GeomGravX, Tartu, Estonia

Lagrangian dynamics and Regge-like trajectories for microstructured test bodies

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Outline

- Metric-Affine Geometry: Conventions/Notation

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- Conservation Laws of Metric-Affine Gravity
- Lagrangian formulation of particle motion in Metric-Affine Geometries
- Mass evolution equation/ Novel Regge-like trajectories with hypermomentum charge
- Conclusions/Further Prospects

The talk is mostly based on the papers

- "Motion of test particles in spacetimes with torsion and nonmetricity" (**Damianos Iosifidis** and Friedrich Hehl), Published in: Phys.Lett.B 850 (2024) 138498
  - e-Print: 2310.15595 [gr-qc]
- Lagrangian Dynamics of Spinning Pole-Dipole-Quadrupole Particles in Metric-Affine Geometries" (**DI**),
  - e-Print: 2509.14757 [gr-qc] [under review at PLB]
- "Novel Regge-like trajectories for spinning, dilating, hadronic particles" (**DI**), Published in Phys. Lett. B (2026): 140656.
  - e-Print: 2604.22845 [physics.gen-ph]?

Metric-Affine Gravity

Metric Gravity

- ${\Gamma^{\alpha}}_{\mu\nu} \rightarrow$ *torsionless* , metric compatibility $\nabla_{\sigma} g_{\mu\nu} = 0$
- $S = S_{Gravity} + S_{Matter} = \int d^n x \sqrt{-g}\, [\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Phi)]$

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Teleparallel/Symmetric Teleparallel Gravity

- ${R^{\alpha}}_{\beta\mu\nu} = 0$, $\nabla_{\sigma} g_{\mu\nu} = 0$ but ${S_{\mu\nu}}^{\alpha} = {\Gamma^{\alpha}}_{[\mu\nu]} \neq 0$
- ${R^{\alpha}}_{\beta\mu\nu} = 0$, ${S_{\mu\nu}}^{\alpha} = 0$ but $Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} \neq 0$

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Metric-Affine Gravity (MAG)

- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}, \Phi) \right] \Rightarrow$ No a priori constraints on the geometry.

Geometrical Objects

Two distinctively different notions on a manifold

- Metric Tensor $g_{\mu\nu}$: Defines distances,lengths and dot products

$$||\alpha||^2 := \alpha^\mu \alpha^\nu g_{\mu\nu} \ , \ \ (\alpha \cdot \beta) := \alpha^\mu \beta^\nu g_{\mu\nu}$$

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The two need not be related a priori! Their relation may be found after solving the field equations!

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Torsion

- $\nabla_{[\mu}\nabla_{\nu]}\phi = S_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi$, Torsion Tensor $\boxed{S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}}$

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Curvature

- $[\nabla_{\alpha}, \nabla_{\beta}]u^{\mu} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu} + 2S_{\alpha\beta}{}^{\nu}\nabla_{\nu}u^{\mu}$

  Curvature Tensor: $\boxed{R^{\mu}{}_{\nu\alpha\beta} := 2\partial_{[\alpha}\Gamma^{\mu}{}_{|\nu|\beta]} + 2\Gamma^{\mu}{}_{\rho[\alpha}\Gamma^{\rho}{}_{|\nu|\beta]}}$

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Non-Metricity

- $\boxed{Q_{\alpha\mu\nu} := -\nabla_{\alpha}g_{\mu\nu}} = -\partial_{\alpha}g_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\nu} + \Gamma^{\lambda}{}_{\nu\alpha}g_{\lambda\mu}$

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Torsion/Non-metricity related vectors

$$S_{\mu} = S_{\mu\lambda}{}^{\lambda}\,, \qquad \tilde{S}^{\mu} = \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho\sigma} \quad (only\ \ for\ \ n = 4)$$

$$Q_{\mu} = g^{\alpha\beta} Q_{\mu\alpha\beta}\,, \qquad q_{\mu} = g^{\rho\alpha} Q_{\rho\alpha\mu}$$

Linear Connection

Linear connection decomposition

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{1}{2} g^{\alpha\lambda}(Q_{\mu\nu\alpha} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda}(S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha})$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$ is the Levi-Civita part of the connection. Distortion: $N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu}$

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Post-Riemannian expansions

Each quantity ⇒ decomposed into Riemannian and non-Riemannian counterparts. Example:

$$\begin{aligned} R = {} & \tilde{R} + S_{\mu\nu\alpha}S^{\mu\nu\alpha} - 2S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 4S_{\mu}S^{\mu} - 4\tilde{\nabla}_{\mu}S^{\mu} \\ & + \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} - \frac{1}{4}Q_{\mu}Q^{\mu} + \frac{1}{2}Q_{\mu}q^{\mu} \\ & + 2Q_{\alpha\mu\nu}S^{\alpha\mu\nu} + 2S_{\mu}(q^{\mu} - Q^{\mu}) + \tilde{\nabla}_{\mu}(q^{\mu} - Q^{\mu} - 4S^{\mu}) \end{aligned}$$

Hypermomentum, Canonical and Metrical Energy Momentum Tensors

Metrical (Hilbert) and Canonical Energy Momentum Tensor

Metrical: $T_{\alpha\beta} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}$. Canonical: $t^{\mu}{}_{c} = \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta e_{\mu}{}^{c}}$

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Hypermomentum Tensor (intrinsic part)

Hypermomentum: $\Delta_{\lambda}{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$

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Relation Between Energy Tensors

$$t^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} - \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}\left(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}\right)$$

where $\hat{\nabla}_{\nu} = 2S_{\nu} - \nabla_{\nu}$.

Kinematics of moving particles

Velocity fields

Consider a curve $\mathcal{C}$ with an arbitrary parametrization $x^{\alpha}(\lambda)$, $\alpha = 0, 1, 2, ...(n-1)$. The un-normalized and normalized n-velocity fields are respectively given by

$$\upsilon^{\mu} := \frac{dx^{\mu}}{d\lambda} \ , \ u^{\mu} := \frac{dx^{\mu}}{d\tau} \ , \ u_{\mu}u^{\mu} = -1 \tag{1}$$

where λ is an arbitrary parameter and τ denoted proper time paramatrization. By construction $u^{\mu}u_{\mu} = -1$ whereas $\upsilon^{\mu}\upsilon_{\mu} = -l^2 \neq 0$.

Absolute derivatives

$$\dot{} = \frac{D}{d\lambda} := \upsilon^{\mu}\nabla_{\mu} \ , \ \frac{\tilde{D}}{d\lambda} := \upsilon^{\mu}\tilde{\nabla}_{\mu} \tag{2}$$

Masses

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- **Dynamical** rest mass

$$\mathcal{M}^2 := -P_\mu P^\mu \tag{4}$$

The latter being the physical one. Let us recall that before imposing a spin supplementary condition, neither of them is a constant of motion for the standard form of the Mathisson-Papapetrou equations.

P_μ= momentum of the particle

Recalling some known facts

The MPD equations

In GR a spinning test particle is described by the so-called Mathisson-Papapetrou-Dixon (MPD) equations :

$$\dot{P}^{\mu} = \frac{1}{2} S^{\alpha\beta} R_{\alpha\beta\gamma\mu} \upsilon^{\gamma} \tag{5}$$

$$\dot{S}^{\mu\nu} = P^{\mu} \upsilon^{\nu} - P^{\nu} \upsilon^{\mu} \tag{6}$$

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Important note

The system is underdetermined; spin supplementary condition needed to fix the centroid of the body and close the system!

Spin Supplementary Conditions (SSC)

The most common SSC's are the

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Either of these 2 SSC, when imposed, also implies that the spin magnitude $S_{\mu\nu} S^{\mu\nu}$ is a constant of motion.

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Note:

Pirani SSC is problematic: leads to unphysical, helical motions of the particle's center of mass (Zitterbewegung) and fails to define a unique worldline.

Lagrangian formulation of MPD equations

Lagrangian description of the MPD equations was developed by I. Bailey and W. Israel (Commun. Math. Phys., 42:65–82, 1975).

There, one starts with a generic Lagrangian

$$L(\upsilon^{\mu}, e_{\mu}{}^{a}, \dot{e}_{\mu}{}^{a}, g_{\mu\nu}) \tag{8}$$

$e_{\mu}{}^{a}$: material frame of particle and $\dot{e}_{\mu}{}^{a} := \upsilon^{\lambda}\tilde{\nabla}_{\lambda}e_{\mu}{}^{a}$ its velocity.

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Defining the canonical momentum and spin tensor according to:

$$P_{\mu} := \frac{\partial L}{\partial \upsilon^{\mu}} \quad , \quad S_{\nu}{}^{\mu} := e_{\nu}{}^{a}\frac{\partial L}{\partial \dot{e}_{\mu}{}^{a}} - (\mu \leftrightarrow \nu) \tag{9}$$

and performing variations wrt path and frame one arrives at MPD.

Lagrangian formulation of microstructured test bodies in generic non-Riemannian backgrounds (D.I. 2025)

Extension to full micro-properties and generic background

Extending to MAG we consider also an independent connection and start with

$$I = \int L(\upsilon^{\mu}, e_{\mu}{}^{a}, \dot{e}_{\mu}{}^{a}, g_{\mu\nu}) d\lambda \tag{10}$$

where now $\dot{e}_{\mu}{}^{a} := \upsilon^{\lambda}\nabla_{\lambda}e_{\mu}{}^{a}$; this is where the connection dependence is hidden!

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Along with the canonical momentum we now define the excitations of hypermomentum and energy-momentum according to

$$\boxed{H_{\nu}{}^{\mu} := 2 e_{\nu}{}^{a} \frac{\partial L}{\partial \dot{e}_{\mu}{}^{a}} \quad , \quad t^{\mu\nu} := 2 \frac{\partial L}{\partial g_{\mu\nu}}} \tag{11}$$

Field Equations

Microstructured particle in generic metric-affine backgrounds

Path and frame variations of (10) yield the eom

$$\boxed{\frac{DP_\mu}{d\lambda} = -2S_{\mu\alpha\beta}\upsilon^\alpha P^\beta + \frac{1}{2}H^{\alpha\beta}R_{\alpha\beta\gamma\mu}\upsilon^\gamma - \frac{1}{2}Q_{\mu\alpha\beta}t^{\alpha\beta}} \tag{12}$$

$$\boxed{\frac{DH_\nu{}^\mu}{d\lambda} = 2(P_\nu\upsilon^\mu - t^\mu{}_\nu)} \tag{13}$$

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$$\boxed{\frac{DH_\nu{}^\mu}{d\lambda} = 2(P_\nu\upsilon^\mu - t^\mu{}_\nu)} \tag{13}$$

The above slightly generalize the existing eom found in the literature by extending $t^{\mu\nu}$ further from the pole approximation (i.e. $t^{\mu\nu} \neq mu^\mu u^\nu$).

The rest is in perfect agreement with previous results of Puetzfeld and Obukhow (2014) and Iosifidis and Hehl (2023)

Conserved Quantities

Definition

Consider a generic Metric-Affine space. The vector field ζ is said to be a generalized Killing vector field if it is simultaneously an isometry and an isoparallelism in the sense that its action leaves invariant the geometric background entities:

$$L_{\zeta} g_{\mu\nu} = 0 \ \ (\textit{isometry}) \tag{14}$$

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Theorem

Let ζ^μ be a generalized Killing vector field. Then the quantity

$$P_\mu \zeta^\mu + \frac{1}{2} H^{\mu\nu} \widetilde{\nabla}_\mu \zeta_\nu - H^{\mu\nu} N_{\mu\nu\kappa} \zeta^\kappa = const. \tag{16}$$

is a constant of motion.

Mass evolution

Now back to our eom. Differentiating the defining relation of dynamical mass and using the eom, after some algebra we find (D.I. 2026, PLB)

$$\boxed{\dot{P}_\nu\left[\frac{D}{d\lambda}\Big(P_\mu S^{\mu\nu}\Big)-\xi^{\mu\nu}P_\mu\right]=\frac{1}{2}Q_{\alpha\mu\nu}\upsilon^\alpha\Big(\mathcal{M}^2 t^{\mu\nu}-mP^\mu P^\nu\Big)-m\mathcal{M}\dot{\mathcal{M}}}\tag{17}$$

where we have defined $\xi^{\mu\nu}=-\xi^{\nu\mu}:=H^{\beta[\nu}Q_{\alpha\beta}{}^{\mu]}\upsilon^\alpha$

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where we have defined $\xi^{\mu\nu} = -\xi^{\nu\mu} := H^{\beta[\nu} Q_{\alpha\beta}{}^{\mu]} \upsilon^\alpha$

Some Observations

- Note that in the Riemannian case and under the Tulczyjew constraint ($P_\mu S^{\mu\nu} = 0$) we get that $\mathcal{M} = const$.
- In the presence of nonmetricity, $\mathcal{M} \neq 0$ in general even when the Tulczyjew constraint is imposed!

Regge-like trajectories with pure dilation charge

Let us consider the case of pure dilation (namely set $S^{\mu\nu} = 0 = \Sigma^{\mu\nu} \Rightarrow H_{\mu}{}^{\nu} = H\delta^{\nu}_{\mu}$). Under these conditions:

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Substitution to the mass evolution eqn (17), then yields

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Regge Trajectory with dilation charge

$$\boxed{\dot{\mathcal{M}} = -\frac{Q_{\mu}\upsilon^{\mu}}{4l}\dot{H}} \tag{19}$$

Mass depends on dilation!

This is a new kind of Regge trajectory where the mass of the particle involves as a function of the dilation charge.

Example: Scale Invariant Case

Precise form of the Regge trajectory (i.e. function $\mathcal{M}(H)$)) depends on the specific properties of matter under consideration. physically relevant example ⇒ scale invariant matter. In this case $t := t^{\mu\nu} g_{\mu\nu} = 0$ and from (18) we infer that

$$\dot{H} = -\frac{2l\mathcal{M}}{n} \tag{20}$$

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Mass Relation

$$\dot{\mathcal{M}} + \frac{Q_\mu \upsilon^\mu}{2n}\mathcal{M} = 0 \Rightarrow \mathcal{M} = \mathcal{M}_0 e^{-\frac{1}{2n}\int Q_\mu \upsilon^\mu d\lambda}. \tag{21}$$

We see that when the particle enters a non-metric region of spacetime its initial mass $\mathcal{M}_0$ starts to shift!

Regge Trajectory with shear charge

A completely analogous situation occurs also in the pure shear case $H^{\mu\nu} = \Sigma^{\mu\nu}$ with one extra condition. Here we need to impose the

'Shear Supplementary Condition':

$$\xi^{\mu\nu} P_{\mu} = 0 \;\; \textit{or} \;\; \xi^{\mu\nu} \upsilon_{\mu} = 0 \tag{22}$$

the former being the analogue of the Tulczyjew constraint and the latter resembling the Pirani constraint.

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$$P^\mu = \frac{\mathcal{M}^2}{m} \upsilon^\mu \tag{23}$$

and after some algebra we find the ($\hat{\mu}_{\mu\nu} = \upsilon^\alpha (Q_{\alpha\mu\nu} - Q_\alpha g_{\mu\nu}/n)$)

Mass-Shear relation

Regge Trajectory with shear charge

A completely analogous situation occurs also in the pure shear case $H^{\mu\nu} = \Sigma^{\mu\nu}$ with one extra condition. Here we need to impose the

'Shear Supplementary Condition':

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$$\boxed{\dot{\mathcal{M}} = -\frac{1}{4l} \hat{\mu}^\mu{}_\nu \dot{\Sigma}_\mu{}^\nu} \tag{24}$$

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Important thing to keep in mind

- If the particle possesses no dilation or shear charges, its mass will be a constant of motion even if the underlying space is nonmetric.

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- Under investigation: Revisiting the equivalence principle in Metric-Affine Geometries.

…Thank you!!!