONGATION OF  
NOETHER VECTOR

$$\begin{aligned} X^{[2]} &= \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i \dot{\xi}) \frac{\partial}{\partial \dot{q}^i} + (\ddot{\eta}^i - \ddot{\xi} \dot{q}^i - 2\dot{\xi} \ddot{q}^i) \frac{\partial}{\partial \ddot{q}^i} \\ &= \xi \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + (\dot{\alpha} - \dot{a} \dot{\xi}) \frac{\partial}{\partial \dot{a}} + (\dot{\beta} - \dot{R} \dot{\xi}) \frac{\partial}{\partial \dot{R}} + (\ddot{\alpha} - \ddot{\xi} \dot{a} - 2\dot{\xi} \ddot{a}) \frac{\partial}{\partial \ddot{a}} + \\ &\quad + (\ddot{\beta} - \ddot{\xi} \dot{R} - 2\dot{\xi} \ddot{R}) \frac{\partial}{\partial \ddot{R}} \\ \eta^i &= \{\alpha(t, a, R), \beta(t, a, R)\} \end{aligned}$$

GENERATORS OF INFINITESIMAL  
TRANSFORMATIONS

# NOETHER SYSTEMS

## CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \quad \alpha = \alpha(t, a, R), \quad \beta = \beta(t, a, R) \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta = 0 \\ \alpha f_R + af_{RR}\beta + 2af_R \frac{\partial \alpha}{\partial a} + a^2 f_{RR} \frac{\partial \beta}{\partial a} - af_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_{RR} + \beta a f_{RRR} + a \frac{\partial \alpha}{\partial a} f_{RR} + 2f_R \frac{\partial \alpha}{\partial R} - af_{RR} \frac{\partial \xi}{\partial t} + a \frac{\partial \beta}{\partial R} f_{RR} = 0 \\ 12f_R \frac{\partial \alpha}{\partial t} + 6af_{RR} \frac{\partial \beta}{\partial t} = 0 \\ 6f_{RR} \frac{\partial \alpha}{\partial t} = 0 \\ f_{RR} \frac{\partial \alpha}{\partial R} = 0 \end{array} \right.$$

## NON-CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \quad \alpha = \alpha(t, a), \quad \beta = \beta(t, a, R) \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta - 6f_R \frac{\partial^2 \alpha}{\partial t^2} = 0 \\ \alpha f_R + af_{RR}\beta + 2af_R \frac{\partial \alpha}{\partial a} + a^2 f_R \frac{\partial^2 \alpha}{\partial a^2} - af_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_R + \beta f_{RR}a + af_R \frac{\partial \alpha}{\partial a} - af_R \frac{\partial \xi}{\partial t} = 0 \\ 2 \frac{\partial \alpha}{\partial t} - a \frac{\partial^2 \xi}{\partial t^2} = 0 \end{array} \right.$$

Note: We assume  $g = g_0$

# NOETHER SYSTEMS

## CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \alpha = \alpha(t, a, R), \beta = \beta(t, a, R) \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta = 0 \\ \alpha f_R + a f_{RR}\beta + 2a f_R \frac{\partial \alpha}{\partial a} + a^2 f_{RR} \frac{\partial \beta}{\partial a} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_{RR} + \beta a f_{RRR} + a \frac{\partial \alpha}{\partial a} f_{RR} + 2f_R \frac{\partial \alpha}{\partial R} - a f_{RR} \frac{\partial \xi}{\partial t} + a \frac{\partial \beta}{\partial R} f_{RR} = 0 \\ 12f_R \frac{\partial \alpha}{\partial t} + 6a f_{RR} \frac{\partial \beta}{\partial t} = 0 \\ 6f_{RR} \frac{\partial \alpha}{\partial t} = 0 \\ f_{RR} \frac{\partial \alpha}{\partial R} = 0 \end{array} \right.$$

## NON-CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \alpha = \alpha(t, a), \beta = \beta(t, a, R) \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta - 6f_R \frac{\partial^2 \alpha}{\partial t^2} = 0 \\ \alpha f_R + a f_{RR}\beta + 2a f_R \frac{\partial \alpha}{\partial a} + a^2 f_R \frac{\partial^2 \alpha}{\partial a^2} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_R + \beta f_{RR}a + a f_R \frac{\partial \alpha}{\partial a} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2 \frac{\partial \alpha}{\partial t} - a \frac{\partial^2 \xi}{\partial t^2} = 0 \end{array} \right.$$

COMPLETELY  
DIFFERENT

# SELECTED MODELS

## CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$
$\xi_0 t + \xi_1$	$\frac{\xi_0}{3} a + \frac{a_1}{\sqrt{a}}$	$\beta(a, R, t)$	$f_0 R$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$
$a_0 t + \xi_1$	$a_0 a$	$-2a_0 R$	$f_0 R^2$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1, 2\}$
$\xi_0 t + \xi_1$	$a_0 a$	$-2\xi_0 R$	$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$
$\xi_1$	$a_0 a^{-1}$	$-2a_0 a^{-2} R$	$f_0 R^{\frac{3}{2}}$

## NON-CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta(a, R)$	$f(R)$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$
$\xi_0 t + \xi_1$	$\frac{\xi_0}{3} a$	$\beta(a, R, t)$	$f_0 R$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$
$a_0 t + \xi_1$	$\alpha_0 a$	$-2a_0 R$	$f_0 R^2$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1\}$
$\xi_0 t + \xi_1$	$\alpha_0 a$	$-2\xi_0 R$	$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$
$\xi_1$	$\alpha_0 a^{-1}$	$-2a_0 a^{-2} R$	$f_0 R^{\frac{3}{2}}$

# SELECTED MODELS

## CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$
$\xi_0 t + \xi_1$	$\frac{\xi_0}{3} a + \frac{a_1}{\sqrt{a}}$	$\beta(a, R, t)$	$f_0 R$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$
$a_0 t + \xi_1$	$a_0 a$	$-2a_0 R$	$f_0 R^2$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1, 2\}$
$\xi_0 t + \xi_1$	$a_0 a$	$-2\xi_0 R$	$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$
$\xi_1$	$a_0 a^{-1}$	$-2a_0 a^{-2} R$	$f_0 R^{\frac{3}{2}}$

## NON-CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta(a, R)$	$f(R)$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$
$\xi_0 t + \xi_1$	$\frac{\xi_0}{3} a$	$\beta(a, R, t)$	$f_0 R$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$
$a_0 t + \xi_1$	$\alpha_0 a$	$-2a_0 R$	$f_0 R^2$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1\}$
$\xi_0 t + \xi_1$	$\alpha_0 a$	$-2\xi_0 R$	$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$
$\xi_1$	$\alpha_0 a^{-1}$	$-2a_0 a^{-2} R$	$f_0 R^{\frac{3}{2}}$

# CONSERVED QUANTITIES

## CANONICAL CASE

## NON-CANONICAL CASE

$f(R)$	$j^{[i]}$	$j^{[i]}$
$f_0 R + \Lambda$	$[6f_0 a \dot{a}^2 - \Lambda a^3] \xi_1 + g_0$	$[6f_0 a \dot{a}^2 - \Lambda a^3] \xi_1 + g_0$
$f_0 R$	$6f_0(\xi_0 t + \xi_1) a \dot{a}^2 - 4f_0 \xi_0 a^2 \dot{a} - 12f_0 a_1 \dot{a} \sqrt{a} + g_0$	$6f_0(\xi_0 t + \xi_1) a \dot{a}^2 - 4f_0 \xi_0 a^2 \dot{a} + g_0$
$f_0 R^2 + f_1 R + f_2$	$12f_0 \xi_1 a^2 \dot{a} R + 6f_1 \xi_1 a \dot{a}^2 + 12f_0 \xi_1 a^2 \dot{a} \dot{R} - \xi_1 f_2 a^3 + \xi_1 f_0 a^3 R^2 + g_0$	$12f_0 \xi_1 a^2 \dot{a} R + 6f_1 \xi_1 a \dot{a}^2 + 12f_0 \xi_1 a^2 \dot{a} \dot{R} - \xi_1 f_2 a^3 + \xi_1 f_0 a^3 R^2 + g_0$
$f_0 R^2$	$(a_0 t + \xi_1) [12f_0 a \dot{a}^2 R + 12f_0 a^2 \dot{a} \dot{R} + f_0 a^3 R^2] - 12f_0 a_0 a^3 \dot{R} + g_0$	$(a_0 t + \xi_1) [12f_0 a \dot{a}^2 R + 12f_0 a^2 \dot{a} \dot{R} + f_0 a^3 R^2] - 12f_0 a_0 a^3 \dot{R} + g_0$
$f_0 R^p + \Lambda$ where $p \neq \{0, 1, 2\}$	$6f_0 \xi_1 p a \dot{a}^2 R^{p-1} + 6f_0 \xi_1 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 \xi_1 (p-1) a^3 R^p - \Lambda \xi_1 a^3 + g_0$	$6f_0 \xi_1 p a \dot{a}^2 R^{p-1} + 6f_0 \xi_1 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 \xi_1 (p-1) a^3 R^p - \Lambda \xi_1 a^3 + g_0$
$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$	$(\xi_0 t + \xi_1) [6f_0 p a \dot{a}^2 R^{p-1} + 6f_0 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 (p-1) a^3 R^p] +$ $-6f_0 p(p-1) a_0 a^3 \dot{R} R^{p-2} - 12f_0 p [a_0 - (p-1)\xi_0] a^2 \dot{a} R^{p-1} + g_0$	$(\xi_0 t + \xi_1) [6f_0 p a \dot{a}^2 R^{p-1} + 6f_0 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 (p-1) a^3 R^p] +$ $-6f_0 p(p-1) a_0 a^3 \dot{R} R^{p-2} + 6f_0 \xi_0 p (a_0 - \xi_0) a^2 \dot{a} R^{p-1} + g_0$
$f_0 R^{\frac{3}{2}}$	$9f_0 \xi_1 a \dot{a}^2 R^{\frac{1}{2}} + \frac{9}{2} f_0 \xi_1 a^2 \dot{a} \dot{R} R^{-\frac{1}{2}} + \frac{1}{2} f_0 \xi_1 a^3 R^{\frac{3}{2}} - 9f_0 a_0 \dot{a} R^{\frac{1}{2}} - \frac{9}{2} f_0 a_0 \dot{R} R^{-\frac{1}{2}} + g_0$	$9f_0 \xi_1 a \dot{a}^2 R^{\frac{1}{2}} + \frac{9}{2} f_0 \xi_1 a^2 \dot{a} \dot{R} R^{-\frac{1}{2}} + \frac{1}{2} f_0 \xi_1 a^3 R^{\frac{3}{2}} - 9f_0 a_0 \dot{a} R^{\frac{1}{2}} - \frac{9}{2} f_0 a_0 \dot{R} R^{-\frac{1}{2}} + g_0$

# CONSERVED QUANTITIES

## CANONICAL CASE

## NON-CANONICAL CASE

$f(R)$	$j^{[i]}$	$j^{[i]}$
$f_0 R + \Lambda$	$[6f_0 a \dot{a}^2 - \Lambda a^3] \xi_1 + g_0$	$[6f_0 a \dot{a}^2 - \Lambda a^3] \xi_1 + g_0$
$f_0 R$	$6f_0(\xi_0 t + \xi_1) a \dot{a}^2 - 4f_0 \xi_0 a^2 \dot{a} - 12f_0 a_1 \dot{a} \sqrt{a} + g_0$	$6f_0(\xi_0 t + \xi_1) a \dot{a}^2 - 4f_0 \xi_0 a^2 \dot{a} + g_0$
$f_0 R^2 + f_1 R + f_2$	$12f_0 \xi_1 a^2 \dot{a} R + 6f_1 \xi_1 a \dot{a}^2 + 12f_0 \xi_1 a^2 \dot{a} \dot{R} - \xi_1 f_2 a^3 + \xi_1 f_0 a^3 R^2 + g_0$	$12f_0 \xi_1 a^2 \dot{a} R + 6f_1 \xi_1 a \dot{a}^2 + 12f_0 \xi_1 a^2 \dot{a} \dot{R} - \xi_1 f_2 a^3 + \xi_1 f_0 a^3 R^2 + g_0$
$f_0 R^2$	$(a_0 t + \xi_1) [12f_0 a \dot{a}^2 R + 12f_0 a^2 \dot{a} \dot{R} + f_0 a^3 R^2] - 12f_0 a_0 a^3 \dot{R} + g_0$	$(a_0 t + \xi_1) [12f_0 a \dot{a}^2 R + 12f_0 a^2 \dot{a} \dot{R} + f_0 a^3 R^2] - 12f_0 a_0 a^3 \dot{R} + g_0$
$f_0 R^p + \Lambda$ where $p \neq \{0, 1, 2\}$	$6f_0 \xi_1 p a \dot{a}^2 R^{p-1} + 6f_0 \xi_1 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 \xi_1 (p-1) a^3 R^p - \Lambda \xi_1 a^3 + g_0$	$6f_0 \xi_1 p a \dot{a}^2 R^{p-1} + 6f_0 \xi_1 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 \xi_1 (p-1) a^3 R^p - \Lambda \xi_1 a^3 + g_0$
$f_0 R^p$ where $p = \frac{1}{2} + \frac{3\alpha_0}{2\xi_0}$	$(\xi_0 t + \xi_1) [6f_0 p a \dot{a}^2 R^{p-1} + 6f_0 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 (p-1) a^3 R^p] +$ $-6f_0 p(p-1) a_0 a^3 \dot{R} R^{p-2} - 12f_0 p [a_0 - (p-1)\xi_0] a^2 \dot{a} R^{p-1} + g_0$	$(\xi_0 t + \xi_1) [6f_0 p a \dot{a}^2 R^{p-1} + 6f_0 p(p-1) a^2 \dot{a} \dot{R} R^{p-2} + f_0 (p-1) a^3 R^p] +$ $-6f_0 p(p-1) a_0 a^3 \dot{R} R^{p-2} + 6f_0 \xi_0 p (a_0 - \xi_0) a^2 \dot{a} R^{p-1} + g_0$
$f_0 R^{\frac{3}{2}}$	$9f_0 \xi_1 a \dot{a}^2 R^{\frac{1}{2}} + \frac{9}{2} f_0 \xi_1 a^2 \dot{a} \dot{R} R^{-\frac{1}{2}} + \frac{1}{2} f_0 \xi_1 a^3 R^{\frac{3}{2}} - 9f_0 a_0 \dot{a} R^{\frac{1}{2}} - \frac{9}{2} f_0 a_0 \dot{R} R^{-\frac{1}{2}} + g_0$	$9f_0 \xi_1 a \dot{a}^2 R^{\frac{1}{2}} + \frac{9}{2} f_0 \xi_1 a^2 \dot{a} \dot{R} R^{-\frac{1}{2}} + \frac{1}{2} f_0 \xi_1 a^3 R^{\frac{3}{2}} - 9f_0 a_0 \dot{a} R^{\frac{1}{2}} - \frac{9}{2} f_0 a_0 \dot{R} R^{-\frac{1}{2}} + g_0$

# WHAT IF WE CONSIDER A MATTER TERM?

## CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \quad \alpha = \alpha(t, a, R), \quad \beta = \beta(t, a, R) \\ \text{EXTRA TERMS} \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta - 3\omega\rho_0\alpha a^{-3\omega-3} + \rho_0 \frac{\partial \xi}{\partial t} a^{-3\omega-2} = 0 \\ \alpha f_R + a f_{RR}\beta + 2a f_R \frac{\partial \alpha}{\partial a} + a^2 f_{RR} \frac{\partial \beta}{\partial a} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_{RR} + \beta a f_{RRR} + a \frac{\partial \alpha}{\partial a} f_{RR} + 2f_R \frac{\partial \alpha}{\partial R} - a f_{RR} \frac{\partial \xi}{\partial t} + a \frac{\partial \beta}{\partial R} f_{RR} = 0 \\ 12f_R \frac{\partial \alpha}{\partial t} + 6a f_{RR} \frac{\partial \beta}{\partial t} = 0 \\ 6f_{RR} \frac{\partial \alpha}{\partial t} = 0 \\ f_{RR} \frac{\partial \alpha}{\partial R} = 0 \end{array} \right.$$

## NON-CANONICAL CASE

$$\left\{ \begin{array}{l} \xi = \xi(t), \quad \alpha = \alpha(t, a), \quad \beta = \beta(t, a, R) \\ \text{EXTRA TERMS} \\ \left(3\alpha + a \frac{\partial \xi}{\partial t}\right) (f - Rf_R) - aRf_{RR}\beta - 6f_R \frac{\partial^2 \alpha}{\partial t^2} - 3\omega\rho_0\alpha a^{-3\omega-3} + \rho_0 \frac{\partial \xi}{\partial t} a^{-3\omega-2} = 0 \\ \alpha f_R + a f_{RR}\beta + 2a f_R \frac{\partial \alpha}{\partial a} + a^2 f_R \frac{\partial^2 \alpha}{\partial a^2} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2\alpha f_R + \beta f_{RR}a + a f_R \frac{\partial \alpha}{\partial a} - a f_R \frac{\partial \xi}{\partial t} = 0 \\ 2 \frac{\partial \alpha}{\partial t} - a \frac{\partial^2 \xi}{\partial t^2} = 0 \end{array} \right.$$

### CANONICAL CASE

$$\mathcal{L} = a^3(f - f_R R) + 6f_{RR} \dot{a} \dot{R} a^2 + 6f_R a \dot{a}^2 + \rho_0 a^{-3\omega}$$

### NON-CANONICAL CASE

$$\mathcal{L} = a^3(f - f_R R) - 6f_R a^2 \ddot{a} - 6f_R a \dot{a}^2 + \rho_0 a^{-3\omega}$$

# SELECTED MODELS

## CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$	$\mathcal{L}_m$
$\xi_1$	0	$\beta(a, R, t)$	$f_0R + \Lambda$	$\rho_0 a^{-3\omega}$
$3a_0t + \xi_1$	$a_0a$	$\beta(a, R, t)$	$f_0R + \Lambda$	$-\Lambda a^3 ; \omega = -1 ; \rho_0 = -\Lambda$
$\xi_1$	0	0	$f_0R^2 + f_1R + f_2$	$\rho_0 a^{-3\omega}$
$\xi_0t + \xi_1$	0	$\xi_0R + \xi_0 \frac{f_1}{2f_0}$	$f_0R^2 + f_1R + f_2$	$\rho_0 a^3 ; \omega = -1$
$a_0t + \xi_1$	$a_0a$	$-2a_0R$	$f_0R^2 + f_2$	$-f_2 a^3 ; \omega = -1 ; \rho_0 = -f_2$
$\xi_1$	0	0	$f_0R^p + \Lambda$ where $p \neq \{0, 1, 2\}$	$\rho_0 a^{-3\omega}$

## NON-CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$	$\mathcal{L}_m$
$\xi_1$	0	$\beta(a, R, t)$	$f_0R + \Lambda$	$\rho_0 a^{-3\omega}$
$\xi_1$	0	0	$f_0R^2 + f_1R + f_2$	$\rho_0 a^{-3\omega}$
$a_0t + \xi_1$	$a_0a$	$-2a_0R$	$f_0R^2$	$\rho_0 a^{-1} ; \omega = \frac{1}{3}$
$\xi_1$	0	0	$f_0R^p + \Lambda$ where $p \neq \{0, 1\}$	$\rho_0 a^{-3\omega}$
$-\frac{\beta_0}{4}t^2 + \xi_1$	$-\frac{\beta_0}{4}at$	$\beta_0Rt$	$f_0R^{\frac{5}{4}}$	$\rho_0 a^{-2} ; \omega = \frac{2}{3}$
$3\omega a_0t + \xi_1$	$a_0a$	$-6a_0\omega R$	$f_0R^p$ w $p = \frac{\omega + 1}{2\omega}$	$\rho_0 a^{-3\omega}$

# SELECTED MODELS

## CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$	$\mathcal{L}_m$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$	$\rho_0 a^{-3\omega}$
$3a_0 t + \xi_1$	$a_0 a$	$\beta(a, R, t)$	$f_0 R + \Lambda$	$-\Lambda a^3 ; \omega = -1 ; \rho_0 = -\Lambda$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$	$\rho_0 a^{-3\omega}$
$\xi_0 t + \xi_1$	0	$\xi_0 R + \xi_0 \frac{f_1}{2f_0}$	$f_0 R^2 + f_1 R + f_2$	$\rho_0 a^3 ; \omega = -1$
$a_0 t + \xi_1$	$a_0 a$	$-2a_0 R$	$f_0 R^2 + f_2$	$-f_2 a^3 ; \omega = -1 ; \rho_0 = -f_2$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1, 2\}$	$\rho_0 a^{-3\omega}$

## NON-CANONICAL CASE

$\xi(t)$	$\alpha(a)$	$\beta$	$f(R)$	$\mathcal{L}_m$
$\xi_1$	0	$\beta(a, R, t)$	$f_0 R + \Lambda$	$\rho_0 a^{-3\omega}$
$\xi_1$	0	0	$f_0 R^2 + f_1 R + f_2$	$\rho_0 a^{-3\omega}$
$a_0 t + \xi_1$	$a_0 a$	$-2a_0 R$	$f_0 R^2$	$\rho_0 a^{-1} ; \omega = \frac{1}{3}$
$\xi_1$	0	0	$f_0 R^p + \Lambda$ where $p \neq \{0, 1\}$	$\rho_0 a^{-3\omega}$
$-\frac{\beta_0}{4} t^2 + \xi_1$	$-\frac{\beta_0}{4} a t$	$\beta_0 R t$	$f_0 R^{\frac{5}{4}}$	$\rho_0 a^{-2} ; \omega = \frac{2}{3}$
$3\omega a_0 t + \xi_1$	$a_0 a$	$-6a_0 \omega R$	$f_0 R^p$ w $p = \frac{\omega + 1}{2\omega}$	$\rho_0 a^{-3\omega}$

# WHAT'S NEXT?

## $f(R)$ COSMOLOGY

- CALCULATE THE CONSERVED QUANTITIES FOR THE CASES WITH MATTER;
- DEVELOP THE HAMILTONIAN DESCRIPTION;
- SOLVE THE WHEELER-DE WITT EQUATION USING THE CONSERVED QUANTITIES;
- CHANGE THE METRIC.

## $f(G)$ COSMOLOGY

- APPLY THE NOETHER SYMMETRY APPROACH AND THEN, FIND SOLUTIONS W/O AND WITH MATTER IN BOTH CANONICAL AND NON CANONICAL CASE;
- DEVELOP THE HAMILTONIAN DESCRIPTION;
- SOLVE THE WHEELER-DE WITT EQUATION USING THE CONSERVED QUANTITIES;
- CHANGE METRIC;
- CONSIDER  $d$  – DIMENSIONAL CASE.

# CONCLUSIONS

1. Despite its successes, General Relativity is not the final theory of gravity.
2. There is no a priori reason to consider a gravitational action linear in the Ricci scalar.
3. The basic idea of modified theories of gravity is changing the gravitational side of the Einstein's field equations rather than the energy-matter content.
4. Too many models can fit observations and experiments because of the large number of free functions and parameters that they contain, so a physical criterion to select viable models is necessary.
5. Any relativistic theory of gravity must satisfy certain minimal requirements from phenomenological point of view.
6. Currently, there is no a modified theory able to adress all the shortcomings and, at the same time, to recover the already valid results we know. We can think of them like toy models. Maybe the solution is a combination of them.

# REFERENCES

- *Francesco Bajardi and Salvatore Capozziello. Noether Symmetries in Theories of Gravity. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11 2022.*
- Francesco Bajardi, Salvatore Capozziello, and Francesca Spinnato. *Symmetry-Based Selection of Gravitational Lagrangians via Noether Approach*. *Symmetry*, 18(4):570, 2026.
- Francesco Bajardi, Salvatore Capozziello, Tiziana Di Salvo, and Francesca Spinnato. *The Noether Symmetry Approach: Foundation and Applications: The Case of Scalar-Tensor Gauss–Bonnet Gravity*. *Symmetry*, 15(9):1625, 2023.

THANKS FOR  
YOUR ATTENTION!



GeomGravX

*10 years of Geometric Foundations  
of Gravity and eXtensions*

29th June 2026

