Gravity actions and symmetry breaking using Clifford algebras

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Outline

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1. Mathematical preliminaries: Clifford algebras
2. Formulations of general relativity
3. Clifford algebra formulation of gravity actions
4. Conclusion

Real Clifford algebras

- Ingredients:
  1. Real vector space $V \cong \mathbb{R}^n$ of dimension $n = p + q$.
  2. Symmetric bilinear form $\eta : V \times V \to \mathbb{R}$ of signature (p, q).

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- Construction:
  1. Free associative tensor algebra generated by V:

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{R} \oplus V \oplus V \otimes V \oplus \ldots \tag{1}$$

  2. Ideal $I(V, \eta)$ generated by elements

$$uv + vu - 2\eta(u, v)\mathbb{1}\,, \quad u, v \in V\,. \tag{2}$$

  3. Clifford algebra:

$$\mathrm{Cl}_{p,q} = \mathrm{Cl}(V, \eta) = T(V)/I(V, \eta)\,. \tag{3}$$

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⇝ Here: consider in particular $(p, q) = (4, 1)$ and $(p, q) = (2, 3)$.

Properties of $\mathrm{Cl}_{p,q}$

- Decomposition into subspaces:

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$\rightsquigarrow$ Possible to choose orthonormal basis $(\mathbb{e}_A, 0 \leq A < p+q)$ of V such that:

- Bilinear form:

$$\eta(\mathbb{e}_A, \mathbb{e}_B) = \eta_{AB}\,, \quad \mathbb{e}_A\mathbb{e}_B + \mathbb{e}_B\mathbb{e}_A = \eta_{AB}\mathbb{1}\,. \tag{7}$$

- Subspaces:

$$\mathrm{Cl}_{p,q}^{k} = \mathrm{span}\{\mathbb{e}_{A_1}\cdots\mathbb{e}_{A_k}|0 \leq A_1 < \ldots < A_k < p+q\}\,. \tag{8}$$

- Volume element:

$$\Omega = \mathbb{e}_0\cdots\mathbb{e}_{p+q-1}\,. \tag{9}$$

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$\Rightarrow$ Relation for left and right eigenspaces depends on k.

Spin algebras & decomposition

- Spin algebra identified with second subspace:

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- Explicit form of projectors for $v \in \mathfrak{g}$:

$$P_+vP_+ + P_-vP_- = \frac{v + YvY}{2} \in \mathfrak{h}\,, \tag{20a}$$

$$P_+vP_- + P_-vP_+ = \frac{v - YvY}{2} \in \mathfrak{z}\,. \tag{20b}$$

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Stelle-West action

- Dynamical fields:
  1. Spin(p, q) connection:
$$A \in \Omega^1_G(P, \mathfrak{g})\,. \tag{21}$$
  2. Symmetry breaking vector field:
$$y \in \Omega^0_G(P, V)\,, \quad \eta(y, y) = 1\,. \tag{22}$$
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⇝ Stelle-West action:
$$S[A, y, \lambda] = \int_M \kappa_y(F \wedge F) + \lambda(\eta(y, y) - 1)\,. \tag{25}$$

MacDowell-Mansouri action

- Dynamical fields:
  - Cartan connection:

$$A \in \Omega^1_H(P, \mathfrak{g})\,. \tag{26}$$

⇒ Algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ allows split $A = \omega + e$:

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$$F_{\mathfrak{h}} = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] + \frac{1}{2}[e \wedge e] = R + \frac{1}{2}[e \wedge e] \in \Omega^2(P, \mathfrak{h})\,, \tag{29a}$$

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$\rightsquigarrow$ MacDowell-Mansouri action:

$$S[A] = \int_M \mathrm{tr}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \,. \tag{30}$$

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⇝ Plebanski action requires only one half of the fields:

$$S[\omega, e] = \int_M \operatorname{tr}\left(R^+ \wedge \Sigma^+ + \Sigma^+ \wedge \Sigma^+ + \Phi(\Sigma^+ \wedge \Sigma^+)\right) \,. \tag{38}$$

Relation between gravity actions

Stelle-West

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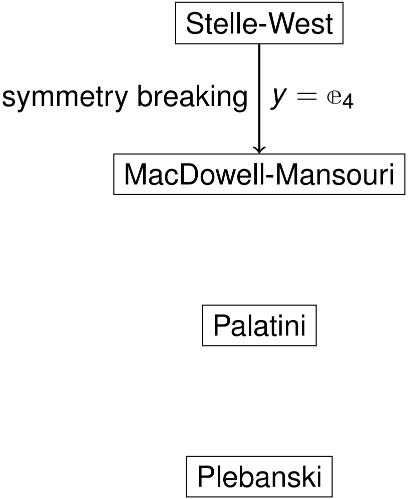
Plebanski

Relation between gravity actions



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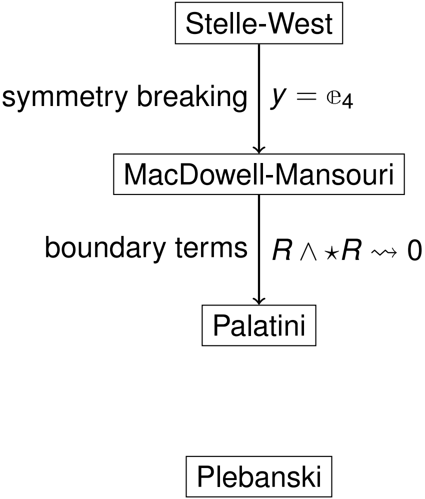

Relation between gravity actions

Stelle-West

↓ symmetry breaking — $y = \mathbb{e}_4$

MacDowell-Mansouri

↓ boundary terms — $R \wedge \star R \rightsquigarrow 0$

Palatini

↓ separation — $e \wedge e = \Sigma^+ + \Sigma^-$

Plebanski

Outline

Stelle-West action

- Interpretation of dynamical fields:
  1. Spin(p, q) connection:
  $$A \in \Omega^1_G(P, \mathrm{Cl}^2_{p,q}) \subset \Omega^1(P, \mathrm{Cl}_{p,q})\,. \tag{39}$$
  2. Symmetry breaking vector field:
  $$y \in \Omega^0_G(P, \mathrm{Cl}^1_{p,q}) \subset \Omega^0(P, \mathrm{Cl}_{p,q})\,. \tag{40}$$
  3. Lagrange multiplier:
  $$\lambda \in \Omega^0(P)\,. \tag{41}$$

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$$F = \mathrm{d}A + \frac{1}{2}[A \wedge A] \in \Omega^2(P, \mathrm{Cl}^2_{p,q}) \subset \Omega^2(P, \mathrm{Cl}_{p,q})\,. \tag{42}$$

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$$S[A, y, \lambda] = \int_M \mathrm{Tr}(\Omega y F \wedge F + \lambda(y^2 - \mathbb{1})) = \int_M \mathrm{Tr}(\Omega_y F \wedge F + \lambda(y^2 - \mathbb{1}))\,. \tag{44}$$

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- Proper subspaces can be ascertained using further Lagrange multipliers.

MacDowell-Mansouri action

- Decomposition of Cartan connection and curvature:

$$\omega = \frac{A + yAy}{2}\,, \qquad F_{\mathfrak{h}} = \frac{F + yFy}{2}\,, \tag{45a}$$

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Palatini action

- Curvature of the spin connection:

$$R = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = \mathrm{d}\frac{A + yAy}{2} + \frac{1}{2}\left[\frac{A + yAy}{2} \wedge \frac{A + yAy}{2}\right]. \tag{48}$$

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Plebanski action

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G-inv. Palatini

Palatini

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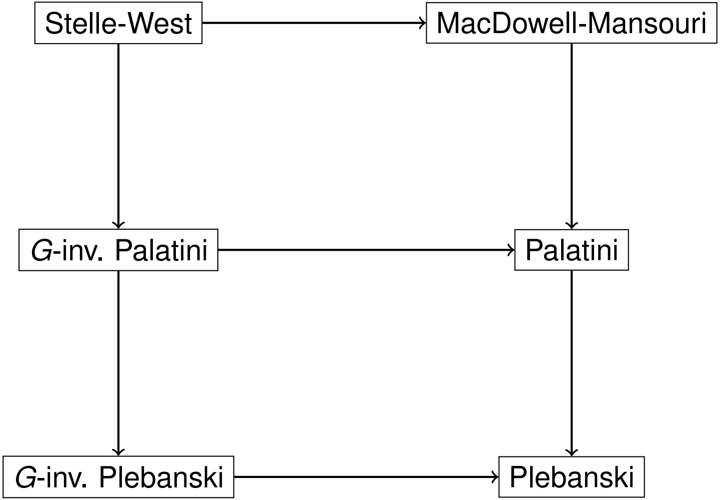

Outline

Conclusion

- Summary:
  1. Different gravity actions expressed through Clifford-algebra-valued forms.
     - ⋆ Stelle-West action.
     - ⋆ MacDowell-Mansouri action.
     - ⋆ Palatini action.
     - ⋆ Plebanski action.
  2. Possible to establish equivalence through Lagrange multipliers or projectors.
  3. All actions show $\mathrm{Spin}(p, q)$ symmetry due to gauge vector field y.
  4. Explicit symmetry breaking through Lagrange multiplier allows gauge fixing.
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- Outlook:
  1. Consider dynamical field y instead of fixed by Lagrange multiplier.
  2. Modified coupling to matter (fermions) through Clifford algebra.
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$\rightsquigarrow$ See upcoming publication with Paul Hafemann.