

SPACETIME ENERGY IN GENERAL PARALLEL GRAVITY

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Electrodynamics in curved spacetime

The electromagnetic field strength is given by the 4-potential as

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

Electrodynamics action consists of the field and matter parts,

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} [L_{\text{em}} + L_{\text{m}}] \quad (2)$$

where the definition of the theory is encoded in the conjugate tensor $H^{\mu\nu}$ as a function of the field strength and encapsulated in the Lagrangian (in Gaussian units)

$$\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} H^{\mu\nu}. \quad (3)$$

In vacuum Maxwell electrodynamics the constitutive relation between the field $F^{\mu\nu}$ and "excitation" $H^{\mu\nu}$ is very simple,

$$H_{\mu\nu} = F_{\mu\nu}, \quad H^{\mu\nu} = F^{\mu\nu}, \quad (4)$$

but in media or in generalised (nonlinear) theories $H^{\mu\nu}$ can take a more elaborate form. Varying the action with respect to the potential A_μ and integrating by parts gives the inhomogeneous Maxwell equation

$$\nabla_\mu H^{\mu\nu} = 4\pi J^\nu. \quad (5)$$

where the electric 4-current density is

$$J_\mu = -\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} L_{\text{m}})}{\partial A^\mu}. \quad (6)$$

The total electric charge contained in a region V on a spacelike hypersurface Σ_t (corresponding to time moment t) is given by the spatial integral over the charge density,

$$Q \equiv \int_V \rho \sqrt{\gamma} d^3x = - \int_V n_\mu J^\mu \sqrt{\gamma} d^3x. \quad (7)$$

where $\gamma_{\mu\nu}$ is the induced 3-metric on Σ_t and n_μ is the future-directed unit normal to Σ_t . Using the Maxwell equation and Gauss (Stokes) theorem we can rewrite this as an integral over the surface S that encloses V ,

$$Q = \frac{1}{4\pi} \oint_S n_\mu H^{\mu i} s_i \sqrt{\sigma} d^2x, \quad (8)$$

where s_{ij} is the induced 2-metric on S and s_i is the outward-directed unit normal to S (note the electric field $E^i = n_\mu H^{\mu i}$).

Point charge in static spherically symmetric spacetime

Let us take the usual spherical coordinates and consider a charge q located at the origin $r = 0$ and at rest. The spacetime around the charge may be described by a static spherically symmetric line element with zero shift,

$$ds^2 = -N(r)^2 dt^2 + A(r)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (9)$$

and the potential

$$A_\mu = \left[\frac{q}{r} \ 0 \ 0 \ 0 \right]. \quad (10)$$

On a spacelike hypersurface Σ_t at a fixed time moment t we can take a 2-sphere of arbitrary radius, and the Gauss law (8) computes the charge,

$$Q = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \left(\frac{q}{Ar^2} \right) A(r^2 \sin\theta) d\theta = q.$$

General relativity

The action of general relativity (GR) is given by the Einstein-Hilbert and matter terms,

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} [L_{\text{EH}} + L_{\text{m}}] = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{1}{16\pi G_{\text{N}}} R + L_{\text{m}} \right], \quad (11)$$

where the Ricci scalar is computed from Levi-Civita connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \quad (12)$$

via the curvature tensor

$$R^\lambda_{\rho\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\alpha} \Gamma^\alpha_{\nu\rho} - \Gamma^\lambda_{\nu\alpha} \Gamma^\alpha_{\mu\rho} \quad (13)$$

and contractions

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (14)$$

Variation of the action w.r.t. the metric gives Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\text{N}} T_{\mu\nu} \quad (15)$$

where the matter energy-momentum tensor is

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial(\sqrt{-g} L_{\text{m}})}{\partial g^{\mu\nu}} = g_{\mu\nu} L_{\text{m}} - 2 \frac{\partial \mathcal{L}_{\text{m}}}{\partial g^{\mu\nu}}. \quad (16)$$

Many theoreticians starting with Einstein have attempted to rewrite the equation in a form similar to (5), but inevitably the result is not unique and can not be made covariant.

General affine connection

We can write the general affine connection as

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + N^\lambda_{\mu\nu}, \quad (17)$$

where the distortion tensor $N^\lambda_{\mu\nu}$ encodes the difference of the general connection from the Levi-Civita connection (12). It can be conveniently decomposed into

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + K^\lambda_{\mu\nu} + L^\lambda_{\mu\nu}, \quad (18)$$

i.e. the Levi-Civita connection, contorsion

$$K^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (T_{\mu\alpha\nu} + T_{\nu\alpha\mu} + T_{\alpha\mu\nu}),$$

and disformation

$$L^\lambda_{\mu\nu} = -\frac{1}{2} g^{\lambda\alpha} (Q_{\mu\alpha\nu} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}). \quad (19)$$

Torsion encodes the antisymmetric part of the connection,

$$T^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\nu\mu}, \quad (20)$$

and nonmetricity measures the metric incompatibility

$$Q_{\rho\mu\nu} = \hat{\nabla}_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \hat{\Gamma}^\alpha_{\rho\mu} g_{\alpha\nu} - \hat{\Gamma}^\alpha_{\rho\nu} g_{\alpha\mu} \quad (21)$$

of the general covariant derivative $\hat{\nabla}$. The general curvature and its contractions are defined like in the Riemannian case (13), (14). It is possible to rewrite the curvature of general affine connection in terms of the Riemannian curvature plus distortion terms as

$$\hat{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + N^\rho_{\mu\lambda} N^\lambda_{\nu\sigma} - N^\rho_{\nu\lambda} N^\lambda_{\mu\sigma} + \nabla_\mu N^\rho_{\nu\sigma} - \nabla_\nu N^\rho_{\mu\sigma},$$

while the contractions of this formula give

$$\begin{aligned} \hat{R} &= R + N^\rho_{\rho\lambda} N^\lambda_{\nu\sigma} g^{\sigma\nu} - N^\rho_{\nu\lambda} N^\lambda_{\rho\sigma} g^{\sigma\nu} \\ &\quad + \nabla_\rho (N^\rho_{\nu\sigma} g^{\sigma\nu}) - \nabla_\nu (N^\rho_{\rho\sigma} g^{\sigma\nu}) \\ &= R + \mathbb{G} + B_{\mathbb{G}}, \end{aligned} \quad (22)$$

distinguishing the bulk and boundary (divergence) terms.

Teleparallel geometry

Teleparallel geometry is defined by the condition that the curvature tensor vanishes,

$$\hat{R}^\rho_{\sigma\mu\nu} \equiv 0. \quad (23)$$

General teleparallel connection can be written in terms of real invertible matrices $L^a_{\mu}(x) \in GL(4, \mathbb{R})$ as

$$\hat{\Gamma}^\lambda_{\mu\nu} = (L^{-1})^\lambda_a \partial_\mu L^a_{\nu}, \quad (24)$$

whereby the terms in the curvature tensor definition all cancel because $(L^{-1})^\lambda_a L^a_{\mu} = \delta^\lambda_{\mu}$. In a given coordinate system the matrices L^a_{μ} are not unique in a sense that arbitrary local transformation by real invertible matrices $\xi^a_b(x) \in GL(4, \mathbb{R})$,

$$L^a_{\mu} = \xi^a_b L^b_{\mu}, \quad (25)$$

will not alter the basic property of vanishing curvature (23) although the value of the connection components $\hat{\Gamma}^\lambda_{\mu\nu}$ does change.

General teleparallel equivalent of general relativity (GTEGR)

The teleparallel condition (23) allows us to rewrite the Einstein-Hilbert action via (22) as the scalar \mathbb{G} and a boundary term $B_{\mathbb{G}}$. The latter does not affect the field equations in spacetime bulk and we drop it, focusing on the action

$$\begin{aligned} S &= \int_{\mathcal{M}} d^4x \sqrt{-g} [L_{\text{g}} + L_{\text{m}}] \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{16\pi G_{\text{N}}} \mathbb{G} + L_{\text{m}} \right]. \end{aligned} \quad (26)$$

Here the geometric Lagrangian L_{g} depends on the metric and connection encoded in torsion and nonmetricity as represented in the scalar \mathbb{G} . The matter Lagrangian L_{m} remains completely unaltered from its GR form, and we can expect the same field equations as in GR.

To mimic the action of electrodynamics, we can rewrite L_{g} in terms of the torsion and nonmetricity fields along with their respective conjugates ("excitations"),

$$L_{\text{g}} = \frac{1}{16\pi G_{\text{N}}} (T^\alpha_{\mu\nu} S^{\mu\nu} + Q^\alpha_{\mu\nu} P^{\mu\nu}), \quad (27)$$

where

$$\begin{aligned} S_\alpha^{\mu\nu} &= -\frac{1}{4} T^\alpha_{\mu\nu} + \frac{1}{2} T^{[\mu\nu]}_\alpha - \delta^{[\mu}_\alpha T^{\nu]} \\ &\quad - \frac{1}{2} Q^{[\mu\nu]}_\alpha - \frac{1}{2} \delta^{[\mu}_\alpha Q^{\nu]} + \frac{1}{2} \delta^{[\mu}_\alpha \tilde{Q}^{\nu]}, \\ P^\alpha_{\mu\nu} &= -\frac{1}{4} Q^\alpha_{\mu\nu} + \frac{1}{2} Q_{(\mu\nu)}^\alpha + \frac{1}{4} Q^\alpha g_{\mu\nu} - \frac{1}{4} (\tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu} Q_{\nu)}) \\ &\quad + \frac{1}{2} T_{(\mu\nu)}^\alpha + \frac{1}{2} T^\alpha g_{\mu\nu} - \frac{1}{2} \delta^\alpha_{(\mu} T_{\nu)}. \end{aligned}$$

Variation of the action w.r.t. the metric yields

$$\left(\hat{\nabla}_\alpha + T_\alpha + \frac{1}{2} Q_\alpha \right) P^\alpha_{\mu\nu} = 4\pi G_{\text{N}} \left(g_{\mu\nu} L_{\text{g}} - 2 \frac{\partial L_{\text{g}}}{\partial g^{\mu\nu}} + T_{\mu\nu} \right). \quad (28)$$

Unravelling all the terms in (28) would make torsion and non-metricity contributions to cancel out and manifest that this equation is completely equivalent to Einstein's equation (15). Variation of the action (27) w.r.t. to the teleparallel connection yields

$$\left(\hat{\nabla}_\alpha + T_\alpha + \frac{1}{2} Q_\alpha \right) (S_\mu^{\alpha\nu} - P^{\alpha\nu}_\mu) \equiv 0, \quad (29)$$

which is not an independent field equation, but a geometric identity reflecting the quasi-invariance of the action under flatness preserving variations of the teleparallel connection as parametrised by the matrices L^a_{μ} .

By using the connection equation we can maneuver the torsion conjugate to replace the nonmetricity conjugate in the metric equation, and then contract the first spacetime index, $S_\alpha^{\mu\nu} = S_\rho^{\mu\nu} (L^{-1})^\rho_\alpha$, to obtain a manifestly antisymmetric object similar to the electromagnetic conjugate tensor. These operations yield

$$\nabla_\mu S_\alpha^{\mu\nu} = 4\pi G_{\text{N}} (t_\alpha^\nu + T_\alpha^\nu), \quad (30)$$

where the geometric terms on r.h.s. are given by

$$t_\mu^\nu = \delta_\mu^\nu L_{\text{g}} - Q_\mu^{\alpha\beta} P^\nu_{\alpha\beta} - 2T_{\alpha\beta\mu} S^{\alpha\beta\nu}. \quad (31)$$

General parallel relativity (G_{||}R)

Using the freedom to make teleparallel frame transformations (25), we can pick a gauge (choose the matrix L^a_{μ} that generates the connection $\hat{\Gamma}^\lambda_{\mu\nu}$) such that $t_\mu^\nu = 0$. In this "canonical" frame the gravitational field equation (30) becomes [1]

$$\nabla_\mu S_\alpha^{\mu\nu} = 4\pi G_{\text{N}} T_\alpha^\nu, \quad (32)$$

which is truly akin to the inhomogeneous Maxwell equation (5). Now analogously to the electric charge (8), the same steps lead to the definition of energy and momentum charge, obtainable by a surface integral at arbitrary radius,

$$P_a = \frac{1}{4\pi} \oint_S n_\mu S_\alpha^{\mu i} s_i \sqrt{\sigma} d^2x. \quad (33)$$

Kerr-Newman spacetime

As a nontrivial example, let us take KN metric, described with the following line element in Boyer-Lindquist coordinates:

$$\begin{aligned} ds^2 &= \left(\frac{2mr - q^2}{\rho^2} - 1 \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &\quad - \frac{(2amr + q^2 a) \sin^2\theta}{\rho^2} dt d\phi \\ &\quad + \frac{a^4 + r^4 + 2a^2 r^2 - a^4 \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2 \\ &\quad - \frac{a^2 r^2 \sin^2\theta + a^2 q^2 \sin^2\theta - 2a^2 m r \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2, \end{aligned} \quad (34)$$

with $\rho = \sqrt{r^2 + a^2 \cos^2\theta}$ and $\Delta = a^2 - 2mr + r^2 + q^2$. The parameters a, q are the angular momentum and the electromagnetic charge, respectively. The electromagnetic 4-potential is given by

$$A_\mu = \left[\frac{3rq}{3\rho^2} \ 0 \ 0 \ -\frac{3arq \sin^2\theta}{3\rho^2} \right]. \quad (35)$$

The canonical frame condition $t_\mu^\nu = 0$ is solved by the matrix

$$(L^{-1})^a_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{r\zeta_+ V \sin(\theta)}{\rho^2} & \frac{r\zeta_+ \sin(\theta)}{\rho^2} & \frac{\zeta_+ \cos(\theta)}{\rho^2} & -\frac{r \sin(\Phi) U - a \cos(\Phi) W}{\Delta \rho^2 \sin(\theta)} \\ -\frac{r\zeta_- V \sin(\theta)}{\rho^2} & \frac{r\zeta_- \sin(\theta)}{\rho^2} & \frac{\zeta_- \cos(\theta)}{\rho^2} & \frac{r \cos(\Phi) U - a \sin(\Phi) W}{\Delta \rho^2 \sin(\theta)} \\ -\frac{ZV \cos(\theta)}{\rho^2} & \frac{Z \cos(\theta)}{\rho^2} & -\frac{r \sin(\theta)}{\rho^2} & -\frac{aV \cos(\theta)}{\rho^2} \end{bmatrix},$$

where $\zeta_\pm = a \sin\Phi \pm r \cos\Phi$, $\tilde{\zeta}_\pm = r \sin\Phi \pm a \cos\Phi$, $Z = a^2 + r^2$, $V = \frac{a^2 + r^2 - \Delta}{\Delta}$, $U = a^2 \sin^2\theta - \Delta$, $W = r^2 \sin^2\theta + \cos^2\theta \Delta$, $\Phi = -\phi - \int \frac{a}{\Delta} dr$. By taking the unit normal $n_\mu = \left[0, \frac{\rho}{\sqrt{r^2 + a^2}}, 0, 0 \right]$ we compute the gravitational excitation energy to be [2]

$$\begin{aligned} P_0 &= \frac{1}{4\pi} \oint n_\mu S_0^{\mu 0} dS \\ &= m - \frac{q^2}{4r} - \frac{aq^2 \operatorname{atan}\left(\frac{a}{r}\right)}{4r^2} - \frac{q^2 \operatorname{atan}\left(\frac{a}{r}\right)}{4a}. \end{aligned} \quad (36)$$

Adding the Maxwell excitation energy, the final result is

$$E = m + \frac{q^2 r}{4(r^2 + a^2)}. \quad (37)$$

References

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