Teleparallel Regularization of Gravitational Action

Martin Krššák

Department of Theoretical Physics
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava, Slovakia

July 1, 2026

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS
Comenius University
Bratislava

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,
  - Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$, $g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$
  - Levi-Civita connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$
- Riemannian geometry Notation: all quantities with ∘ are related to Riemannian geometry

$$
\begin{aligned}
0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\
\overset{\circ}{R}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)}
\end{aligned}
$$

- **Teleparallel geometry**

$$
\begin{aligned}
0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\
T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)}
\end{aligned}
$$

- Leads to the pure-gauge teleparallel connection

$$
\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)
$$

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,

  Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$, $\quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$

  Linear connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$

- Riemannian geometry Notation: all quantities with $\circ$ are related to Riemannian geometry

$$0 = dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, \qquad \text{(zero torsion)}$$

$$\overset{\circ}{R}{}^a{}_b = d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b \qquad \text{(non-zero curvature)}$$

- **Teleparallel geometry**

$$0 = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad \text{(zero curvature)}$$

$$T^a = dh^a + \omega^a{}_b \wedge h^b \qquad \text{(non-zero torsion)}$$

- Leads to the pure-gauge teleparallel connection

$$\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)$$

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,

  Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$ $\quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$

  Linear connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$

- Riemannian geometry Notation: all quantities with ∘ are related to Riemannian geometry

$$0 = dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, \qquad \text{(zero torsion)}$$

$$\overset{\circ}{R}{}^a{}_b = d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b \qquad \text{(non-zero curvature)}$$

- **Teleparallel geometry**

$$0 = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad \text{(zero curvature)}$$

$$T^a = dh^a + \omega^a{}_b \wedge h^b \qquad \text{(non-zero torsion)}$$

- Leads to the pure-gauge teleparallel connection

$$\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)$$

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,

  Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$ $\quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$

  Linear connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$

- Riemannian geometry Notation: all quantities with $\circ$ are related to Riemannian geometry

$$
\begin{aligned}
0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\
\overset{\circ}{R}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)}
\end{aligned}
$$

- **Teleparallel geometry**

$$
\begin{aligned}
0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\
T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)}
\end{aligned}
$$

- Leads to the pure-gauge teleparallel connection

$$
\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)
$$

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,

  Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$ $\quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$

  Linear connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$

- Riemannian geometry Notation: all quantities with $\circ$ are related to Riemannian geometry

$$
\begin{aligned}
0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\
\overset{\circ}{R}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)}
\end{aligned}
$$

- **Teleparallel geometry**

$$
\begin{aligned}
0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\
T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)}
\end{aligned}
$$

- Leads to the pure-gauge teleparallel connection

$$
\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)
$$

Teleparallel Gravity: Gravity as Torsion

- Alternative formulation of general relativity formulated in **tetrad formalism**,

  Metric $g_{\mu\nu}$ $\Longrightarrow$ tetrad 1-form $h^a = h^a{}_\mu dx^\mu$ $\quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu$

  Linear connection $\Gamma^\rho{}_{\mu\nu}$ $\Longrightarrow$ spin connection 1-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$

- Riemannian geometry Notation: all quantities with ∘ are related to Riemannian geometry

$$
\begin{aligned}
0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\
\overset{\circ}{R}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)}
\end{aligned}
$$

- **Teleparallel geometry**

$$
\begin{aligned}
0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\
T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)}
\end{aligned}
$$

- Leads to the pure-gauge teleparallel connection

$$
\omega^a{}_b = \Lambda^a{}_c d(\Lambda^{-1})^c{}_b, \qquad \Lambda^a{}_b \in SO(1,3)
$$

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

- Allows to rewrite any Riemannian expression in terms of teleparallel ones

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_{\nu} \right)$$

- where T is torsion scalar

$$T = \frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}$$

- **Teleparallel Equivalent of General Relativity (TEGR)**

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

- Allows to rewrite any Riemannian expression in terms of teleparallel ones

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_\nu \right)$$

where T is torsion scalar

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

- Allows to rewrite any Riemannian expression in terms of teleparallel ones

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_\nu \right)$$

where T is torsion scalar

$$T = \frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}$$

- **Teleparallel Equivalent of General Relativity (TEGR)**

$$\boxed{\mathcal{S}_{\mathrm{TG}} = \frac{1}{2\kappa} \int_{\mathcal{M}} h\, T}$$

Teleparallel Equivalent of General Relativity

- Relation between Riemannian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connection

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b} \qquad T^a = K^a{}_b \wedge h^b$$

- Allows to rewrite any Riemannian expression in terms of teleparallel ones

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_\nu \right)$$

where T is torsion scalar

$$T = \frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}$$

- **Teleparallel Equivalent of General Relativity (TEGR)**

$$\boxed{\mathcal{S}_{\text{TG}} = \frac{1}{2\kappa} \int_{\mathcal{M}} h\, T}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_{\mu}\left(hT^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T$$

is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_{\sigma}\left(hS_{\mu}{}^{\rho\sigma}\right) + \kappa j_{\mu}{}^{\rho} = 0$$

are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity ...
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi]\, e^{-S^{E}}$$

.

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu\left(hT^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(hS_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^{\rho} = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity ...
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi]e^{-S^E}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa} \int_{\mathcal{M}} h\, T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma \left(h S_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^\rho = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity ...
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi] e^{-S^E}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_{\nu} \right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa} \int_{\mathcal{M}} h\, T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma \left(h S_\mu{}^{\rho\sigma} \right) + \kappa t_\mu{}^{\rho} = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity ...
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(hT^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(hS_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^{\rho} = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity ...
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi]\,e^{-S^E}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu\left(hT^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(hS_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^{\rho} = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity . . .
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities . . .
    - Finite Euclidean action solutions (Instantons) dominating the [illegible] approximation

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu\left(hT^{\nu\mu}{}_\nu\right)$

$$\mathcal{S}_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\, T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(hS_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^\rho = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity . . .
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities . . .
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi] e^{-\mathcal{S}^E}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(h T^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\, T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(h S_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^\rho = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity . . .
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi] e^{-\mathcal{S}^E}$$

(Non-) Triviality of Teleparallel Gravity

- Due to $-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu \left(hT^{\nu\mu}{}_{\nu}\right)$

$$\mathcal{S}_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T$$

  is dynamically equivalent to the standard general relativity
- Field equations

$$h^{-1}\partial_\sigma\left(hS_\mu{}^{\rho\sigma}\right) + \kappa t_\mu{}^{\rho} = 0$$

  are just Einstein field equations
- To obtain something non-trivial we need to
  - Modify gravity . . .
  - Look where equivalence is non-trivial: global aspects
    - Conserved quantities ...
    - Finite Euclidean action solutions (Instantons): dominating the path integral in semi-classical approximation

$$Z = \int_{\mathcal{M}} d[\psi]e^{-\mathcal{S}^E}$$

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-g}\,\overset{\circ}{R}$$

- Vanishes on-shell for all vacuum spacetimes?

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Vanishes on-shell for all vacuum spacetimes? No instantons?
- Different from YM theory where we evaluate $\int \mathrm{Tr}\, F \wedge \star F$ and have a rich structure
- Not the full action! $\mathcal{S}_{\text{EH}}(g, \partial g, \partial^2 g)$ requires both $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ and $n^\rho \delta\partial_\rho g_{\mu\nu}|_{\partial\mathcal{M}} = 0$
- **Gibbons-Hawking-York solution**

$$\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\,K$$

where K is the trace of the extrinsic curvature $K = \nabla_\mu n^\mu$

- For path integrals we need to evaluate the action, but $\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}}$ diverges

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Vanishes on-shell for all vacuum spacetimes? No instantons?
- Different from YM theory where we evaluate $\int \mathrm{Tr} F \wedge \star F$ and have a rich structure
- Not the full action! $\mathcal{S}_{\text{EH}}(g, \partial g, \partial^2 g)$ requires both $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ and $n^\rho \delta\partial_\rho g_{\mu\nu}|_{\partial\mathcal{M}} = 0$
- **Gibbons-Hawking-York solution**

$$\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\,K$$

where K is the trace of the extrinsic curvature $K = \nabla_\mu n^\mu$

- For path integrals we need to evaluate the action, but $\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}}$ diverges

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Vanishes on-shell for all vacuum spacetimes? No instantons?
- Different from YM theory where we evaluate $\int \text{Tr} F \wedge \star F$ and have a rich structure
- Not the full action! $\mathcal{S}_{\text{EH}}(g, \partial g, \partial^2 g)$ requires both $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ and $n^\rho \delta\partial_\rho g_{\mu\nu}|_{\partial\mathcal{M}} = 0$
- **Gibbons-Hawking-York solution**

$$\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma} K$$

where K is the trace of the extrinsic curvature $K = \nabla_\mu n^\mu$

- For path integrals we need to evaluate the action, but $\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}}$ diverges

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\overset{\circ}{R}$$

- Vanishes on-shell for all vacuum spacetimes? No instantons?
- Different from YM theory where we evaluate $\int \text{Tr} F \wedge \star F$ and have a rich structure
- Not the full action! $\mathcal{S}_{\text{EH}}(g, \partial g, \partial^2 g)$ requires both $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ and $n^\rho \delta\partial_\rho g_{\mu\nu}|_{\partial\mathcal{M}} = 0$
- **Gibbons-Hawking-York solution** 1977&1972

$$\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\overset{\circ}{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\,K$$

where K is the trace of the extrinsic curvature $K = \nabla_\mu n^\mu$

- For path integrals we need to evaluate the action, but $\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}}$ diverges

What is the Gravitational Action?

- (Einstein-)Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Vanishes on-shell for all vacuum spacetimes? No instantons?
- Different from YM theory where we evaluate $\int \text{Tr} F \wedge \star F$ and have a rich structure
- Not the full action! $\mathcal{S}_{\text{EH}}(g, \partial g, \partial^2 g)$ requires both $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$ and $n^\rho \delta\partial_\rho g_{\mu\nu}|_{\partial\mathcal{M}} = 0$
- **Gibbons-Hawking-York solution** 1977&1972

$$\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\,K$$

where K is the trace of the extrinsic curvature $K = \nabla_\mu n^\mu$

- For path integrals we need to evaluate the action, but $\mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}}$ diverges

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = +\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right),$$

  - Asymptotically flat case Mann, Marolf 2006

[illegible]

- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = +\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\,K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right).$$

  - Asymptotically flat case Mann, Marolf 2006

- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma} K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999
  - Asymptotically flat case Mann, Marolf 2006
- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian, Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}\left(\frac{2}{l}-\frac{l}{2}\mathcal{R}\right),$$

  - Asymptotically flat case Mann, Marolf 2006

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}}\sqrt{-\gamma}\hat{K}$$

    where $\hat{K}$ is a solution to $\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m\hat{K}_{mk}$

- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma} K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right),$$

  - Asymptotically flat case Mann, Marolf 2006

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\hat{K}$$

  where $\hat{K}$ is a solution to $\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m \hat{K}_{mk}$

- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right),$$

  - Asymptotically flat case Mann, Marolf 2006

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\hat{K}$$

where $\hat{K}$ is a solution to $\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m\hat{K}_{mk}$

- **Kounterterms, topological regularization...** Olea 2007

Counterterms for the Gravitational Action

- The standard solution in general relativity is adding a **counterterm**

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$$

- **Background Subtraction** Gibbons& Hawking 1977

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma} K^0,$$

- **Holographic Renormalization:**
  - AdS case Balasubramanian Kraus 1999

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\left(\frac{2}{l} - \frac{l}{2}\mathcal{R}\right),$$

  - Asymptotically flat case Mann, Marolf 2006

$$\mathcal{S}_{\text{counter}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}\hat{K}$$

where $\hat{K}$ is a solution to $\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m \hat{K}_{mk}$

- **Kounterterms, topological regularization...** Olea 2007

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

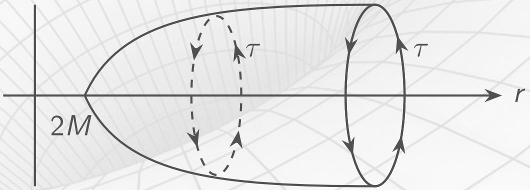
- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$



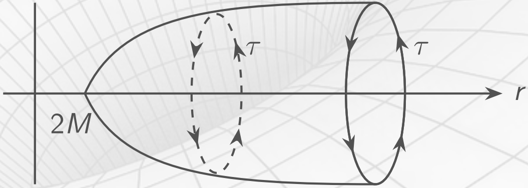
- Euclidean Schwarzschild manifold $\mathcal{E}$ corresponds only to the exterior of a black hole
- Leads to $\boxed{\mathcal{S}^E_{\text{GR}} = \frac{1}{2}\beta M}$ and the entropy-area law $\frac{A}{4}$

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$



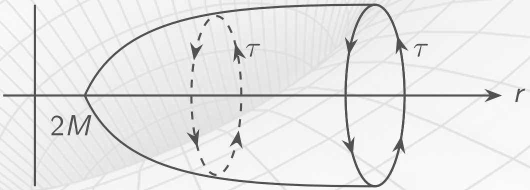
- Euclidean Schwarzschild manifold $\mathcal{E}$ corresponds only to the exterior of a black hole
- Leads to $\mathcal{S}^E_{\text{GR}} = \frac{1}{2}\beta M$ and the entropy-area law $\frac{A}{4}$

Example: Gibbons-Hawking Instanton

- Wick rotated $t \longrightarrow -i\tau$ Schwarzschild solution Gibbons-Hawking 1977

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

- Periodic time $\beta = 8\pi M$ with associated temperature $T = 1/\beta$



- Euclidean Schwarzschild manifold $\mathcal{E}$ corresponds only to the exterior of a black hole
- Leads to $\boxed{\mathcal{S}^E_{\text{GR}} = \frac{1}{2}\beta M}$ and the entropy-area law $\frac{A}{4}$

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives! Naively does not require GHY and counterterms
  depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathrm{TG}}(h^a, \omega^a{}_b)$$

Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations

Must be matched to the tetrad: determining the **proper frame**

$\omega^a{}_b$ enters the action as Krssak 2016

$$\mathcal{S}_{\mathrm{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathrm{TG}}(h^a, 0) + \int$$

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives! Naively does not require GHY and counterterms

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^a{}_{\mu\nu}T_a{}^{\mu\nu} + \frac{1}{2}T^a{}_{\mu\nu}T^{\nu\mu}{}_a - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: the **proper frame** as a pair $\{h^a{}_\mu, \omega^a{}_{b\mu}\}$

enters the action as Krssak 2016

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: determining the **proper frame** as a pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- $\omega^a{}_b$ enters the action as

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu(h\omega^{\rho\mu}{}_\rho)$$

- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm.

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: determining the **proper frame** as a pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- $\omega^a{}_b$ enters the action as Krssak 2016

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu (h\,\omega^{\rho\mu}{}_\rho)$$

- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm.

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^a{}_{\mu\nu}T_a{}^{\mu\nu} + \frac{1}{2}T^a{}_{\mu\nu}T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: determining the **proper frame** as a pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- $\omega^a{}_b$ enters the action as Krssak 2016

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu(h\,\omega^{\rho\mu}{}_\rho)$$

- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm: **teleparallel regularization**

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^a{}_{\mu\nu}T_a{}^{\mu\nu} + \frac{1}{2}T^a{}_{\mu\nu}T^{\nu\mu}{}_a - T^{\rho\mu}{}_\rho T^\nu{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: determining the **proper frame** as a pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- $\omega^a{}_b$ enters the action as Krssak 2016

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu(h\,\omega^{\rho\mu}{}_\rho)$$

- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm: **teleparallel regularization**
- Fundamental law of preservation of difficulty!

Regularization of Teleparallel Action

- Teleparallel action

$$\mathcal{S}_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\rho\mu}{}_{\rho}T^{\nu}{}_{\mu\nu}\right]$$

- Does not contain second derivatives!Naively does not require GHY and counterterms
- Action depends on both h^a and $\omega^a{}_b$

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b)$$

- Teleparallel spin connection $\omega^a{}_b$ is not determined by any field equations
- Must be matched to the tetrad: determining the **proper frame** as a pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- $\omega^a{}_b$ enters the action as Krssak 2016

$$\mathcal{S}_{\mathsf{TG}}(h^a, \omega^a{}_b) = \mathcal{S}_{\mathsf{TG}}(h^a, 0) + \int_{\mathcal{M}} \partial_\mu(h\,\omega^{\rho\mu}{}_\rho)$$

- Finding $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ is equivalent to finding the counterterm: **teleparallel regularization**
- Fundamental law of preservation of difficulty!

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu = \mathrm{diag}(f, f^{-1}, r, r\sin\theta) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \overset{\circ}{\omega}{}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu = \mathrm{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu = \text{diag}(f, f^{-1}, r, r\sin\theta) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \tilde{\omega}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu = \text{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu \equiv \text{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \tilde{\omega}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu \equiv \text{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu \equiv \mathrm{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \overset{\circ}{\omega}{}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu = \mathrm{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu \equiv \mathrm{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \overset{\circ}{\omega}{}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu \equiv \mathrm{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu \equiv \mathrm{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \overset{\circ}{\omega}{}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu \equiv \mathrm{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Proper Diagonal Frame

- Require that torsion vanishes at infinity $r \to \infty$ Obukhov&Pereira 2002, Krssak&Pereira 2015
- Diagonal tetrad of the **static** observer

$$h^a{}_\mu \equiv \mathrm{diag}\big(f, f^{-1}, r, r\sin\theta\big) \qquad f^2 = 1 - \frac{2M}{r}$$

- Teleparallel spin connection is always possible to write as

$$\tilde{\omega}^a{}_b = \overset{\circ}{\omega}{}^a{}_b({}^0h^a)$$

- where ${}^0h^a$ is the reference tetrad

$${}^0h^a{}_\mu \equiv \mathrm{diag}(1, 1, r, r\sin\theta)$$

- Leads to the spin connection

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta.$$

- Naive Euclidean action Krssak&Pereira 2015

$$\mathcal{S}^E_{proper} = \mathcal{S}^E_{TG}(h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = \beta M = 2\,\mathcal{S}^E_{GR}$$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- [illegible] class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- [illegible] construct tetrad either in the Cartesian coordinate system

$$^{\mathrm{CC}}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a.$$

- Canonical frame: $\{^{\mathrm{CC}}h^a{}_\mu, 0\}$
- [illegible] tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0$$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$$^{CC}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{^{CC}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$$^{C}h^a{}_\mu = {}^0h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{^{C}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$${}^{CC}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{CC}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$${}^{C}h^a{}_\mu = {}^{0}h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{C}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$${}^{CC}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{CC}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$${}^{C}h^a{}_\mu = {}^{0}h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{C}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$${}^{CC}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{CC}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$${}^{C}h^a{}_\mu = {}^{0}h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{C}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$${}^{\mathrm{CC}}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{\mathrm{CC}}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$${}^{\mathrm{C}}h^a{}_\mu = {}^0h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{\mathrm{C}}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Canonical Frame

- **Canonical frame** Beltran-Jimenez, Heisenberg, Koivisto 2019

$$h^{-1}\partial_\sigma(hS_\mu{}^{\rho\sigma}) + \kappa t_\mu{}^\rho = 0 \qquad \boxed{t_\mu{}^\rho = 0}$$

- In GR or STEGR: Kerr-Schild class of spacetimes in the Cartesian coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Fk_\mu k_\nu, \qquad k_\mu = (1, x^i/r) \qquad F = M/r$$

- In TEGR: construct tetrad either in the Cartesian coordinate system

$${}^{\mathrm{CC}}h^a{}_\mu = \delta^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{\mathrm{CC}}h^a{}_\mu, 0\}$
- Or tetrad adapted to Schwarzchild in Eddington-Finkelstein coordinates

$${}^{\mathrm{C}}h^a{}_\mu = {}^0h^a{}_\mu + Fk_\mu k^a,$$

- Canonical frame: $\{{}^{\mathrm{C}}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$

Euclidean Canonical Frame

- Several issues applying the Wick rotation for non-diagonal tetrads Kontsevich, Witten, Visser 2021
- Requires Wick rotating the metric and then finding a corresponding tetrad

$$ {}^{EC}h^{a}{}_{\mu} = \begin{pmatrix} 1 - \frac{M}{r} & \frac{iM}{r} & 0 & 0 \\ \frac{iM}{r} & 1 + \frac{M}{r} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}. $$

- Euclidean canonical proper frame $\{{}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}\}$
- Naive Euclidean canonical action

$$ \mathcal{S}^{E}_{\text{canonical}} = \mathcal{S}^{E}_{\text{TG}}({}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}) = 0 $$

Euclidean Canonical Frame

- Several issues applying the Wick rotation for non-diagonal tetrads Kontsevich, Witten, Visser 2021
- Requires Wick rotating the metric and then finding a corresponding tetrad

$$ {}^{EC}h^{a}{}_{\mu} = \begin{pmatrix} 1 - \frac{M}{r} & \frac{iM}{r} & 0 & 0 \\ \frac{iM}{r} & 1 + \frac{M}{r} & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} . $$

- Euclidean canonical proper frame $\{{}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}\}$
- Naive Euclidean canonical action

$$ \mathcal{S}^{E}_{\text{canonical}} = \mathcal{S}^{E}_{\text{TG}}({}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}) = 0 $$

Euclidean Canonical Frame

- Several issues applying the Wick rotation for non-diagonal tetrads Kontsevich, Witten, Visser 2021
- Requires Wick rotating the metric and then finding a corresponding tetrad

$$
{}^{EC}h^a{}_\mu = \begin{pmatrix} 1-\frac{M}{r} & \frac{iM}{r} & 0 & 0 \\ \frac{iM}{r} & 1+\frac{M}{r} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.
$$

- Euclidean canonical proper frame $\{{}^{EC}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Naive Euclidean canonical action

$$
\mathcal{S}^E_{\text{canonical}} = \mathcal{S}^E_{\text{TG}}\left({}^{EC}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\right) = 0
$$

Euclidean Canonical Frame

- Several issues applying the Wick rotation for non-diagonal tetrads Kontsevich, Witten, Visser 2021
- Requires Wick rotating the metric and then finding a corresponding tetrad

$$
{}^{EC}h^{a}{}_{\mu} = \begin{pmatrix} 1-\frac{M}{r} & \frac{iM}{r} & 0 & 0 \\ \frac{iM}{r} & 1+\frac{M}{r} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.
$$

- Euclidean canonical proper frame $\{{}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}\}$
- Naive Euclidean canonical action

$$
\mathcal{S}^{E}_{\text{canonical}} = \mathcal{S}^{E}_{\text{TG}}({}^{EC}h^{a}{}_{\mu}, \tilde{\omega}^{a}{}_{b\mu}) = 0
$$

Euclidean Canonical Frame

- Several issues applying the Wick rotation for non-diagonal tetrads Kontsevich, Witten, Visser 2021
- Requires Wick rotating the metric and then finding a corresponding tetrad

$$
{}^{EC}h^a{}_\mu = \begin{pmatrix} 1-\frac{M}{r} & \frac{iM}{r} & 0 & 0 \\ \frac{iM}{r} & 1+\frac{M}{r} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.
$$

- Euclidean canonical proper frame $\{{}^{EC}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Naive Euclidean canonical action

$$
\mathcal{S}^E_{\text{canonical}} = \mathcal{S}^E_{\text{TG}}({}^{EC}h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}) = 0
$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{\text{GR}} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{2M}^{\infty} dr\, hT,$$

- For vacuum spacetimes, we have $\boxed{hT = -2\,\partial_\mu(\dots)}$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$S_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\boxed{\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}}$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{GR} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{2M}^{\infty} dr\, hT,$$

- For vacuum spacetimes, we have $\boxed{hT = -2\,\partial_\mu(hT^\mu)}$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$S_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\boxed{\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}}$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{GR} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^\beta d\tau \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{2M}^\infty dr\, hT,$$

- For vacuum spacetimes, we have $hT = 2\partial_\mu($[illegible]$)$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$S_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{GR} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{2M}^{\infty} dr\, hT,$$

- For vacuum spacetimes, we have $\boxed{h\,T = -2\,\partial_\mu(hT^\mu)}$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$S_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\boxed{\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}}$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{GR} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{2M}^{\infty} dr\, hT,$$

- For vacuum spacetimes, we have $\boxed{h\,T = -2\,\partial_\mu(hT^\mu)}$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$\tilde{\mathcal{S}}^E_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\boxed{\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}}$$

How to evaluate the gravitational action

- "Naive" actions

$$\mathcal{S}^E_{proper} = \beta M = 2\,\mathcal{S}^E_{GR} \qquad \mathcal{S}^E_{\text{canonical}} = 0$$

- Are **bulk actions**

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT = \frac{1}{2\kappa}\int_0^\beta d\tau \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{2M}^\infty dr\, hT,$$

- For vacuum spacetimes, we have $\boxed{h\,T = -2\,\partial_\mu(hT^\mu)}$
- Using the Stokes' theorem we can calculate the **quasi-local action**

$$\tilde{\mathcal{S}}^E_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu,$$

- Leading to an overall agreement

$$\boxed{\tilde{\mathcal{S}}^E_{\text{proper}} = \tilde{\mathcal{S}}^E_{\text{canonical}} = \frac{\beta M}{2} = \mathcal{S}^E_{\text{GR}}}$$

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho\, d^3x = \int_V \nabla . \mathbf{E}\, d^3x = \oint_{\partial V} \mathbf{E} . \mathbf{A}$$

- For a point charge

$$\mathbf{E} = [illegible] \frac{1}{4\pi\epsilon_0} \frac{[illegible]}{r^3}$$

- Total charge is Q, while $\nabla . \mathbf{E}$ [illegible]
- We can
  1. Add the Dirac delta function as a source
  2. [illegible] the volume integral [illegible] $\mathbf{E}$ [illegible]
     [illegible]

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho d^3x = \int_V \nabla.\mathbf{E} d^3x = \oint_{\partial V} \mathbf{E}.\mathbf{A}$$

- For a point charge
- Total charge is Q, while $\nabla.\mathbf{E}$
- We can
  - Add the Dirac delta function as a source

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla.\mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E}.\mathbf{A}$$

- For a point charge
- Total charge is Q, while $\nabla.\mathbf{E}$
- We can
  1. Add the Dirac delta function as a source

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla . \mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E} . \mathbf{A}$$

- For a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}$$

- Total charge is Q, while $\nabla . \mathbf{E}$
- We can
  - Add the Dirac delta function as a source

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla.\mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E}.\mathbf{A}$$

- For a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}$$

- Total charge is Q, while $\nabla.\mathbf{E} = 0$
- We can
  - Add the Dirac delta function as a source

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla . \mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E} . \mathbf{A}$$

- For a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}$$

- Total charge is Q, while $\nabla . \mathbf{E} = 0$
- We can
  - Add the Dirac delta function as a source
  - Consider the volume integral as a charge where $\mathbf{E}$ is regular, while surface integral being a total charge bounded by ∂V (including singularity)

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla . \mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E} . \mathbf{A}$$

- For a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}$$

- Total charge is Q, while $\nabla . \mathbf{E} = 0$
- We can
  - Add the Dirac delta function as a source
  - Consider the volume integral as a charge where $\mathbf{E}$ is regular, while surface integral being a total charge bounded by ∂V (including singularity)

Undregrad Flashbacks: Electrostatics

- Gauss's Law in differential form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- Total charge

$$Q = \int_V \rho \, d^3x = \int_V \nabla.\mathbf{E} \, d^3x = \oint_{\partial V} \mathbf{E}.\mathbf{A}$$

- For a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}$$

- Total charge is Q, while $\nabla.\mathbf{E} = 0$
- We can
  - Add the Dirac delta function as a source
  - Consider the volume integral as a charge where $\mathbf{E}$ is regular, while surface integral being a total charge bounded by ∂V (including singularity)

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{S}^E_{\mathrm{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma} n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$

- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathsf{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$

- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathrm{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$

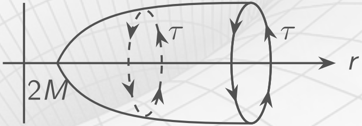
- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathsf{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathsf{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$



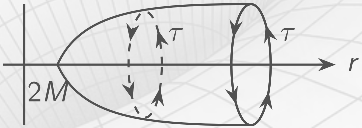
- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\text{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\text{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$



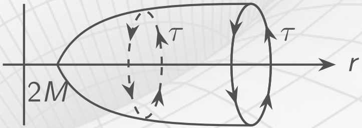
- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathrm{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$



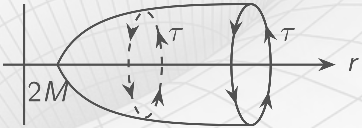
- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathrm{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma} n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$



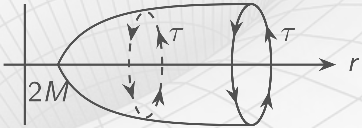
- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Role of Singularities in Euclidean Action

- Similarly

$$S^E_{\mathrm{TG}} = \frac{1}{2\kappa}\int_{\mathcal{E}} hT \qquad \tilde{\mathcal{S}}^E_{\mathrm{TG}} = -\frac{1}{\kappa}\oint_{\partial\mathcal{E}} \sqrt{\gamma}\, n_\mu T^\mu$$

- are equal iff T^μ is regular and if $\partial\mathcal{E}$ is a boundary of $\mathcal{E}$
- Euclidean Schwarzchild manifold $\mathcal{E}$



- Vectorial torsions for both proper and canonical frames are singular at $r = 0$
- Bulk and quasilocal actions agrees when $2M$ is treated as other boundary
- If $2M$ is not treated as a boundary: black hole interior (which is not in $\mathcal{E}$) does contribute to the Euclidean action
- We detect a hole in the manifold

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW) Susskind 2016

$$\boxed{\text{Action growth} = \frac{d\mathcal{S}}{dt}\Big|_{WdW}}$$

- For proper diagonal frame both bulk and quasi-local action growth agree with GR Krššák 2024

$$\frac{d\mathcal{S}_{\text{proper}}}{dt}\Big|_{WdW} = \frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\Big|_{WdW} = \frac{d\mathcal{S}_{\text{GR}}}{dt}\Big|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\frac{d\mathcal{S}_{\text{canonical}}}{dt}\Big|_{WdW} = \frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\Big|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW) Susskind 2016

$$\text{Action growth} = \left.\frac{d\mathcal{S}}{dt}\right|_{WdW}$$

- For proper diagonal frame both bulk and canonical action growth agree with GR Krssak 2024

$$\left.\frac{d\mathcal{S}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\mathcal{S}_{\text{GR}}}{dt}\right|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\left.\frac{d\mathcal{S}_{\text{canonical}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\right|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW)Susskind 2016

$$\boxed{\text{Action growth} = \left.\frac{d\mathcal{S}}{dt}\right|_{WdW}}$$

- For proper diagonal frame both bulk and quasi-local action growth agree with GR Krssak 2024

$$\left.\frac{d\mathcal{S}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\mathcal{S}_{\text{GR}}}{dt}\right|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\left.\frac{d\mathcal{S}_{\text{canonical}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\right|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW) Susskind 2016

$$\boxed{\text{Action growth} = \left.\frac{d\mathcal{S}}{dt}\right|_{WdW}}$$

- For proper diagonal frame both bulk and quasilocal action growth agree with GR Krssak 2024

$$\left.\frac{d\mathcal{S}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\mathcal{S}_{\text{GR}}}{dt}\right|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\left.\frac{d\mathcal{S}_{\text{canonical}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\right|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW)Susskind 2016

$$\boxed{\text{Action growth} = \left.\frac{d\mathcal{S}}{dt}\right|_{WdW}}$$

- For proper diagonal frame both bulk and quasilocal action growth agree with GR Krssak 2024

$$\left.\frac{d\mathcal{S}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\mathcal{S}_{\text{GR}}}{dt}\right|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\left.\frac{d\mathcal{S}_{\text{canonical}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\right|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Lorentzian actions on Wheeler-de Witt patch

- We can study not only the amount information stored in a BH but also how fast this information gets processed: complexity
- **CA duality:** Holographic complexity is given by the Lorentzian gravitational action growth on Wheeler-de Witt patch (WdW)Susskind 2016

$$\text{Action growth} = \left.\frac{d\mathcal{S}}{dt}\right|_{WdW}$$

- For proper diagonal frame both bulk and quasilocal action growth agree with GR Krssak 2024

$$\left.\frac{d\mathcal{S}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{proper}}}{dt}\right|_{WdW} = \left.\frac{d\mathcal{S}_{\text{GR}}}{dt}\right|_{WdW} = 2M$$

- While for canonical frame action growth vanishes

$$\left.\frac{d\mathcal{S}_{\text{canonical}}}{dt}\right|_{WdW} = \left.\frac{d\tilde{\mathcal{S}}_{\text{canonical}}}{dt}\right|_{WdW} = 0$$

- Does canonical frame always lead to correct results ???

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
  Consider local rotations

$$\tilde{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta \frac{M}{2} \longrightarrow \beta \frac{M - \alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider local rotations

$$\tilde{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- With asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta \frac{M}{2} \longrightarrow \beta \frac{M - \alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean gravitational action

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta\frac{M}{2} \longrightarrow \beta\frac{M - \alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta\frac{M}{2} \longrightarrow \beta\frac{M-\alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta \frac{M}{2} \longrightarrow \beta \frac{M - \alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Teleparallel regularization gives us frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$. How unique is $\tilde{\omega}^a{}_{b\mu}$?
- Consider transformations

$$\bar{\Lambda}^a{}_b = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- with asymptotic behaviour

$$\alpha \sim \frac{\alpha_0}{r^p}, \qquad r \to \infty$$

- We can classify ambiguity in $\tilde{\omega}^a{}_{b\mu}$ into three categories:
  1. $p > 1/2$: leave the action invariant.
  2. $p < 1/2$: cause the action to diverge.
  3. $p = 1/2$: shift the value of the Euclidean quasilocal action

$$\beta \frac{M}{2} \longrightarrow \beta \frac{M - \alpha_0^2}{2}$$

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
  Examples of fall-off for
  - **vectorial torsion**

$$T^{\mu} = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large "gauge" transformations
  - **torsion**

$$T^{\rho}{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

    - $p = 1/2$ boosts are excluded and not a symmetry

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^{a}{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough!

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^{a}{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes,

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^{a}{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

    $p = 1/2$ boosts are then large “gauge” transformations
  - **torsion**

$$\mathcal{T}^\rho{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^{a}{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

$$T^{\mu} = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large “gauge” transformations
  - **torsion**

$$T^{\rho}{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

$$T^\mu = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large "gauge" transformations
  - **torsion**

$$T^\rho{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

$$T^\mu = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large "gauge" transformations
  - **torsion**

$$T^\rho{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

    - $p = 1/2$ boosts are excluded and not a symmetry

Uniqueness of Teleparallel Regularization

- Original proper frame proposal: $\tilde{\omega}^a{}_{b\mu}$ in a such way that torsion vanishes at $r \longrightarrow \infty$
- Is not enough! We should say how fast it vanishes, i.e. give fall-off conditions for torsion
- Two options, fall-off for:
  - **vectorial torsion**

$$T^\mu = \mathcal{O}\left(\frac{1}{r^2}\right),$$

    - $p = 1/2$ boosts are then large "gauge" transformations
  - **torsion**

$$T^\rho{}_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right).$$

    - $p = 1/2$ boosts are excluded and not a symmetry

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries (Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries: Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries: Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)

M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].

Conclusions

- Viable alternative to $\mathcal{S}_{\text{total}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}}$ is to reformulate general relativity in terms of teleparallel geometry and evaluate $\mathcal{S}_{\text{TG}}$
- Difficulty preserving task: $\mathcal{S}_{\text{TG}}$ depends on the choice of a frame $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$
- Interplay between bulk and quasilocal actions helps us to understand the role of singularity
- In Euclidean case: agreement with the standard general relativity for both proper diagonal and canonical frames
- Proper diagonal frame is determined only up to some ambiguity related to remnant symmetries Nester, Ong, Ferraro, Fiorini, Guzman, Golovnev

More details

M. Krššák, M. Stano, *Teleparallel regularization of the gravitational action* (this/next week)
M. Stano, *Black Hole Action in Einstein's other Gravity*, Msc. thesis, Comenius University (2024) [arXiv:2503.11746].