Gravitational Instantons in Spin(4) Gauge Formulation of Space-Time with Cartan-Khronon.

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Introduction - Instantons

Instantons are finite-action, classical solutions to the equations of motion in field theory, formulated on Euclidean spacetime.

Coined by physicist Gerard 't Hooft in 1976, they act as **"pseudoparticles"** that represent a field system locally "jumping" or tunneling between different energy states.

They represent configurations of a field not originating from field sources, but rather from the "shape or topology of space with respect to those fields".

Mimicking methods of deriving them, Eguchi and Hanson found a "vacuum solution with gravity" in 1977[1] and Hawking and Gibbons began classifying those Euclidean vacuum solutions according to the fixed-point structure of *Killing vectors*.[2]

Physicists use them to probe non-perturbative solutions as they represent heavy, stable, localized configurations of pure gravity.[3]

Theoretical Background - Spin(4) gauge theory of space, time and gravitation.[4]

The formalism of a standard Euclidean gravity theory is followed, using the classical first-order **Palatini–Cartan formalism** with a $spin(4)$ *Lie-algebra valued one-form connection* $A = A^{ij}{}_{\mu} J_{ij} dx^{\mu}$.

The Generators of the $so(4)$ algebra J^{ij} with the (inconvenient) commutation relations, are used to form the **topological invariants** and **action terms** of the theory.

The Chiral Decomposition and Reducibility of the $spin(4)$ algebra, comes by forming the basis of the generators so that they satisfy the $su(2)_+ \oplus su(2)_-$ algebra over $\mathbb{R}$, avoiding the complexification that occurs in the Lorentz $SO(1,3)$ case.

This construction leads to the decomposition of the $spin(4)$ connection to ${}^{\pm}A^{ij}$ in $SO(4)$ and ${}^{\pm}A^{i}$ in $SU(2)_{\pm}$ representation. By extension, we have the decomposition of the ***field strength*** in ${}^{\pm}F^{ij}$ and hence the invariants forming the action of the theory.

In *Palatini formulation* the soldering co-frame is usually defined as $e = e^{i}{}_{\mu} V_i dx^{\mu}$. Here, instead, we use the **Khronon** field ϕ^i, carrying the $SO(4)$ index i and can be regarded as a bilinear formed from the primordial spinor and the symmetry group taken as the double-cover $Spin(4)$.

The Khronon field selects a preferred Euclidean direction, interpreted as a **temporal direction** after symmetry reduction and through the foliation defined by its level sets.

The emergence of a preferred temporal direction can be understood as a reduction of symmetry at the level of solutions.

Method and Approach: Reproducing GR Instantons

Instanton solutions are intrinsically tied to the global shape and topology of the manifold, allowing us to classify allowed spacetime structures.

Confirming $Spin(4)$ instanton solutions indicates whether this gauge theory replicates and predicts classical general relativity backgrounds (Einstein manifolds) through a purely connection-based, geometric framework. The method can be described concretely as:

Choose a Khronon ansatz, construct the co-frame $e^i = D\phi^i$, derive the induced metric, and impose the Spin(4) constraints.

The Riemannian metric to confirm is:

$$ds^2 = f_0^2(x)dt^2 + g_0^2(x)(dx^1)^2 + h_0^2(x)d\Omega^2 \tag{1}$$

The Khronon scalar Field: $\phi^I = \kappa^{-1/2}\phi\delta^I_4$

Euclidean manifold M with: $x = \{x^1, x^2, x^3, x^4\} = \{x^1, x^2, x^3, t\}$

The $Spin(4)$ connection:

Co-frame by the electric components.

The Electric components

$$A^{14} = A^{14}(g,h,\phi),\quad A^{24} = A^{24}(g,h,\phi),\quad A^{34} = A^{34}(g,h,\phi).$$

The Magnetic components

$$A^{12} = A^{12}(X),\quad A^{13} = A^{13}(Y),\quad A^{23} = A^{23}(Z).$$

The Co-frame forms as:

$$e^I = D\phi^I = d\phi^I + A^I{}_J\phi^J,\ \ e^I = (A^I{}_4\phi, d\phi)$$

Set time coordinate t as:

$$d\phi = f(x)dt$$

with f, g, h, X, Y, Z scalar functions of spacetime x. The metric formed by this process is given by $g_{\mu\nu} = e^I{}_{\mu} e^J{}_{\nu}\delta_{IJ}$, and the line element is:

$$ds^2 = d\phi^2 + g^2(x)(dx^1)^2 + h^2(x)d\Omega^2 = f^2(x)dt^2 + g^2(x)(dx^1)^2 + h^2(x)d\Omega^2 \tag{2}$$

Choosing Right-Hand Model with $g_- = 0$:

$$I_{(2)} = \frac{1}{2}\int \epsilon_{ijkl} D\phi^i \wedge D\phi^j \left(g_+ {}^{+}F^{kl} + g_- {}^{-}F^{kl}\right) \tag{3}$$

Variation with respect **to the connection** constrain $X(x), Y(x), Z(x)$ with respect to $f(x), g(x), h(x)$.

$$T^i \wedge d\phi + \epsilon^i{}_{jk} T^j \wedge D\phi^k = 0 \tag{4}$$

Variation with respect to the **Khronon field** ϕ^I yields *energy* and *momentum* constraints.[4]

$${}^{+}F_i \wedge D\phi^i = 0, \qquad {}^{+}F^i \wedge D\phi - \epsilon^i{}_{jk} {}^{+}F^j \wedge D\phi^k = 0 \tag{5}$$

Confirming these equations with the ansatz $(f, g, h) = (\pm f_0, \pm g_0, \pm h_0)$, shows that the corresponding Riemannian metric is reproduced by the theory.

Results

While GR's topological invariants are derived from the curvature of the metric-compatible connection, the Spin(4) gauge theory produces a richer bundle topology. Predicted solutions will yield values for **topological invariants**, such as the Euler characteristic χ and the Hirzebruch signature τ.[5]

The underlying spacetime manifold is recovered as a specific linear combination of the Spin(4) instanton numbers. The GR sector is therefore identified as the reduction of the gauge topology to the Riemannian curvature of the broken phase.

Eguchi-Hanson Gravitational Instanton

Asymptotically Locally Euclidean Metric (ALE) Self Dual Solution

$$ds^2 = \frac{1}{(1-\frac{a^4}{r^4})}dr^2 + r^2(1-\frac{a^4}{r^4})(\sigma^3)^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \tag{6}$$

$$GR\ Topological\ Numbers:\ c_2 = p_1/2 = -3/2 \quad \chi = 2 \quad \tau = -1$$

$$Khronon\ field\ \phi^I = \delta^I_4 R\ \ with\ \ dR = f(r)dr$$

$$Electric\ \ A^{14} = \frac{r(R)}{R}\sigma^1 \quad A^{24} = \frac{r(R)}{R}\sigma^2 \ \ A^{34} = \frac{g(R)}{R}\sigma^3$$

$$Magnetic\ \ A^{12} = Z(r)\sigma^3\ \ A^{13} = -Y(r)\sigma^2\ \ A^{23} = X(r)\sigma^1$$

$$ds^2 = f^2(r)dr^2 + r^2g^2(r)(\sigma^3)^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \tag{7}$$

$$Ansatz:\ f_0 = \left(1 - \frac{a^4}{r^4}\right)^{-1/2} \quad g_0 = \left(1 - \frac{a^4}{r^4}\right)^{1/2}$$

4 Different Branches (f, g) confirming the constraints:

$$(+f_0, +g_0)\ \ (+f_0, -g_0)\ \ (-f_0, g_0)\ \ (-f_0, -g_0) \tag{8}$$

Overall, 2 different ${}^{+}A^i$ (with one collapsing to ${}^{+}A = 0$) and 1 set of topological invariants (reproducing GR) and 4 different ${}^{-}A^i$ with 4 divergent sets of topological invariants.

$${}^{+}A:\ \ c_2^+ = -3/2\ \ \chi^+ = 2 \quad \tau = -1$$
$${}^{-}A:\ \ c_2^- = \infty \quad \chi^- = \infty\ \ \tau^- = \infty$$

Taub-NUT Gravitational Instanton

Asymptotically Locally Flat Metric (ALF) Anti-Self-Dual solution

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}dr^2 + 4m^2\frac{r-m}{r+m}(\sigma^3)^2 + (r+m)(r-m)\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \tag{9}$$

$$GR\ Topological\ Numbers:\ c_2 = p_1/2 = 0 \quad \chi = 1 \quad \tau = 0$$

$$Khronon\ field\ \phi^I = \delta^I_4 R\ \ with\ \ dR = f(r)dr$$

$$Electric\ \ A^{14} = \frac{h(R)}{R}\sigma^1 \quad A^{24} = \frac{h(R)}{R}\sigma^2 \ \ A^{34} = \frac{g(R)}{R}\sigma^3$$

$$Magnetic\ \ A^{12} = Z(r)\sigma^3\ \ A^{13} = -Y(r)\sigma^2\ \ A^{23} = X(r)\sigma^1$$

$$ds^2 = f^2(r)dr^2 + g^2(r)(\sigma^3)^2 + h^2(r)\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \tag{10}$$

$$Ansatz:\ f_0(r) = \frac{1}{2}\sqrt{\frac{r+m}{r-m}},\ \ g_0(r) = 2m\sqrt{\frac{r-m}{r+m}}, \quad h_0(r) = \sqrt{r^2 - m^2},$$

8 Different Branches (f, g, h) confirming the constraints:

$$\begin{matrix} (+f_0, +g_0, +h_0) & (+f_0, -g_0, +h_0) & (+f_0, +g_0, -h_0) & (+f_0, -g_0, -h_0) \\ (-f_0, +g_0, +h_0) & (-f_0, -g_0, +h_0) & (-f_0, +g_0, -h_0) & (-f_0, -g_0, -h_0) \end{matrix} \tag{11}$$

Overall: 4 Different ${}^{+}A$ with 2 sets of topological numbers.

$$c_2^+ = 0 \quad \chi^+ = 2/3\ \ \tau = 2/3$$
$$c_2^+ = -1\ \ \chi^+ = 5/3\ \ \tau = 1/3$$

8 Different ${}^{-}A^i$ with only 1 set of topological numbers (reproducing GR):

$$c_2^- = 0\ \ \chi^- = 1\ \ \tau^- = 0$$

Conclusions

Confirming $Spin(4)$ instanton solutions indicates whether this gauge theory replicates and predicts classical general relativity backgrounds (Einstein manifolds) through a purely connection-based, geometric framework. **Overall**, known GR instantons are reproduced in one chiral sector, while the remaining $Spin(4)$ sectors reveal additional, sometimes divergent, topological structures. That nontrivial behavior of the remaining sectors is motivation for further investigation and interpretation.

References

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