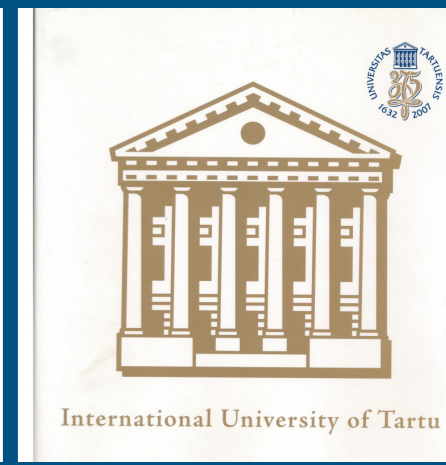
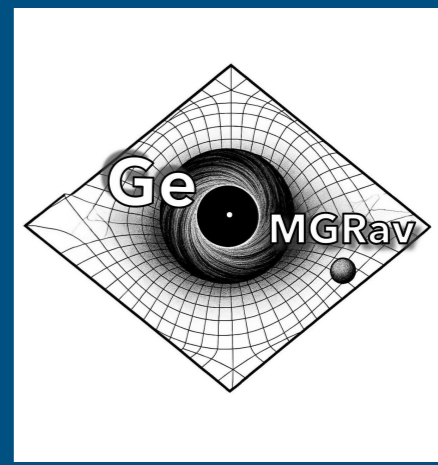


Gravitational Instantons in Spin(4) Gauge Formulation of Space-Time with Cartan-Khronon.

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Introduction - Instantons

Instantons are finite-action, classical solutions to the equations of motion in field theory, formulated on Euclidean spacetime.

Coined by physicist Gerard 't Hooft in 1976, they act as **"pseudoparticles"** that represent a field system locally "jumping" or tunneling between different energy states.

They represent configurations of a field not originating from field sources, but rather from the "shape or topology of space with respect to those fields".

Theoretical Background - Spin(4) gauge theory of space, time and gravitation.[4]

The formalism of a standard Euclidean gravity theory is followed, using the classical first-order **Palatini-Cartan formalism** with a *spin(4)* Lie-algebra valued one-form connection $A = A^{ij} J_{ij} dx^\mu$.

The **Generators** of the *so(4)* algebra J^{ij} with the (inconvenient) commutation relations, are used to form the **topological invariants** and **action terms** of the theory.

The **Chiral Decomposition and Reducibility** of the *spin(4)* algebra, comes by forming the basis of the generators so that they satisfy the $su(2)_+ \oplus su(2)_-$ algebra over \mathbb{R} , avoiding the complexification that occurs in the Lorentz $SO(1,3)$ case.

This construction leads to the decomposition of the *spin(4)* connection to $\pm A^{ij}$ in $SO(4)$ and $\pm A^i$ in $SU(2)_\pm$ representation. By extension, we have the decomposition of the **field strength** in $\pm F^{ij}$ and hence the invariants forming the action of the theory.

In **Palatini formulation** the soldering co-frame is usually defined as $e = e^i_\mu V_i dx^\mu$. Here, instead, we use the **Khronon** field ϕ^i , carrying the $SO(4)$ index i and can be regarded as a bilinear formed from the primordial spinor and the symmetry group taken as the double-cover *Spin(4)*.

The emergence of a preferred temporal direction can be understood as a reduction of symmetry at the level of solutions.

Method and Approach: Reproducing GR Instantons

Instanton solutions are intrinsically tied to the global shape and topology of the manifold, allowing us to classify allowed spacetime structures.

Confirming *Spin(4)* instanton solutions indicates whether this gauge theory replicates and predicts classical general relativity backgrounds (Einstein manifolds) through a purely connection-based, geometric framework. The method can be described concretely as:

Choose a Khronon ansatz, construct the co-frame $e^i = D\phi^i$, derive the induced metric, and impose the Spin(4) constraints.

The Riemannian metric to confirm is:

$$ds^2 = f_0^2(x) dt^2 + g_0^2(x) (dx^1)^2 + h_0^2(x) d\Omega^2 \quad (1)$$

The Khronon scalar Field:

$$\phi^I = \kappa^{-1/2} \phi^I_4$$

Euclidean manifold M with:

$$x = \{x^1, x^2, x^3, x^4\} = \{x^1, x^2, x^3, t\}$$

The *Spin(4)* connection:

Co-frame by the electric components.

The Electric components

$$A^{14} = A^{14}(g, h, \phi),$$

$$A^{24} = A^{24}(g, h, \phi),$$

$$A^{34} = A^{34}(g, h, \phi).$$

The Magnetic components

$$A^{12} = A^{12}(X),$$

$$A^{13} = A^{13}(Y),$$

$$A^{23} = A^{23}(Z).$$

The Co-frame forms as:

Set time coordinate t as:

$$e^I = D\phi^I = d\phi^I + A^I_J \phi^J, \quad e^I = (A^I_4 \phi, d\phi)$$

$$d\phi = f(x) dt$$

with f, g, h, X, Y, Z scalar functions of spacetime x . The metric formed by this process is given by $g_{\mu\nu} = e^I_\mu e^J_\nu \delta_{IJ}$, and the line element is:

$$ds^2 = d\phi^2 + g^2(x) (dx^1)^2 + h^2(x) d\Omega^2 = f^2(x) dt^2 + g^2(x) (dx^1)^2 + h^2(x) d\Omega^2 \quad (2)$$

Choosing Right-Hand Model with $g_- = 0$:

$$I_{(2)} = \frac{1}{2} \int \epsilon_{ijkl} D\phi^i \wedge D\phi^j (g_+^+ F^{kl} + g_-^- F^{kl}) \quad (3)$$

Variation with respect to the connection constrain $X(x), Y(x), Z(x)$ with respect to $f(x), g(x), h(x)$.

$$T^i \wedge d\phi + \epsilon^i_{jk} T^j \wedge D\phi^k = 0 \quad (4)$$

Variation with respect to the **Khronon field** ϕ^I yields **energy** and **momentum** constraints.[4]

$$+F_i \wedge D\phi^i = 0, \quad +F^i \wedge D\phi - \epsilon^i_{jk} +F^j \wedge D\phi^k = 0 \quad (5)$$

Confirming these equations with the ansatz $(f, g, h) = (\pm f_0, \pm g_0, \pm h_0)$, shows that the corresponding Riemannian metric is reproduced by the theory.

Mimicking methods of deriving them, Eguchi and Hanson found a "vacuum solution with gravity" in 1977[1] and Hawking and Gibbons began classifying those Euclidean vacuum solutions according to the fixed-point structure of *Killing vectors*. [2]

Physicists use them to probe non-perturbative solutions as they represent heavy, stable, localized configurations of pure gravity. [3]

Results

While GR's topological invariants are derived from the curvature of the metric-compatible connection, the Spin(4) gauge theory produces a richer bundle topology. Predicted solutions will yield values for **topological invariants**, such as the Euler characteristic χ and the Hirzebruch signature τ . [5]

The underlying spacetime manifold is recovered as a specific linear combination of the Spin(4) instanton numbers. The GR sector is therefore identified as the reduction of the gauge topology to the Riemannian curvature of the broken phase.

Eguchi-Hanson Gravitational Instanton

Asymptotically Locally Euclidean Metric (ALE) Self Dual Solution	
$ds^2 = \frac{1}{(1 - \frac{a^4}{r^4})} dr^2 + r^2 (1 - \frac{a^4}{r^4}) (\sigma^3)^2 + r^2 ((\sigma^1)^2 + (\sigma^2)^2) \quad (6)$	
$\underline{GR \ Topological \ Numbers : \ c_2 = p_1/2 = -3/2 \quad \chi = 2 \quad \tau = -1}$	
$Khronon \ field \ \phi^I = \delta^I_4 R \ \text{with} \ dR = f(r) dr$	
$Electric \ A^{14} = \frac{r(R)}{R} \sigma^1 \quad A^{24} = \frac{r(R)}{R} \sigma^2 \quad A^{34} = \frac{g(R)}{R} \sigma^3$	
$Magnetic \ A^{12} = Z(r) \sigma^3 \quad A^{13} = -Y(r) \sigma^2 \quad A^{23} = X(r) \sigma^1$	
$ds^2 = f^2(r) dr^2 + r^2 g^2(r) (\sigma^3)^2 + r^2 ((\sigma^1)^2 + (\sigma^2)^2) \quad (7)$	
$Ansatz : \ f_0 = \left(1 - \frac{a^4}{r^4}\right)^{-1/2} \quad g_0 = \left(1 - \frac{a^4}{r^4}\right)^{1/2}$	
4 Different Branches (f, g) confirming the constraints:	
$(+f_0, +g_0) \quad (+f_0, -g_0) \quad (-f_0, g_0) \quad (-f_0, -g_0) \quad (8)$	
Overall, 2 different $+A^i$ (with one collapsing to $+A = 0$) and 1 set of topological invariants (reproducing GR) and 4 different $-A^i$ with 4 divergent sets of topological invariants.	
$+A : \ c_2^+ = -3/2 \quad \chi^+ = 2 \quad \tau = -1$	
$-A : \ c_2^- = \infty \quad \chi^- = \infty \quad \tau^- = \infty$	

Taub-NUT Gravitational Instanton

Asymptotically Locally Flat Metric (ALF) Anti-Self-Dual solution	
$ds^2 = \frac{1}{4r - m} dr^2 + 4m^2 \frac{r - m}{r + m} (\sigma^3)^2 + (r + m)(r - m) ((\sigma^1)^2 + (\sigma^2)^2) \quad (9)$	
$\underline{GR \ Topological \ Numbers : \ c_2 = p_1/2 = 0 \quad \chi = 1 \quad \tau = 0}$	
$Khronon \ field \ \phi^I = \delta^I_4 R \ \text{with} \ dR = f(r) dr$	
$Electric \ A^{14} = \frac{h(R)}{R} \sigma^1 \quad A^{24} = \frac{h(R)}{R} \sigma^2 \quad A^{34} = \frac{g(R)}{R} \sigma^3$	
$Magnetic \ A^{12} = Z(r) \sigma^3 \quad A^{13} = -Y(r) \sigma^2 \quad A^{23} = X(r) \sigma^1$	
$ds^2 = f^2(r) dr^2 + g^2(r) (\sigma^3)^2 + h^2(r) ((\sigma^1)^2 + (\sigma^2)^2) \quad (10)$	
$Ansatz : \ f_0(r) = \frac{1}{2} \sqrt{\frac{r+m}{r-m}}, \quad g_0(r) = 2m \sqrt{\frac{r-m}{r+m}}, \quad h_0(r) = \sqrt{r^2 - m^2},$	
8 Different Branches (f, g, h) confirming the constraints:	
$\begin{aligned} & (+f_0, +g_0, +h_0) \quad (+f_0, -g_0, +h_0) \quad (+f_0, +g_0, -h_0) \quad (+f_0, -g_0, -h_0) \\ & (-f_0, +g_0, +h_0) \quad (-f_0, -g_0, +h_0) \quad (-f_0, +g_0, -h_0) \quad (-f_0, -g_0, -h_0) \end{aligned} \quad (11)$	
Overall: 4 Different $+A$ with 2 sets of topological numbers.	
$c_2^+ = 0 \quad \chi^+ = 2/3 \quad \tau = 2/3$	
$c_2^+ = -1 \quad \chi^+ = 5/3 \quad \tau = 1/3$	
8 Different $-A^i$ with only 1 set of topological numbers (reproducing GR):	
$c_2^- = 0 \quad \chi^- = 1 \quad \tau^- = 0$	

Conclusions

Confirming *Spin(4)* instanton solutions indicates whether this gauge theory replicates and predicts classical general relativity backgrounds (Einstein manifolds) through a purely connection-based, geometric framework. **Overall**, known GR instantons are reproduced in one chiral sector, while the remaining *Spin(4)* sectors reveal additional, sometimes divergent, topological structures. That nontrivial behavior of the remaining sectors is motivation for further investigation and interpretation.

References

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