Geometric foundations of gravity and eXtended geometry

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General relativity is conventionally described by the Einstein-Hilbert action in terms of the curvature of the Levi-Civita connection. However, this formulation does not come without foundational issues. It needs to be supplemented by boundary terms in order for it to have a well-defined variational principle as well as finite sensible derivations of quantities such as mass. Boundary terms can be added ad-hoc, or by changing the starting point for the formulation of general relativity by the geometric condition of teleparallelism. Thus, changing the geometric foundations of gravity. It also turns out that in the field of extended geometry the Eintein-Hilbert action is insufficient as a starting point, while teleparallel theories of gravity works. This poster presentation will give an overview of the different formulations of teleparallel formulations of general relativity, what extended geometry is and why it is motivated.

General teleparallel equivalent to GR

General relativity described by the Einstein-Hilbert action assumes a connection that is torsion-free (symmetric in its last two indices) and metric compatible:

$$\overset{L.C}{\nabla}_{\rho} g_{\mu\nu} = 0, \; \overset{L.C}{\Gamma}{}^{\rho}{}_{[\mu\nu]} = 0. \tag{1}$$

The Einstein-Hilbert action is then given by

$$S_{\text{EH}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R(\overset{L.C}{\Gamma}). \tag{2}$$

Definition: The *teleparallel condition* assumes that the Riemannian curvature tensor of an affine connection vanishes:

$$R^{\alpha}{}_{\beta\mu\nu} = 2\partial_{[\mu}\Gamma^{\alpha}{}_{|\beta|\nu]} + 2\Gamma^{\alpha}{}_{\lambda|\mu}\Gamma^{\lambda}{}_{|\beta|\nu} = 0. \tag{3}$$

Defining the *distortion* tensor $N^{\rho}{}_{\mu\nu}$ via the relation

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{L.C}{\Gamma}{}^{\rho}{}_{\mu\nu} + N^{\rho}{}_{\mu\nu}, \tag{4}$$

it can after some calculations be shown that **under the teleparallel condition**

$$S_{\text{EH}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(2N^{\mu}{}_{\tau[\rho}N^{\tau}{}_{|\nu|\sigma]} - 2\overset{L.C}{\nabla}{}_{[\mu}N^{\mu\nu}{}_{\nu]}\right). \tag{5}$$

Note here that the last term is a boundary term. If one supplement the Einstein-Hilbert action by a counterterm to cancel it we get a theory which:

- Has a well-defined variational principle
- Have a finite value, which means that there are sensible definition for quantities such as **mass** in GR
- The formalism of **extended geometry works!**

General relativity as a gauge theory

The teleparallel condition implies that the connection becomes integrable and can for $L^{\mu}{}_{\nu} \in GL(d,\mathbb{R})$ be written as

$$\Gamma^{\rho}{}_{\mu\nu} = L^{\rho}{}_{\lambda}\partial_{\mu}L^{\lambda}{}_{\nu}. \tag{6}$$

The distortion tensor N contains non-metricity and torsion and relates in the following way

$$2N^{\rho}{}_{\mu\nu} = P^{\rho}{}_{\mu\nu} - P_{\mu\nu}{}^{\rho} - P_{\nu\mu}{}^{\rho} + \Theta^{\rho}{}_{\nu\mu} + \Theta^{\rho}{}_{\mu\nu} + \Theta_{\mu\nu}{}^{\rho} \tag{7}$$

Torsion is simply the antisymmetric part of the connection $\Theta_{\mu}{}^{\rho}{}_{\nu} := \Gamma^{\rho}_{[\mu\nu]}$, and non-metricity is given by

$$P_{\rho\mu\nu} = \nabla_{\rho}g_{\mu\nu} = \partial_{\rho}g_{\mu\nu} - 2\Gamma_{(\mu\nu)\rho}. \tag{8}$$

Note two things of this framework for describing gravity

1. There always exists a gauge such that Γ trivializes known as the coincident gauge
2. Our canonical fields are the metric $g_{\mu\nu}$ and the matrices $L^{\rho}{}_{\sigma}$. The 16 extra degrees of freedom compared to standard Einstein-Hilbert formulation are purely gauge.

Trinity of gravity

If we neglect the boundary term of (5) and expand distortion and write it explicitly in terms of non-metricity and torsion, we get the teleparallel equivalent to GR

$$S_{||} = S^{(P)} + S^{(\Theta)} + S^{(M)}, \tag{9}$$

where $S^{(P)}$ only depends on non-metricity, S^{Θ} only depends on torsion and $S^{(M)}$ represents terms mixing torsion with non-metricity:

$$S^{(P)} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{4}P_{\rho\mu\nu}P^{\rho\mu\nu} - \frac{1}{2}P_{\rho\mu\nu}P^{\mu\nu\rho} - \frac{1}{4}p_{\mu}p^{\mu} + \frac{1}{2}p_{\mu}\bar{p}^{\mu}\right) \tag{10}$$

$$S^{(\Theta)} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{4}\Theta_{\mu\nu\rho}\Theta^{\mu\nu\rho} + \frac{1}{2}\Theta_{\mu\nu\rho}\Theta^{\mu\rho\nu} - \theta_{\mu}\theta^{\mu}\right) \tag{11}$$

$$S^{(M)} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(-P_{\rho\mu\nu}\Theta^{\rho\mu\nu} + p_{\mu}\theta^{\mu} - \bar{p}_{\mu}\theta^{\mu}\right) \tag{12}$$

Note that (5) will still hold if we restrict our geometry to be either metric compatible or torsion free. (10) assumes the teleparallel condition and zero torsion and is known as the *symmetric teleparallel equivalent to general relativity*, while (11) assumes the teleparallel condition and a metric compatible connection (no non-metricity) and is known as the (metric) *teleparallel equivalent to general relativity.*

Extended geometry

Extended geometry is a framework based on the idea of "geometrising" duality symmetries that appear in dimensional reduction of supergravity theories. This requires generalisations of geometrical concepts, such as vector fields and Lie derivatives, that are used to describe pure gravity in d spacetime dimensions. The role of $\mathfrak{sl}(d)$ and its d-dimensional highest-weight representation are in these generalisations played by any Kac–Moody algebra $\mathfrak{g}$ and any irreducible highest-weight representation of $\mathfrak{g}$ with a dominant integral weight λ as the highest weight. In the special case of double geometry, $\mathfrak{g}$ is the Lie algebra $\mathfrak{so}(d,d)$, and λ is the highest weight of its $2d$-dimensional vector representation.

Notation for parallel extended geometry

In order two incorporate the indices taking values in higher dimensions, we denote them with latin letters $\{m,n,p,..\}$. For convenience we assume that they are flat. We use hats on torsion and non-metricity $\hat{\Theta}, \hat{P}$ to denote the corresponding trace-free tensors. Traces are denoted by θ, π, p. Factors depending on the dimensions appear when taking traces of the metric making it convenient to introduce

$$(\lambda,\lambda) = \frac{d-1}{d}. \tag{13}$$

We use the matrices

$$t^{p}{}_{qm}{}^{n} = \delta^{p}_{m}\delta^{n}_{q}, \qquad G^{m}{}_{n}{}^{p}{}_{q} = \eta^{mp}{}_{nq}, \qquad \eta^{m}{}_{n}{}^{p}{}_{q} = \delta^{p}_{n}\delta^{m}_{q}. \tag{14}$$

Lagrangian for parallel extended geometry

We propose the Lagrangian given by

$$\mathcal{L}_{||} = \mathcal{L}^{(\Pi)} + \mathcal{L}^{(\Theta)} + \mathcal{L}^{(M)}, \tag{15}$$

where we have the STEGR extended geometry Lagrangian given by

$$\mathcal{L}^{(\Pi)} = \frac{1}{2}G^{MN}\eta^{\alpha\beta}\Pi_{M\alpha}\Pi_{N\beta} - G^{PQ}t^{\alpha}{}_{P}{}^{M}t^{\beta}{}_{Q}{}^{N}\Pi_{N\alpha}\Pi_{M\beta} - 2\left(G^{-1}t^{\alpha}\right)^{MN}\Pi_{M\alpha}\pi_{N} - \frac{(\lambda,\lambda)}{(\lambda,\lambda)-\frac{1}{2}}G^{MN}\pi_{M}\pi_{N}, \tag{16}$$

with the TEGR extended geometry Lagrangian given by

$$\mathcal{L}^{\Theta} = \frac{1}{2}G^{mn}\left(\eta_{s}{}^{p}{}_{q}{}^{r} + \frac{1}{2}G_{s}{}^{p}{}_{q}{}^{r}\right)\hat{\Theta}_{m}{}^{s}{}_{p}\hat{\Theta}_{n}{}^{q}{}_{r} - \left(2 - \frac{1}{(\lambda,\lambda)}\right)G^{mn}\theta_{m}\theta_{n}, \tag{17}$$

and we hypothize the mix term of the Lagrangian to be

$$\mathcal{L}^{(M)} = \hat{P}_{mnp}\hat{\Theta}^{mnp} + \frac{1}{(\lambda,\lambda)}\theta^{m}\left(p_{m} - \bar{p}_{m}\right) \tag{18}$$

It is yet to be investigated for what this formulation entails. Interesting things to note was that the going to the metric teleparallel formulation was useful[2]. The general parallel extended geometry may enhirit similar benefits. In addition we get a gauge formulation.

References

1. "Primary constraints in general teleparallel quadratic gravity´´ F. Bajardi and D.B.
2. "Teleparallelism in the algebraic approach to extended geometry´´M. Cederwall and J. Palmkvist

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