Bunched-beam transverse coherent direct space charge modes

TMCI with space charge for a Gaussian bunch

https://journals.aps.org/prab/pdf/10.1103/PhysRevAccelBeams.24.060101

X. BUFFAT *et al.*

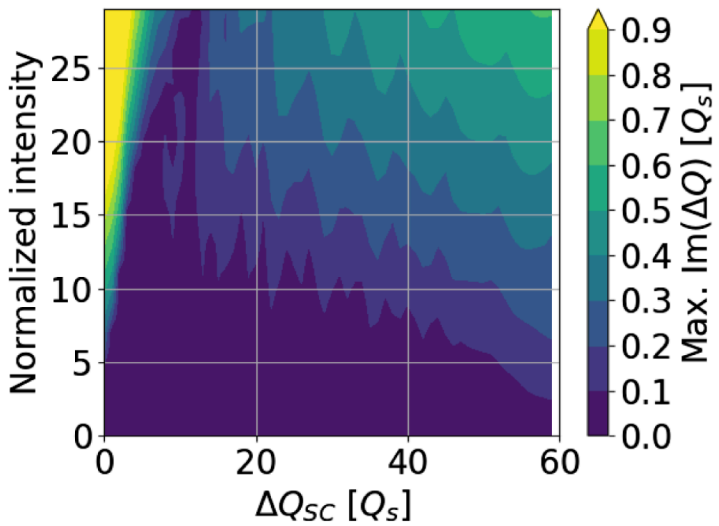

FIG. 8. Maximum imaginary tune shift for a Gaussian distribution obtained with 100 slices and 50 rings for different strengths of the wakefields and the space charge forces.

Seminal paper from M. Blaskiewicz in 1998

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PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS, VOLUME **1**, 044201 (1998)

Fast head-tail instability with space charge

M. Blaskiewicz

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(Received 29 April 1998; published 13 August 1998)

The fast head-tail instability with space charge is studied using series expansion techniques, numerical simulations, and a new formulation which allows for precise estimates of growth rates and thresholds. In regimes where they are reliable, all three techniques predict that space charge suppresses the fast head-tail instability. It is found that the series expansion techniques are unreliable for parameter regimes commonly employed in hadron accelerators. The numerical techniques are less prone to error, but the computational requirements become severe as space charge tune shifts increase. The new model has neither of these problems, but it underestimates the benefits of chromaticity, at least in its simplest form.

Physical model

Physical model

$$\frac{d^2\tau}{d^2\theta} + Q_s^2\tau = 0, \qquad (1)$$

$$\frac{d^2x}{d^2\theta} + Q_x^2 x = C_{sc}\rho(\theta,\tau)\left[x - \langle x(\theta,\tau)\rangle\right] + \int_{-T}^{\tau} d\tau' W(\tau-\tau')\rho(\theta,\tau') \times \langle x(\theta,\tau')\rangle d\tau'. \qquad (2)$$

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Without wake field

Quadrupolar force

Dipolar force

Physical model

$$\frac{d^2\tau}{d^2\theta} + Q_s^2\tau = 0,$$

Variations in transverse beam size are neglected => Linear space charge forces in the transverse direction (like KV distribution)

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- N.B.: it has the same form as the one to be solved for instance in my GALACTIC (impedance) Vlasov solver

$$\frac{\Delta Q}{Q_s} a_{kl} = H a_{ij} = \sum_{i,j=-\infty}^{+\infty} H_{kl,ij}\, a_{ij}$$

Longitudinal distribution

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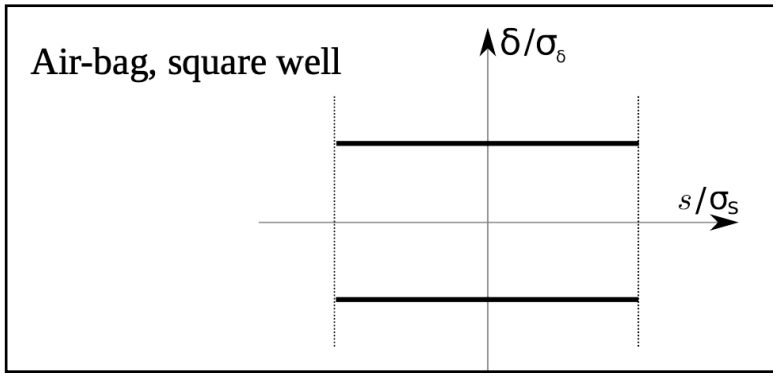

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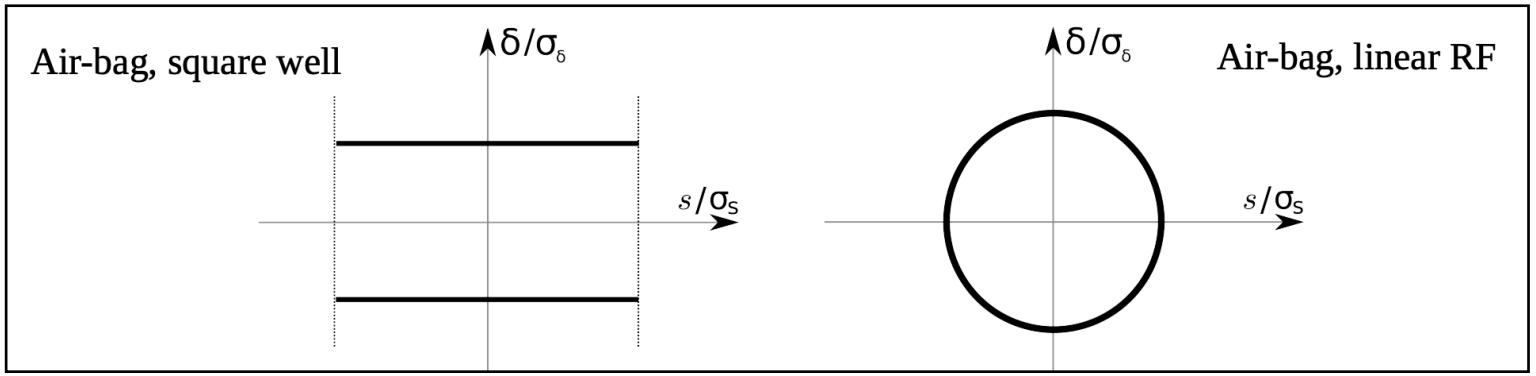
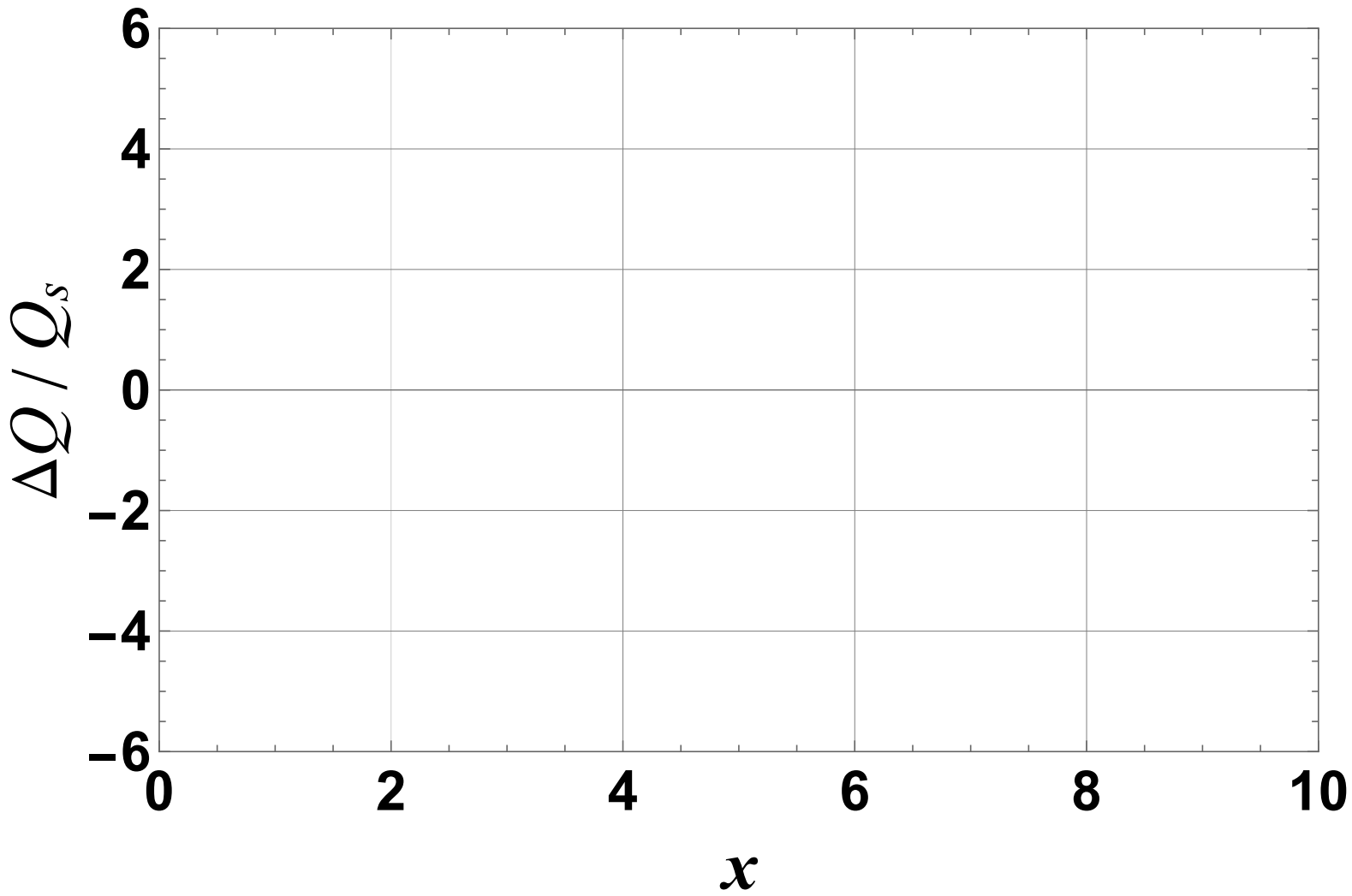
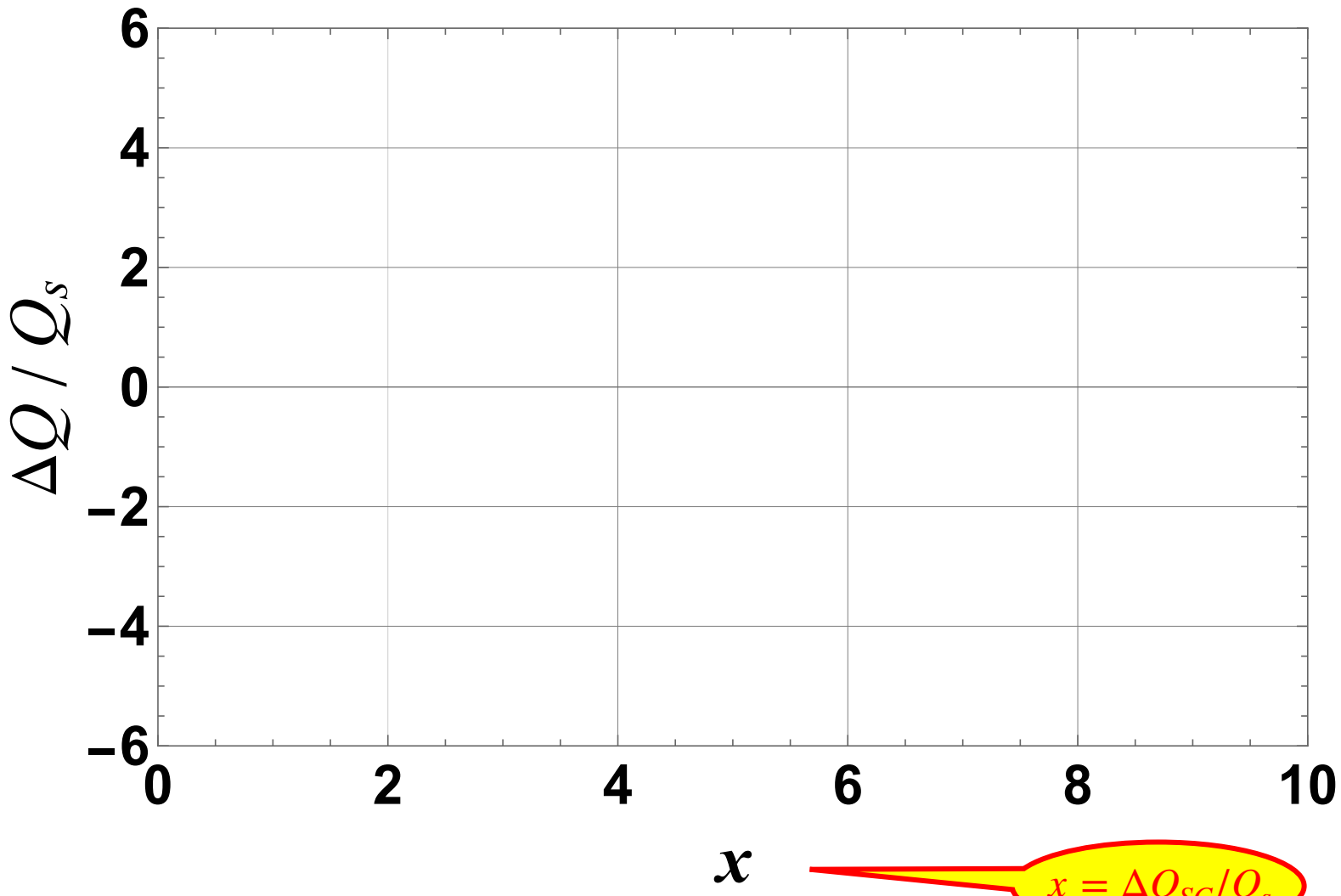


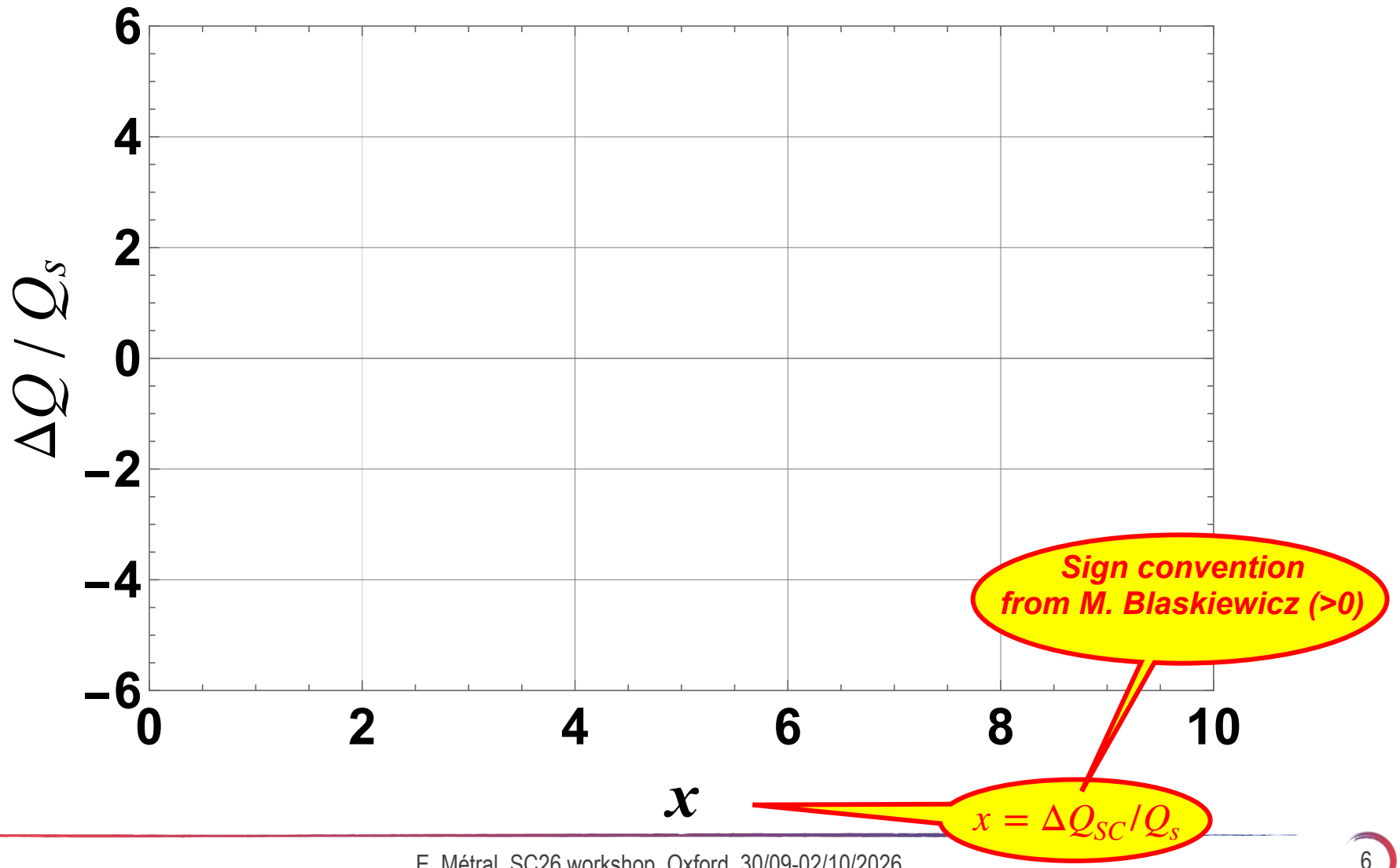
ABS



ABS



ABS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−6
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8
10
$\Delta Q / Q_s$
x
Sign convention
from M. Blaskiewicz (>0)
$x = \Delta Q_{SC} / Q_s$

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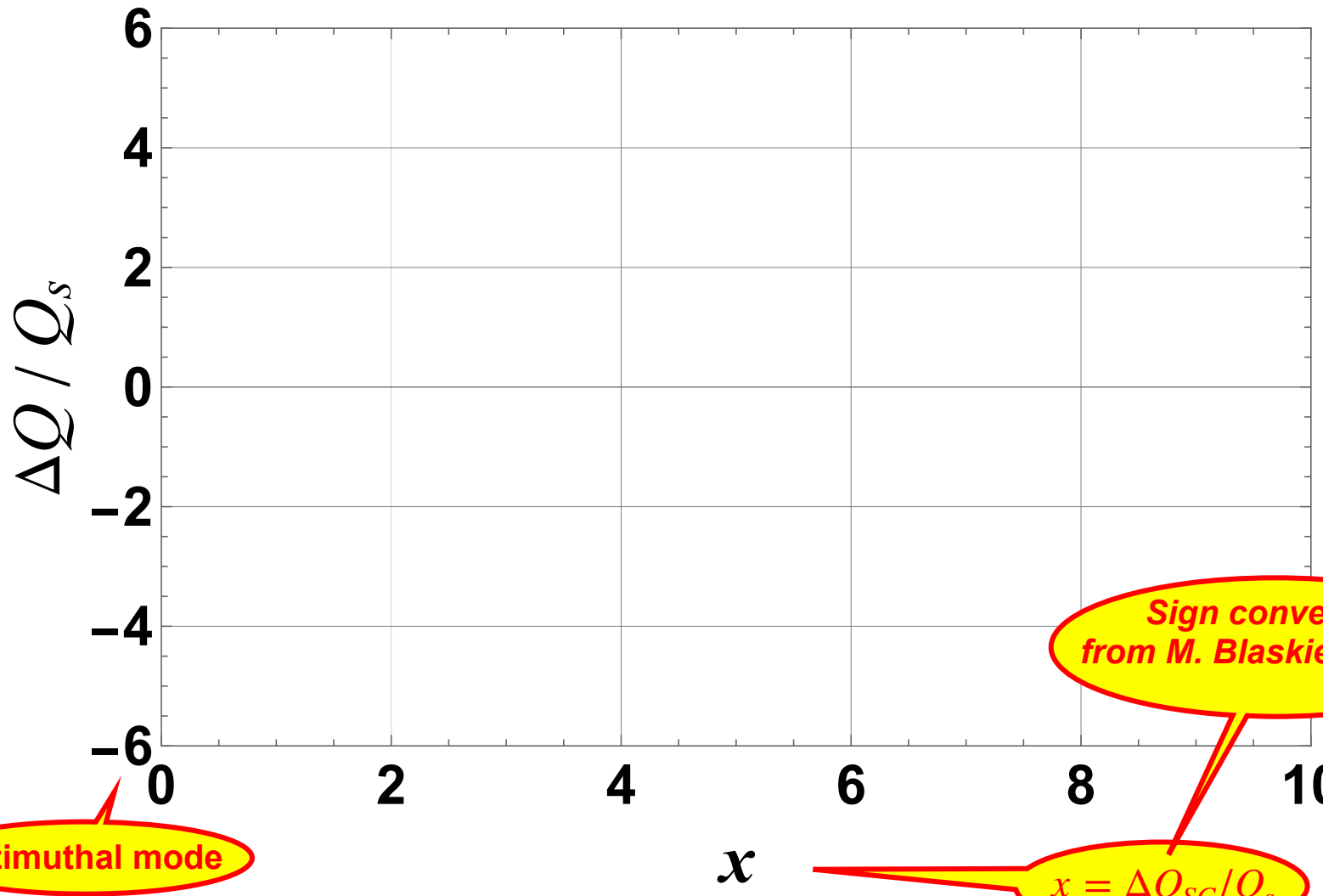
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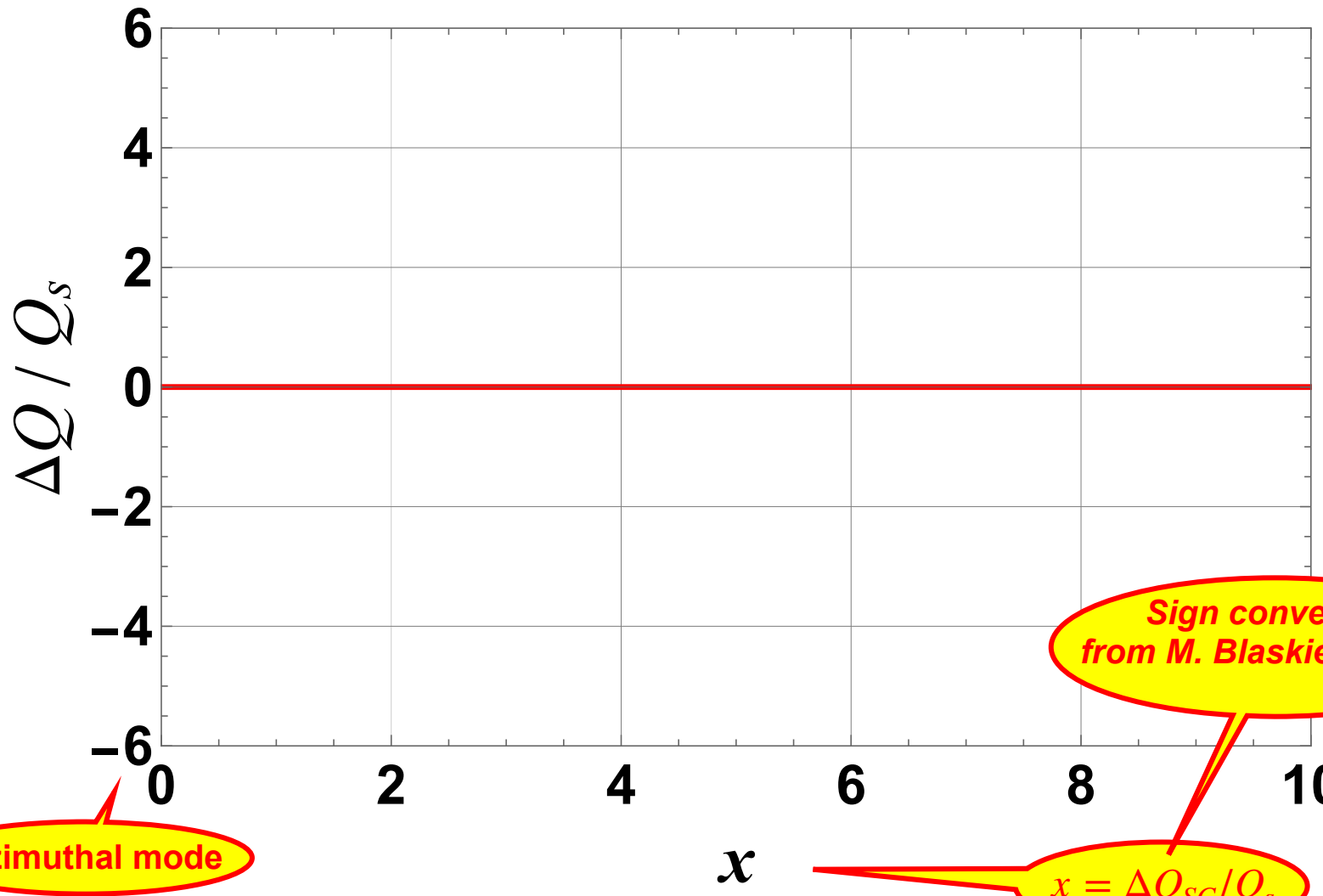
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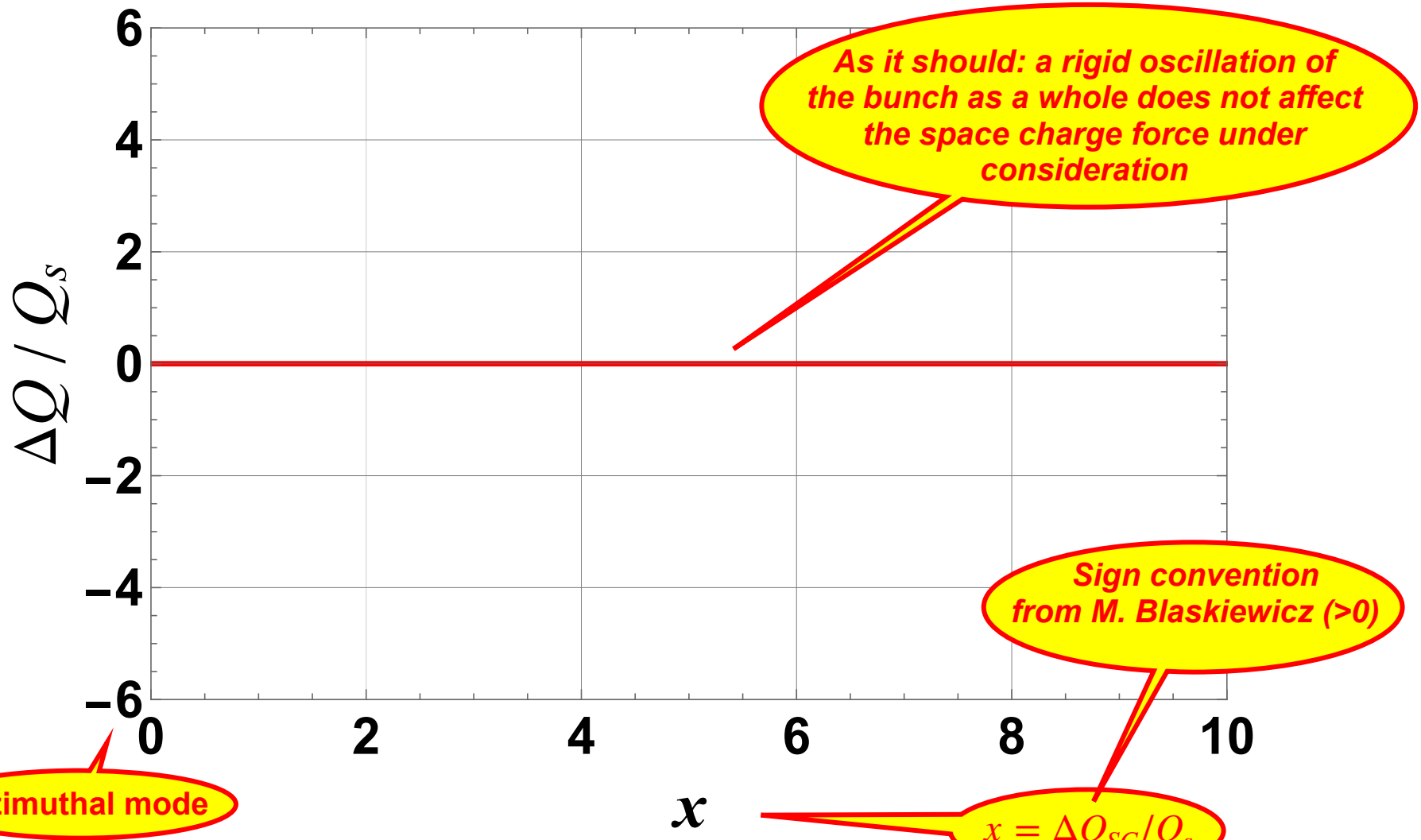
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10
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As it should: a rigid oscillation of the bunch as a whole does not affect the space charge force under consideration
Sign convention from M. Blaskiewicz (>0)
Azimuthal mode
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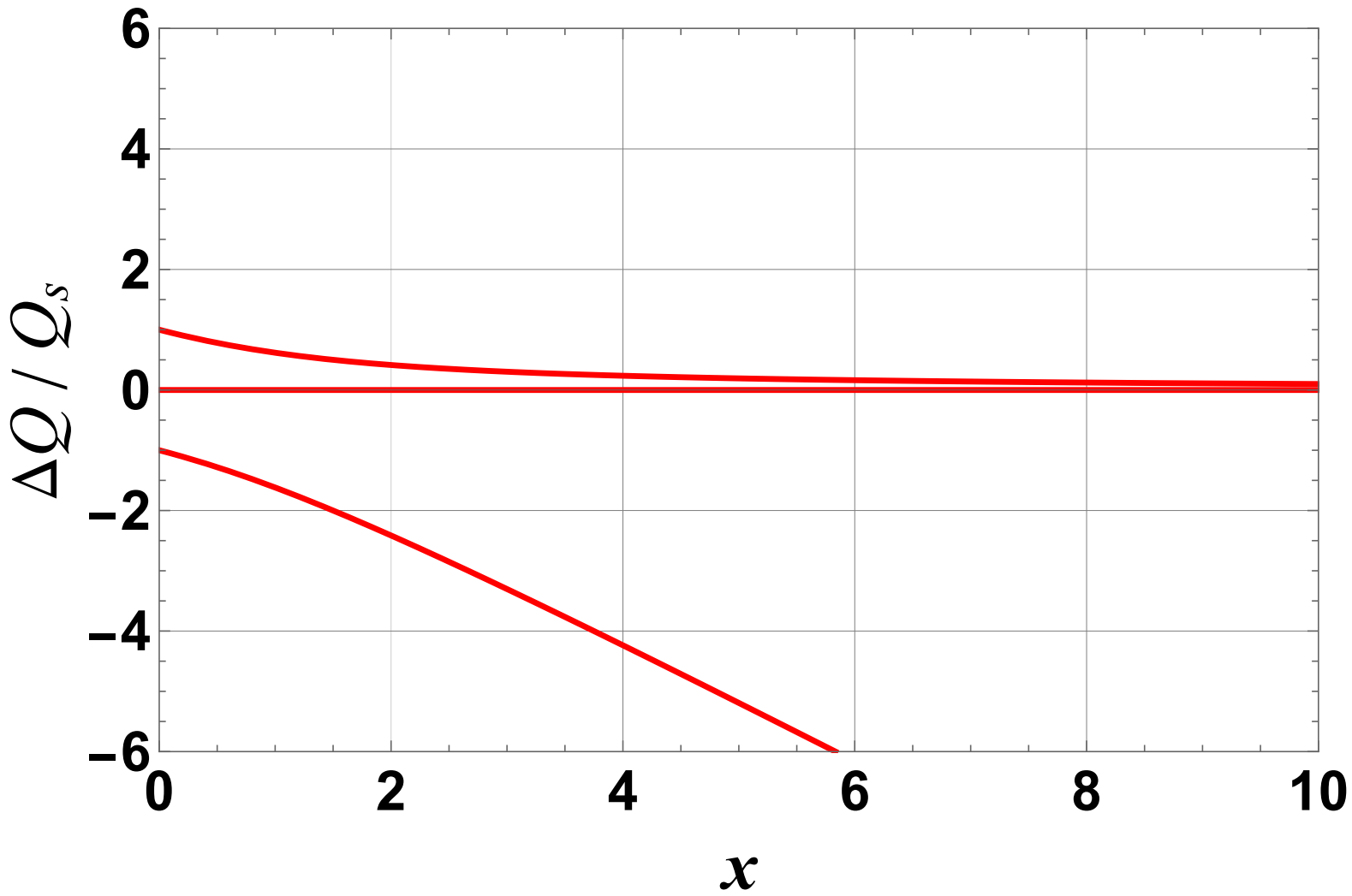
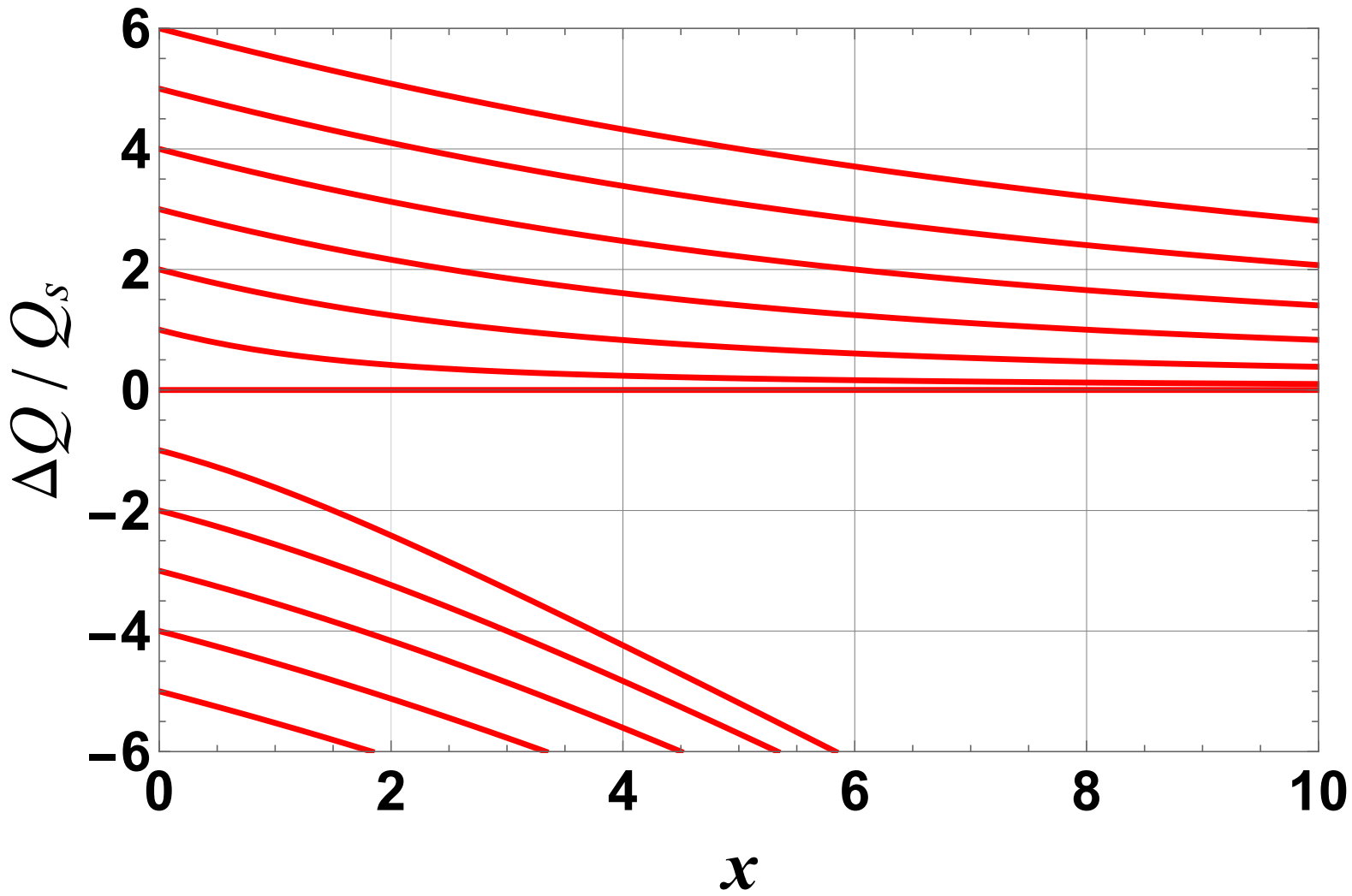
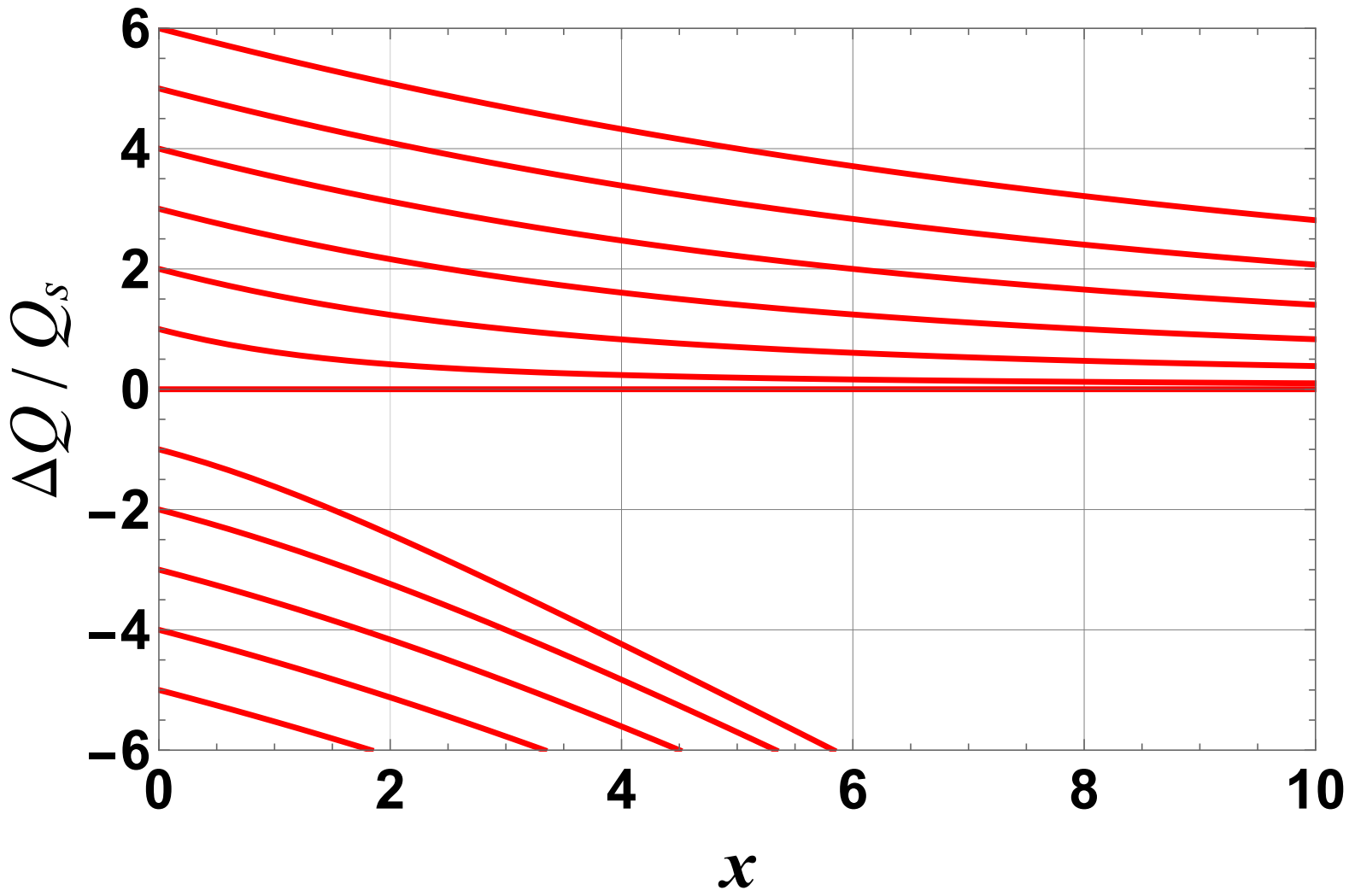
ABS



ABS



ABS => No radial modes



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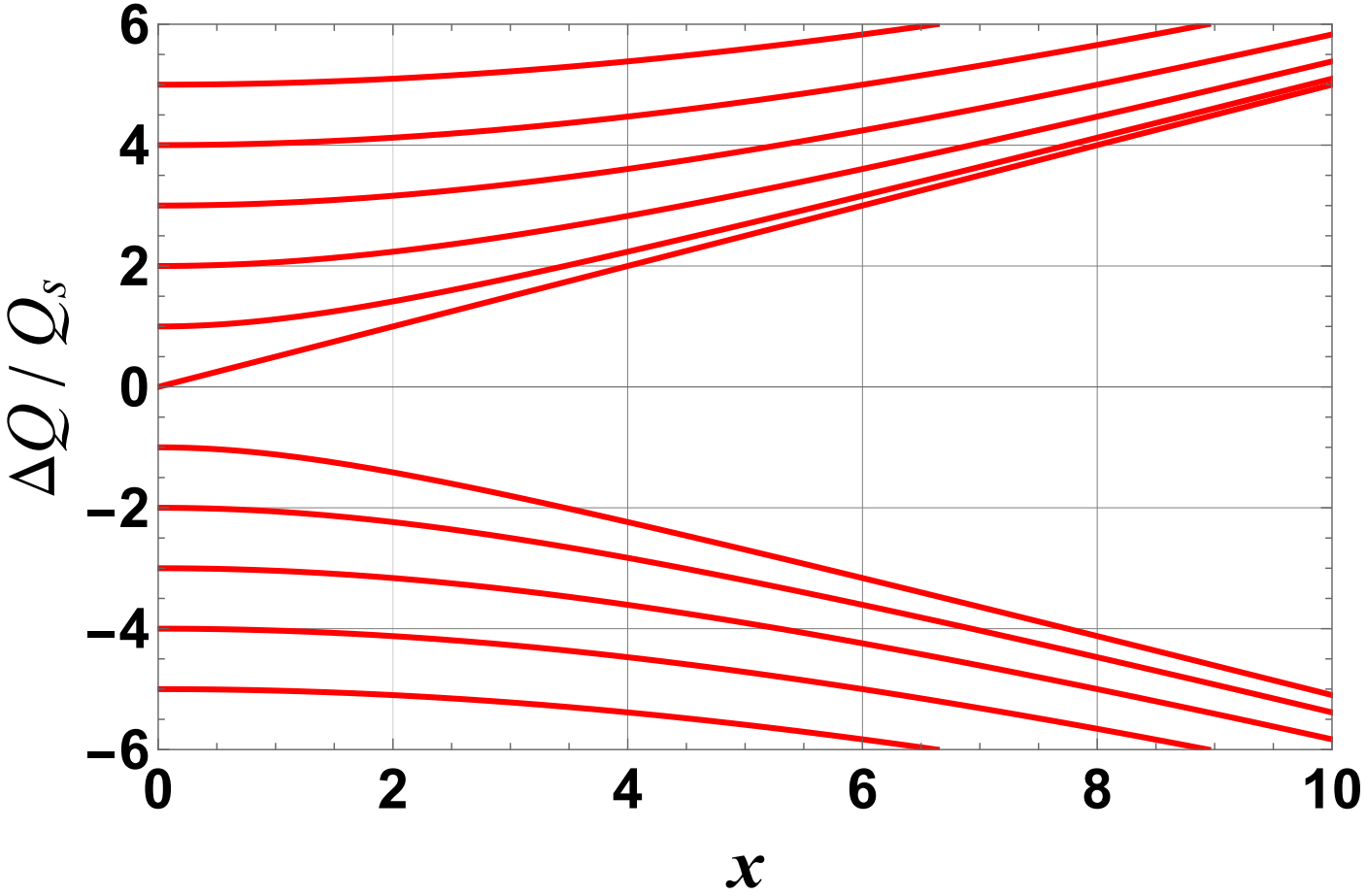

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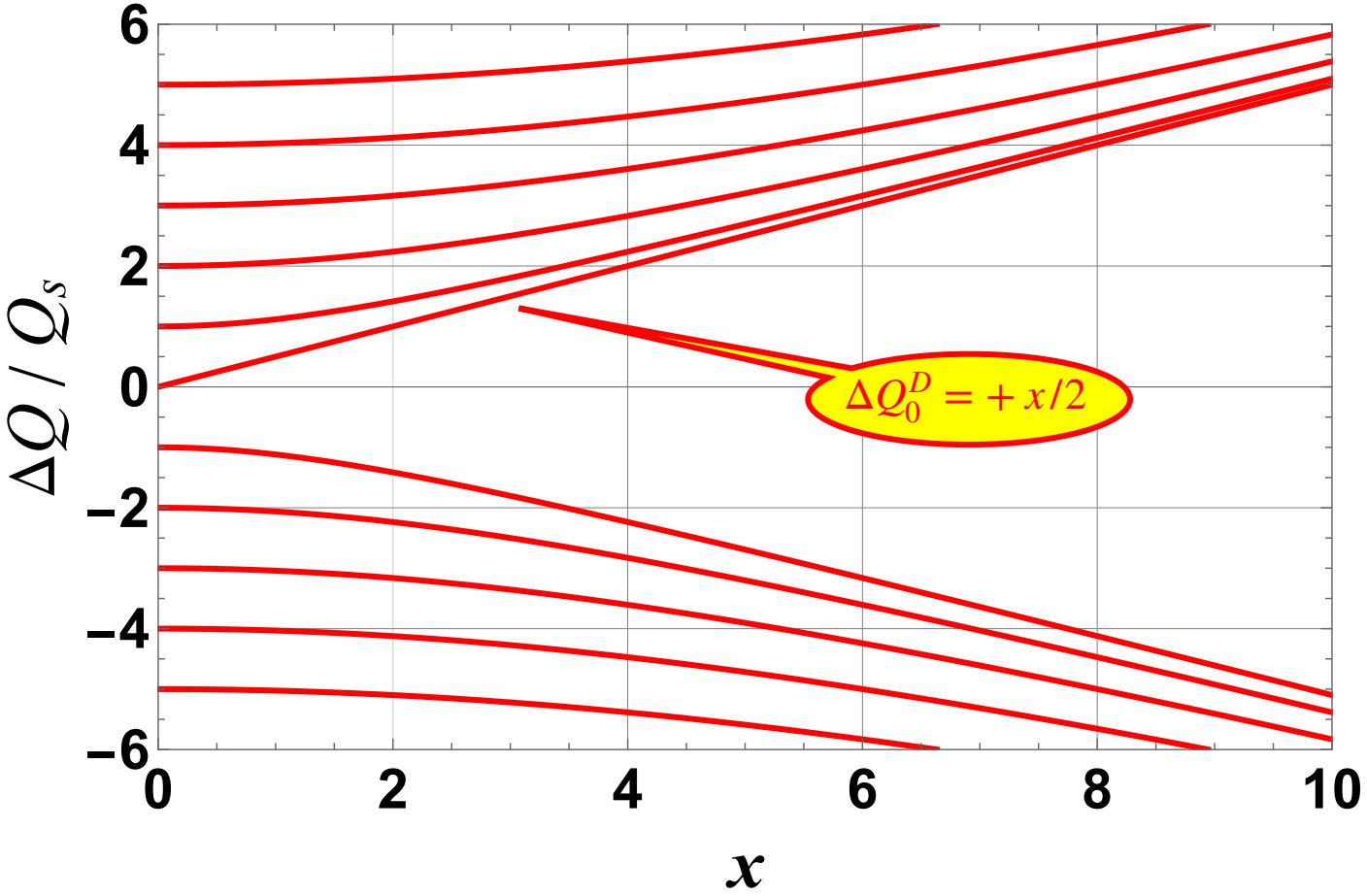

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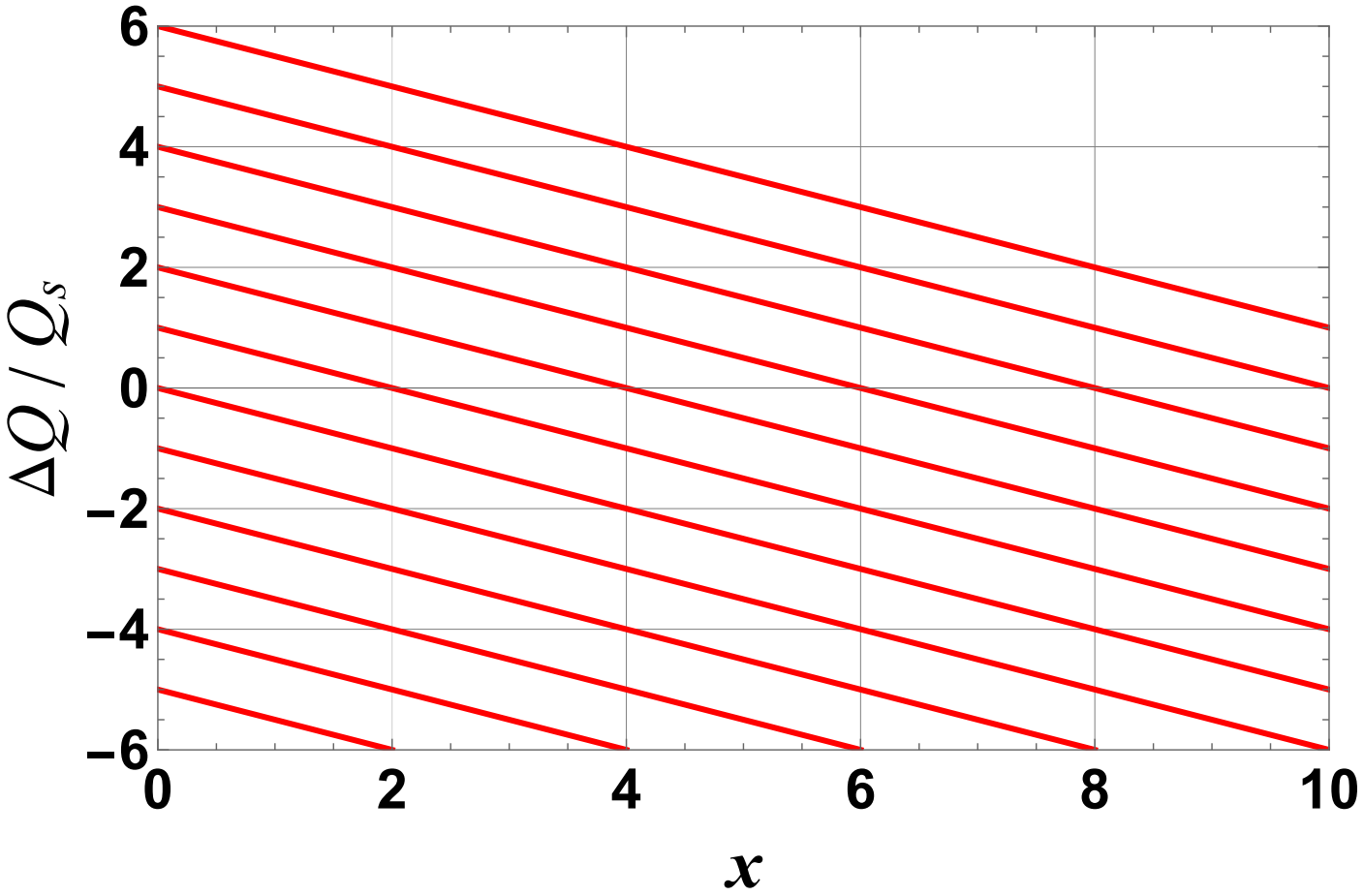

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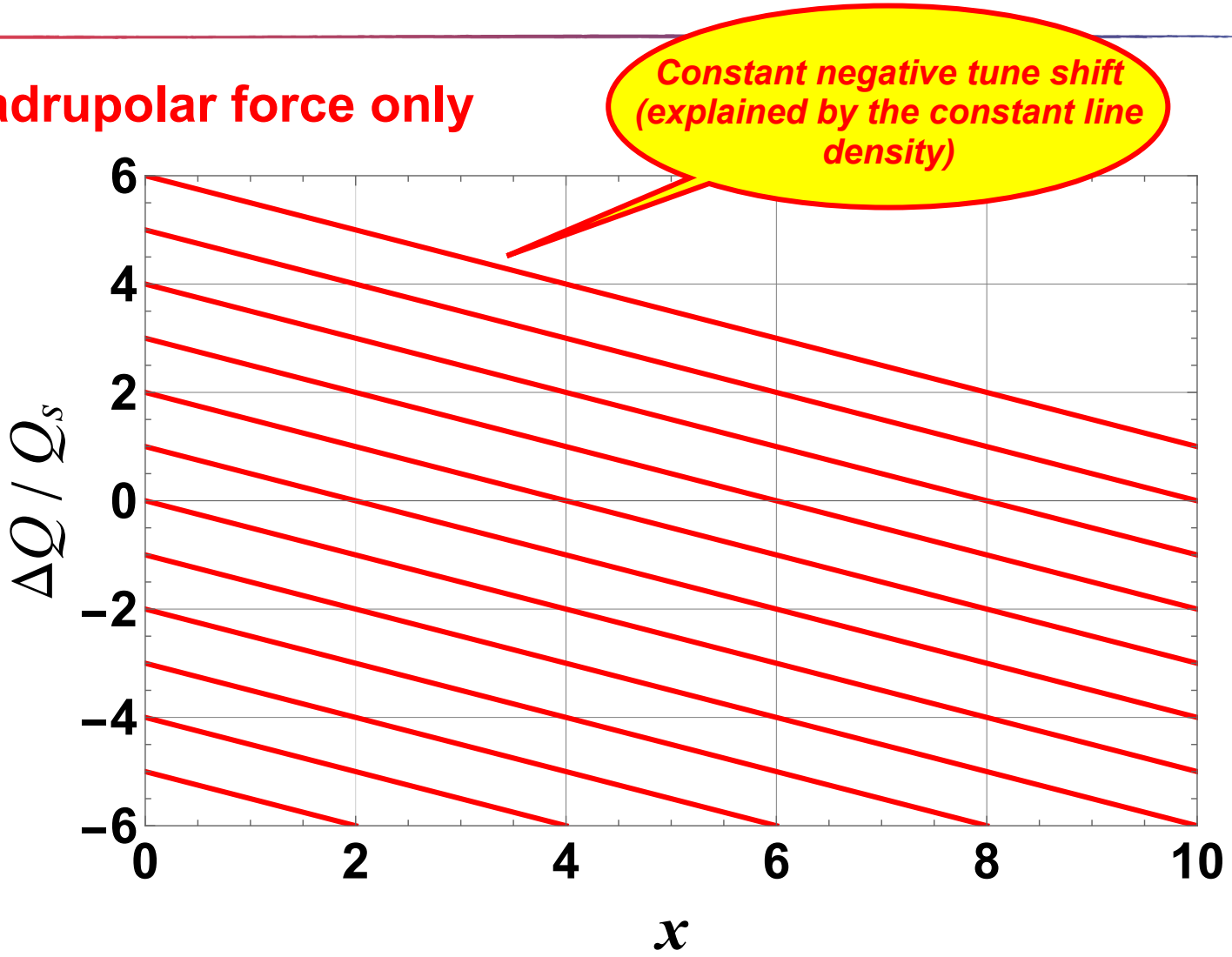

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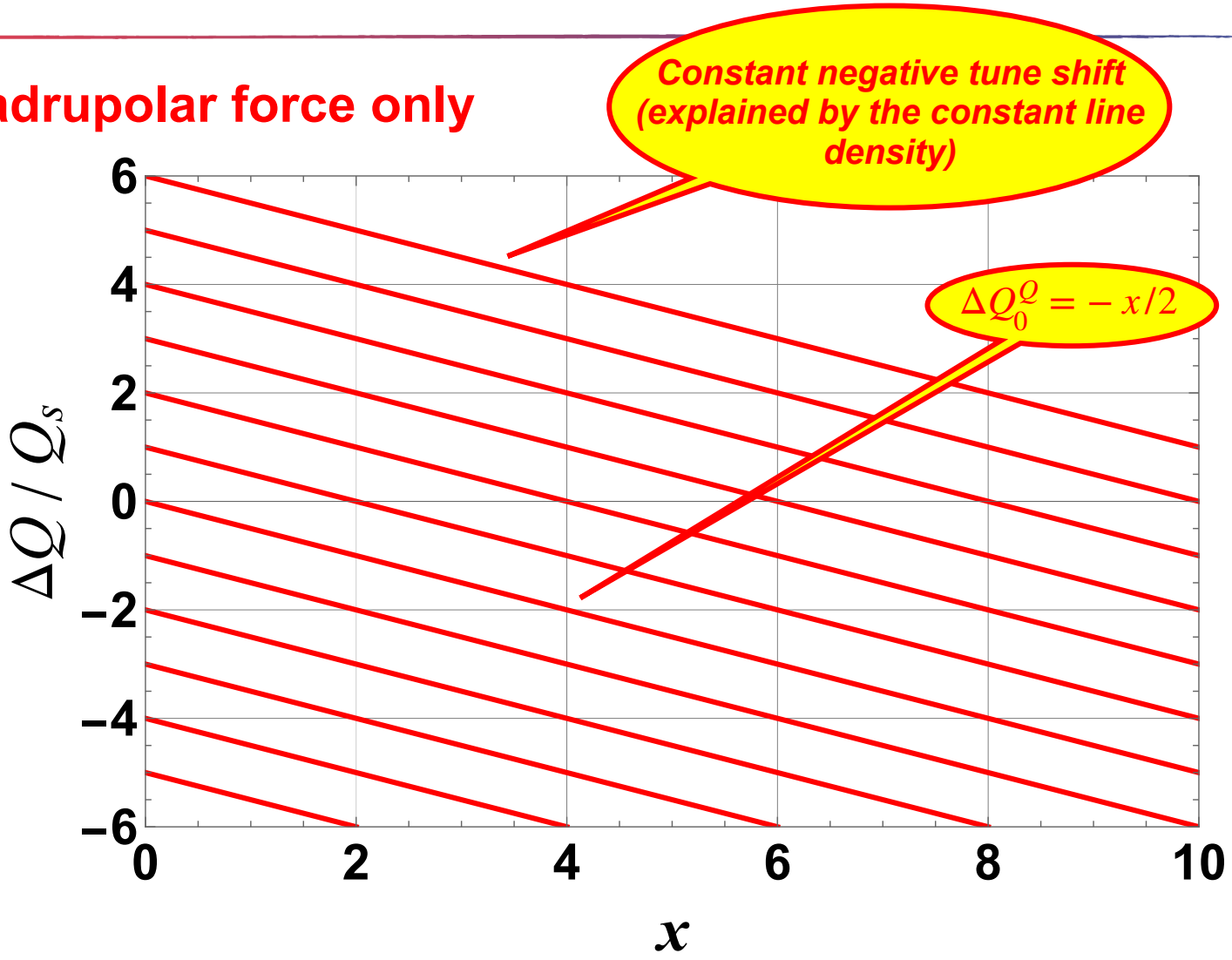

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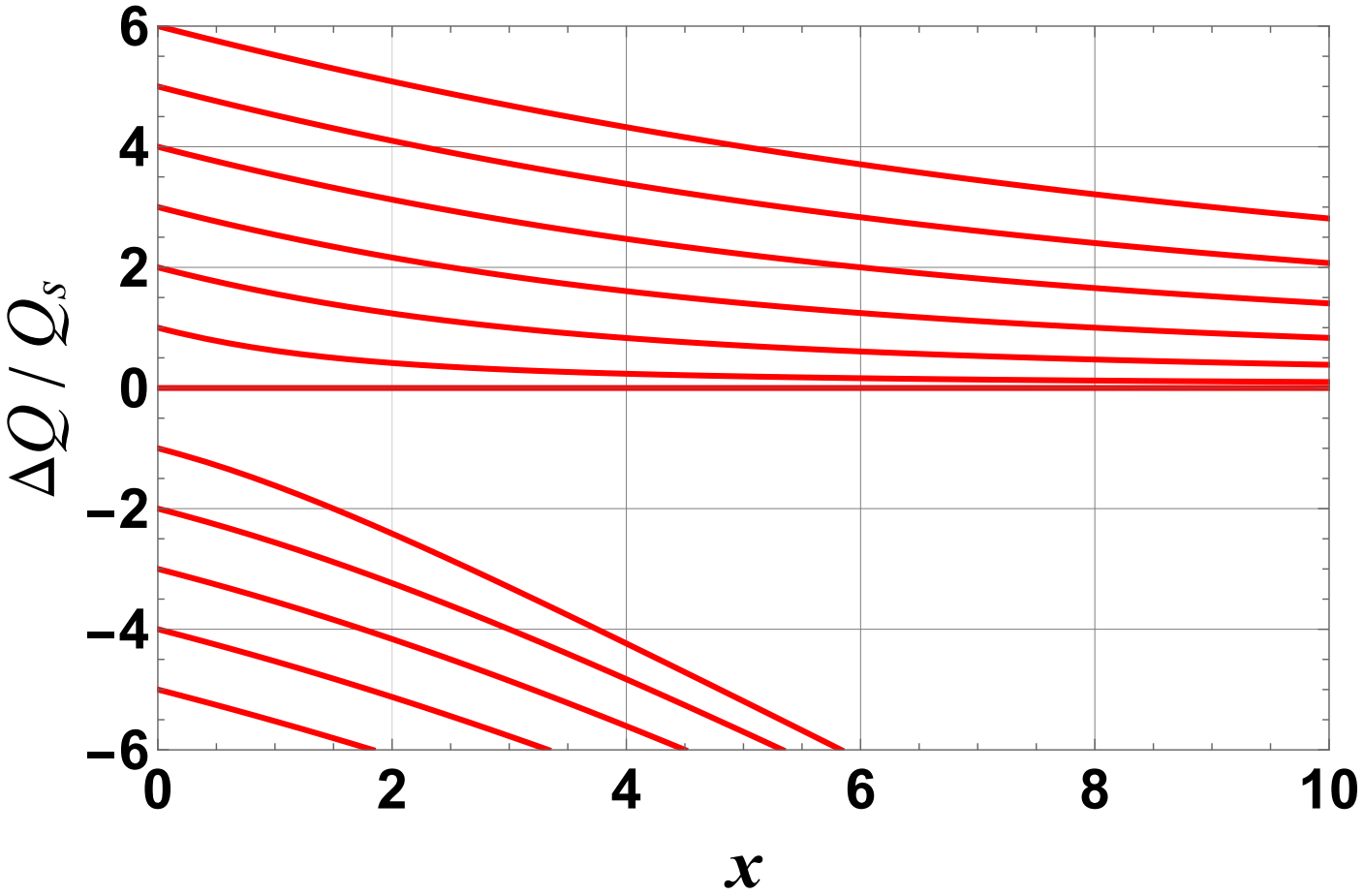

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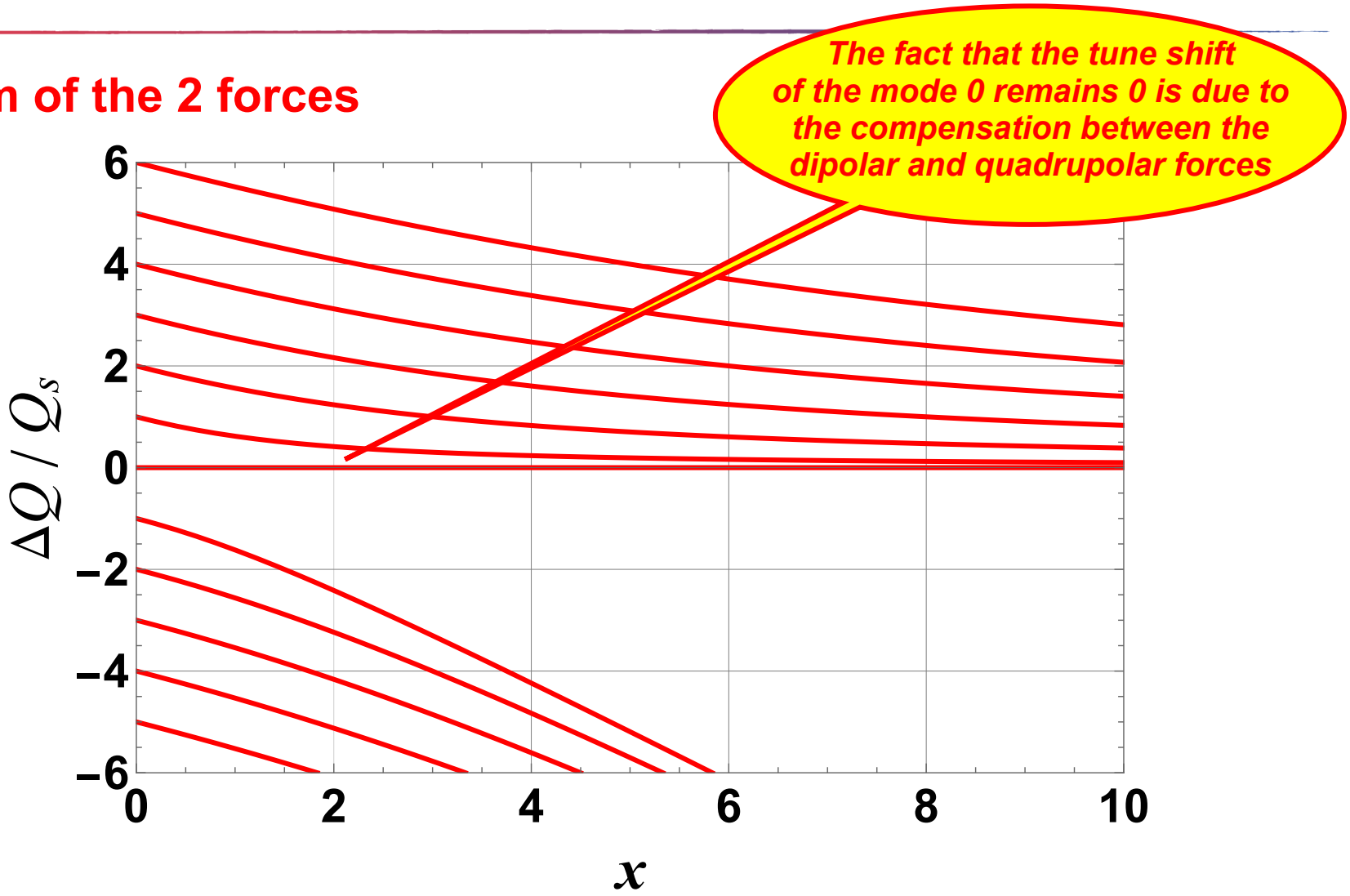

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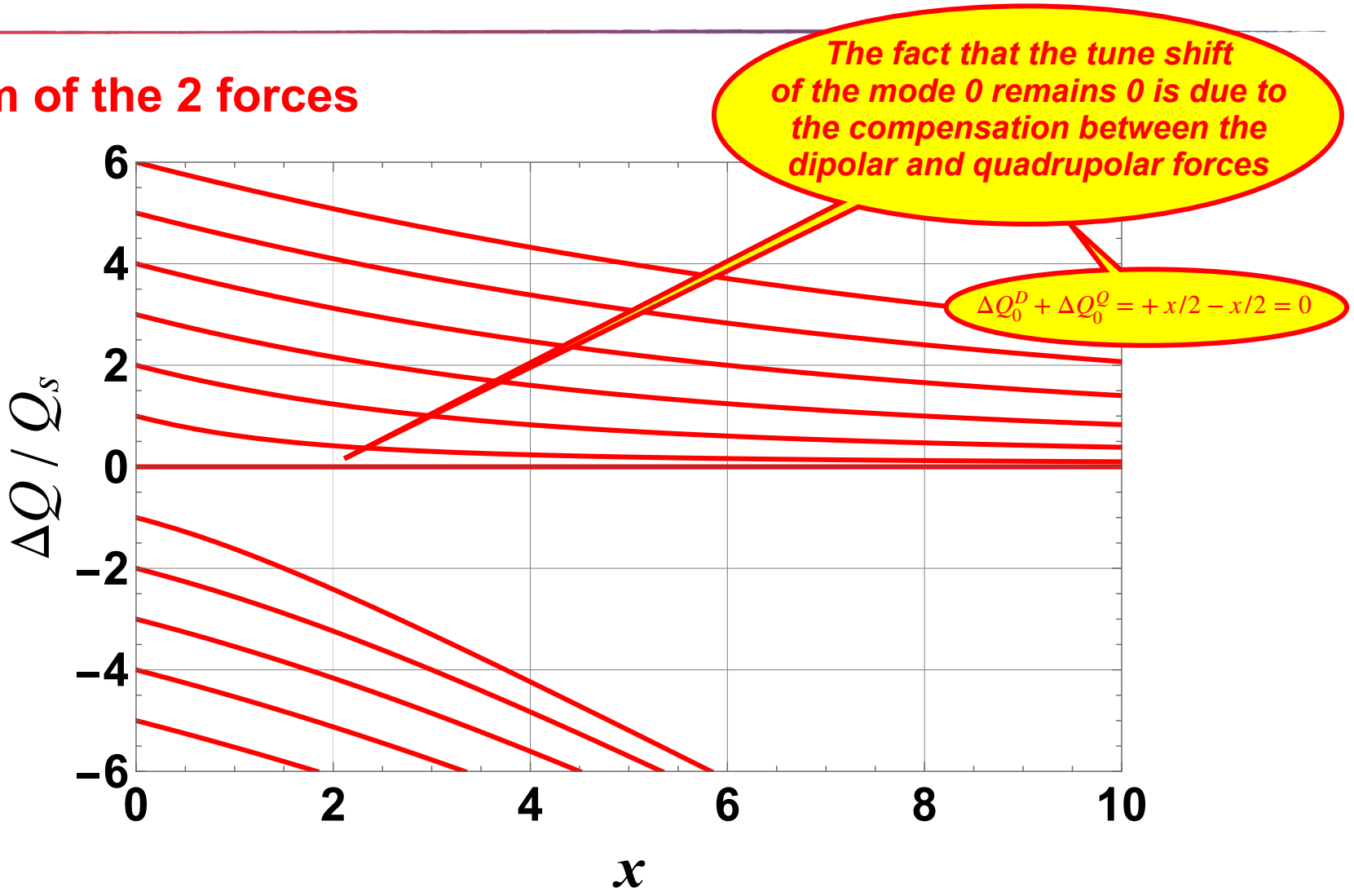
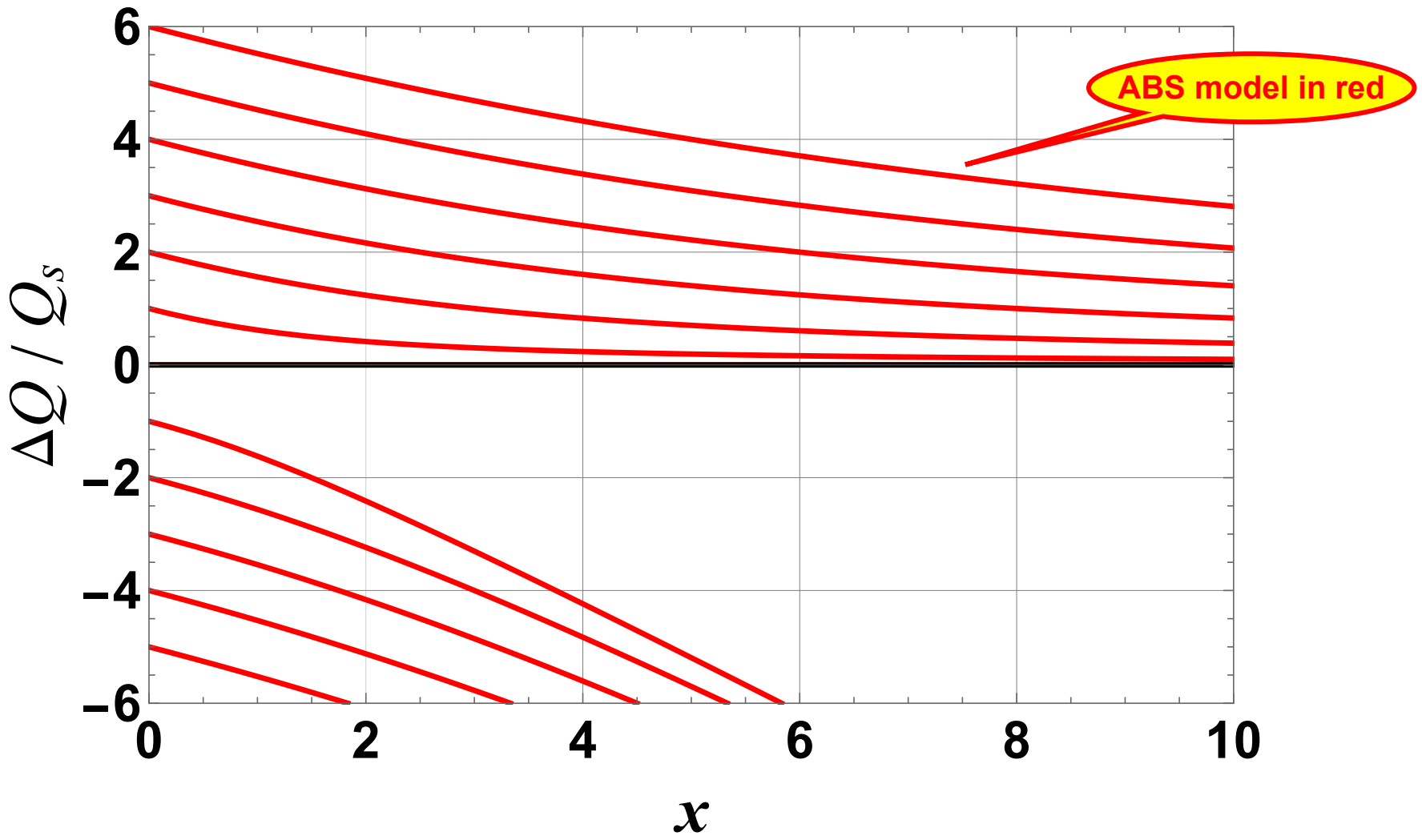


BOXCAR: radial mode = 0



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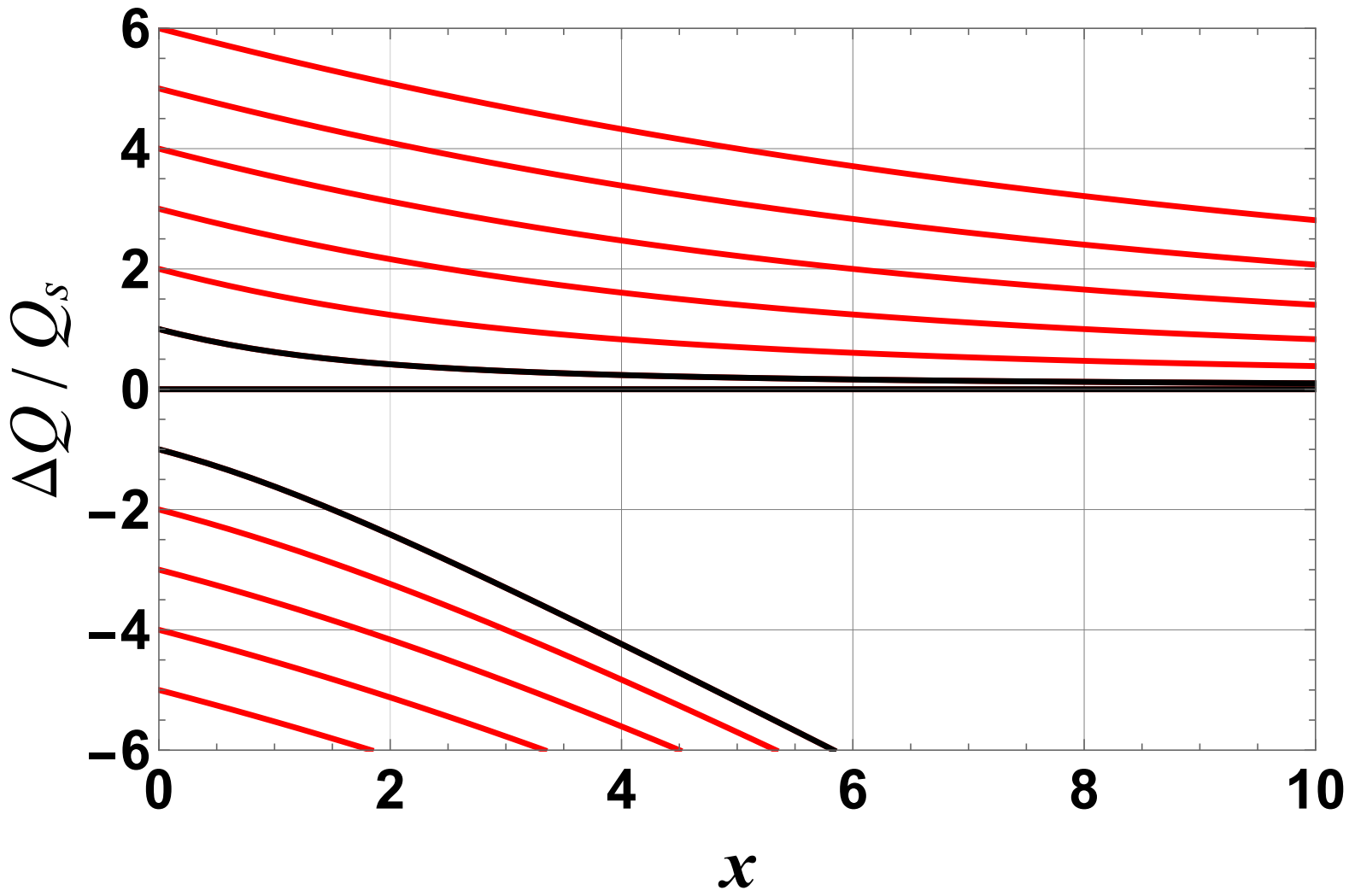

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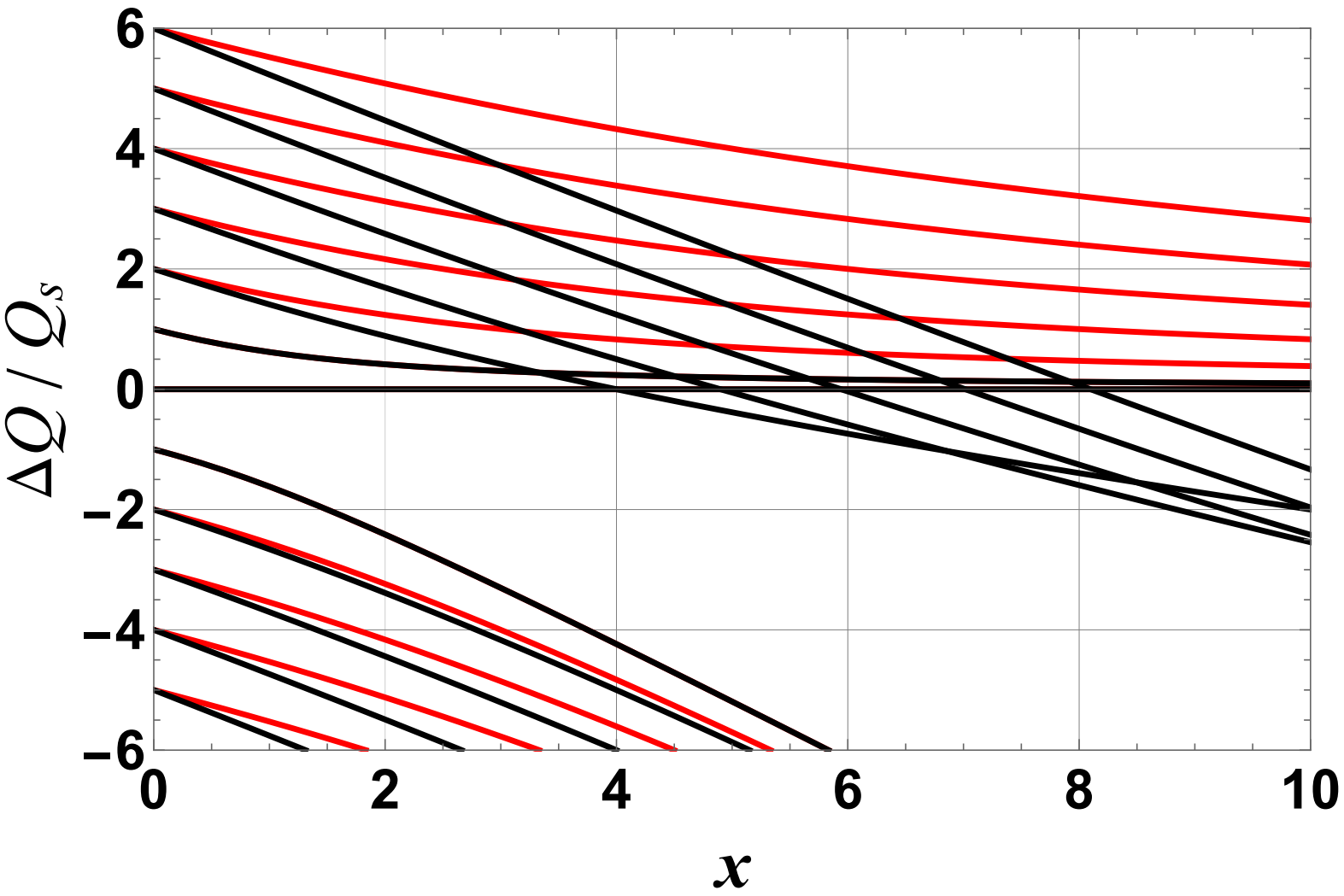
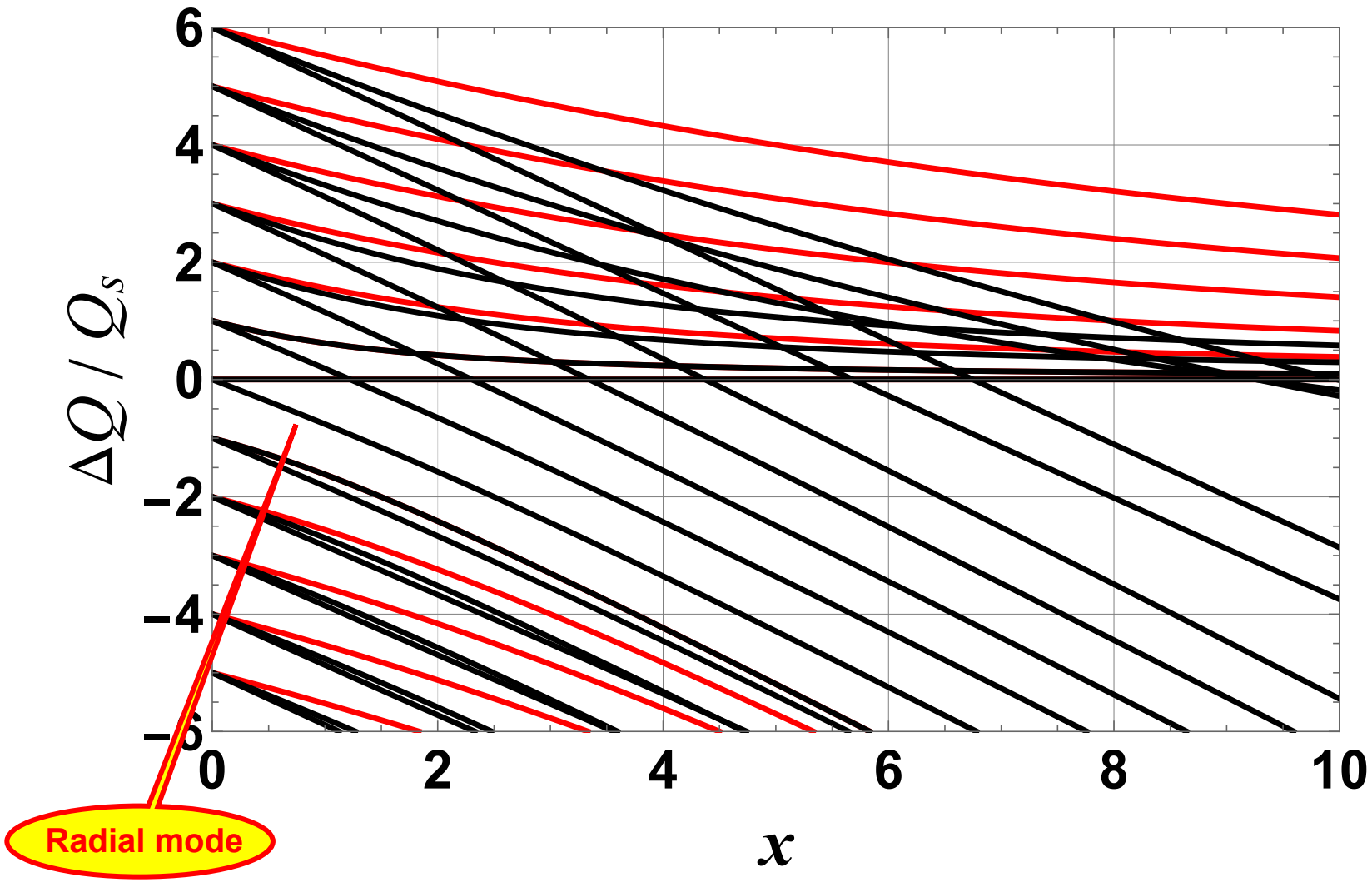
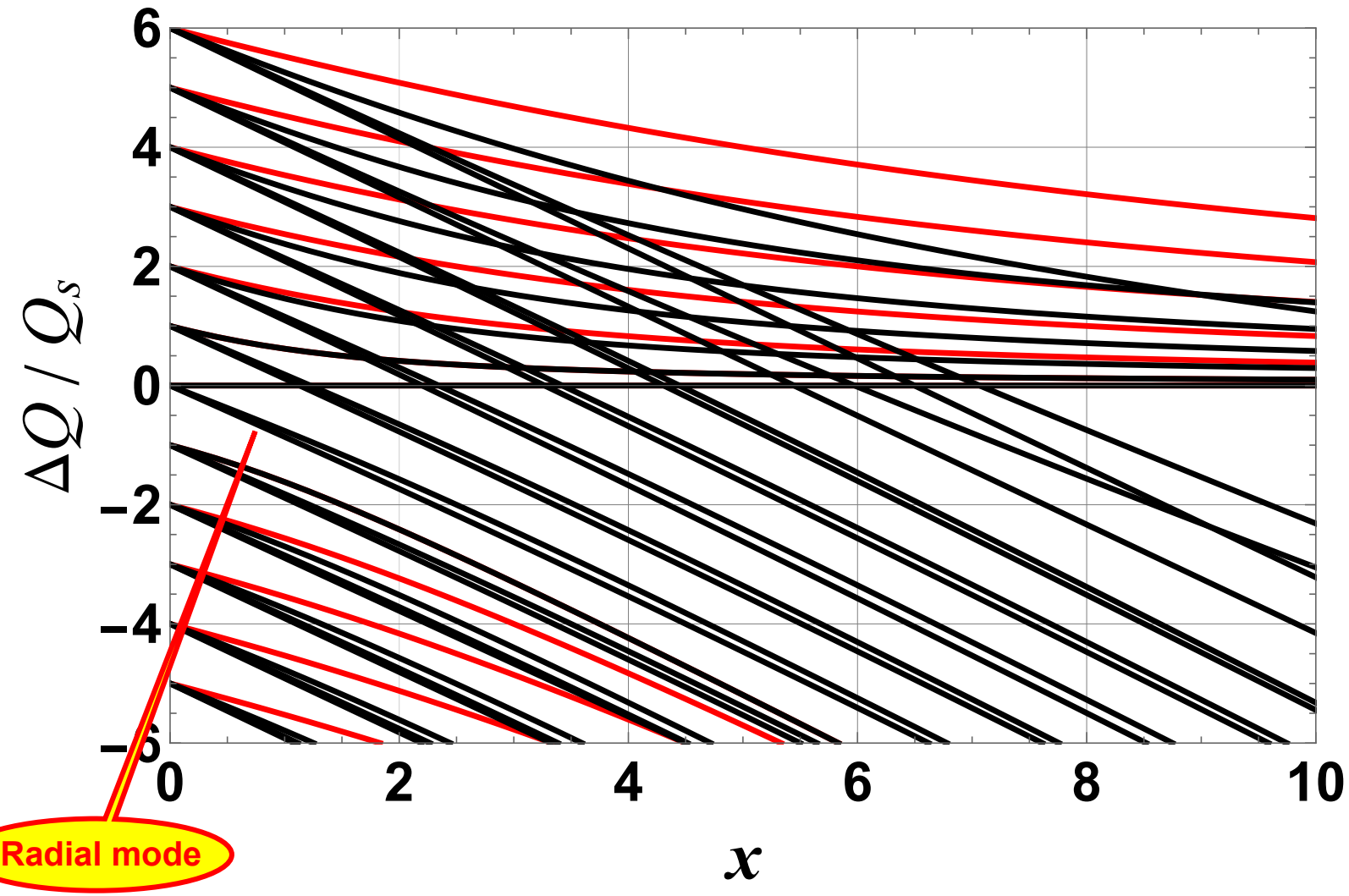


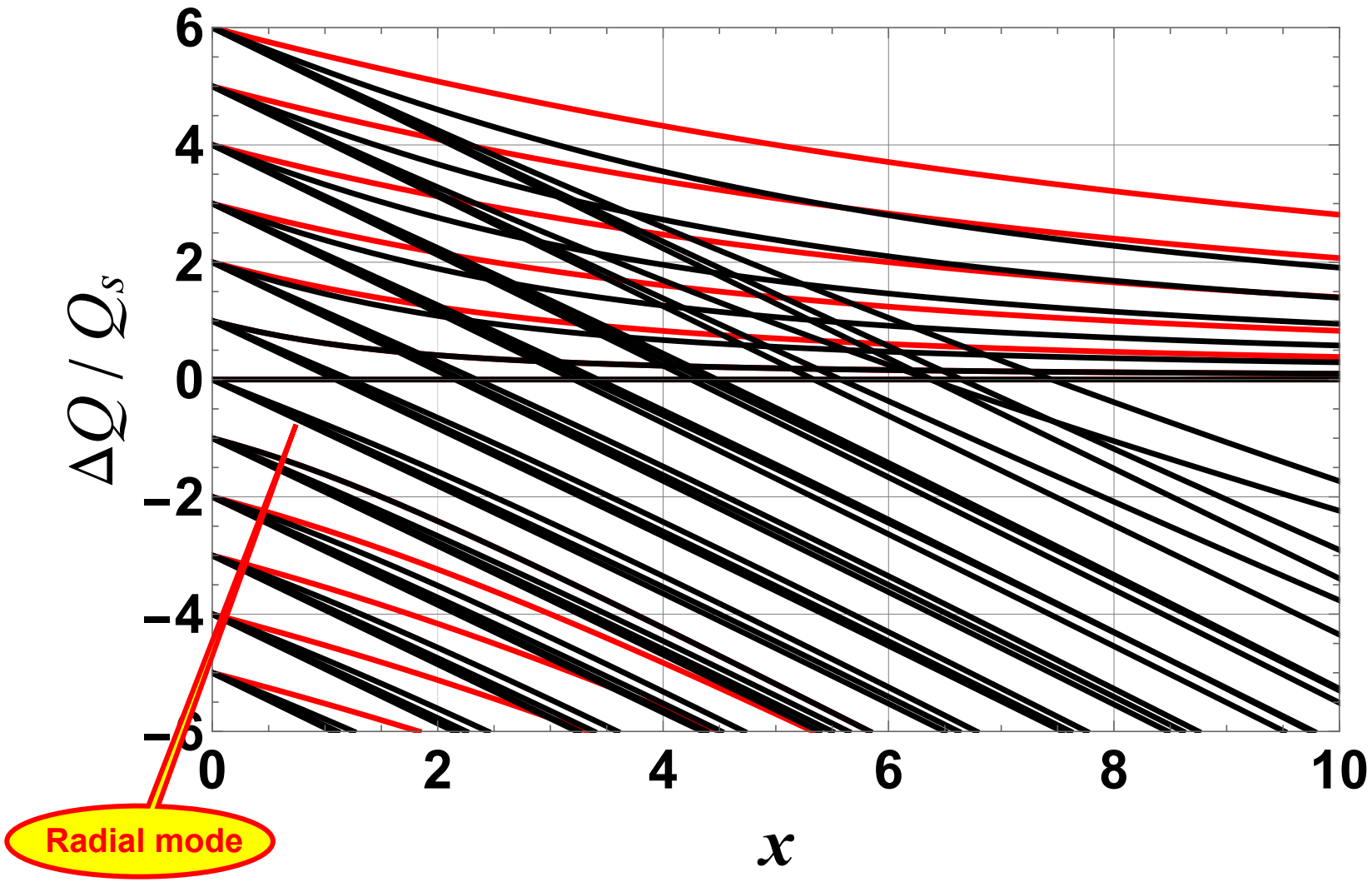
BOXCAR: radial mode = 1



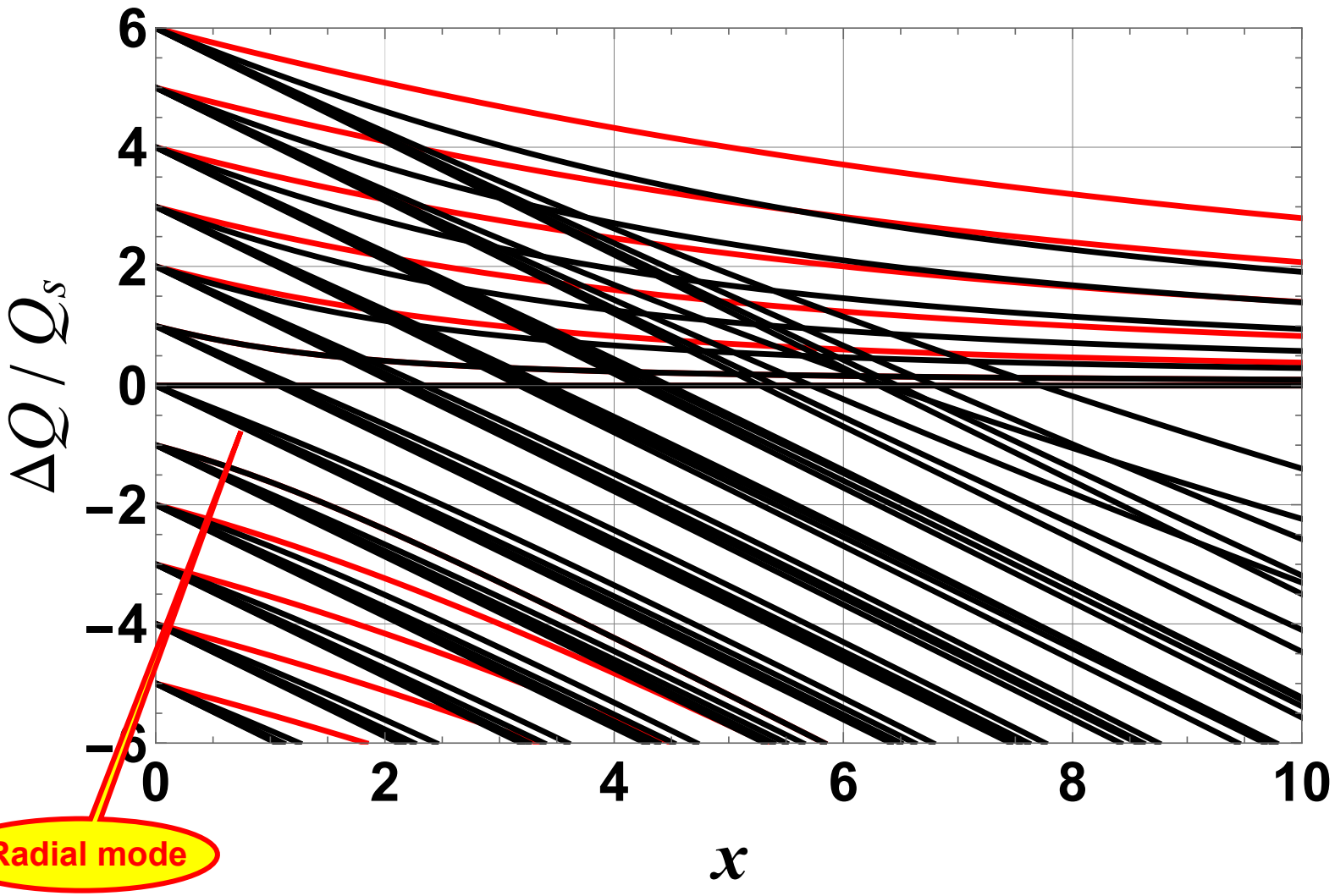
BOXCAR: radial mode = 2



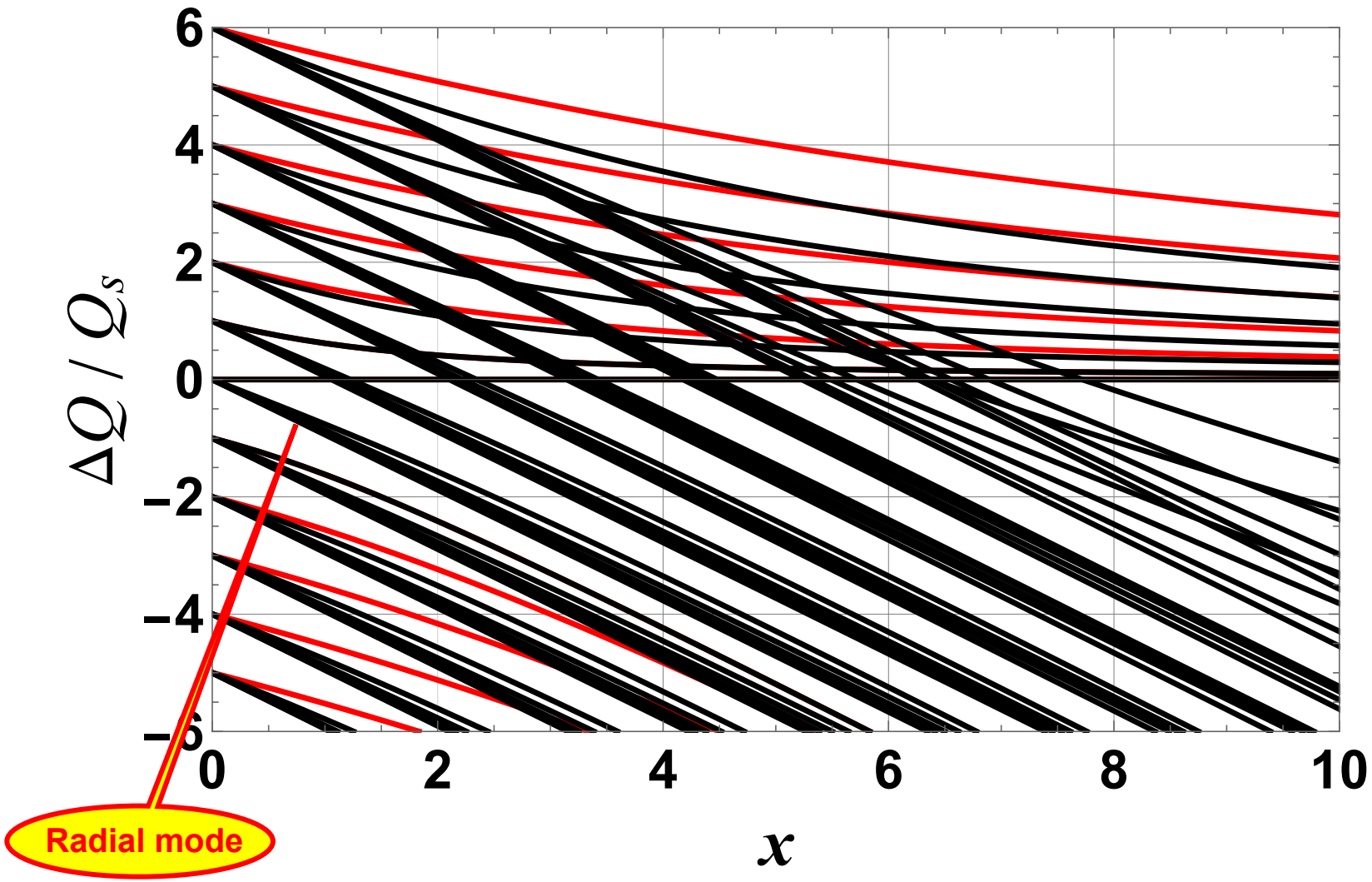
BOXCAR: radial mode = 3



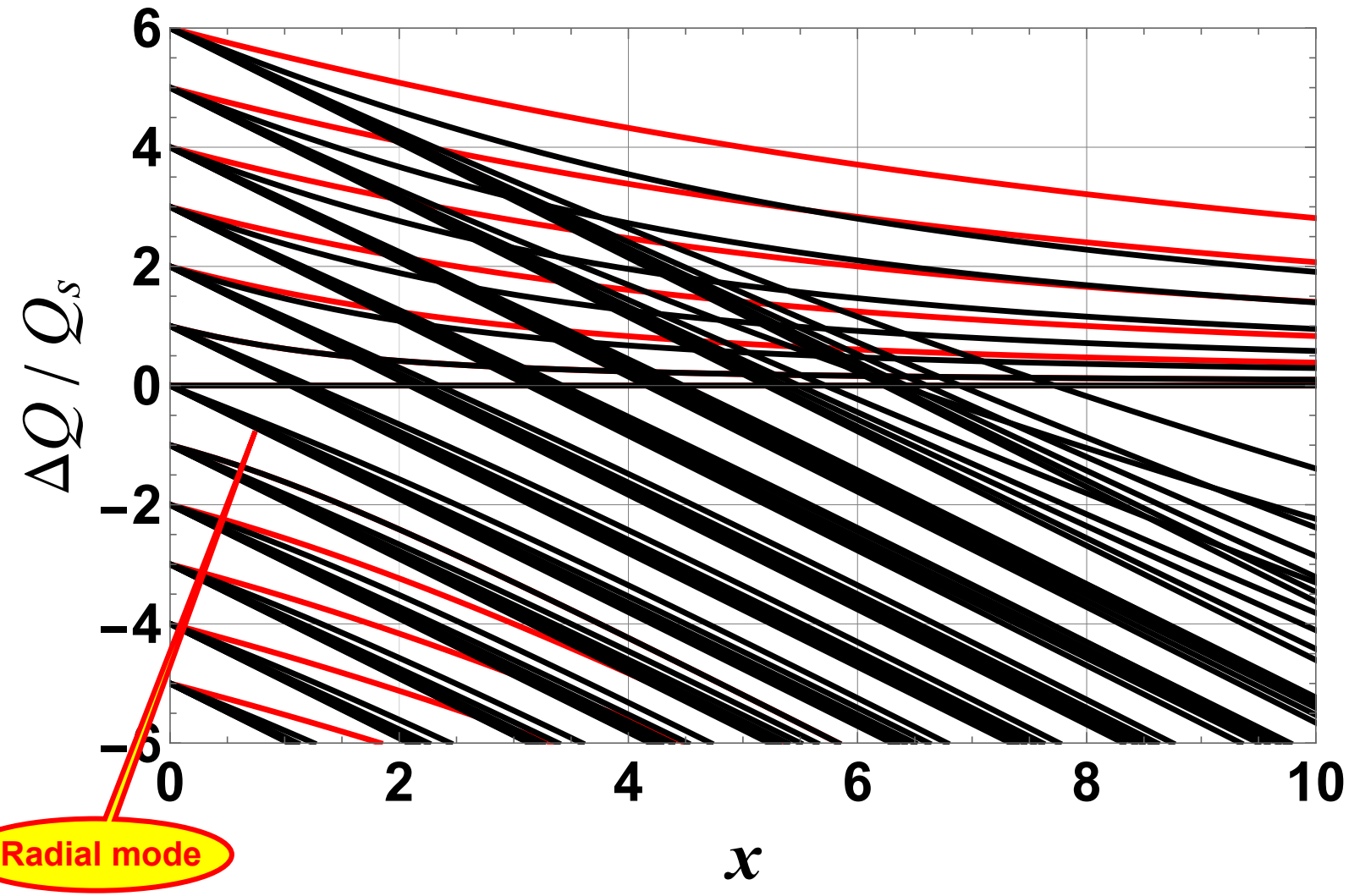
BOXCAR: radial mode = 4



BOXCAR: radial mode = 5

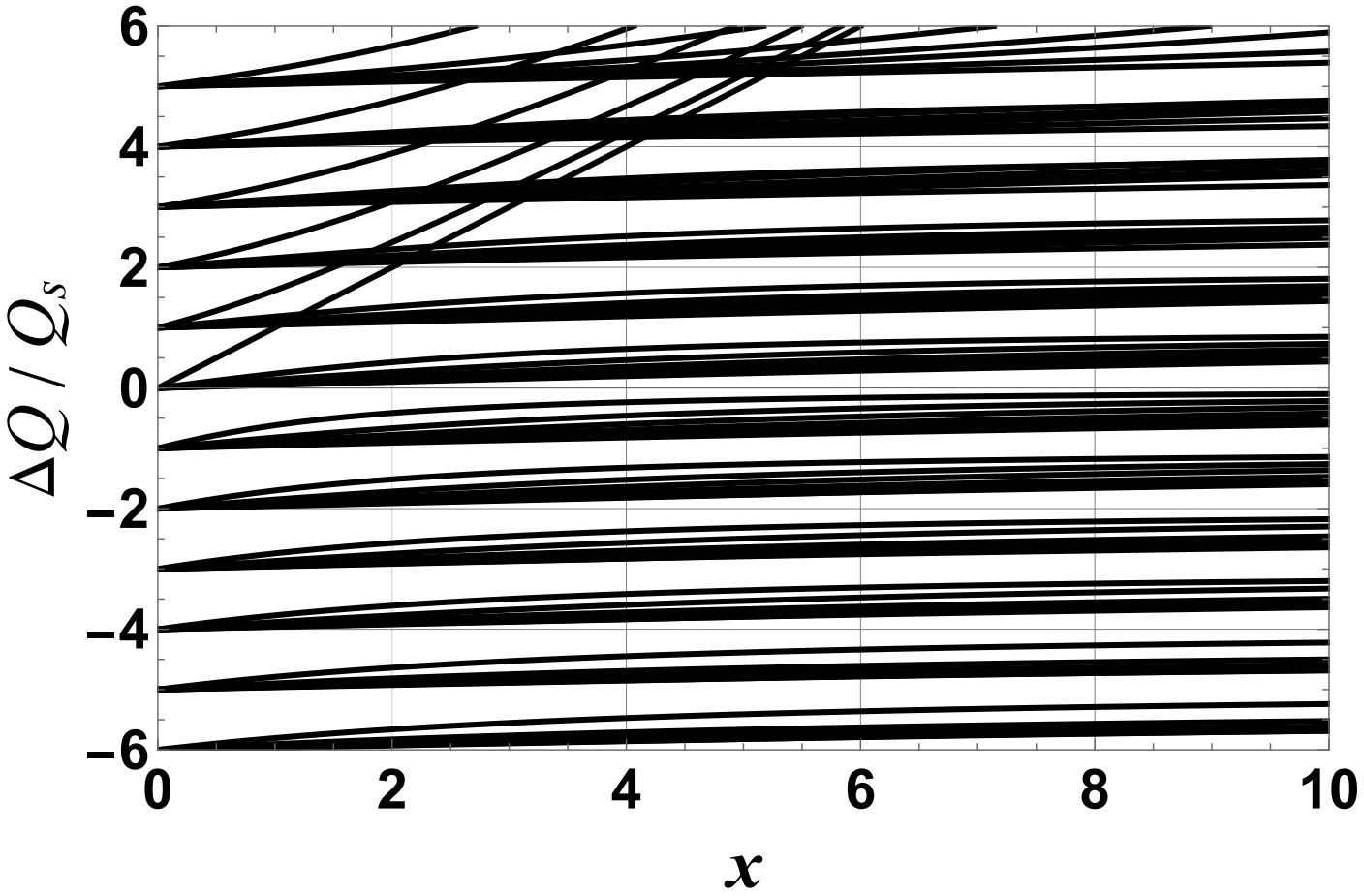


BOXCAR: radial mode = 6



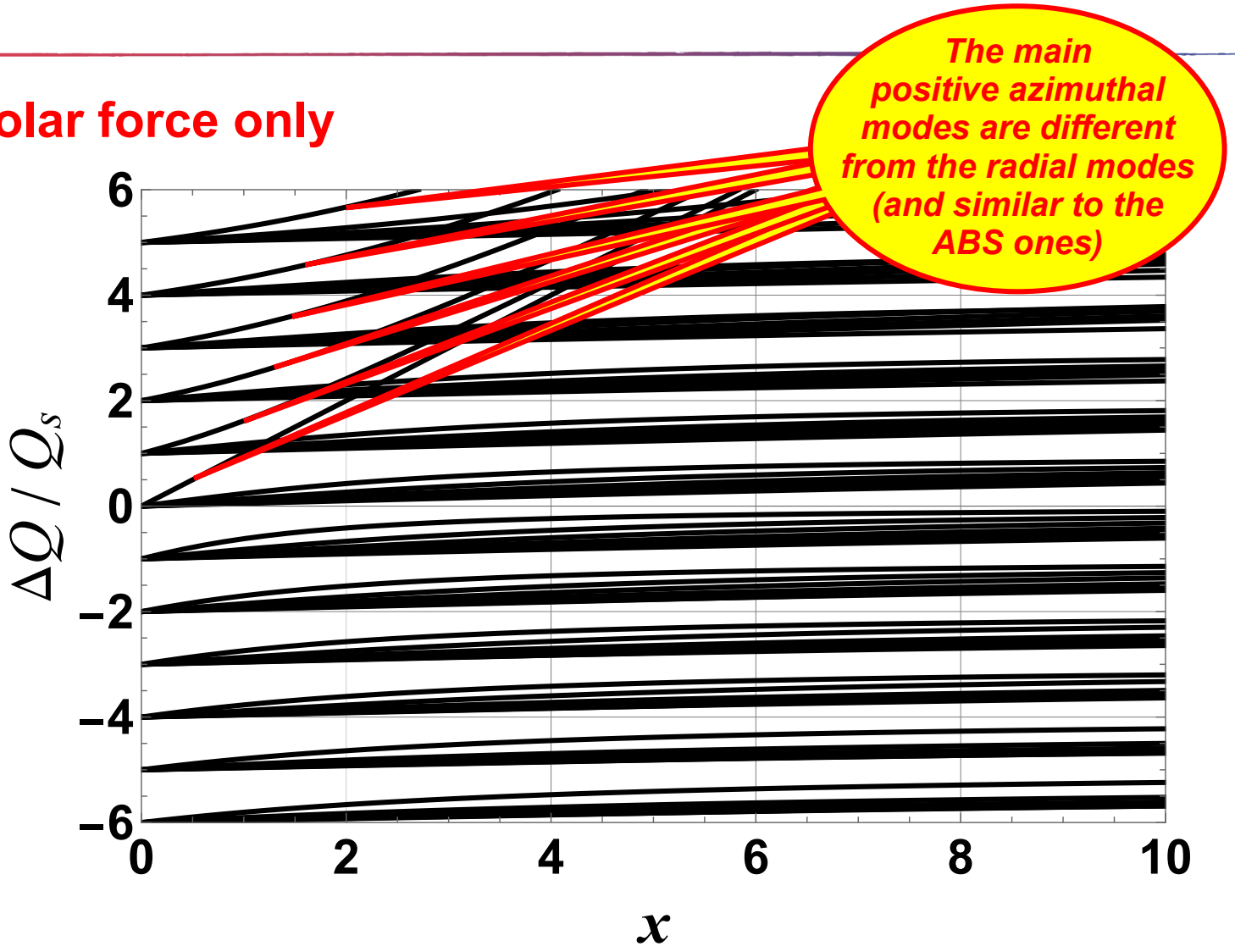
BOXCAR: radial mode = 6

- **Dipolar force only**



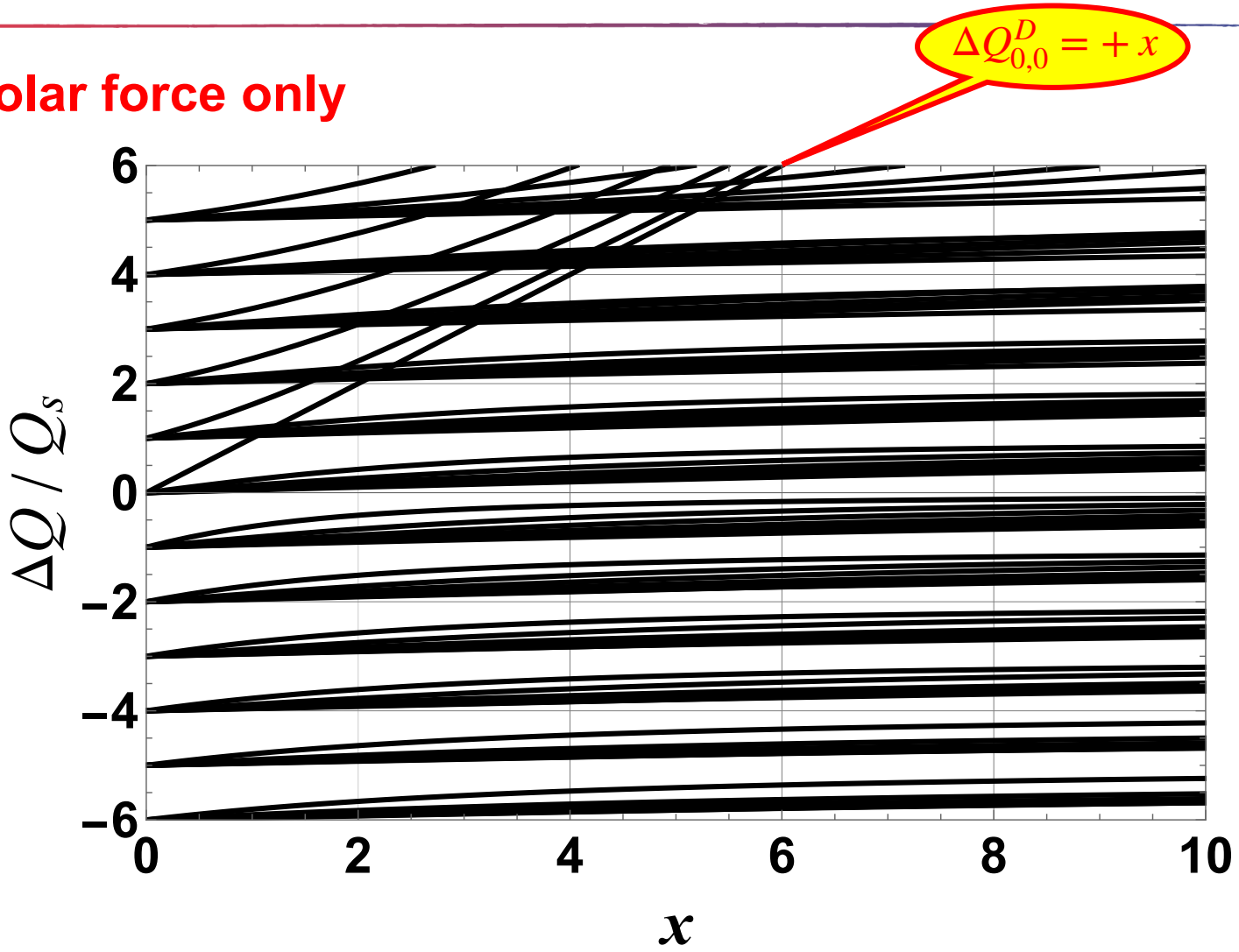
BOXCAR: radial mode = 6

- Dipolar force only



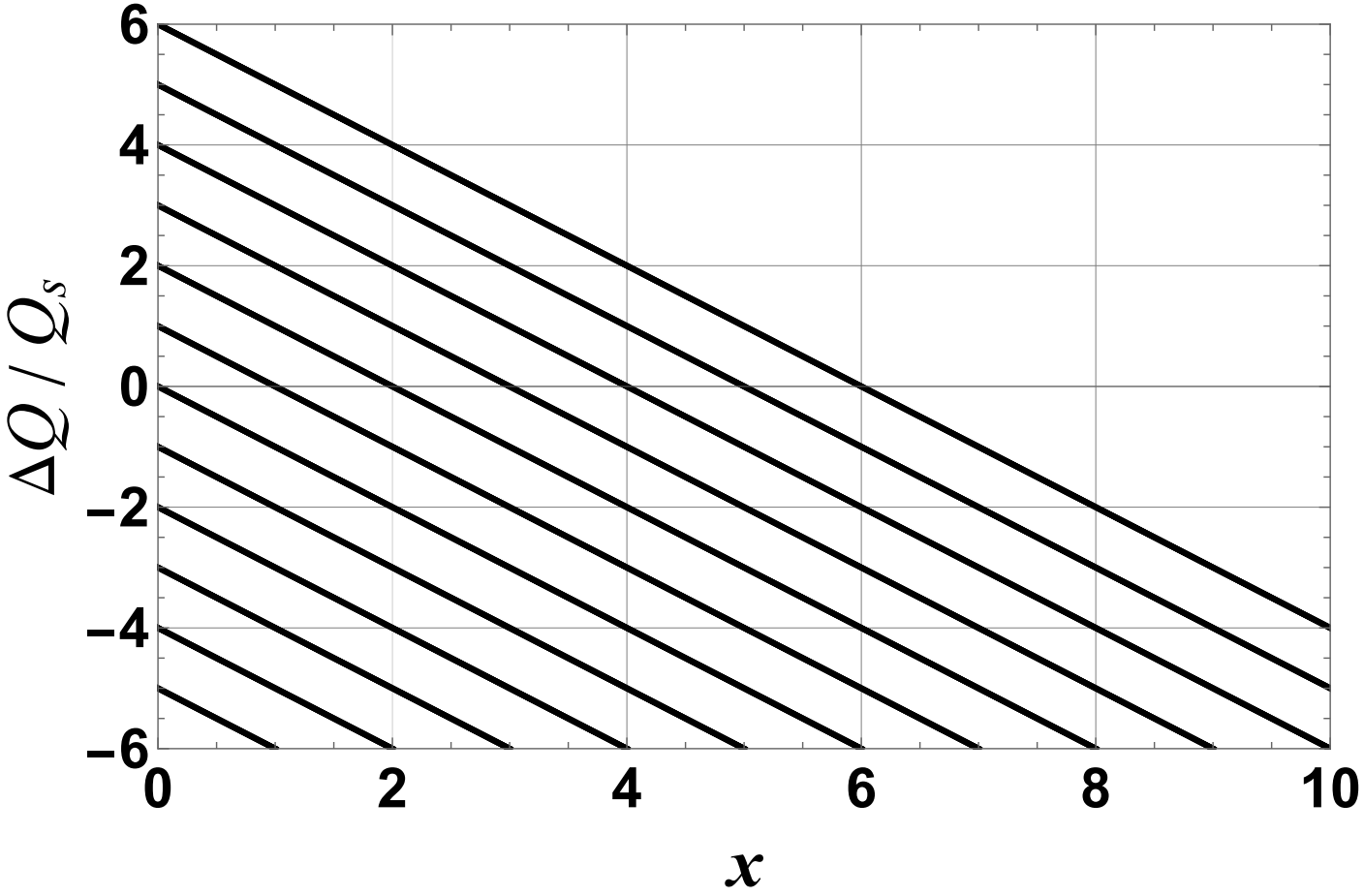
BOXCAR: radial mode = 6

- **Dipolar force only**



BOXCAR: radial mode = 6

- **Quadrupolar force only**



BOXCAR: radial mode = 6

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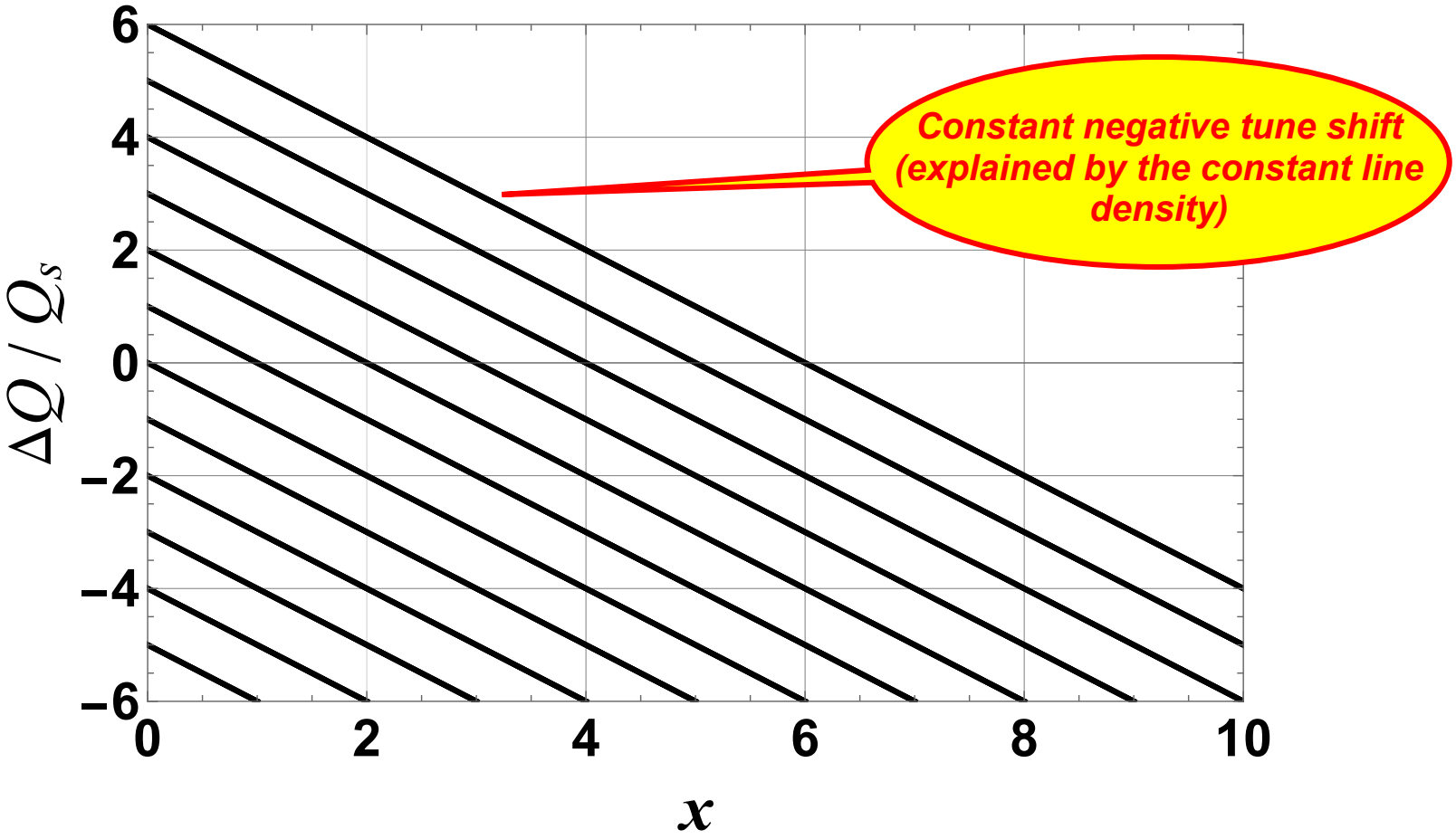

BOXCAR: radial mode = 6

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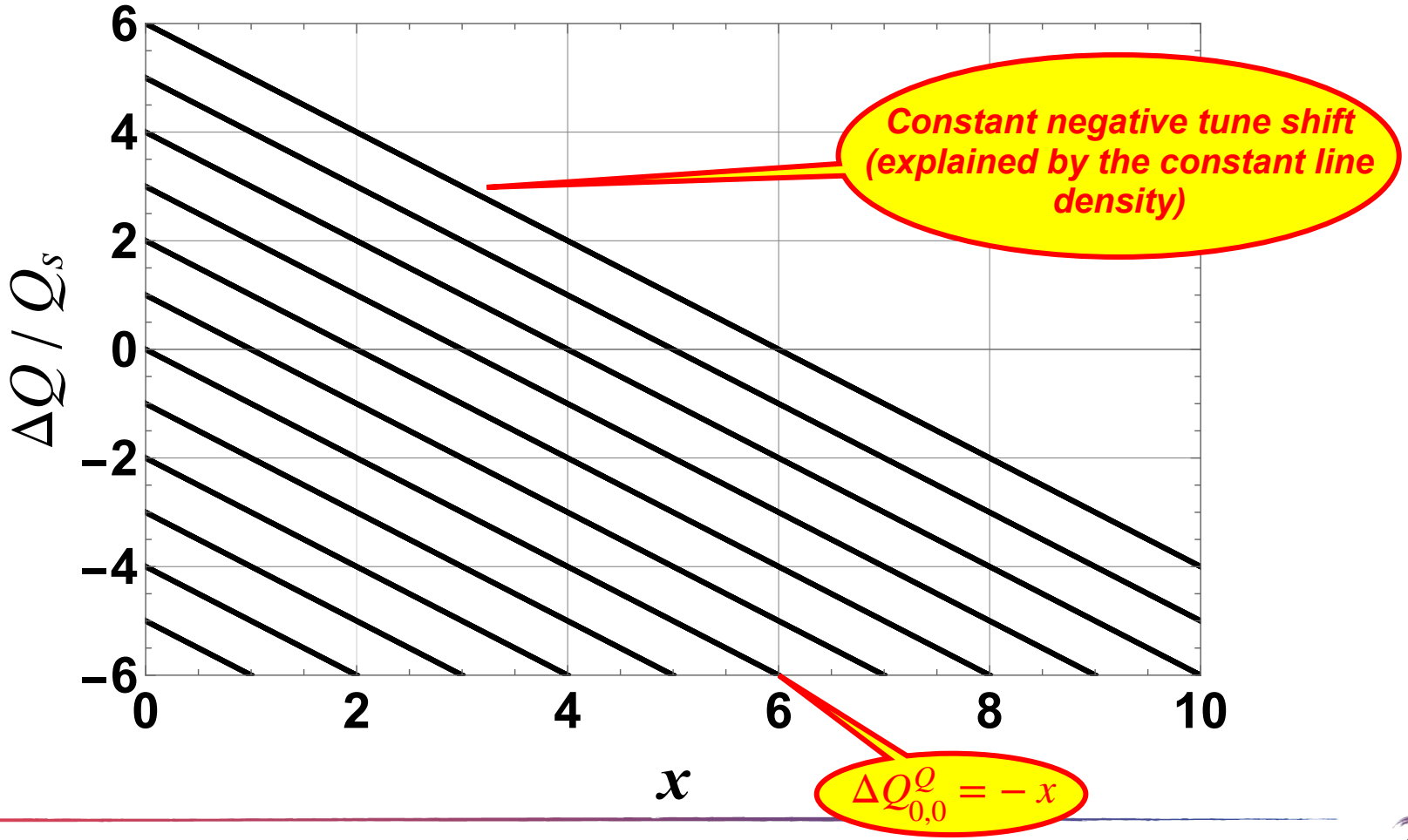
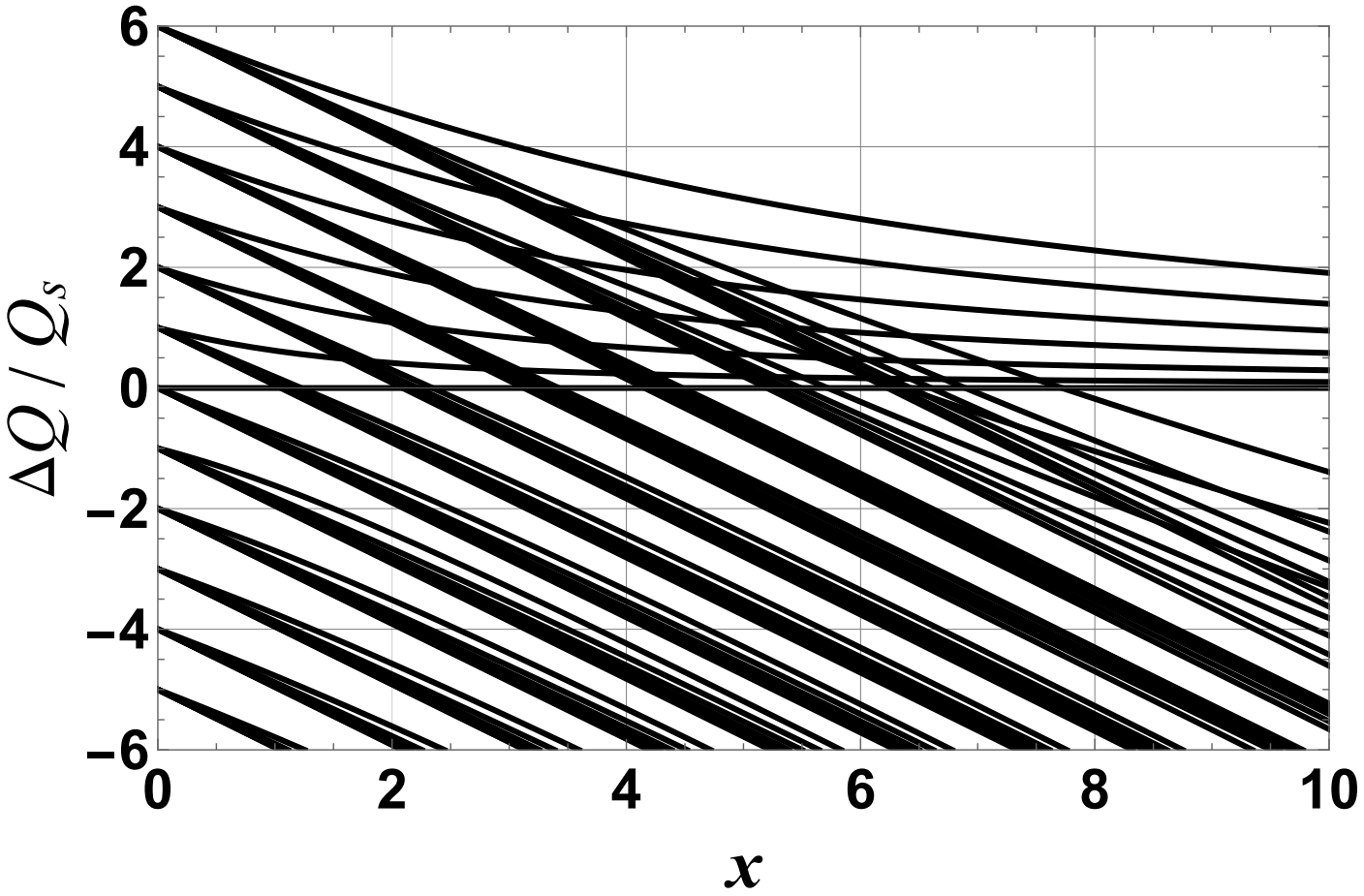


BOXCAR: radial mode = 6

- **Sum of the 2 forces**



BOXCAR: radial mode = 6

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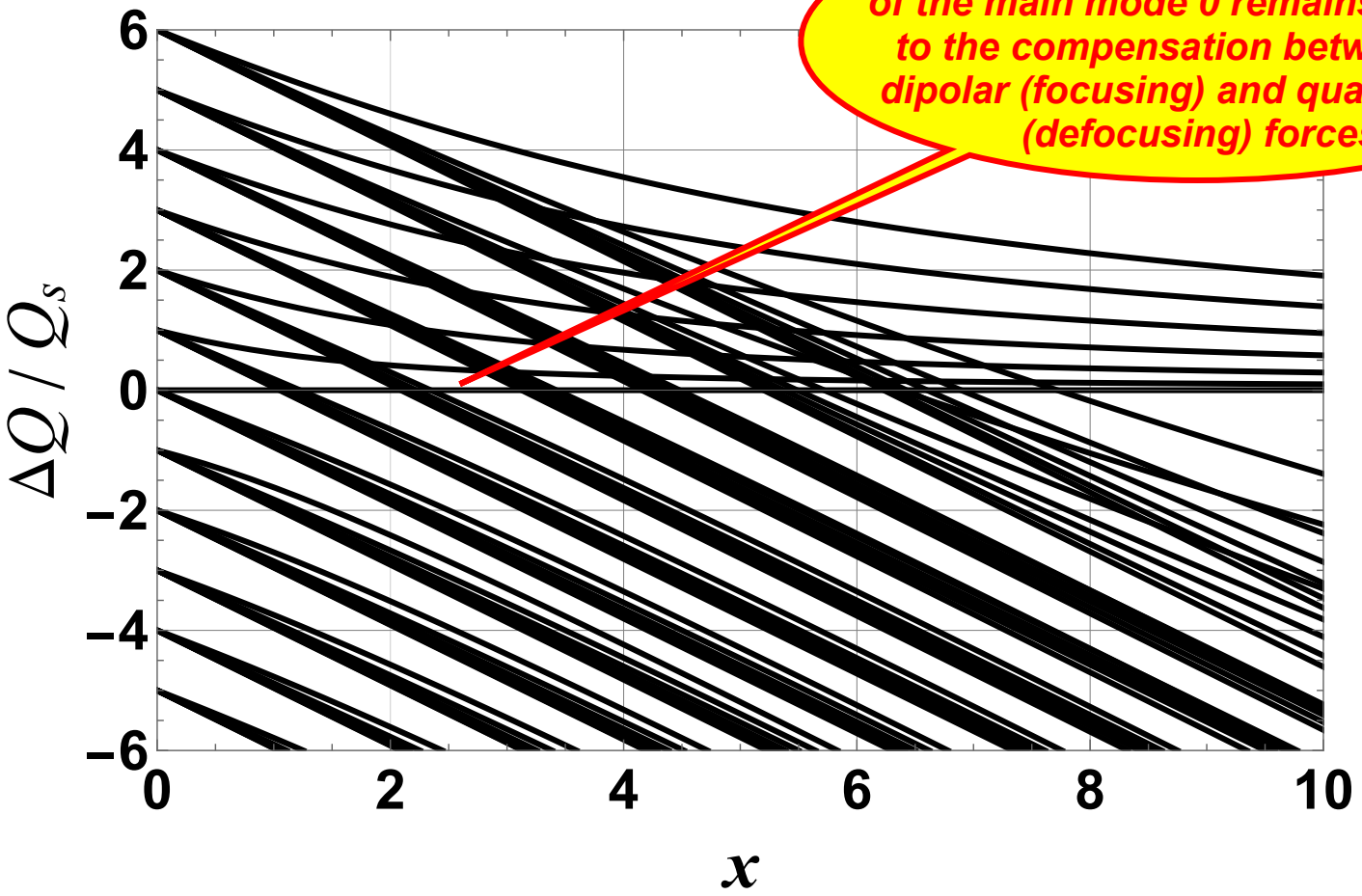

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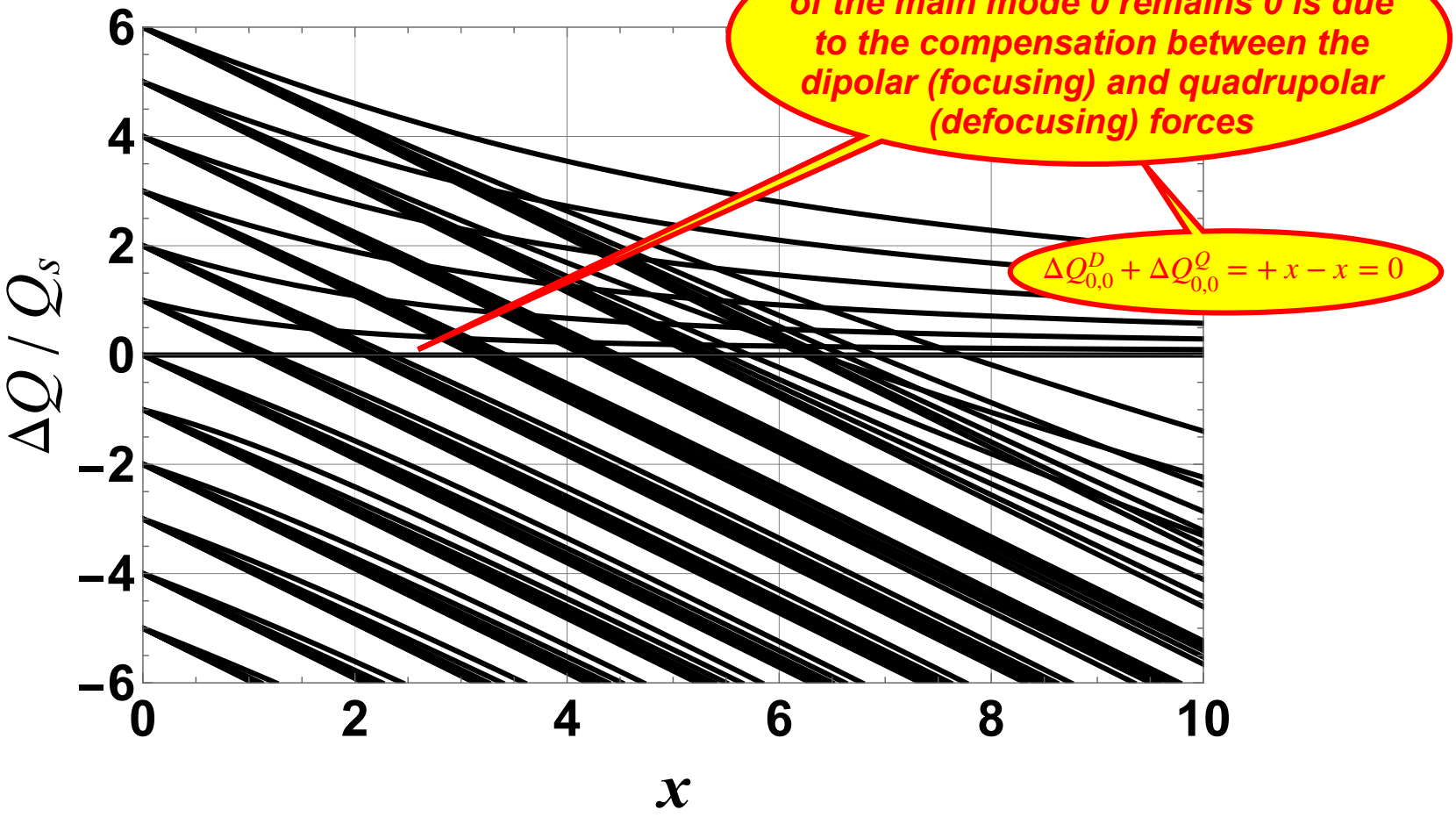

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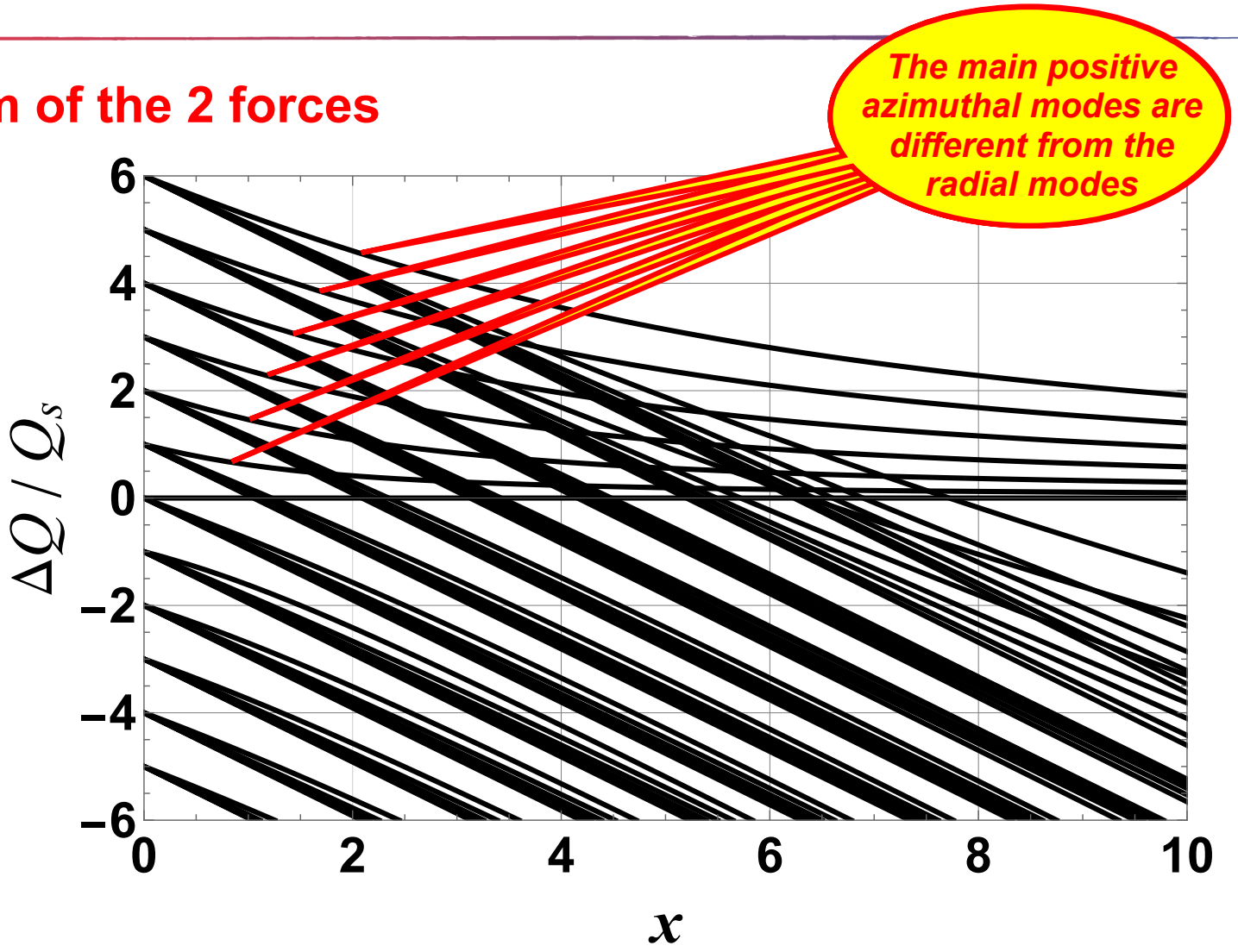

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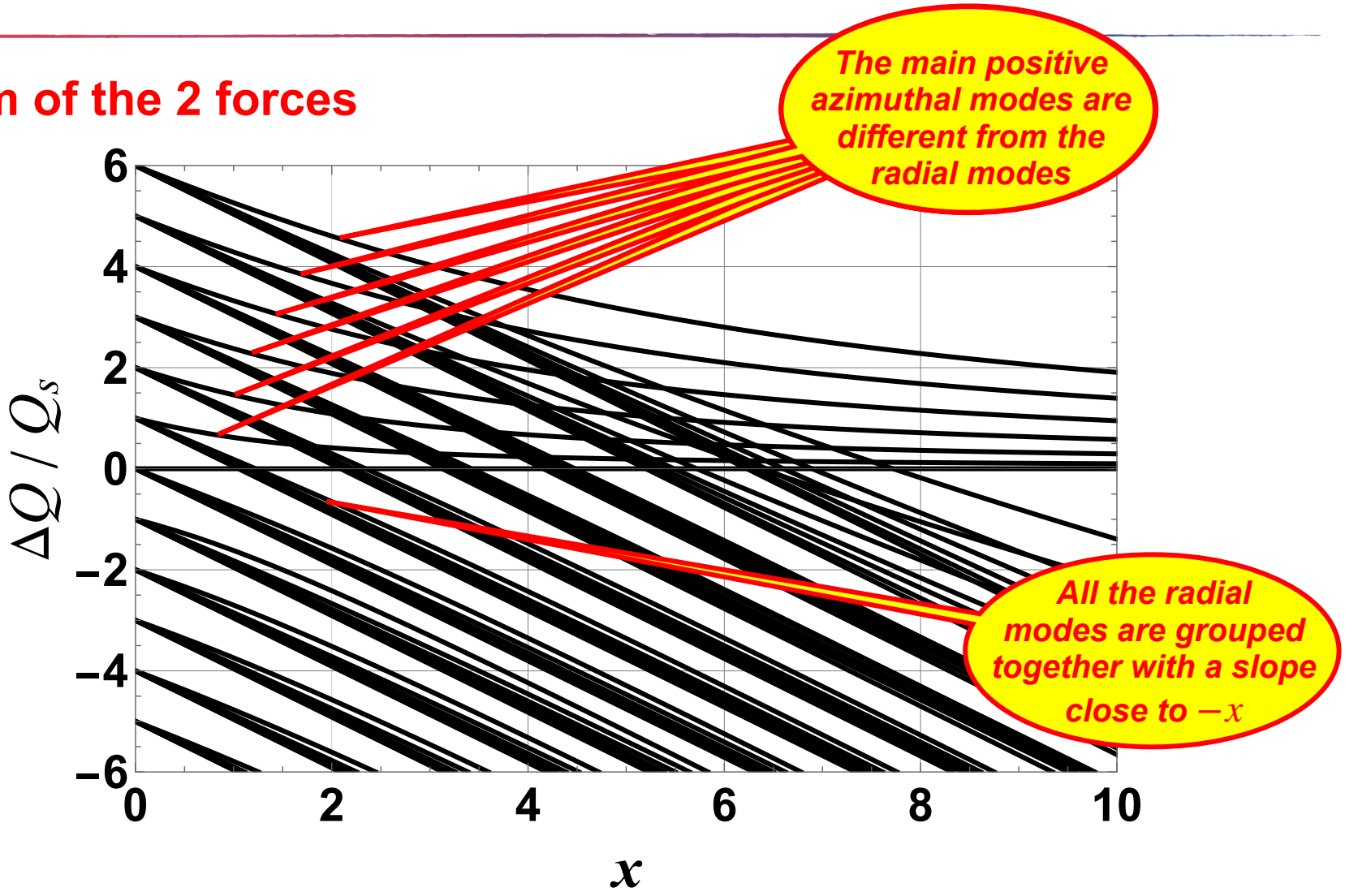
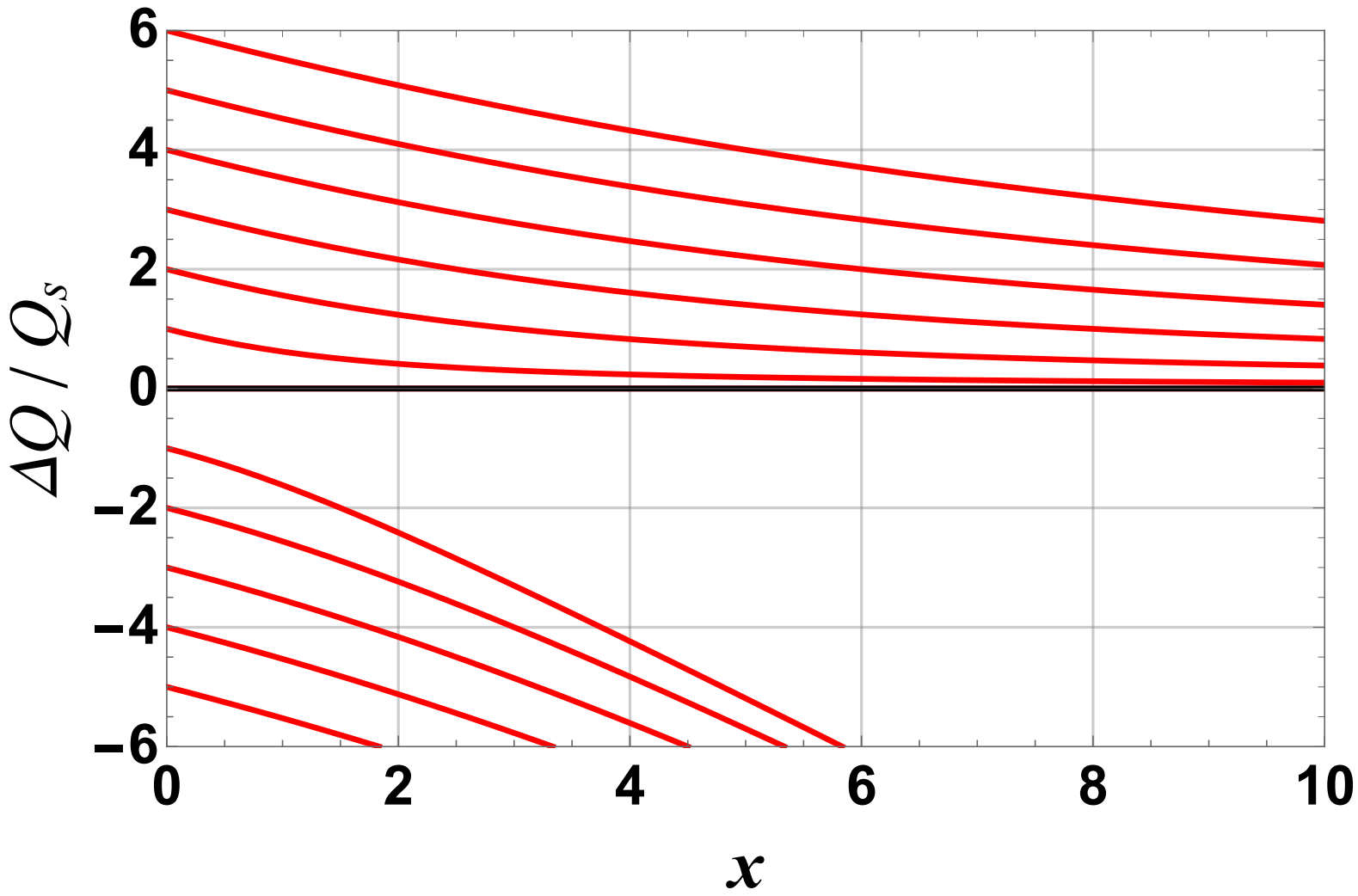


GAUSSIAN: m_{max} = 0



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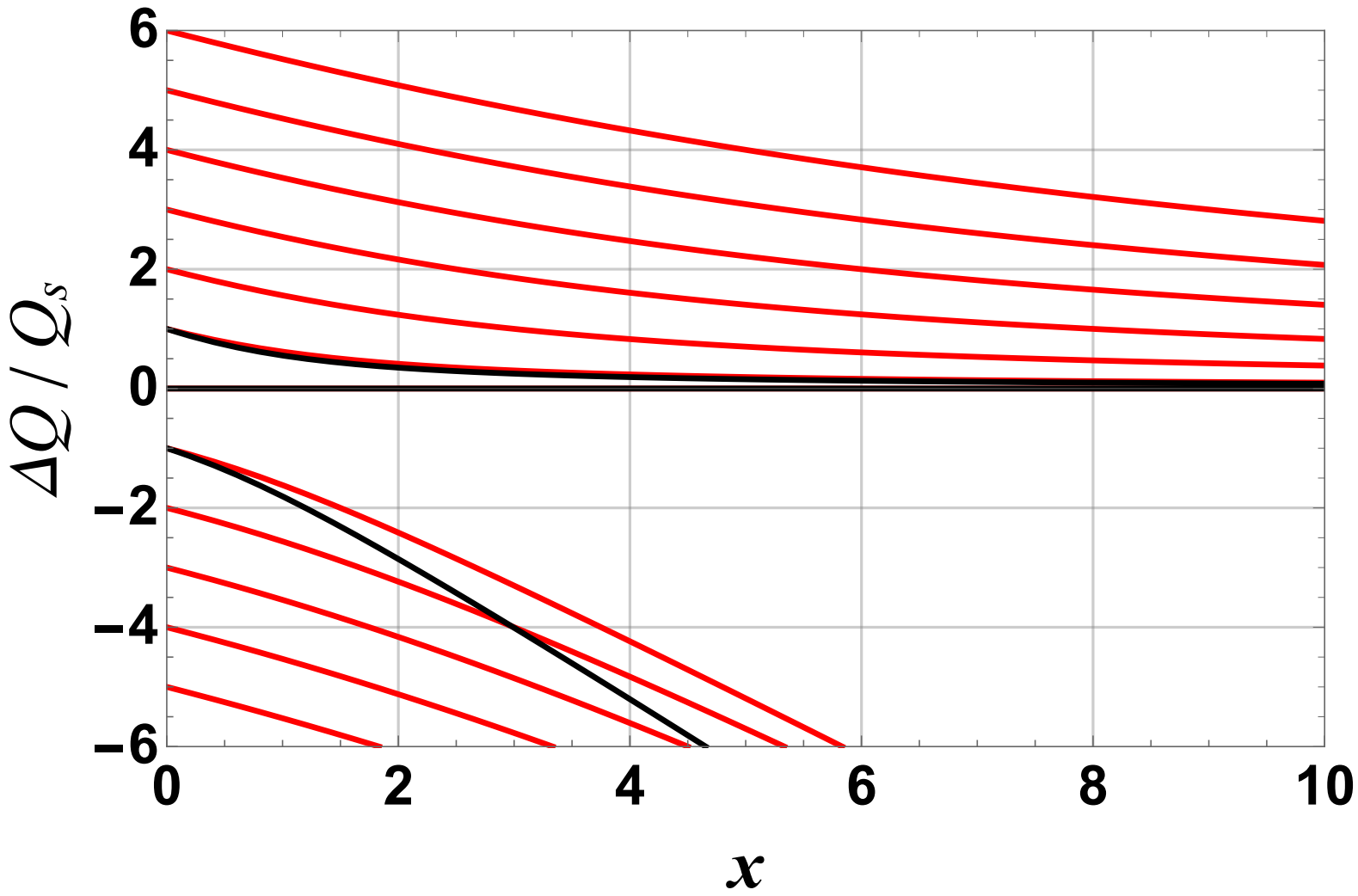
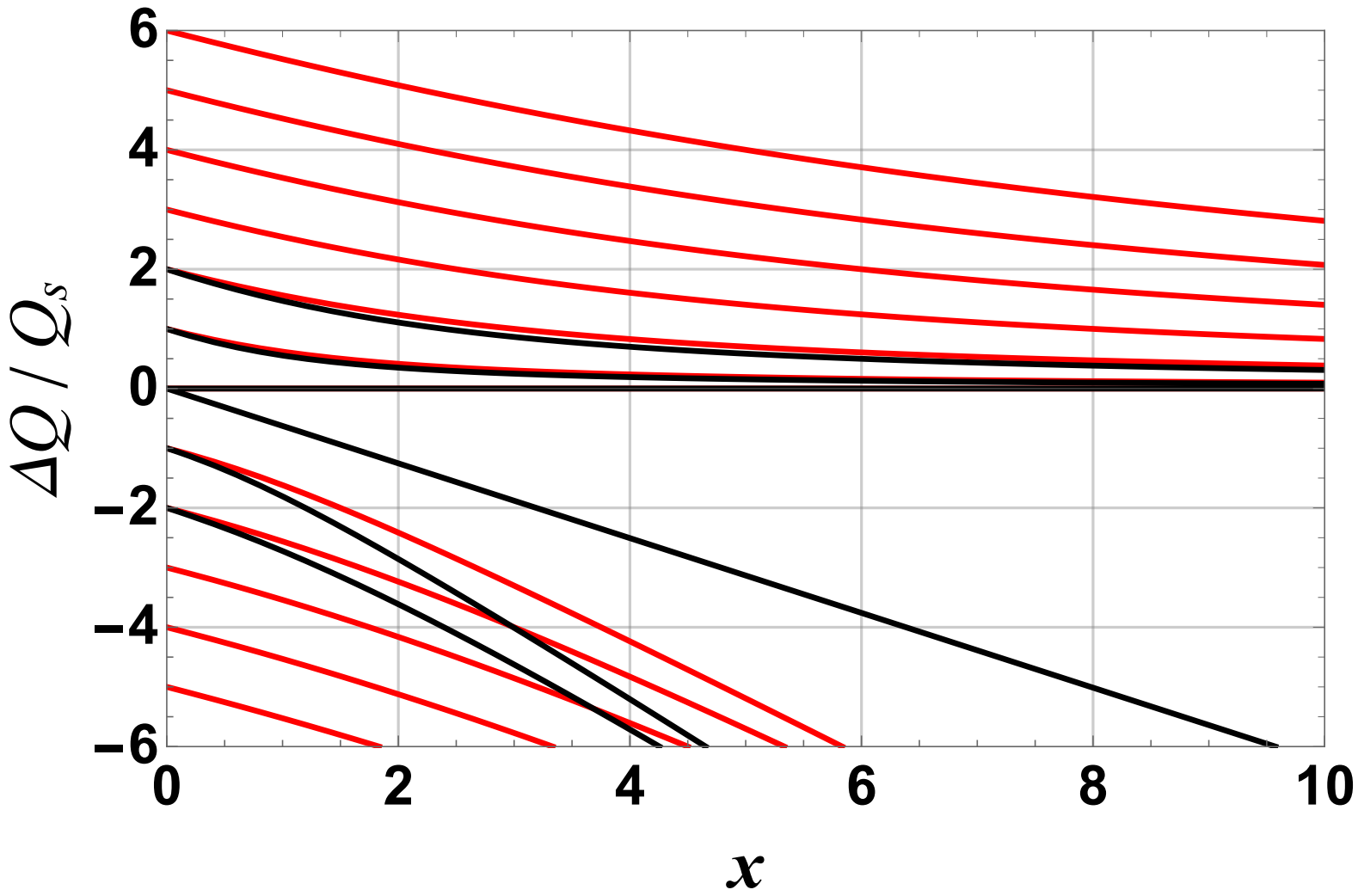
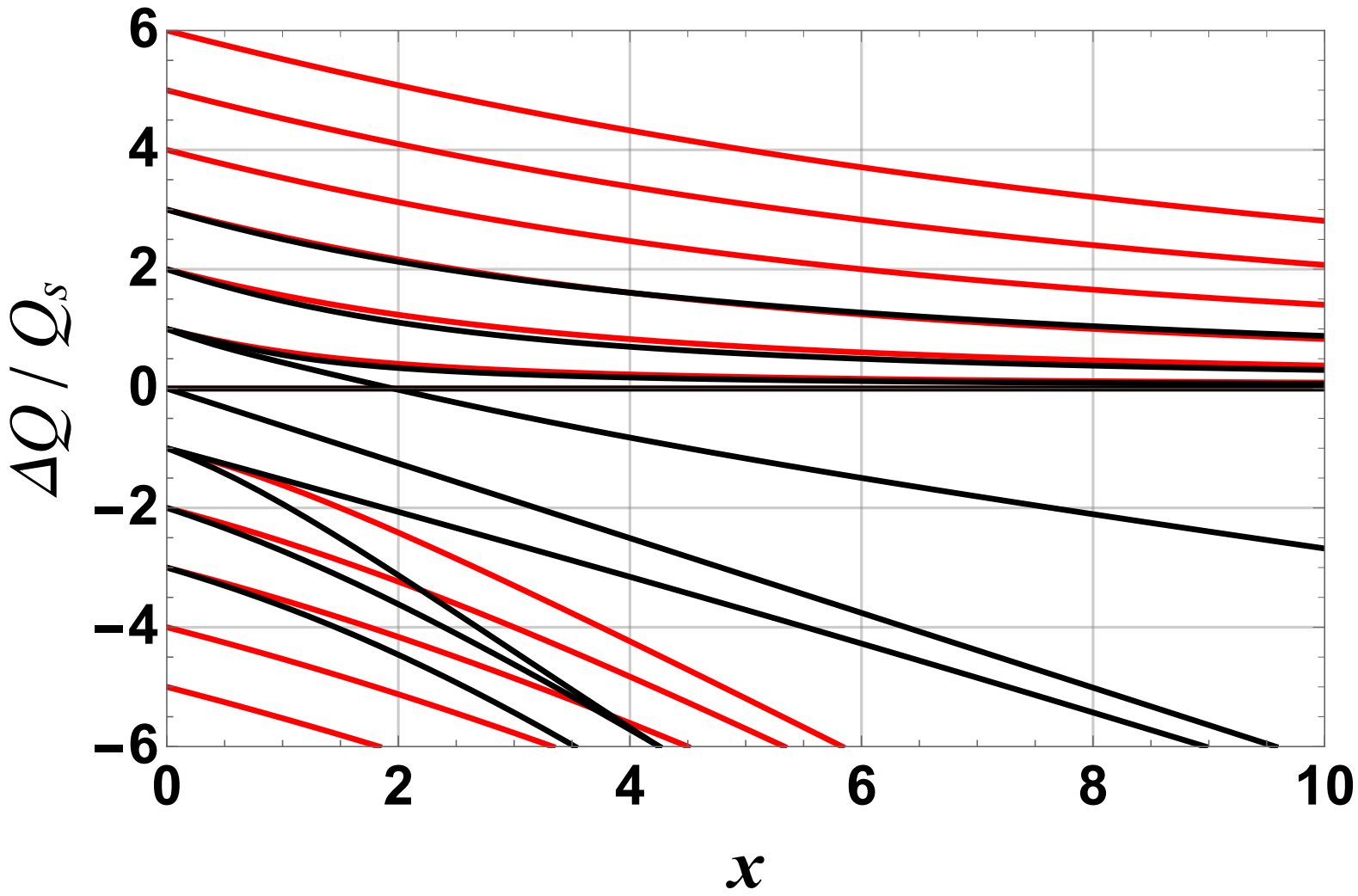


GAUSSIAN: m_{max} = 2



GAUSSIAN: m_{max} = 3



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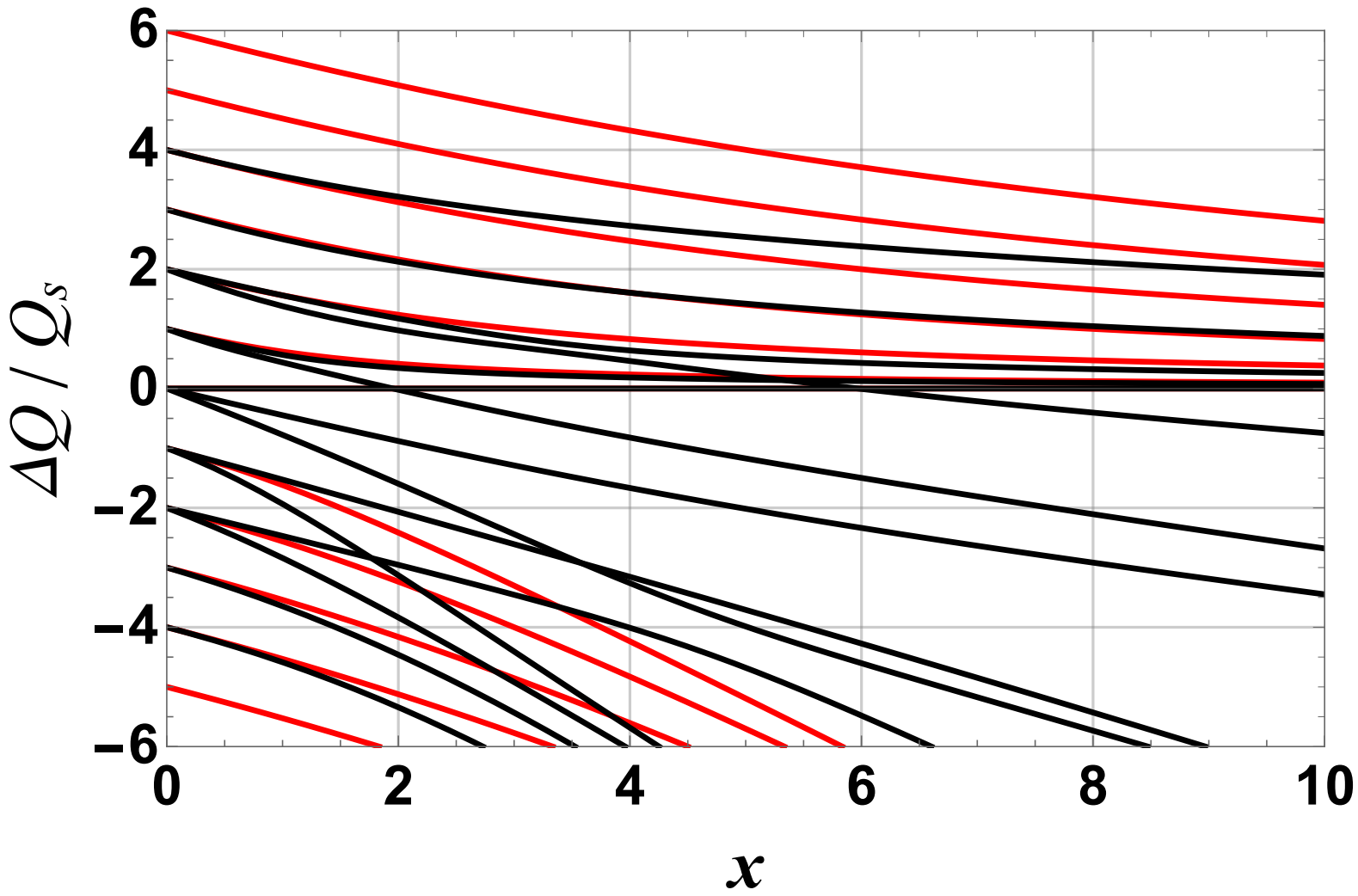
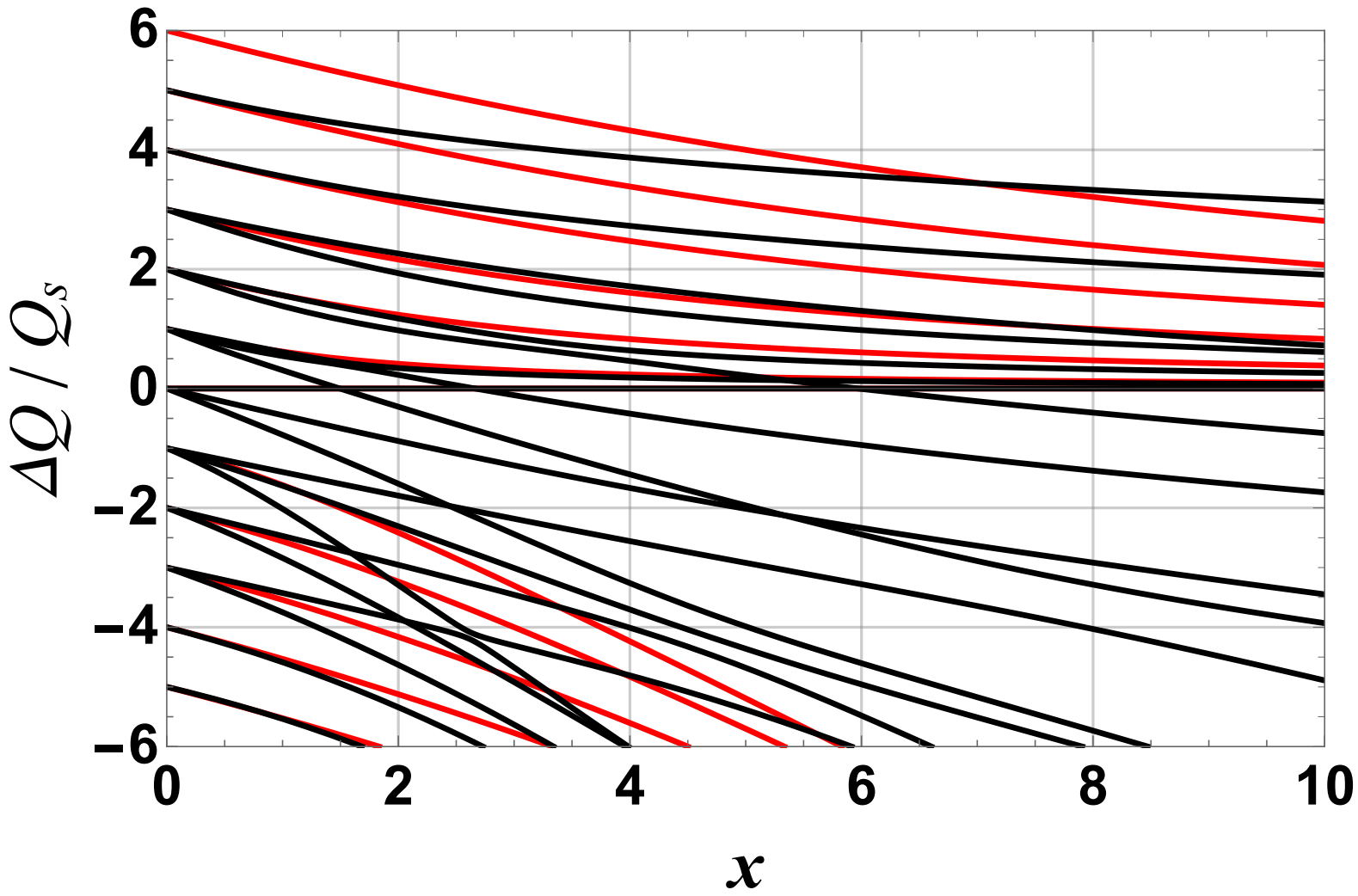
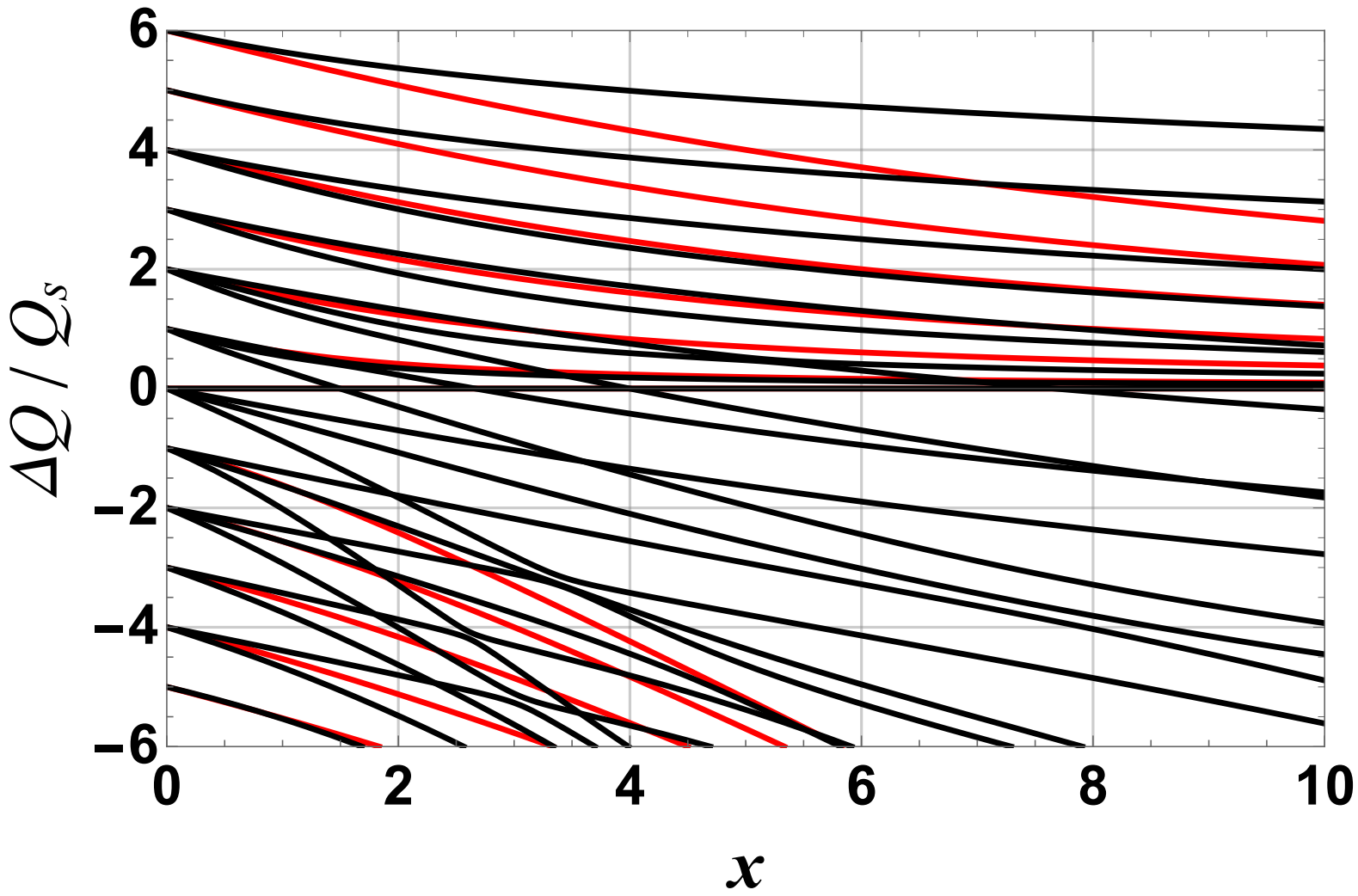


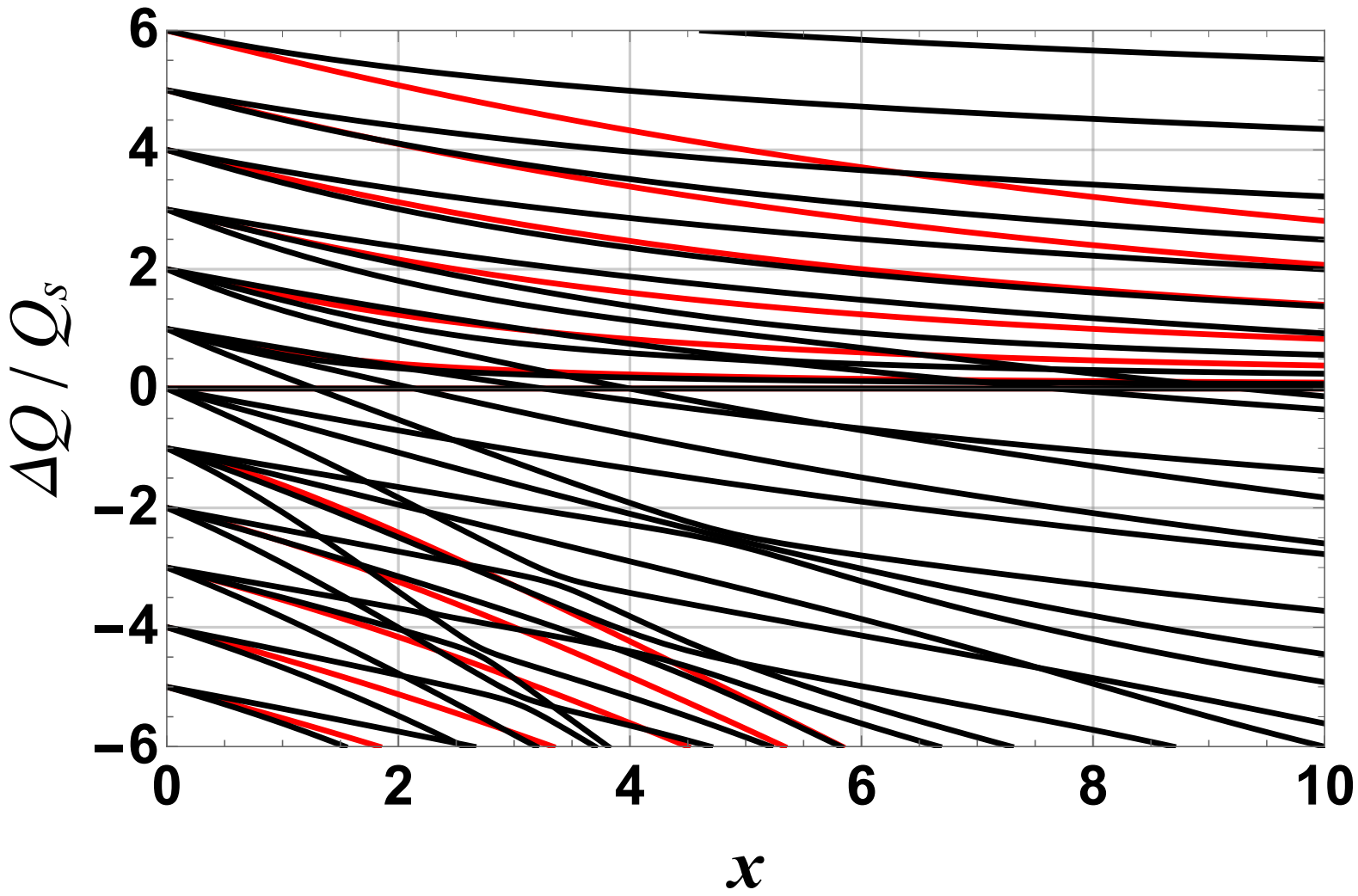
GAUSSIAN: m_{max} = 5



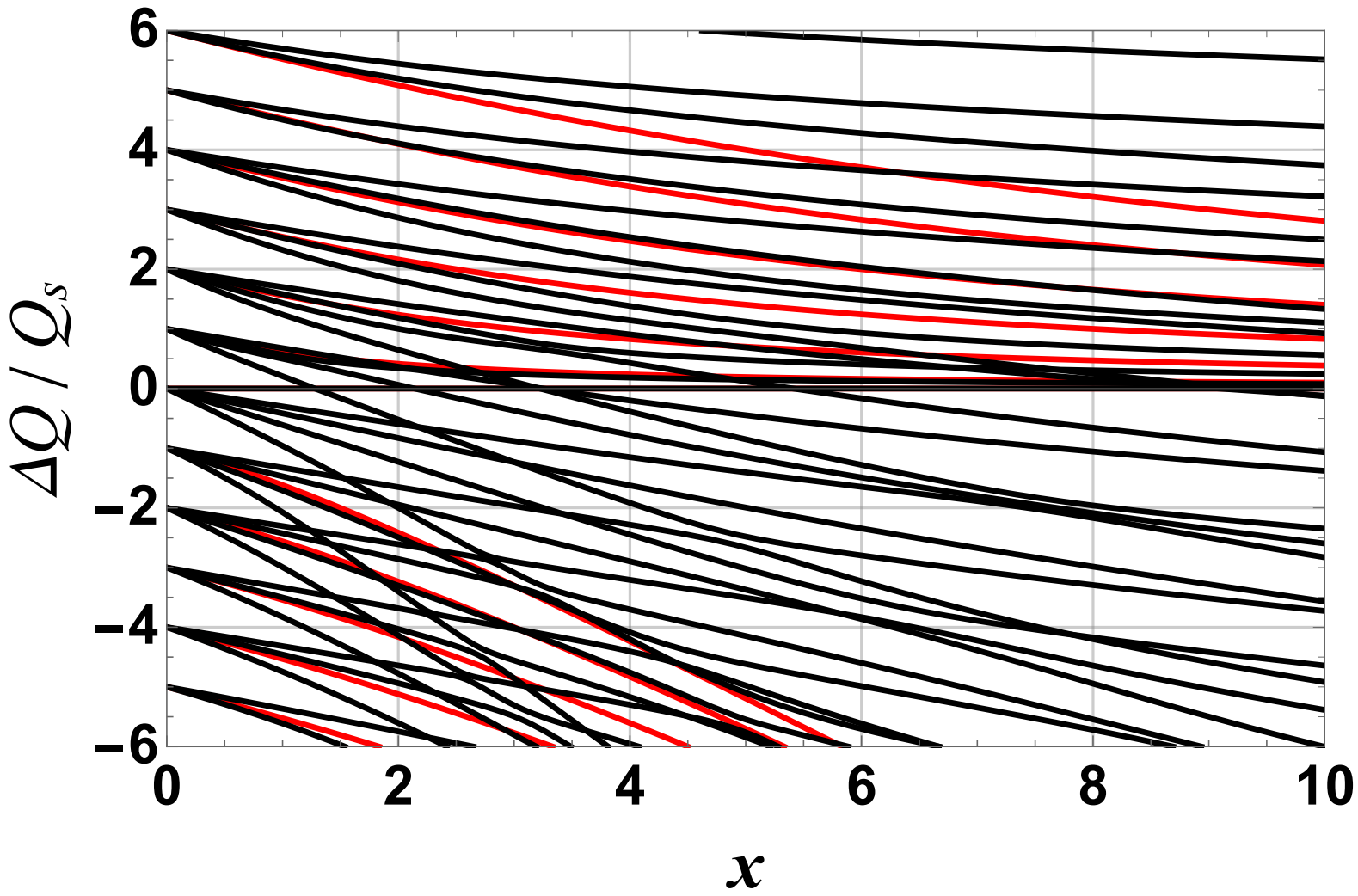
GAUSSIAN: m_{max} = 6



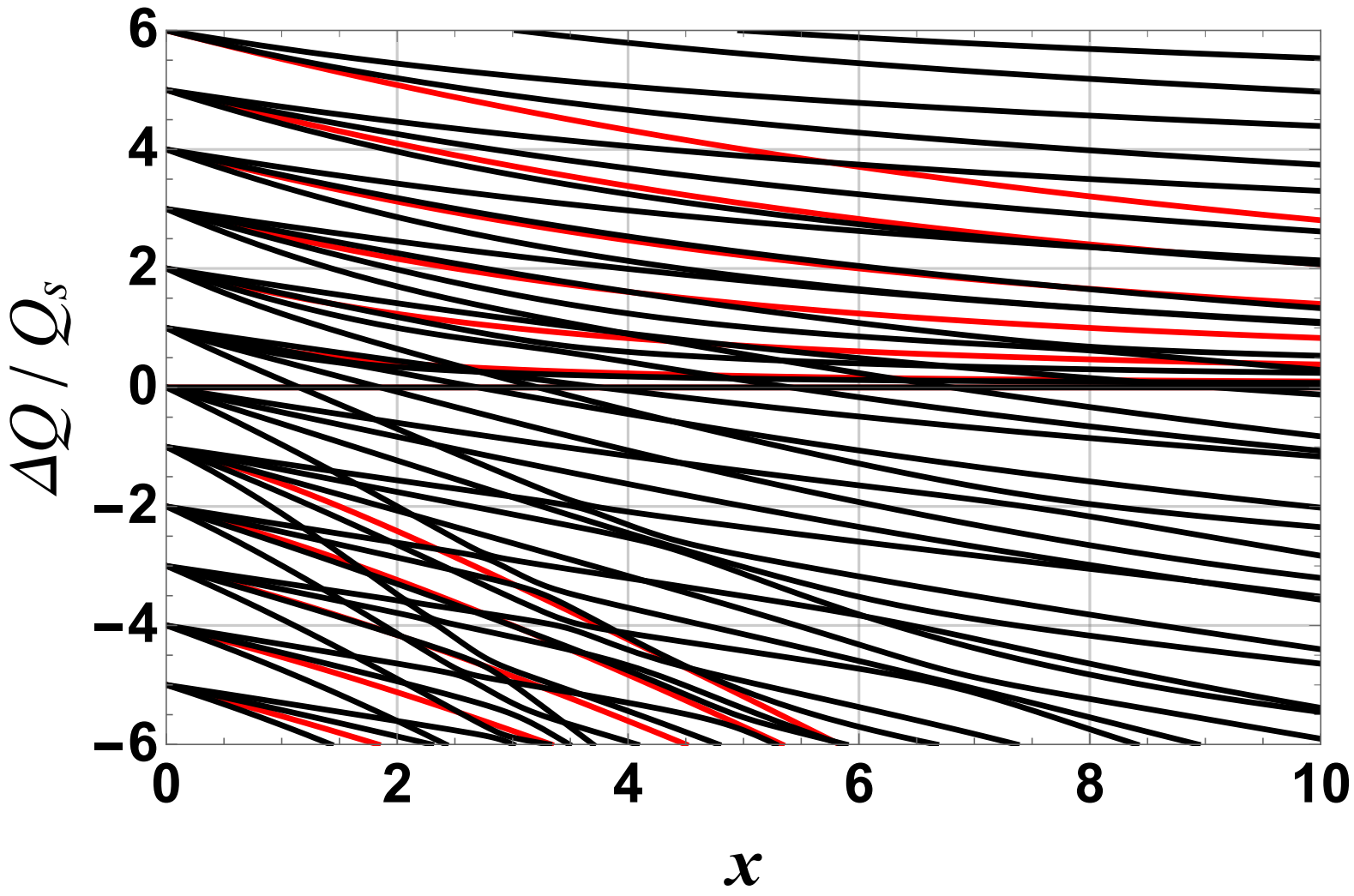
GAUSSIAN: m_{max} = 7



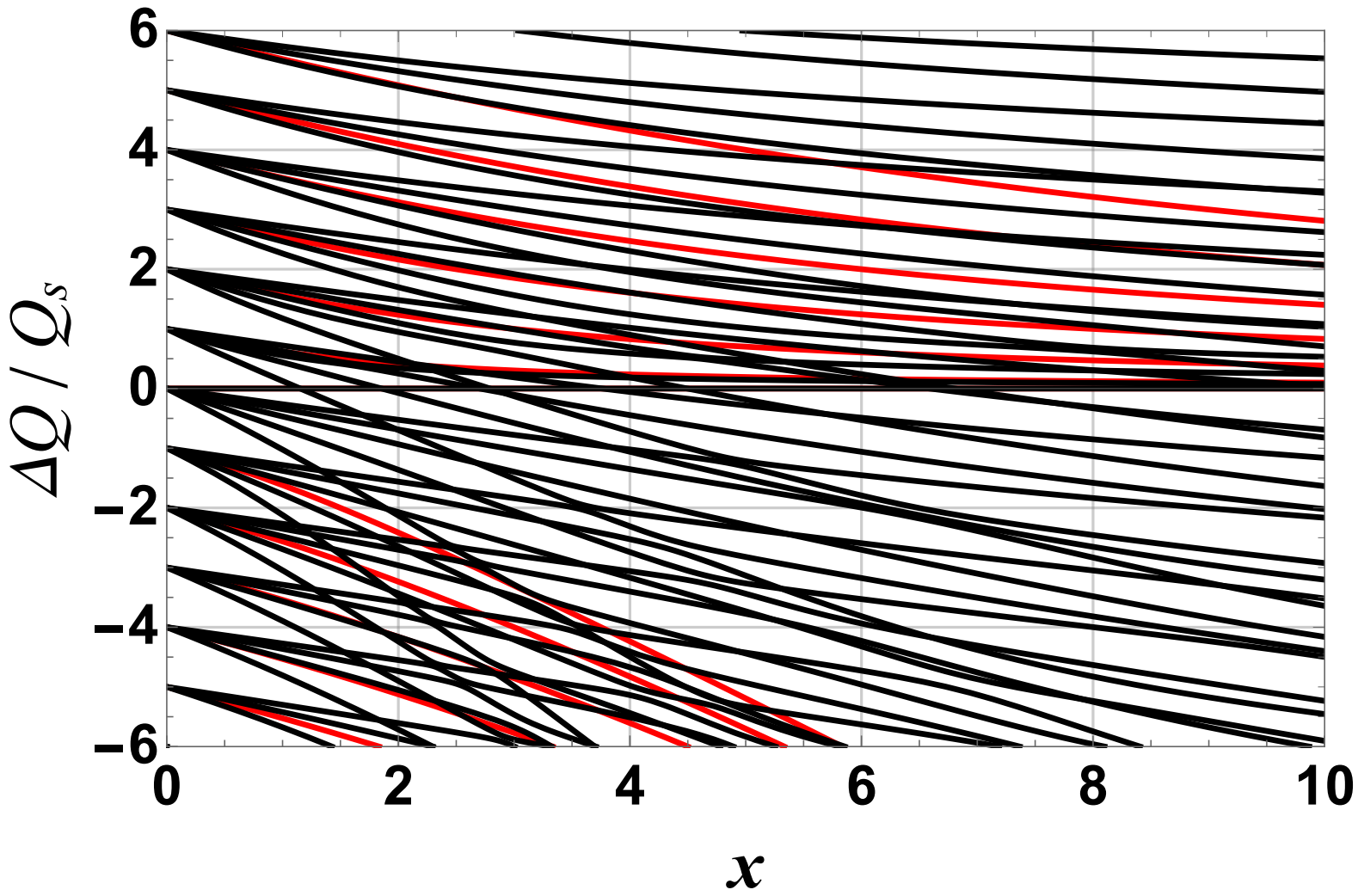
GAUSSIAN: m_{max} = 8



GAUSSIAN: m_{max} = 9

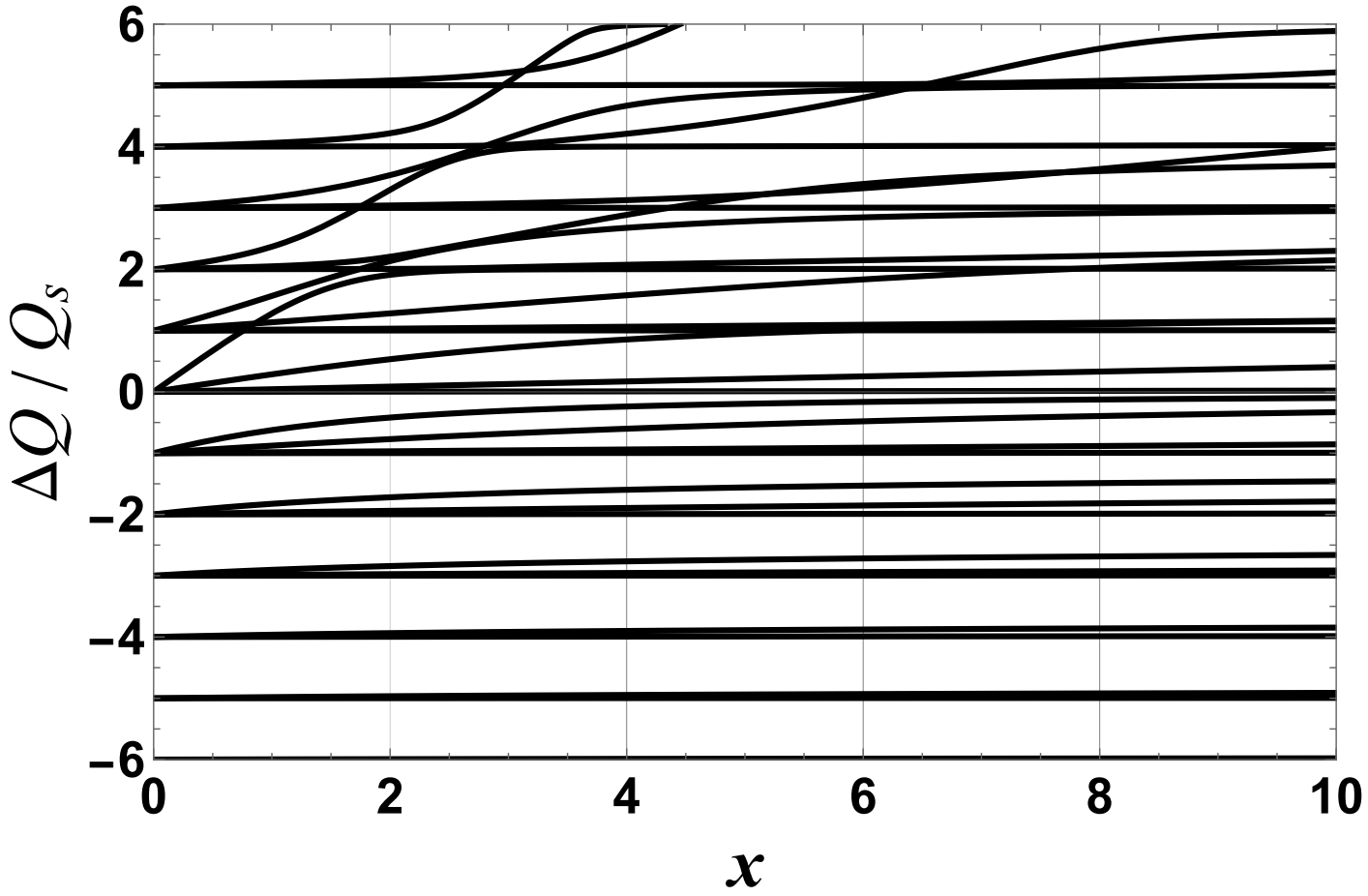


GAUSSIAN: m_{max} = 10



GAUSSIAN: m_{max} = 7

- **Dipolar force only**



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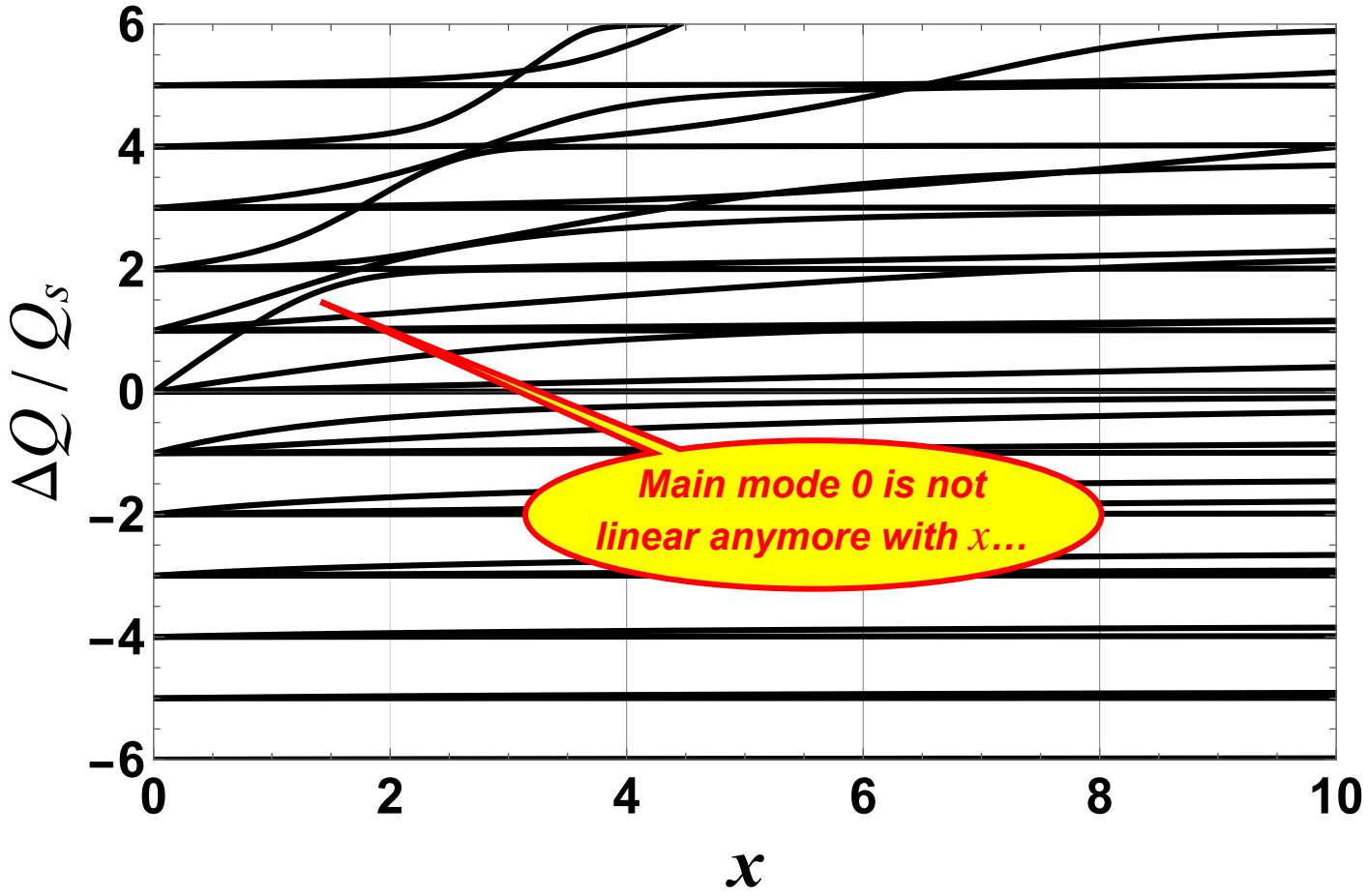
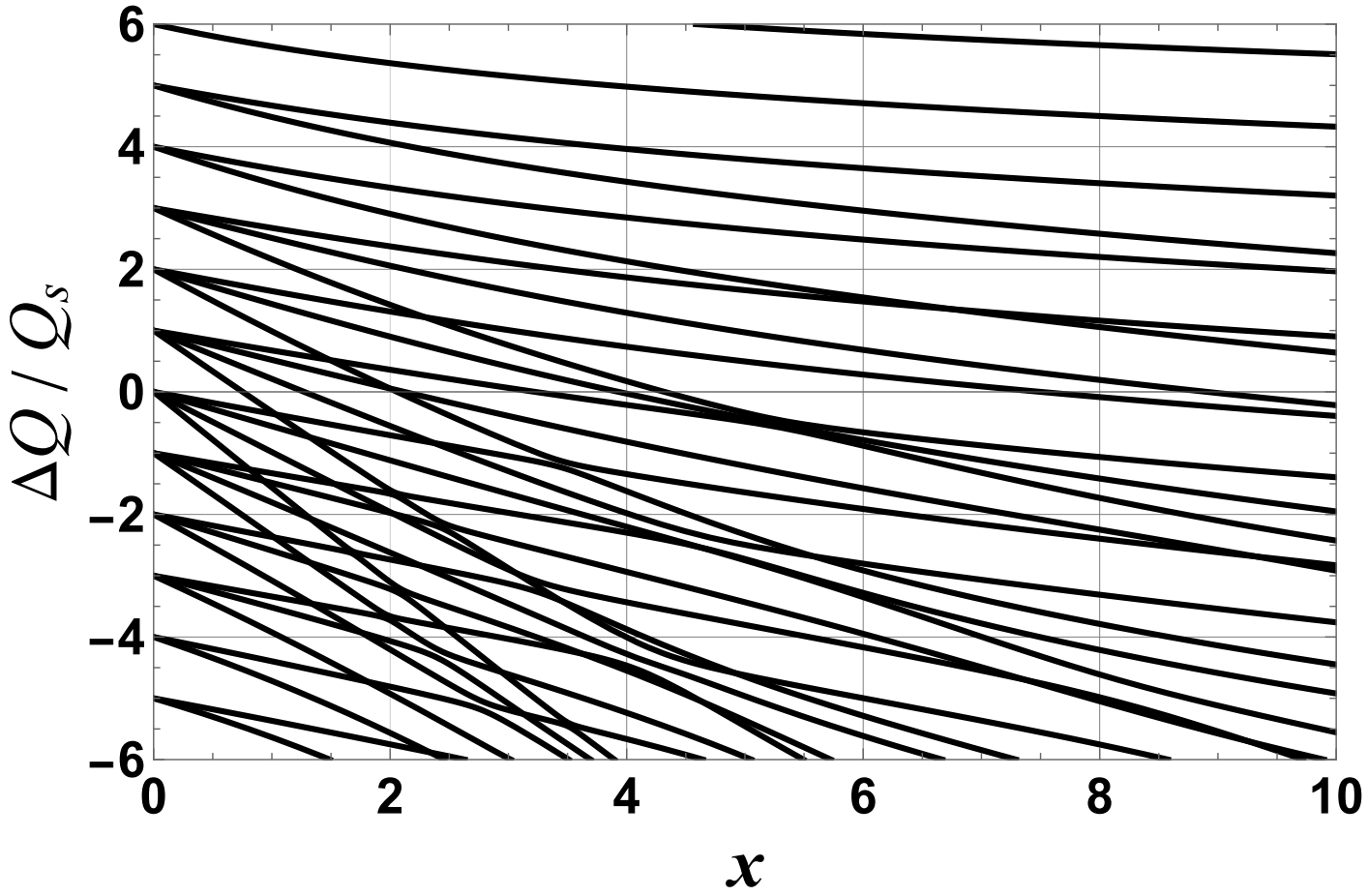


GAUSSIAN: m_{max} = 7

- **Quadrupolar force only**



GAUSSIAN: m_{max} = 7

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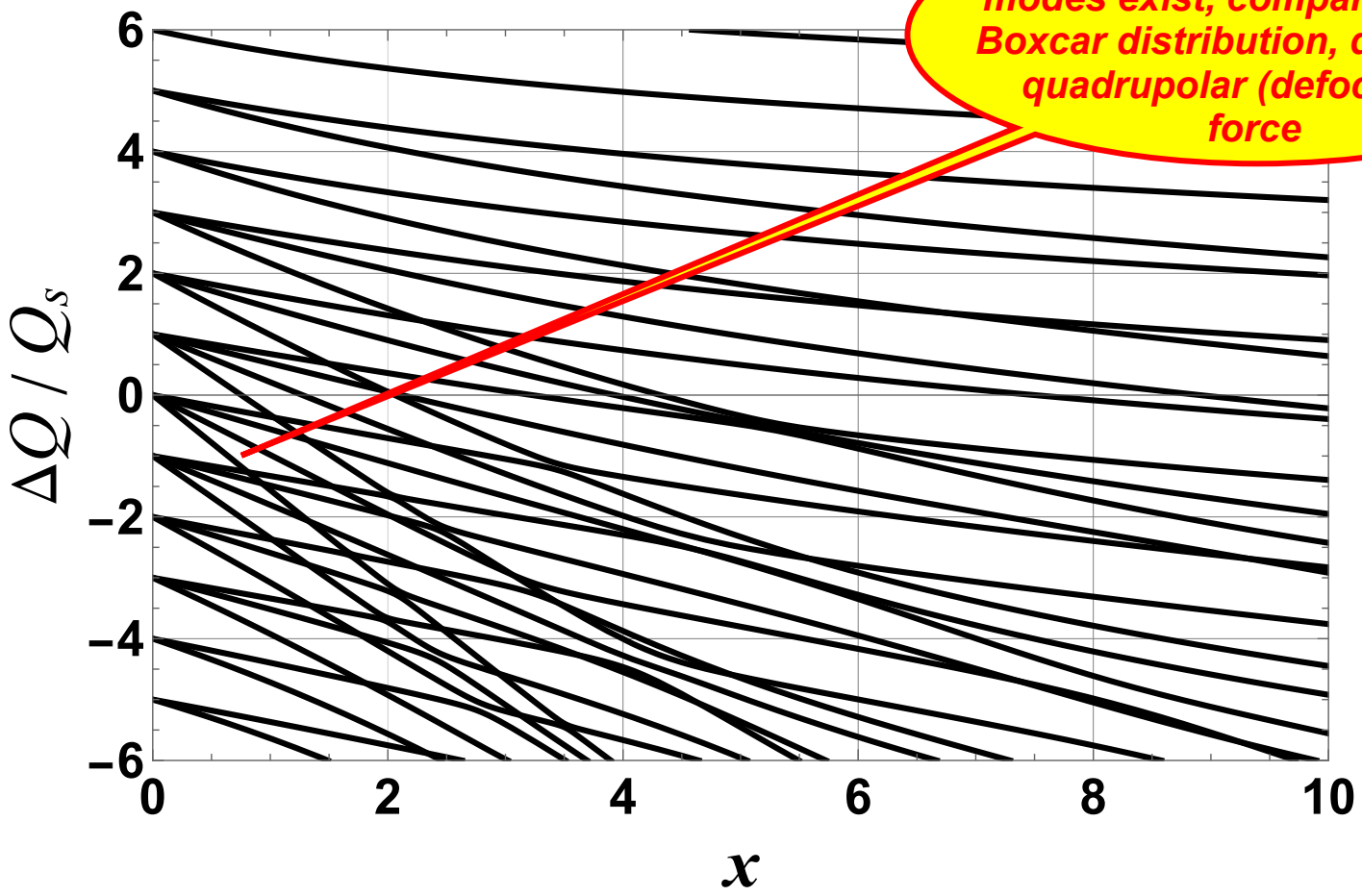
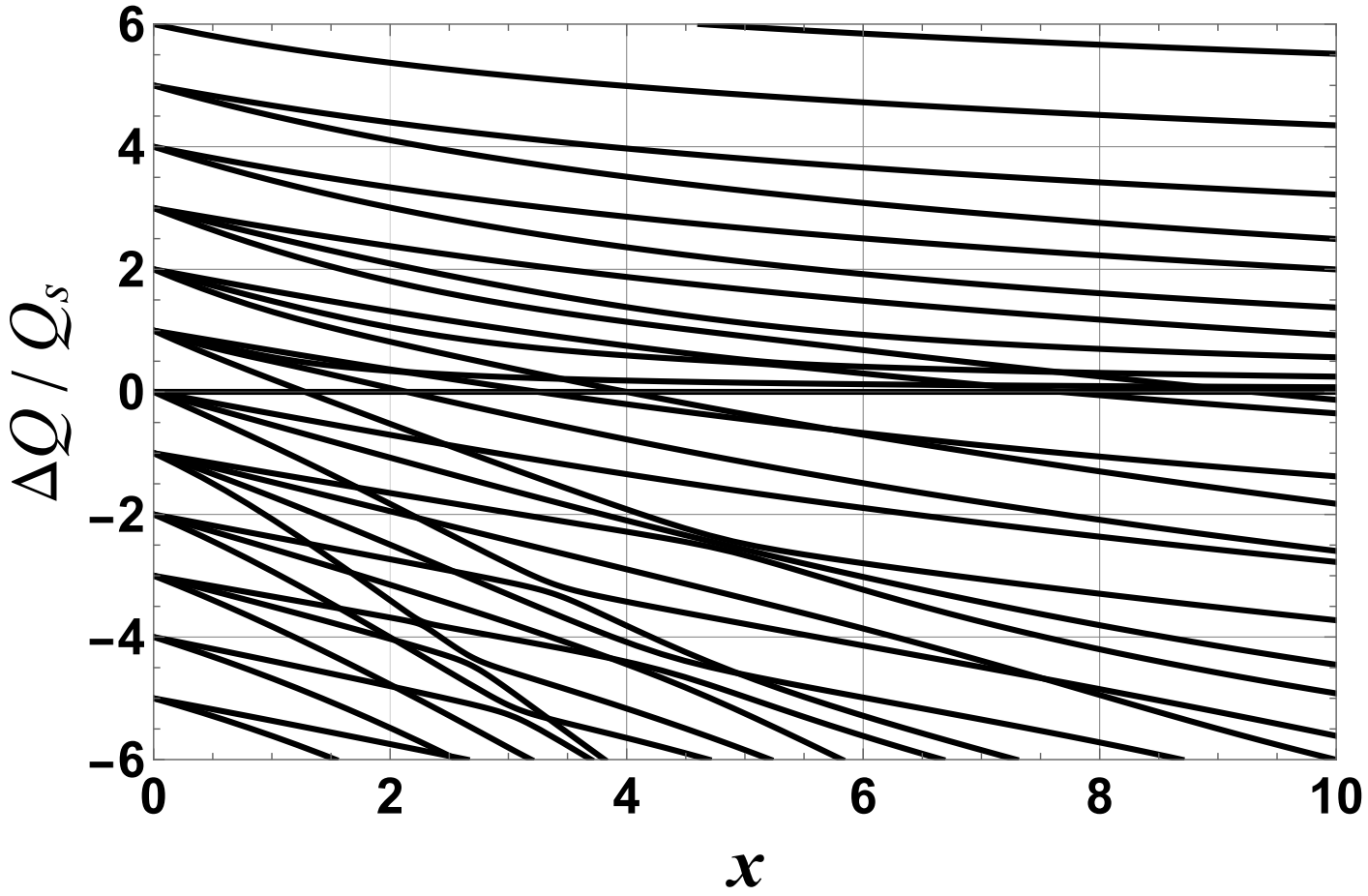


GAUSSIAN: m_{max} = 7

- **Sum of the 2 forces**



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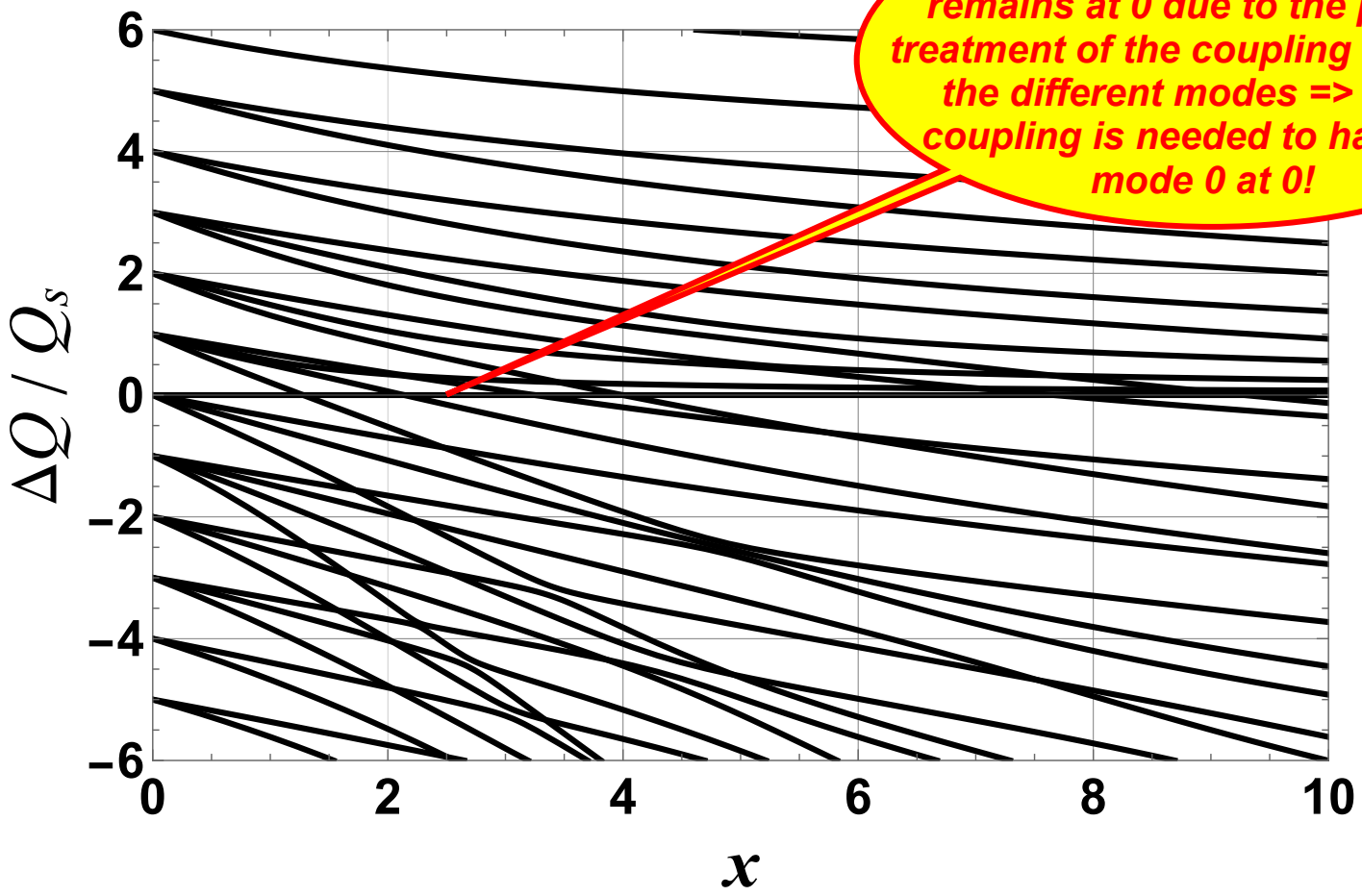

BENCHMARK 1: Boxcar

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- M. Blaskiewicz's solution for the Boxcar line density is (without wake field)

$$T_{p,q,n,m} = (qQ_s - \Delta Q_{sc})\delta_q^m \delta_p^n + \Delta Q_{sc} B_n^m B_p^q \frac{2\delta_{|m|+2n}^{|q|+2p}}{2(|m| + 2n) + 1}$$
$$- \Delta Q_W \frac{B_n^m B_p^q}{2(|m| + 2n) + 1} \left\{ \delta_{|q|+2p}^0 \delta_{|m|+2n}^0 + \frac{+\delta_{|q|+2p}^{|m|+2n+1} - \delta_{|q|+2p}^{|m|+2n-1}}{2(|q| + 2p) + 1} \right\}, \quad (12)$$

$$B_n^m = \left\{ \frac{\Gamma(n + 1/2)\,(2|m| + 1 + 4n)\Gamma(|m| + n + 1/2)}{\pi\Gamma(n + 1)\Gamma(|m| + n + 1)} \right\}^{1/2}$$

BENCHMARK 1: Boxcar

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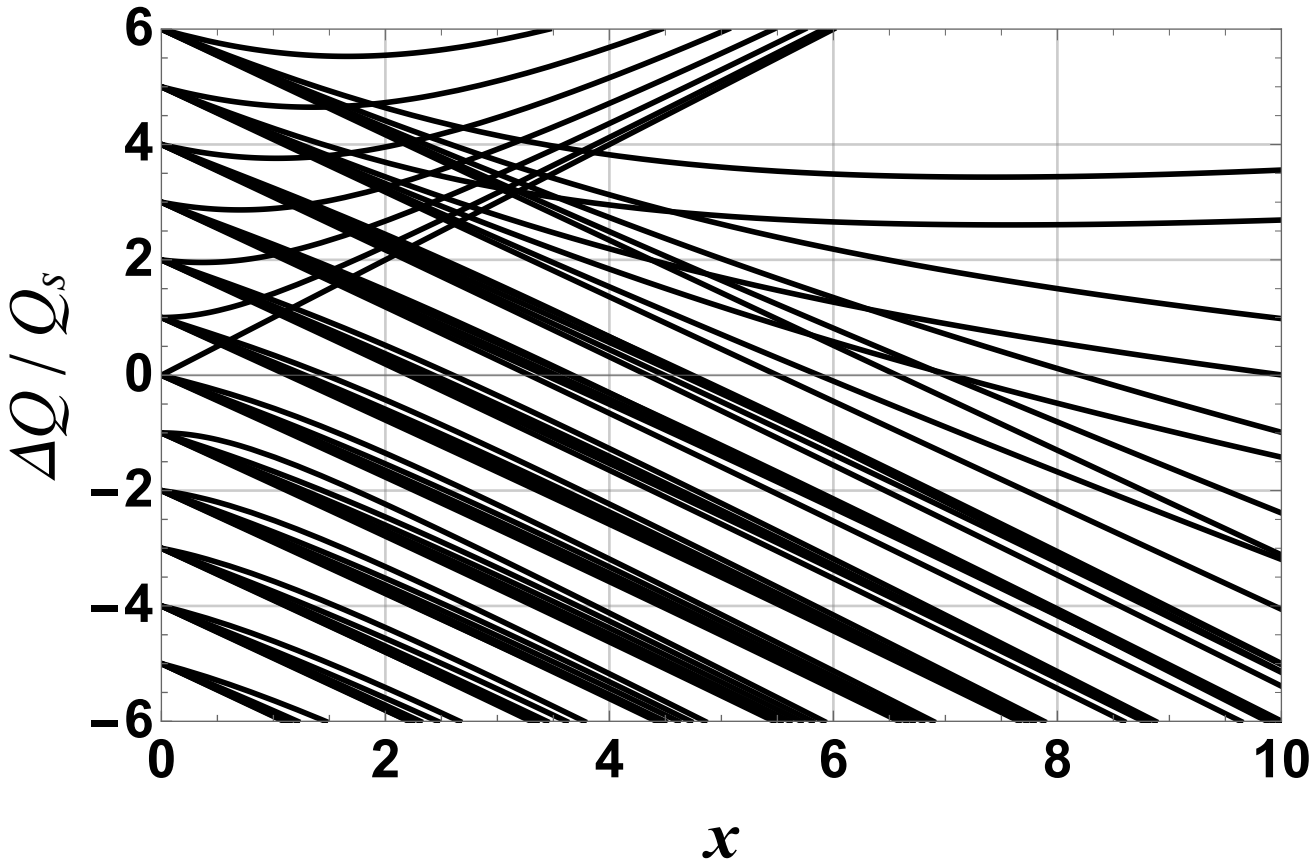

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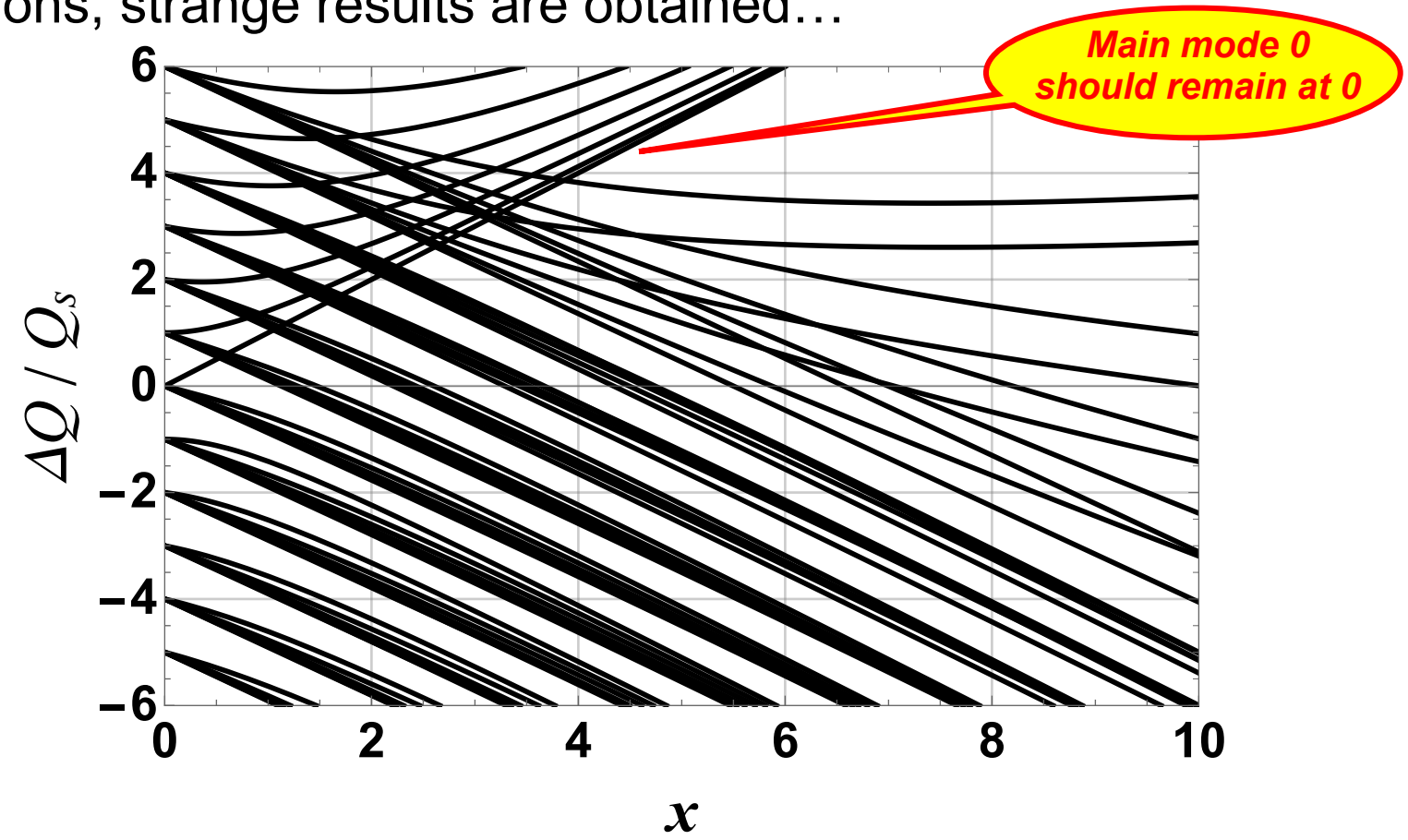

BENCHMARK 1: Boxcar

- Typo:

A factor 1/2 is missing here

$$T_{p,q,n,m} = (qQ_s - \Delta Q_{sc})\delta_q^m \delta_p^n + \Delta Q_{sc} B_n^m B_p^q \frac{2\delta_{|m|+2n}^{|q|+2p}}{2(|m| + 2n) + 1}$$

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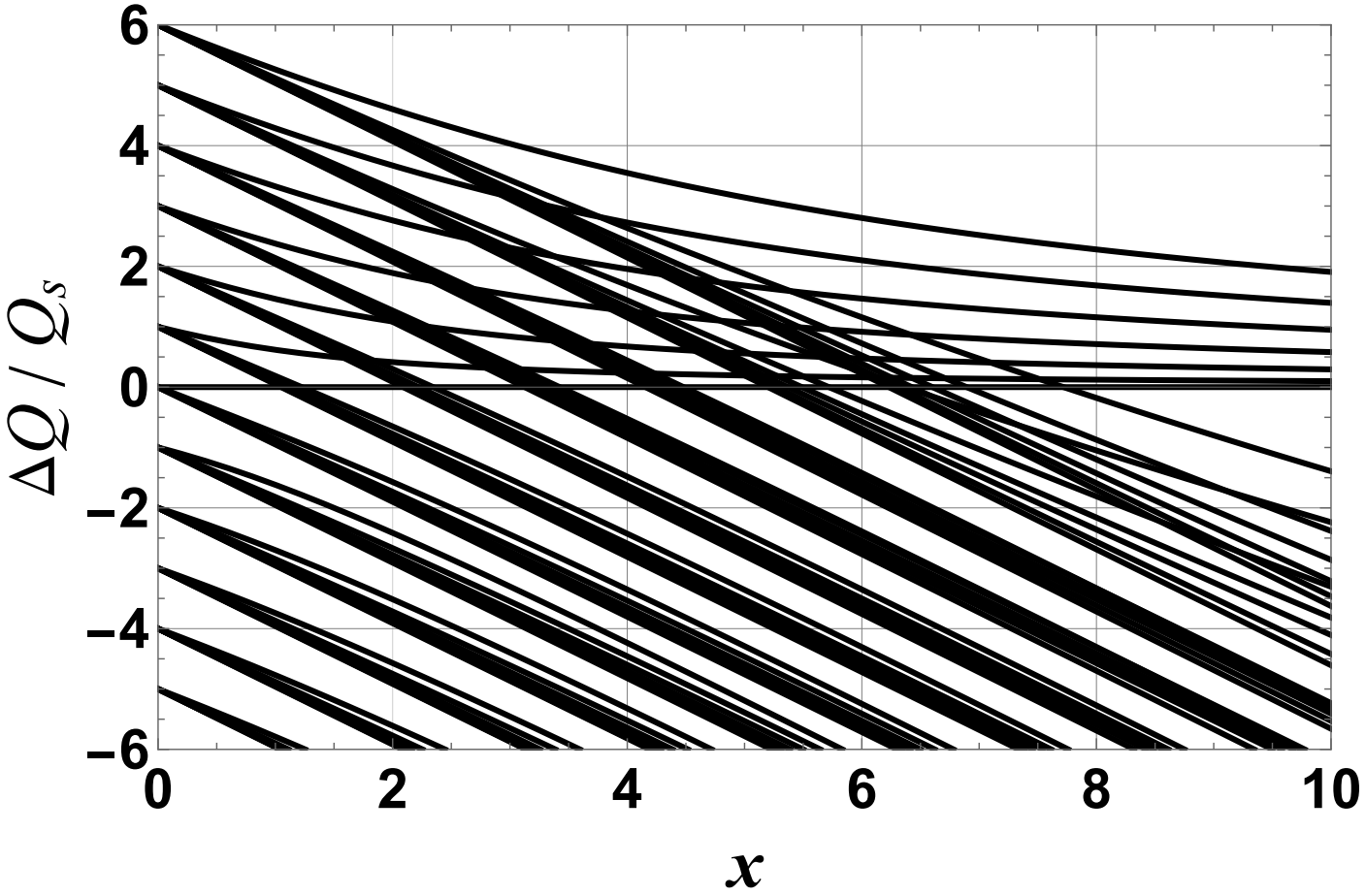
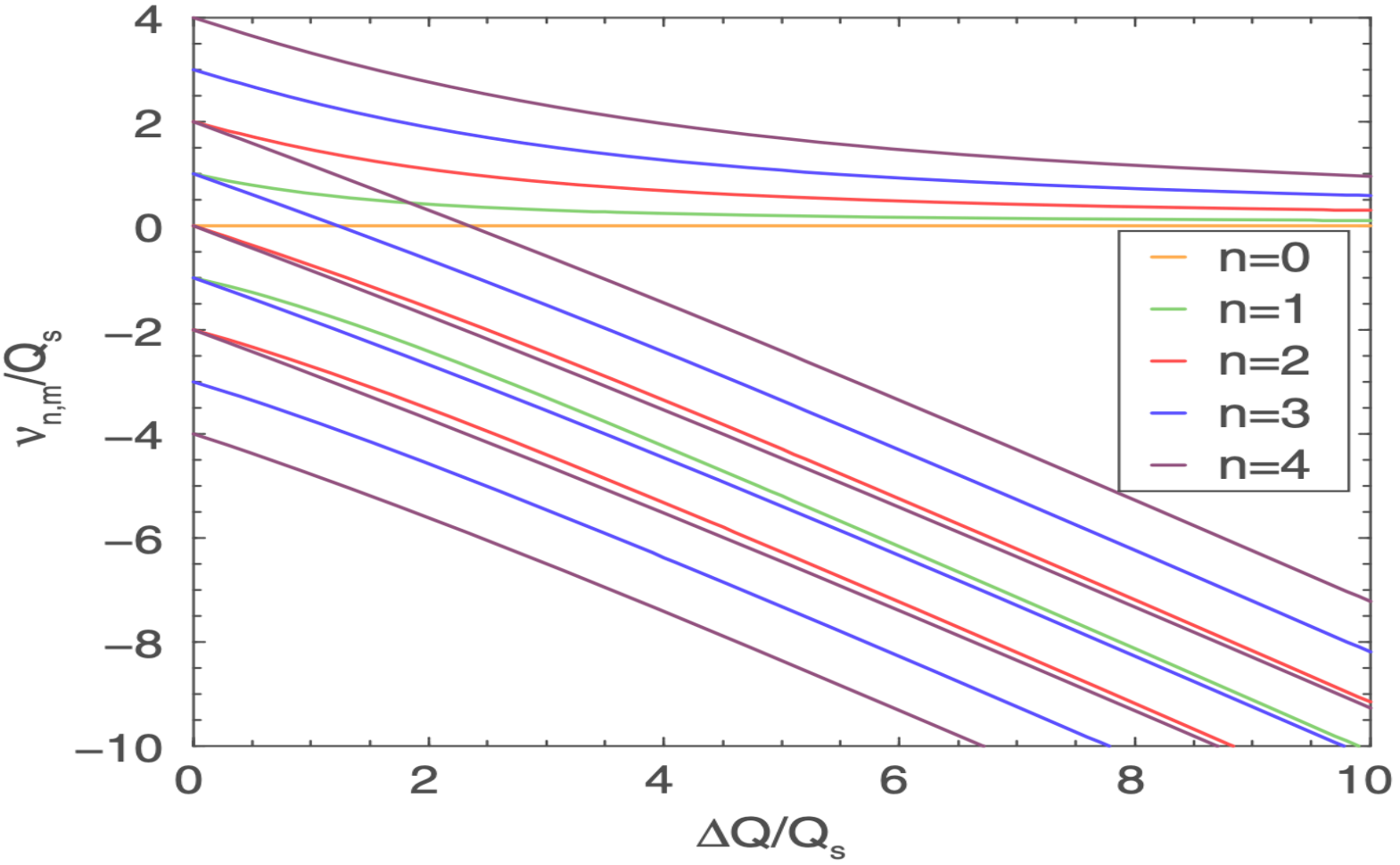


Comparison to the Boxcar model from V. Balbekov in 2017 (https://journals.aps.org/prab/pdf/10.1103/PhysRevAccelBeams.20.034401)



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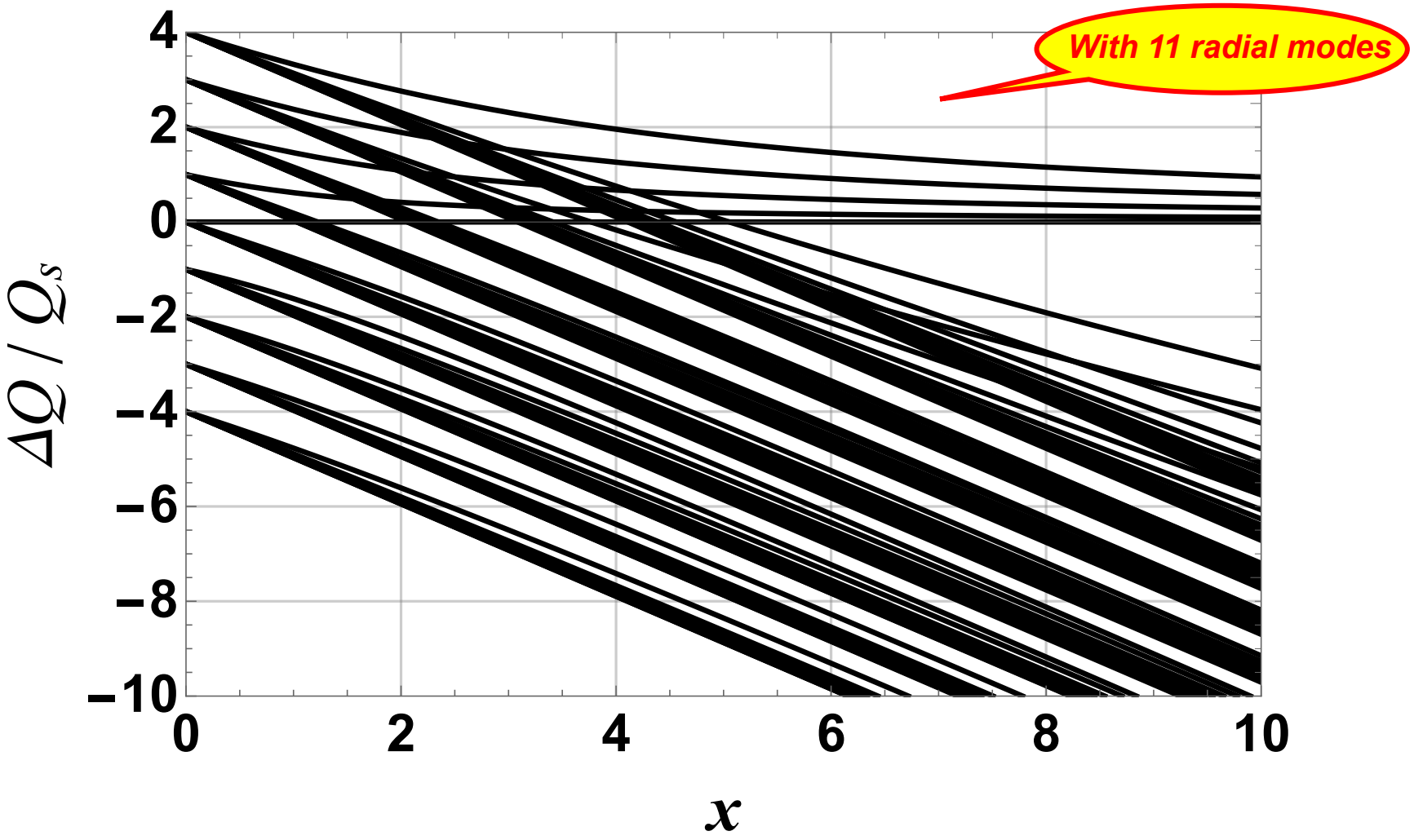

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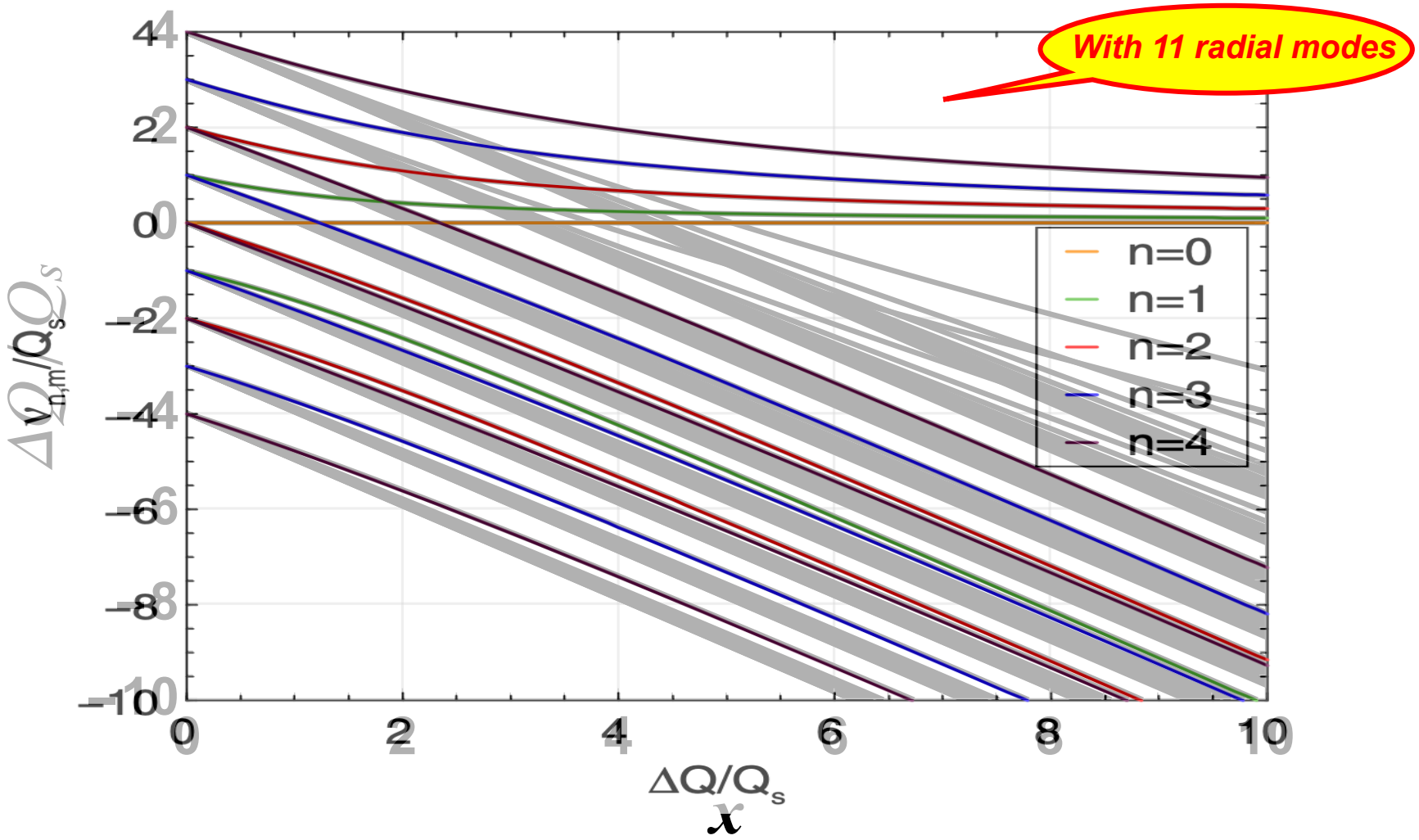

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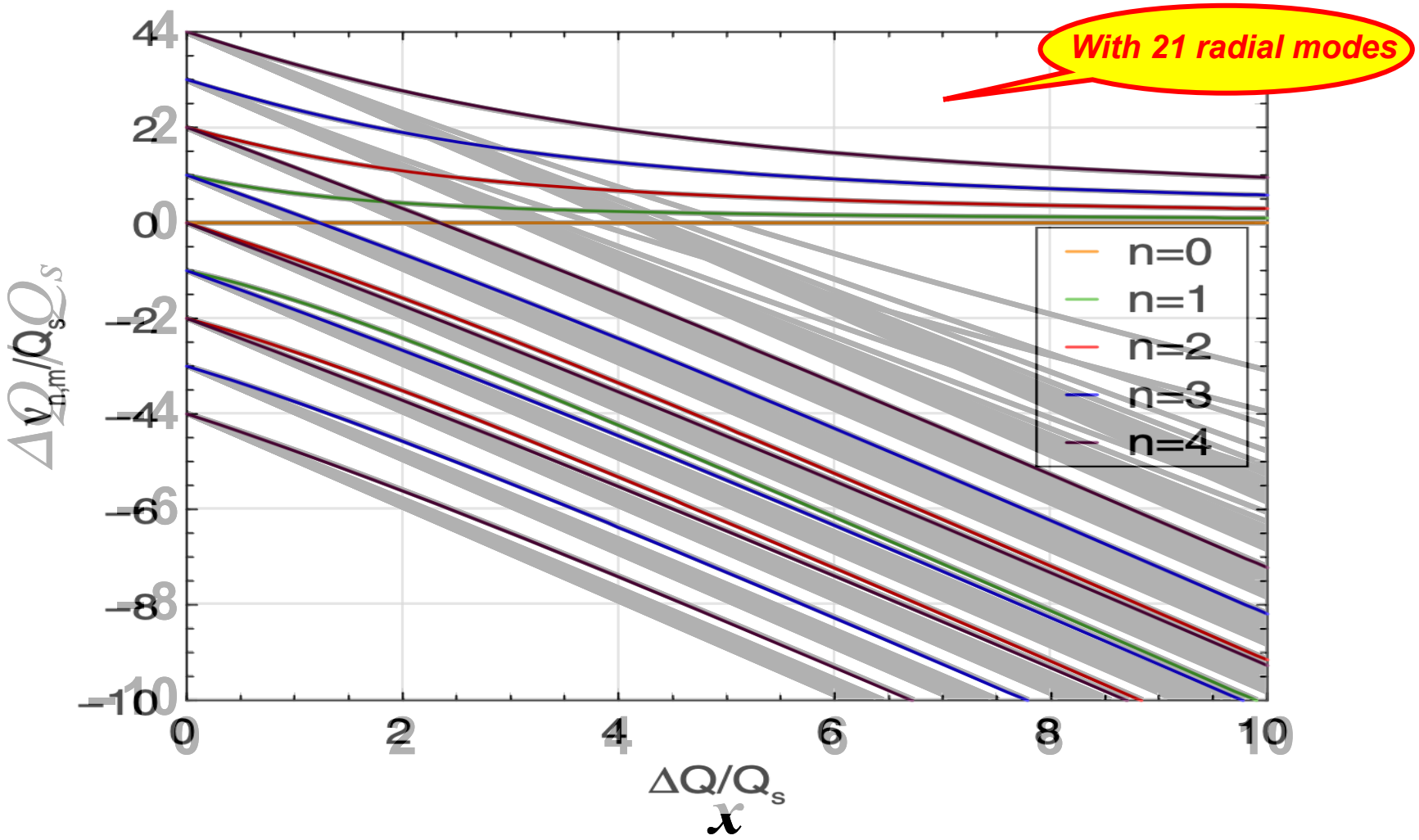

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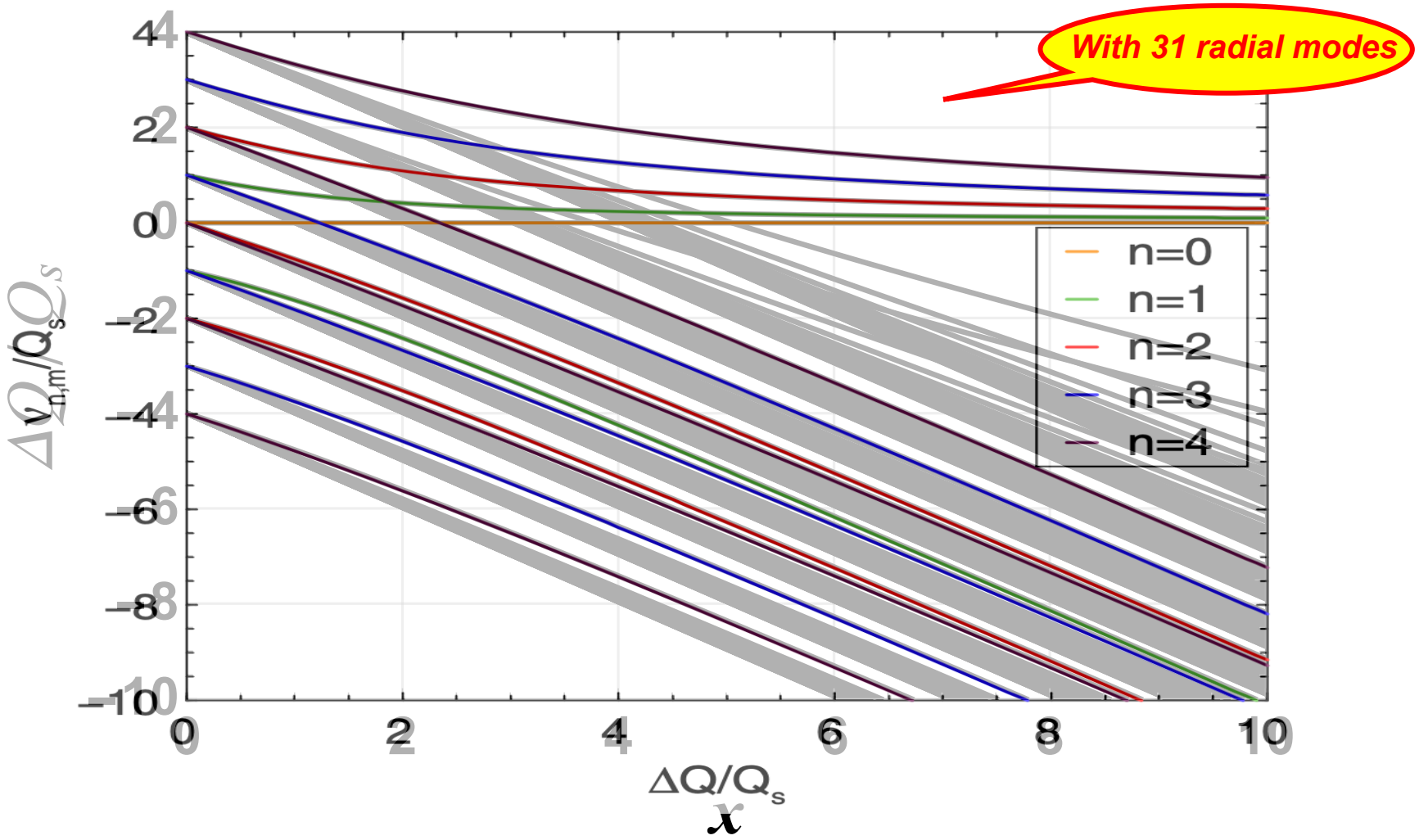

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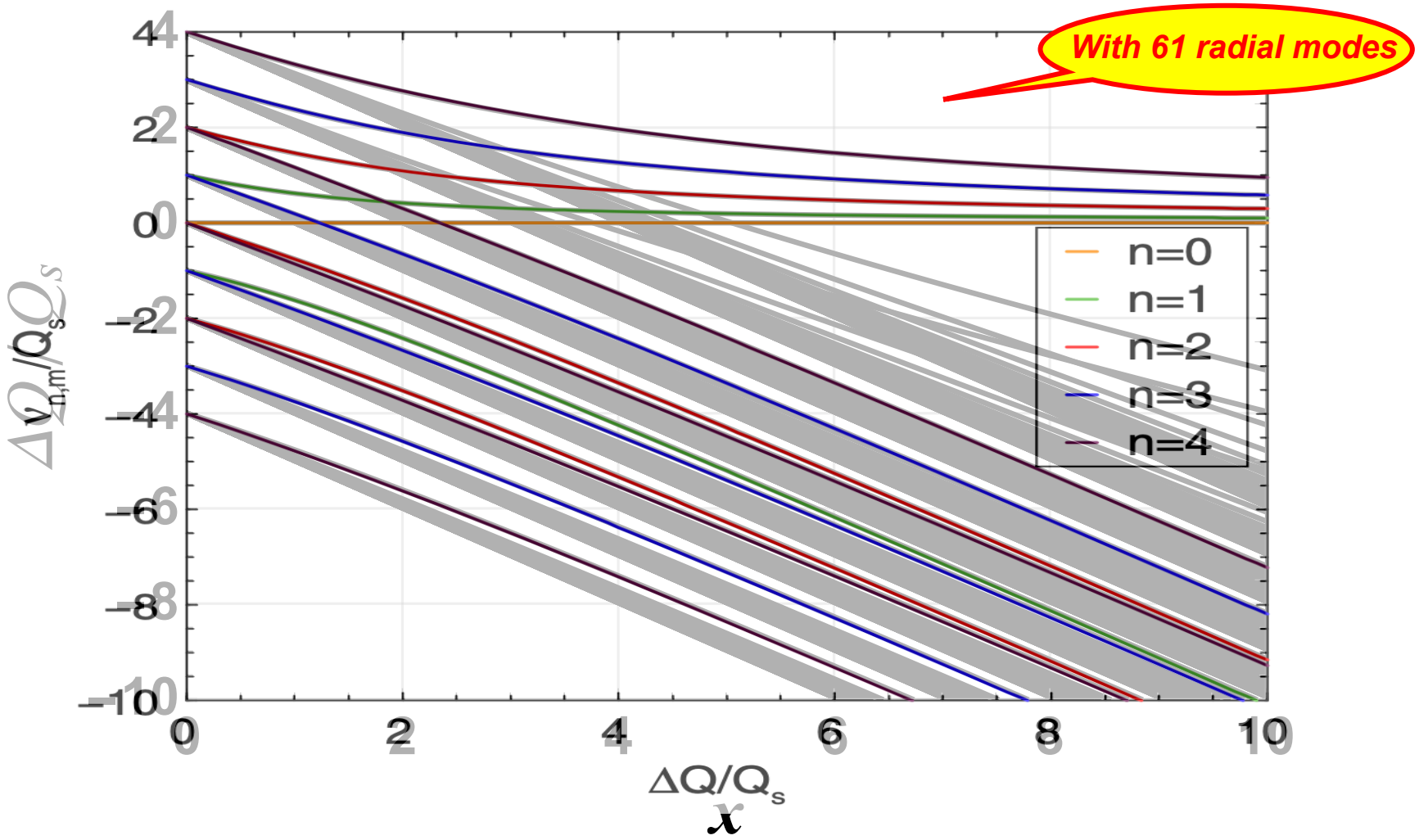

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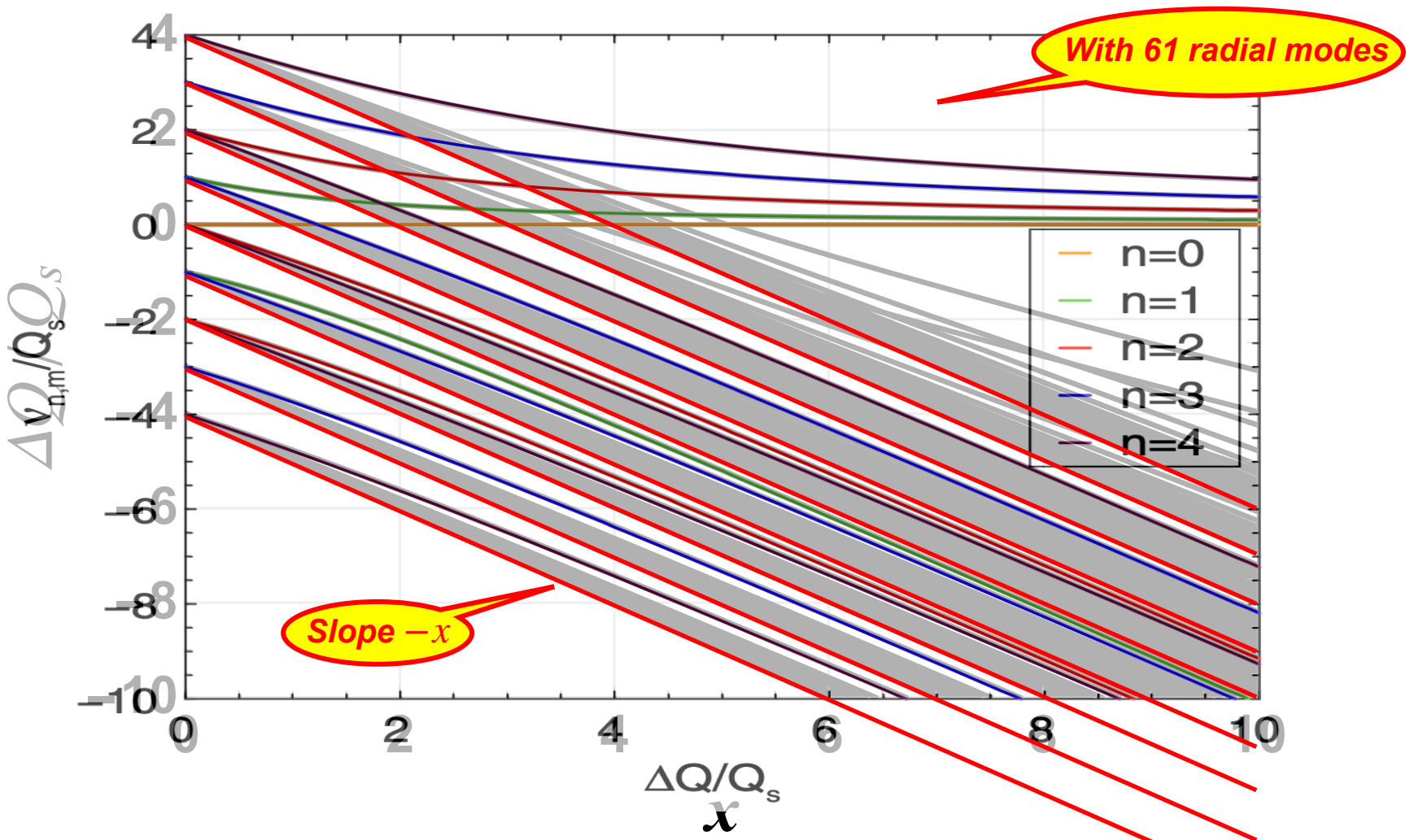

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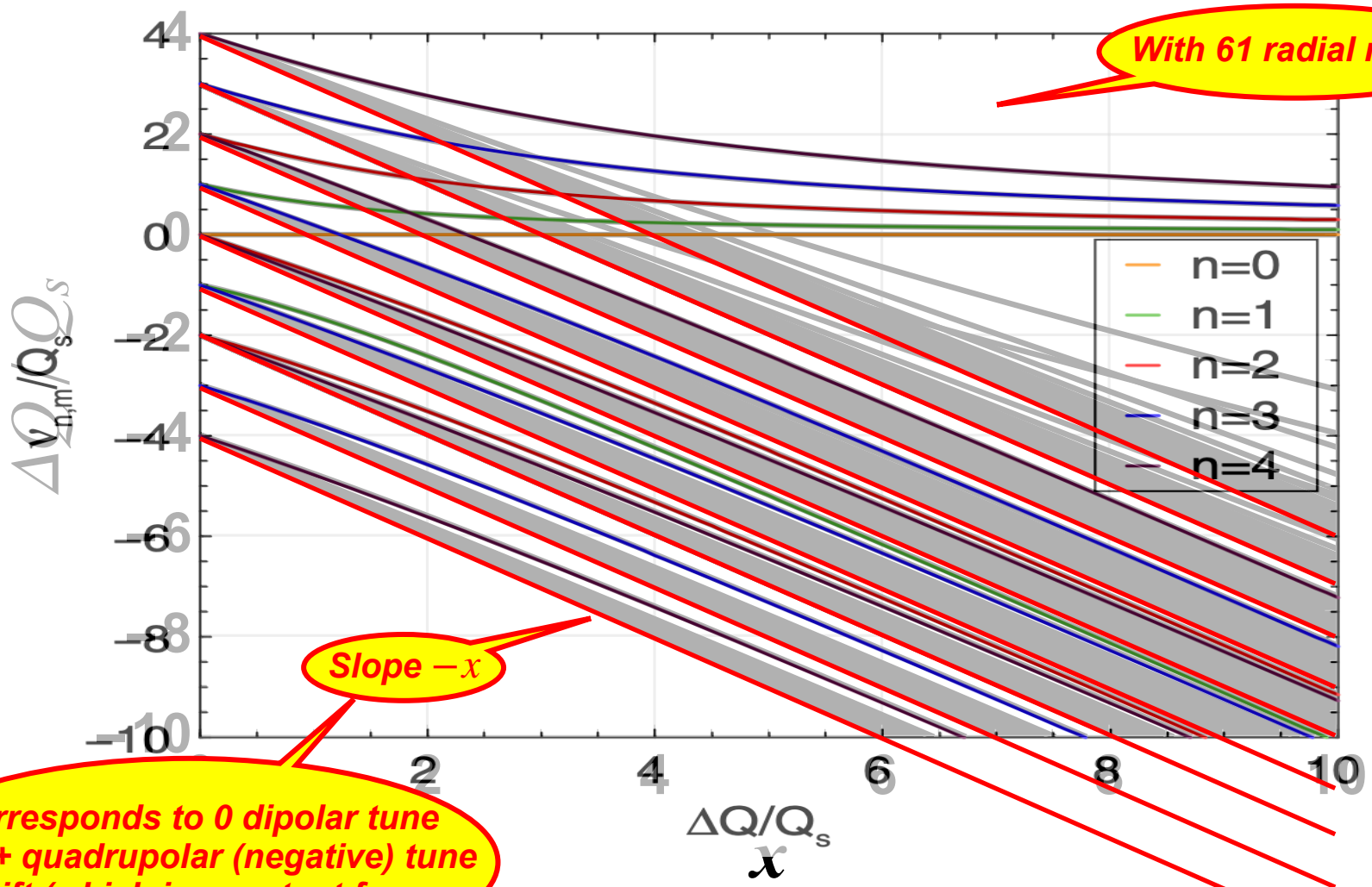

BENCHMARK 2: Gaussian

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- The main azimuthal modes 0, +1, +2, +3 and +4 were checked in 2010 by V. Kornilov and O. Boine-Frankenheim using PATRIC and HEADTAIL (https://proceedings.jacow.org/HB2010/papers/weo1a02.pdf)

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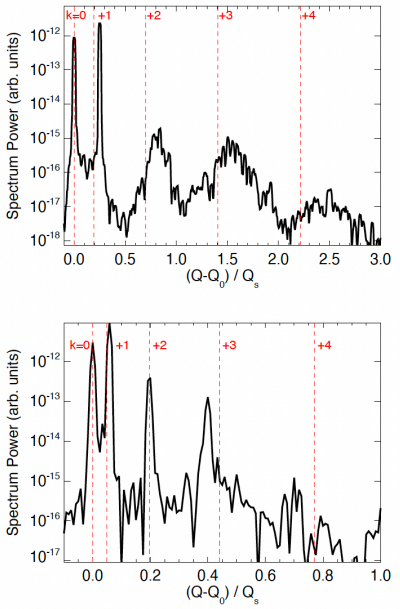

Figure 1: Transverse bunch spectrum from simulations for a Gaussian bunch with space charge: top plot $q = 5$, bottom plot $q = 20$. Red dashed lines are head-tail modes for the airbag bunch, Eq. (1). To clarify the notation we note that the eigenfrequencies without space charge are $(Q - Q_0)/Q_s = 1$ for $k = 1$, $(Q - Q_0)/Q_s = 2$ for $k = 2$, etc.

BENCHMARK 3: Gaussian

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- The main azimuthal modes +1, +2 and +3 were checked by Burov et al. in 2015 with simulations from SYNERGIA (https://journals.aps.org/prab/pdf/10.1103/PhysRevSTAB.18.074401)

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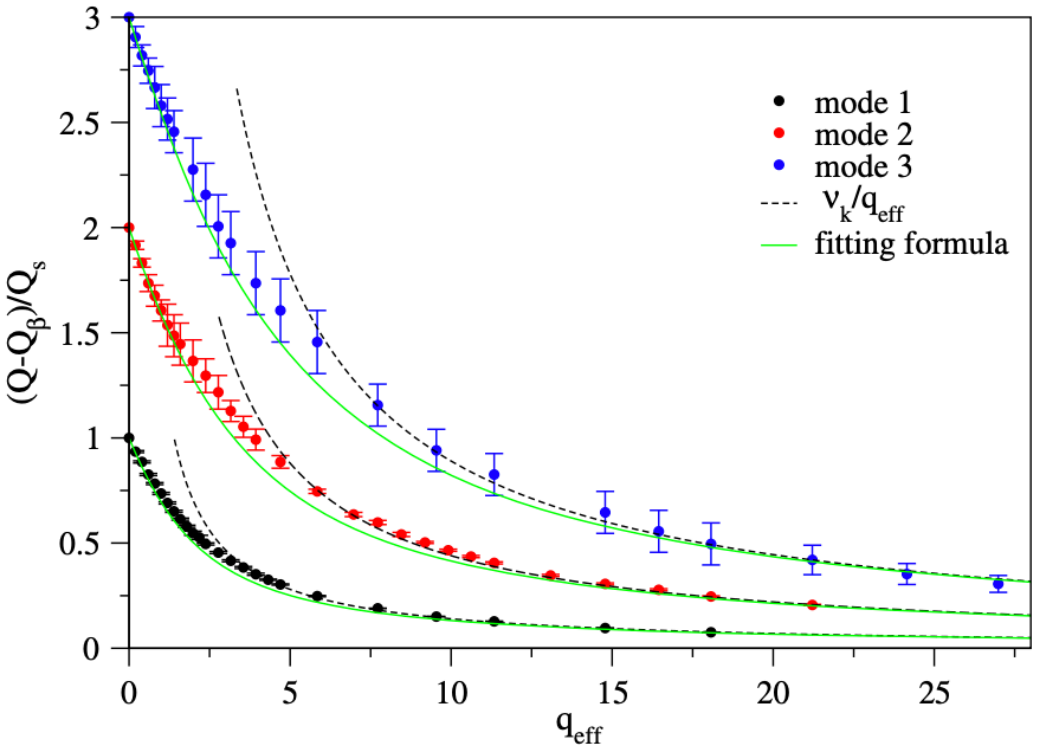

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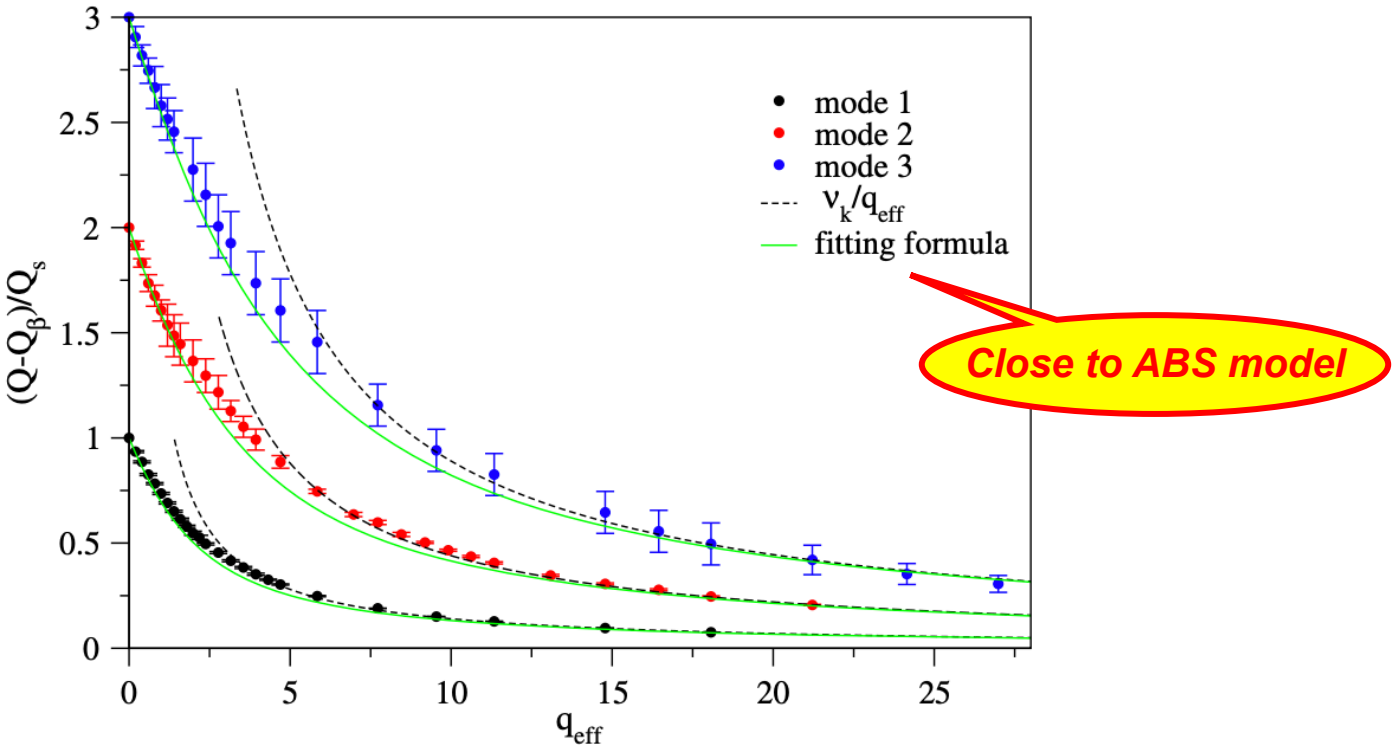

BENCHMARK 4: Gaussian

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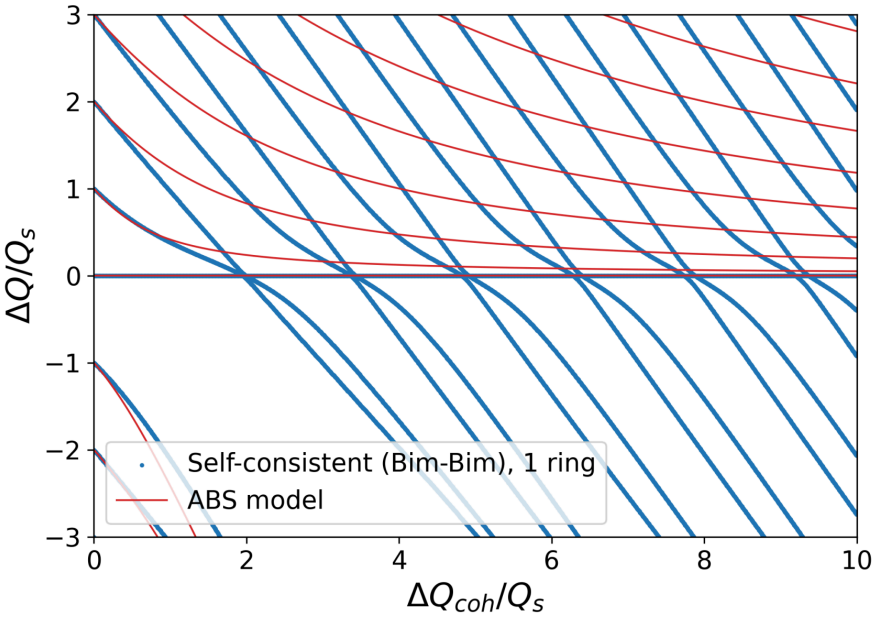
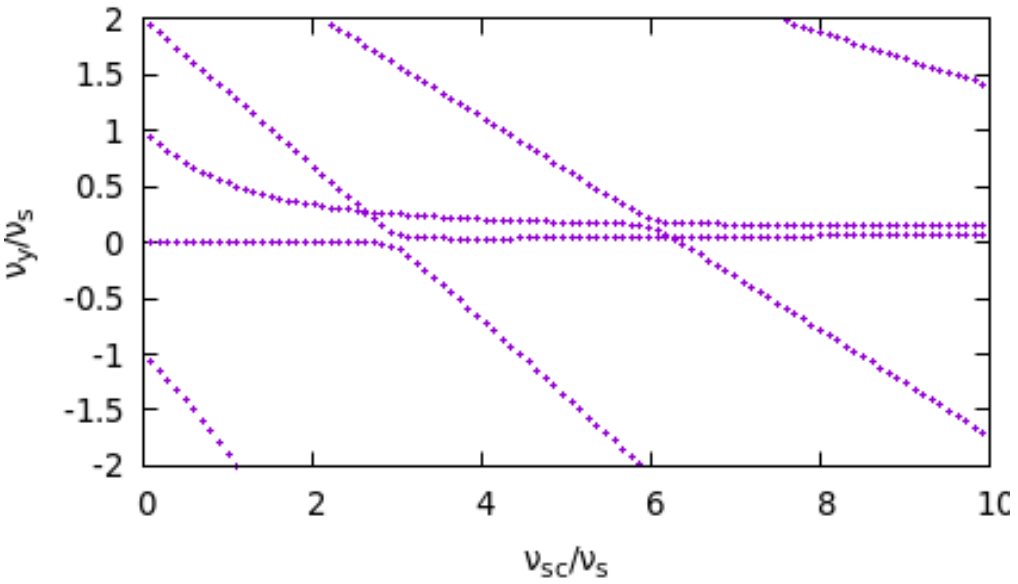
BimBim's (2021) result with 0 radial mode
(https://accelconf.web.cern.ch/ipac2024/pdf/MOPS35.pdf)



Ohmi's (2025) result with 0 radial mode
(https://indico.global/event/9305/contributions/90661/attachments/41536/77818/bbscmc_ohmi_slide.pdf)



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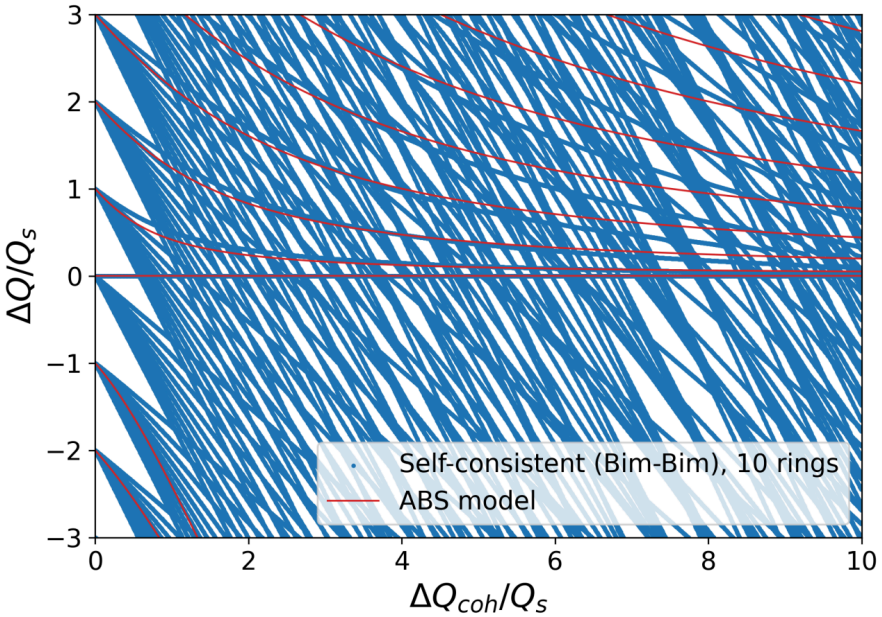

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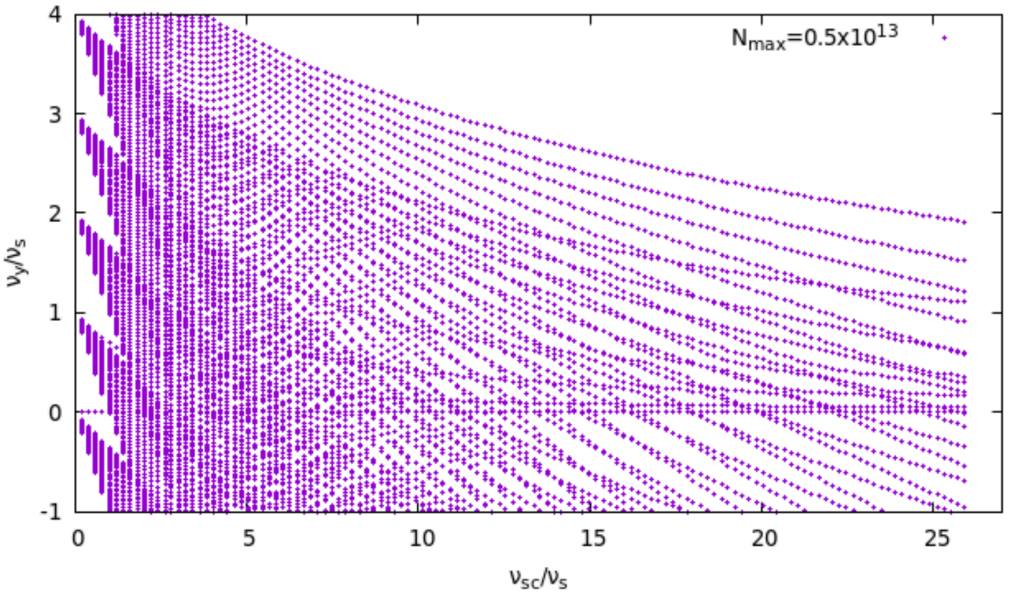

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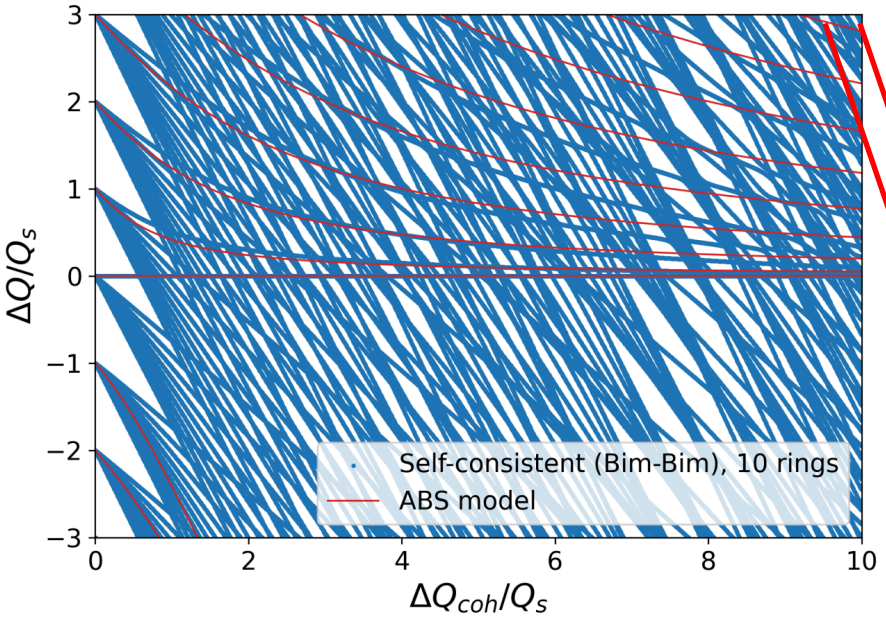

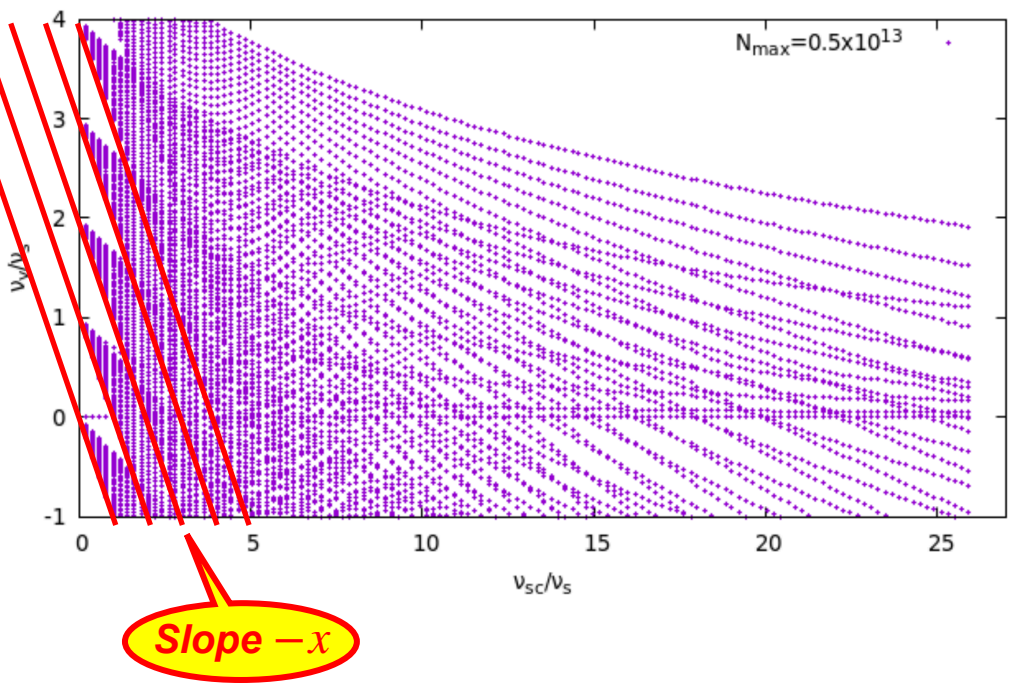

BENCHMARK 4: Gaussian

See also D. Amorim et al., IPAC2025

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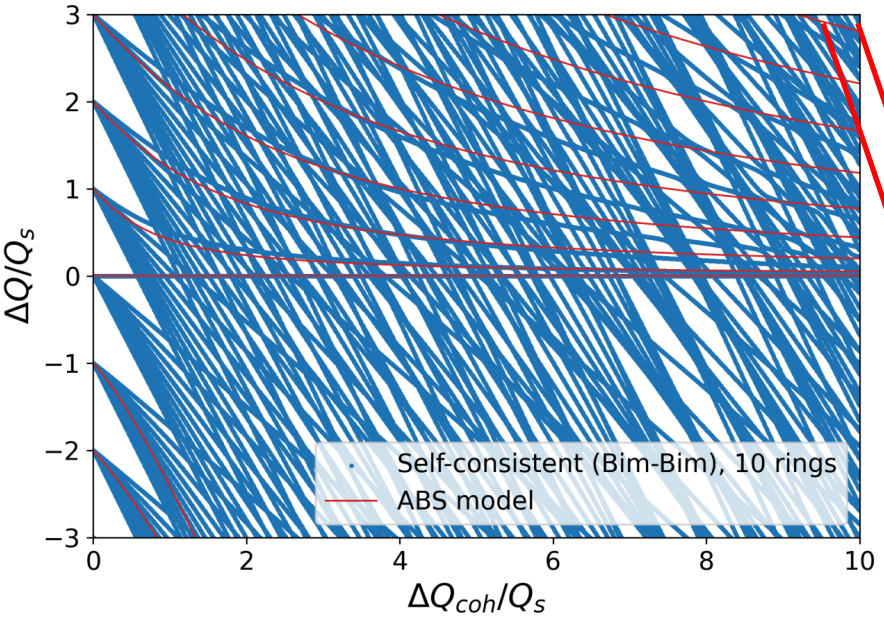

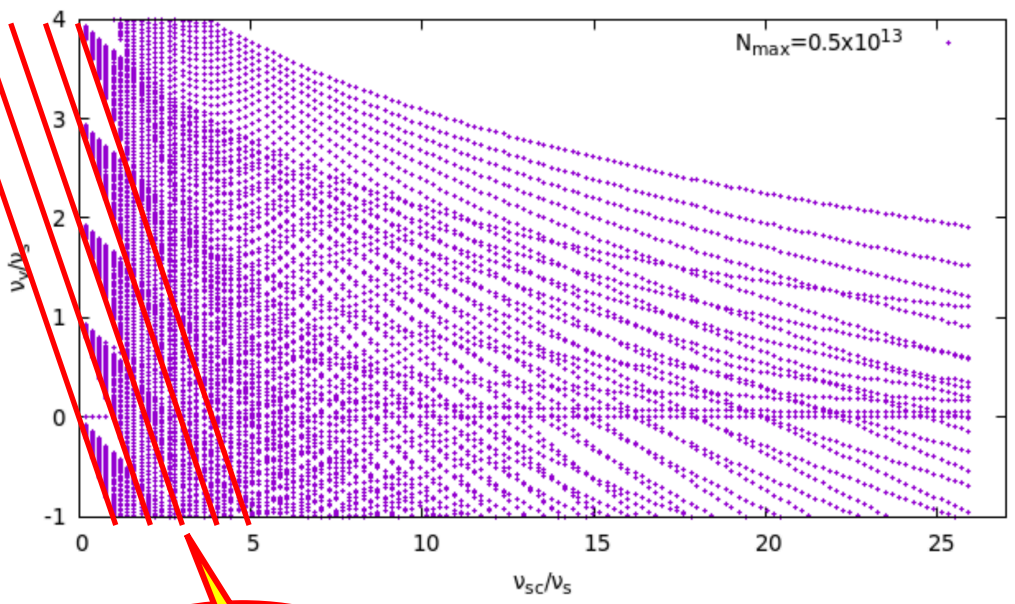

Slope $-x$

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  - The fact that the tune shift of the main mode 0 remains 0 is due to the compensation between the dipolar (focusing) and quadrupolar (defocusing) forces
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- BOXCAR distribution
  - An excellent agreement is found between M. Blaskiewicz (with the corrected typo of the factor 1/2 missing) and V. Balbekov for all the few modes shown by V. Balbekov
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  - The main positive azimuthal modes are different from the radial modes and this is due to the dipolar (focusing) force => The quadrupolar (negative) tune shift is constant
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  - All the radial modes are grouped together with a slope close to $-x$

Conclusions

- GAUSSIAN distribution
  - ✳ Many radial modes exist due to the dipolar (focusing) force

Conclusions

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  - Many additional radial modes exist due to the quadrupolar (defocusing) force

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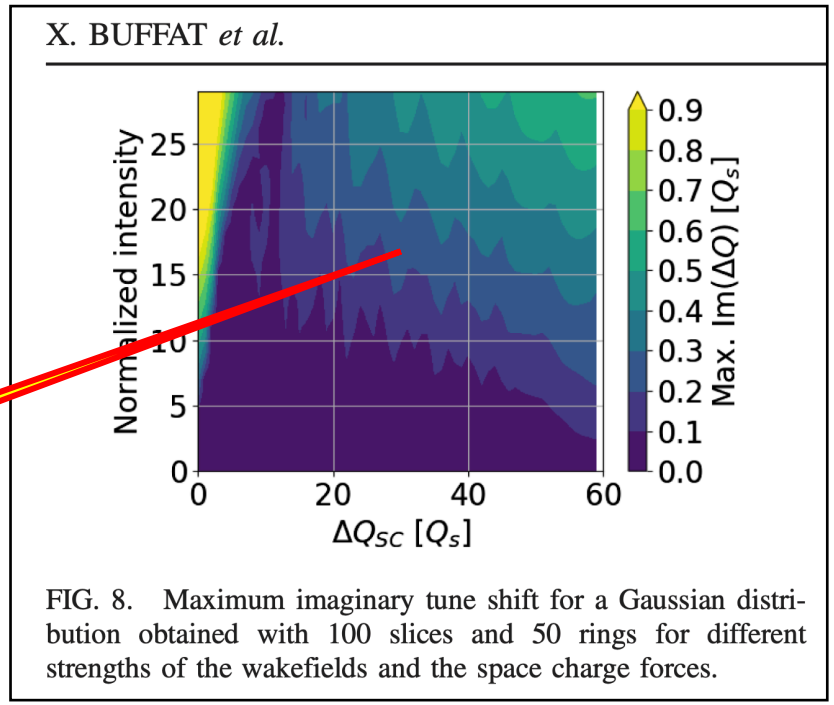

FIG. 8. Maximum imaginary tune shift for a Gaussian distribution obtained with 100 slices and 50 rings for different strengths of the wakefields and the space charge forces.

Due to radial modes!

Thank you for your attention!

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