Discrete 4D Geometry from Integer Arithmetic

Hurwitz Quaternions, Gauge Symmetries, and Two Compatible Metric Signatures

Petri Muinonen *Vantaa, Finland* *natpole@petrimuinonen.eu*
independent researcher

Abstract

We present a framework, Logarithmic Pole Model (LPM) in which two compatible 4D spatial geometries emerge from sheer arithmetic properties of a growing set of positive integers. Adding integers, conceived as time, combined with Lagrange's four-square theorem provides the dimensional foundation.

Integer distances admit decomposition into four coordinates, with multiple decompositions related by Hurwitz quaternion rotations. The 24 unit Hurwitz integers — eight with integer coefficients (±1, ±i, ±j, ±k) and sixteen with half-integer coefficients ((±1 ± i ± j ± k)/2) — generate the binary tetrahedral group 2T without postulation.

Discrete symmetries yield the Standard Model U(1) and SU(2) gauge holonomy through quaternion products in closed vertex loops, providing a arithmetic realization without field continuum. Results from this source are given as a table to demonstrate the power of the approach.

1 Integer arithmetic forces four dimensions

By Lagrange's four-square theorem, every positive integer N decomposes as:

$$N = a^2 + b^2 + c^2 + d^2$$

The integers (a,b,c,d) are not arbitrary labels: all decompositions of the same N sit on a common S^3 of radius √N, and this set is closed under multiplication by the 24 unit elements of 2T which is precisely what makes them coordinates in a physically meaningful sense.

S_N is invariant under 2T

For any integer N > 0, define $S_n = \{q \in \mathcal{H} : N(q) = N\}$ — the set of all Hurwitz quaternions of norm N. Since N(uq) = N(u)N(q) = 1·N = N for any unit u ∈ 2T, and likewise N(qu) = N, the set S_n is stable under left and right multiplication by 2T. The case N = 1 is special: S_1 = 2T is itself a group. For N > 1, S_N is not a group but decomposes into finite orbits under the 2T action — the discrete quaternionic symmetry compatible with the binary tetrahedral / 24-cell structure.

Four-dimensional geometry thus becomes a theorem of integer arithmetic, not a postulate.

2 Hurwitz quaternions relate all decompositions

The 24 unit Hurwitz quaternions:

8 with integer coefficients: ±1, ±i, ±j, ±k

16 with half-integer coefficients: (±1 ± i ± j ± k)/2

generate the binary tetrahedral group 2T for every decomposition of N (in panel 1) and can be rotated into every other by elements of 2T acting on (a,b,c,d).

2T is a finite subgroup of SU(2) — the group of unit quaternions, isomorphic to S^3. The 2T ⊂ SU(2) embedding is an algebraic fact, independent of any continuum assumption. It is applied here to integer arithmetic rather than a continuum [Jansson arXiv:2409.15385 for 2T within SU(2) and electroweak quantum numbers].

3 Metric signatures as distinct projections

Euclidean (all-positive): $D = a^2 + b^2 + c^2 + d^2$

Describes non-observable spatial distances across simultaneity — two poles at the same vertex-count N on S^3.

Minkowski (one-minus): $D = a^2 + b^2 + c^2 - d^2$

Describes distance into an observable past — one pole receiving a causal signal from an earlier epoch at smaller N. This is an appearance: what we observe is not there anymore.

Neither signature requires a symmetry-breaking field. No Hurwitz element ever produces a minus sign — all 24 preserve $a^2 + b^2 + c^2 + d^2 = D$ exactly. The minus sign enters only after a decomposition is chosen, when one coordinate is designated as the elapsed-causal one by a new causal relation between the two poles. If d is so designated, the past observation reads $a^2 + b^2 + c^2 - d^2 = D - 2d^2$, while the Euclidean form remains $a^2 + b^2 + c^2 + d^2 = D$.

The half-integer Hurwitz elements permute the three imaginary coordinates (b,c,d), some with sign change, leaving the real part fixed. This cycling — not a sign change — is the LPM counterpart of t → it in the continuum: **Wick rotation** as geometric designation, not analytic continuation.

4 Gauge symmetries & particle quantum numbers

Particles under LPM view are aggregates in superposition of all causal orderings of constituent poles.

U(1) and SU(2) are subgroups of 2T acting on *internal* pole-to-pole vectors between the poles — not on external causal relations between particles. Gauge transformations permutate internal geometry of the particle. Aggregates settle into 4D symmetry, e.g. pentatope for proton (5 nits).

Electric charge (fundamental fermions):
Q = (#pits / 3) × (−1)^(#pits) × e
Composite particles sum their constituent quark charges.

Mass (factorial from counting causal permutations , **no free parameters** apart from proton 9! as reference):
m = m_p × (N−1)! / 9! × [pentatopic reduction]

* Higgs charge: open problem under the pit-parity rule. The Higgs mass carries a $(25/24)^{1/4}$ circumradius correction arising from its 6 pits forming a 5-simplex rather than the proton's 4-simplex pentatope, the exact rational ratio of their circumradii. Without it, Higgs mass prediction would match only to within 1%.

Particle	pits	nits	charge	LPM mass	observed	error
Leptons						
electron	3	3	−1 e	0.517 MeV	0.511 MeV	+1.2%
muon	3	6	−1 e	104.25 MeV	105.66 MeV	−1.3%
tau	3	9	−1 e	1720 MeV	1776.9 MeV	−3.2%
Quarks (confined — no free mass)						
up	2	1	+2/3 e	—	—	
charm	2	2	+2/3 e	—	—	
top	2	3	+2/3 e	—	—	
down	1	3	−1/3 e	—	—	
strange	1	4	−1/3 e	—	—	
bottom	1	5	−1/3 e	—	—	
Baryons						
proton (uud)	5	5	+1 e	reference (9!)	938.27 MeV	—
neutron (udd)	4	7	0	938.3 MeV	939.57 MeV	−0.1%
Boson						
Higgs	6	7	—*	125.12 GeV	125.25 GeV	−0.10%