Space-Charge Impedance in Material-Filled Beam Regions for Ionization Cooling in a Future Muon Collider

Space Charge 2026, 7th ICFA BD Mini-workshop

Erik Kvikne*, Elias Métral, Erik Adli

Erik Kvikne · CERN BE-ABP-CEI · University of Oslo
*erik.kvikne@cern.ch

02.10.2026

Space Charge in Matter

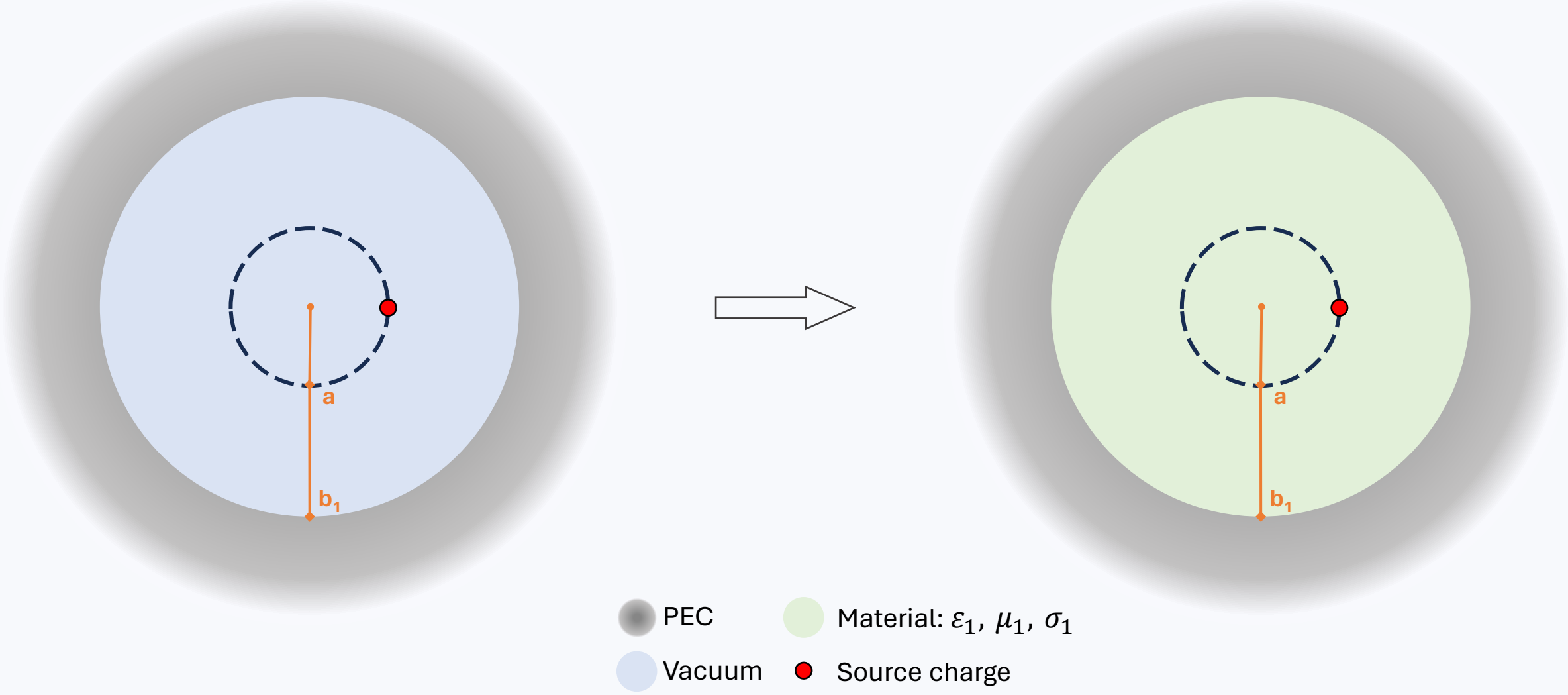
Wake Potential and Electromagnetic benchmark

Conclusion and Outlook

From a vacuum to a material-filled beam region

Standard SC geometry

Material-filled SC geometry



Conventional space-charge: vacuum beam region

Conventional Space-Charge Impedance

[1]

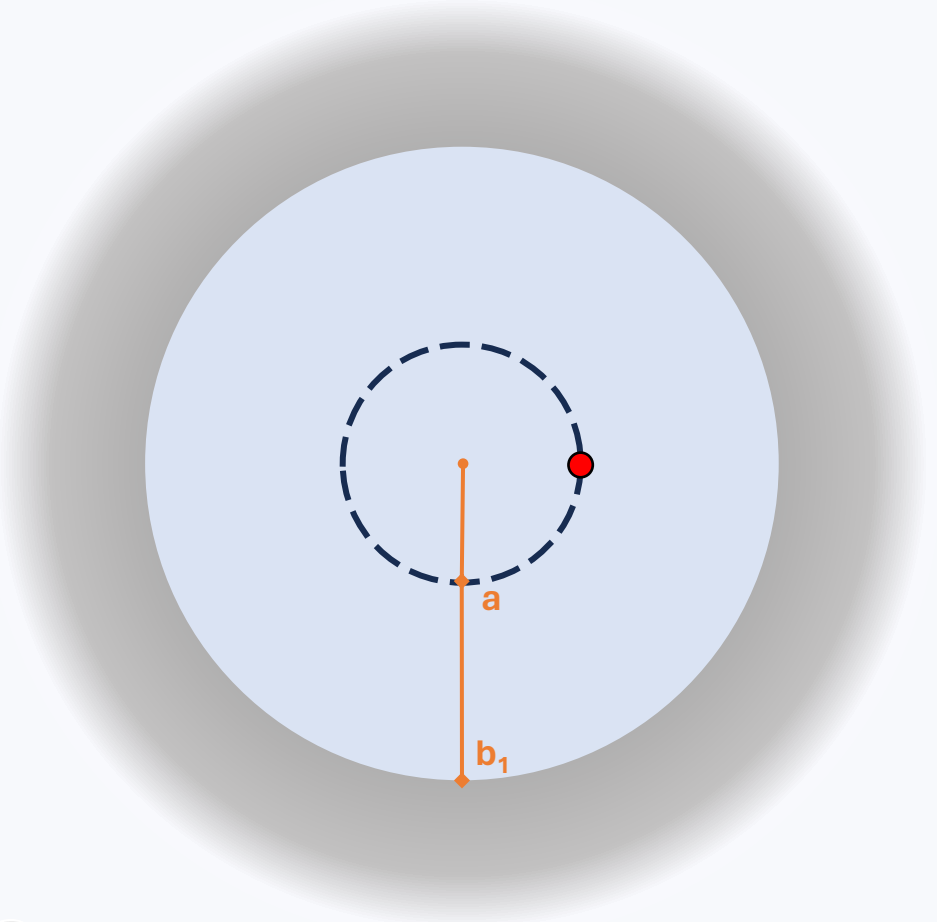
$$Z_{\parallel}^{\mathrm{SC}} = -j\frac{L\omega\mu_0}{2\pi\beta^2}\quad\frac{1}{\gamma^2}\quad I_0^2(x_0)\left[\frac{K_0(x_0)}{I_0(x_0)} - \frac{K_0(x_1)}{I_0(x_1)}\right]$$

Direct SC *Indirect SC*

$$Z_x^{\mathrm{SC}} = -j\frac{LZ_0}{\pi\beta}\quad\frac{1}{\gamma^2}\quad\frac{I_1^2(x_0)}{a^2}\left[\frac{K_1(x_0)}{I_1(x_0)} - \frac{K_1(x_1)}{I_1(x_1)}\right]$$

$$x_0 = \frac{k}{\gamma}\,a, \qquad x_1 = \frac{k}{\gamma}\,b_1$$

Definitions →



- **Purely imaginary** for a vacuum region with PEC boundary
- Overall space-charge strength **suppressed by** $1/\gamma^2$
- Radial field behaviour is **governed by** k/γ

Conventional space-charge: vacuum beam region

Conventional Space-Charge Impedance

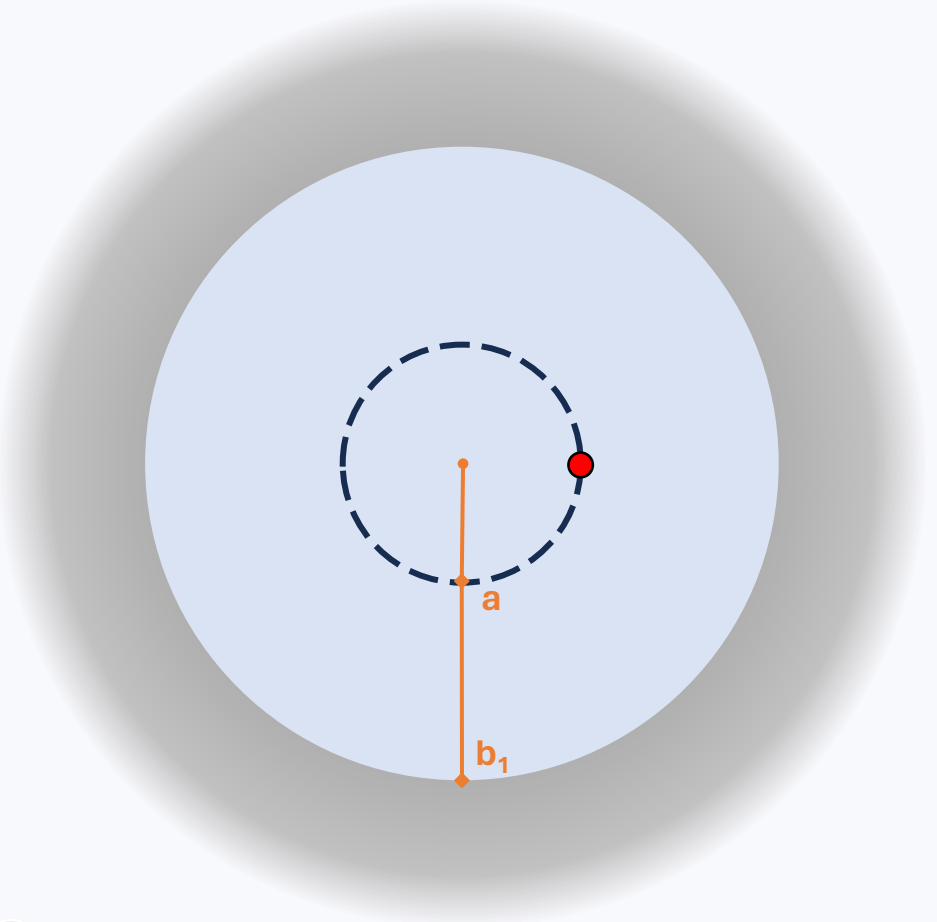
[1]

$$Z_{\parallel}^{\mathrm{SC}} = -j\frac{L\omega\mu_0}{2\pi\beta^2}\underbrace{\frac{1}{\gamma^2}}_{\text{vacuum}} I_0^2(x_0)\left[\frac{K_0(x_0)}{I_0(x_0)} - \frac{K_0(x_1)}{I_0(x_1)}\right]$$

$$Z_{x}^{\mathrm{SC}} = -j\frac{LZ_0}{\pi\beta}\underbrace{\frac{1}{\gamma^2}}_{\text{vacuum}} \frac{I_1^2(x_0)}{a^2}\left[\frac{K_1(x_0)}{I_1(x_0)} - \frac{K_1(x_1)}{I_1(x_1)}\right]$$

$$x_0 = \underbrace{\frac{k}{\gamma}}_{\text{vacuum radial scale}} a, \qquad x_1 = \underbrace{\frac{k}{\gamma}}_{\text{vacuum radial scale}} b_1$$

Definitions →



- **Purely imaginary** for a vacuum region with PEC boundary
- Overall space-charge strength **suppressed by** $1/\gamma^2$
- Radial field behaviour is **governed by** k/γ

Space-charge in a material-filled beam region

[2]

Space-Charge Impedance in Matter

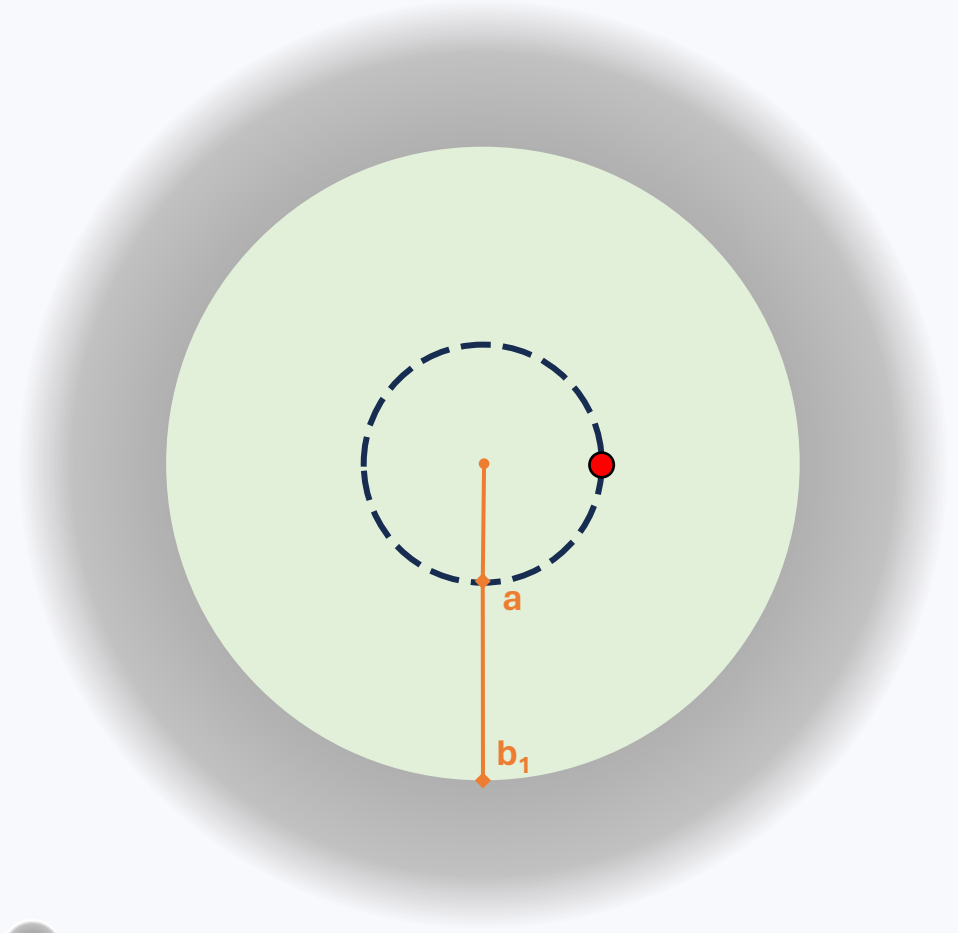
$$\frac{1}{\gamma^2} \longrightarrow F = \frac{1}{\varepsilon_1} - \mu_1 \beta^2$$

$$\frac{k}{\gamma} \longrightarrow \nu_1 = k\sqrt{1 - \beta^2 \varepsilon_1 \mu_1}$$

Definitions →

- The vacuum factors $1/\gamma^2$ and k/γ are replaced by **material-dependent quantities**
- The formalism is extended to material-filled beam regions
- These modifications lead to qualitatively new behaviour in **conductive and dielectric media**

General formalism: arbitrary cylindrically symmetric multilayer structures
Here: material-filled beam region bounded by PEC



Material response through F

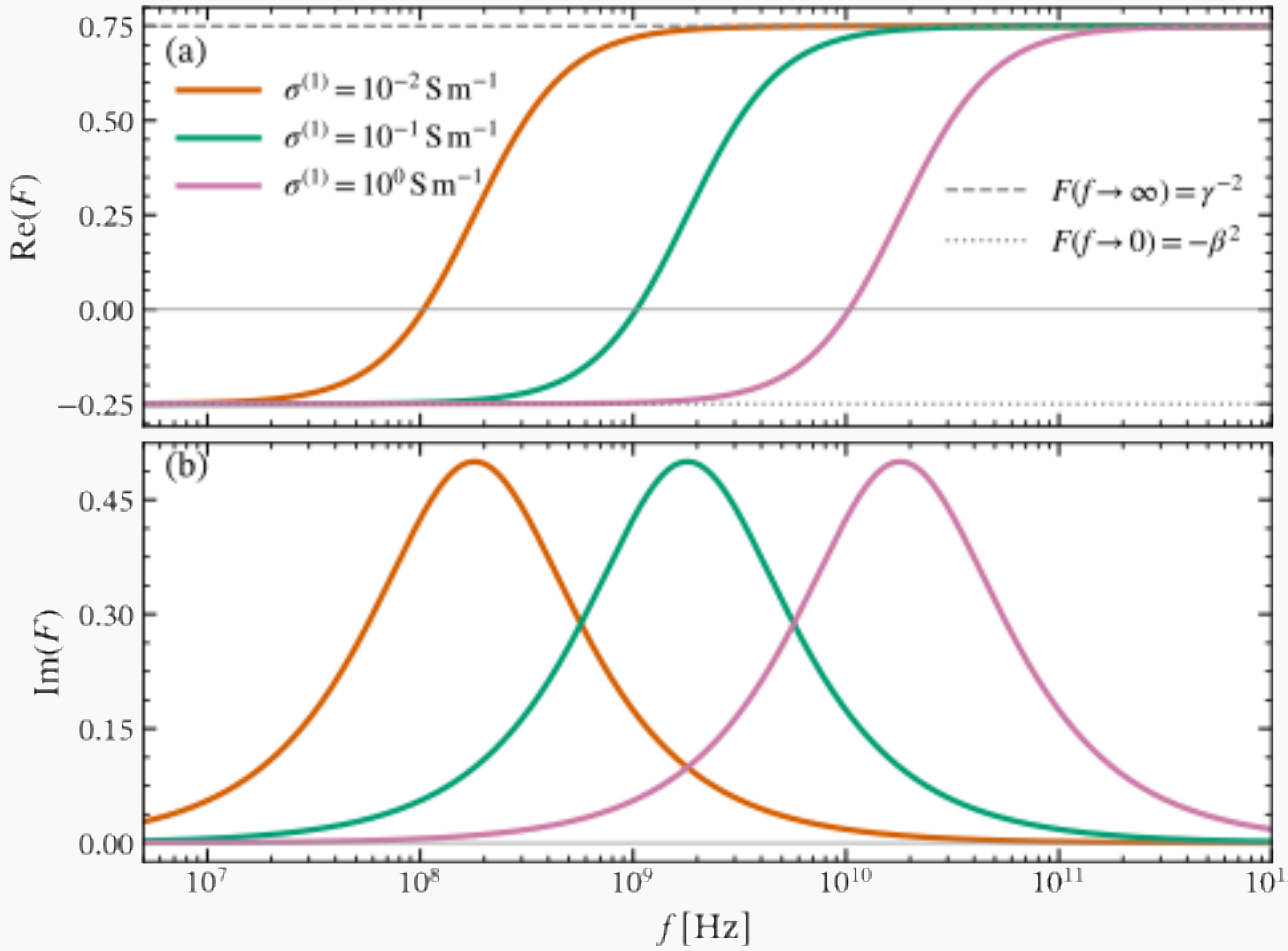
$$F = \frac{1}{\varepsilon_1} - \mu_1\beta^2 \qquad \varepsilon_1 = \varepsilon_r + \frac{\sigma}{j\varepsilon_0\omega}$$

$$F = \frac{\varepsilon_r}{\varepsilon_r^2 + (\sigma/\varepsilon_0\omega)^2} - \mu_1\beta^2 + j\,\frac{\sigma/\varepsilon_0\omega}{\varepsilon_r^2 + (\sigma/\varepsilon_0\omega)^2}$$

- Finite conductivity makes F **complex and frequency dependent**
- $\mathrm{Re}(F)$ connects the **low- and high-frequency limits**, while $\mathrm{Im}(F)$ peaks around

$$\omega \sim \frac{\sigma}{\varepsilon_0\varepsilon_r}$$

- Increasing conductivity shifts the material response to **higher frequencies**



$\beta = 0.5, \qquad \varepsilon_r^{(1)} = \mu_1^{(1)} = 1$

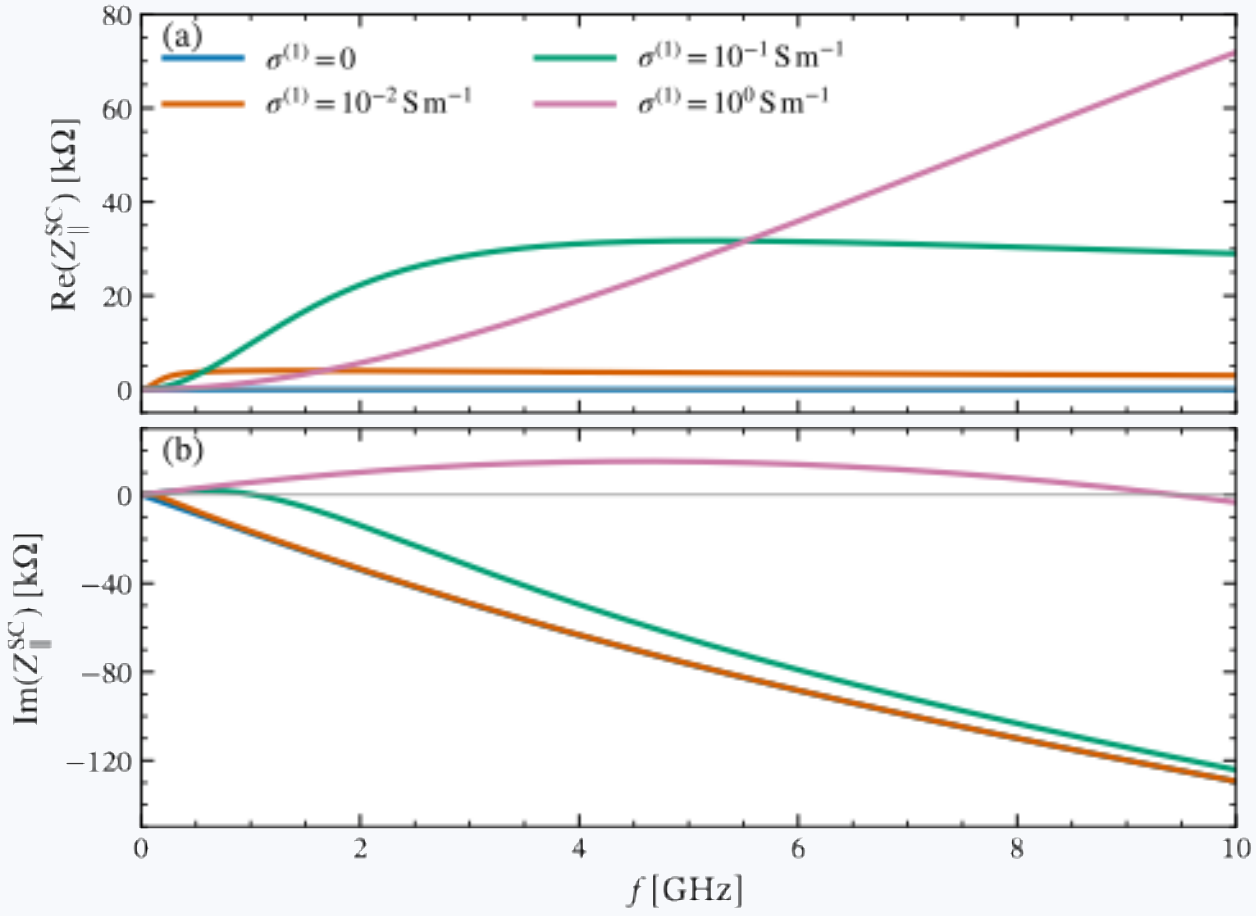
Conductive beam region: impact on longitudinal impedance

- Finite conductivity introduces a **real part** of the impedance
- The **imaginary part can change sign** as the material response changes
- Increasing conductivity shifts these effects to **higher frequencies**

$$Z_{\parallel}^{\mathrm{SC}} \propto -j\omega F$$

$$\mathrm{Im}(F) \neq 0 \quad \Rightarrow \quad \mathrm{Re}\left(Z_{\parallel}^{\mathrm{SC}}\right) \neq 0$$

$$\mathrm{Re}(F) : + \rightarrow - \quad \Rightarrow \quad \mathrm{Im}\left(Z_{\parallel}^{\mathrm{SC}}\right) : - \rightarrow +$$



$\beta = 0.5, \qquad \varepsilon_r^{(1)} = \mu_1^{(1)} = 1, \qquad a = 0.1\ \mathrm{mm}, \qquad b_1 = 1\ \mathrm{cm}, \qquad L = 1\ \mathrm{m}$

Transverse Impedance →

Dielectric media can change the radial field behaviour

$$\nu_1 = k\sqrt{1 - \beta^2 \varepsilon_r^{(1)} \mu_1^{(1)}} \qquad (\sigma = 0)$$

Evanescent radial fields

$$\beta^2 \varepsilon_r^{(1)} \mu_1^{(1)} < 1 \quad \Rightarrow \quad \nu_1 \in \mathbb{R}$$

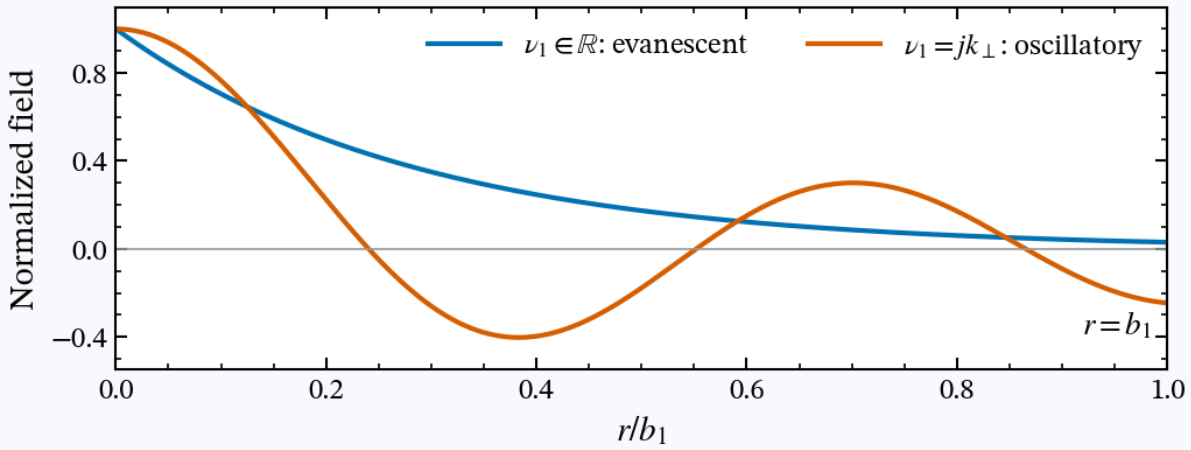
Modified Bessel functions I_m, K_m - non-oscillatory

Oscillatory radial fields

$$\beta^2 \varepsilon_r^{(1)} \mu_1^{(1)} > 1 \quad \Rightarrow \quad \nu_1 = jk_\perp$$

Ordinary Bessel functions J_m, Y_m - oscillatory

Schematic radial field behaviour

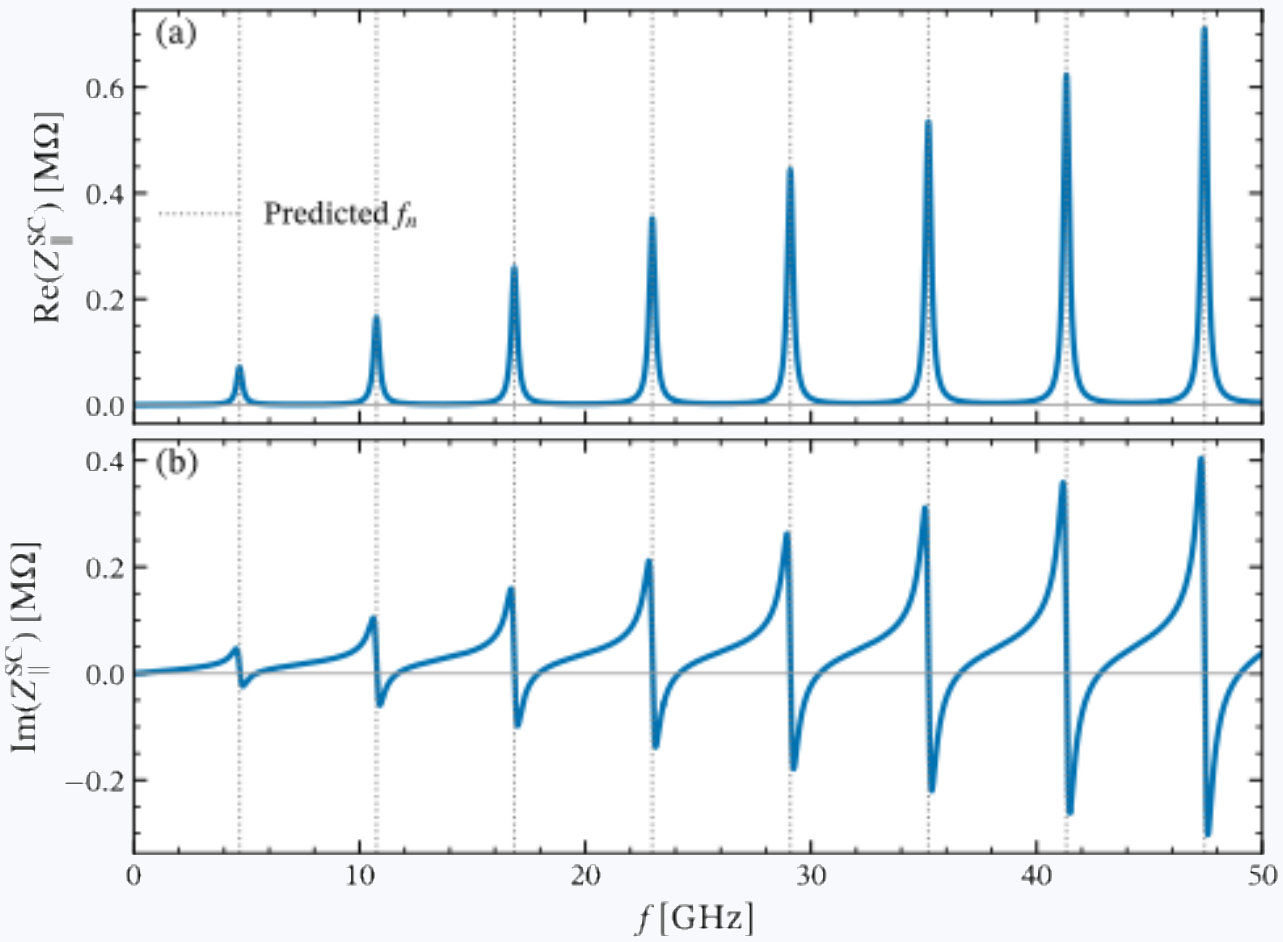


Equivalent condition

$$v > \frac{c}{\sqrt{\varepsilon_r^{(1)} \mu_1^{(1)}}}$$

Beam velocity exceeds the electromagnetic phase velocity in the medium

Oscillatory radial fields give rise to resonant modes



$$\beta = 0.5, \qquad \varepsilon_r^{(1)} = 10, \qquad \mu_1^{(1)} = 1, \qquad \sigma^{(1)} = 10^{-1}\,\mathrm{S\,m^{-1}}, \qquad b_1 = 1\,\mathrm{cm}$$

Resonance condition

$$J_0(k_\perp b_1) = 0$$

$$k_\perp = k\sqrt{\beta^2 \varepsilon_r^{(1)} \mu_1^{(1)} - 1}$$

$$f_n = \frac{\beta c}{2\pi b_1 \sqrt{\beta^2 \varepsilon_r^{(1)} \mu_1^{(1)} - 1}}\, j_{0,n}$$

- The PEC boundary condition permits only **discrete radial field patterns**
- Resonance frequencies follow the **zeros of J_0**
- Finite conductivity **broadens the resonances**

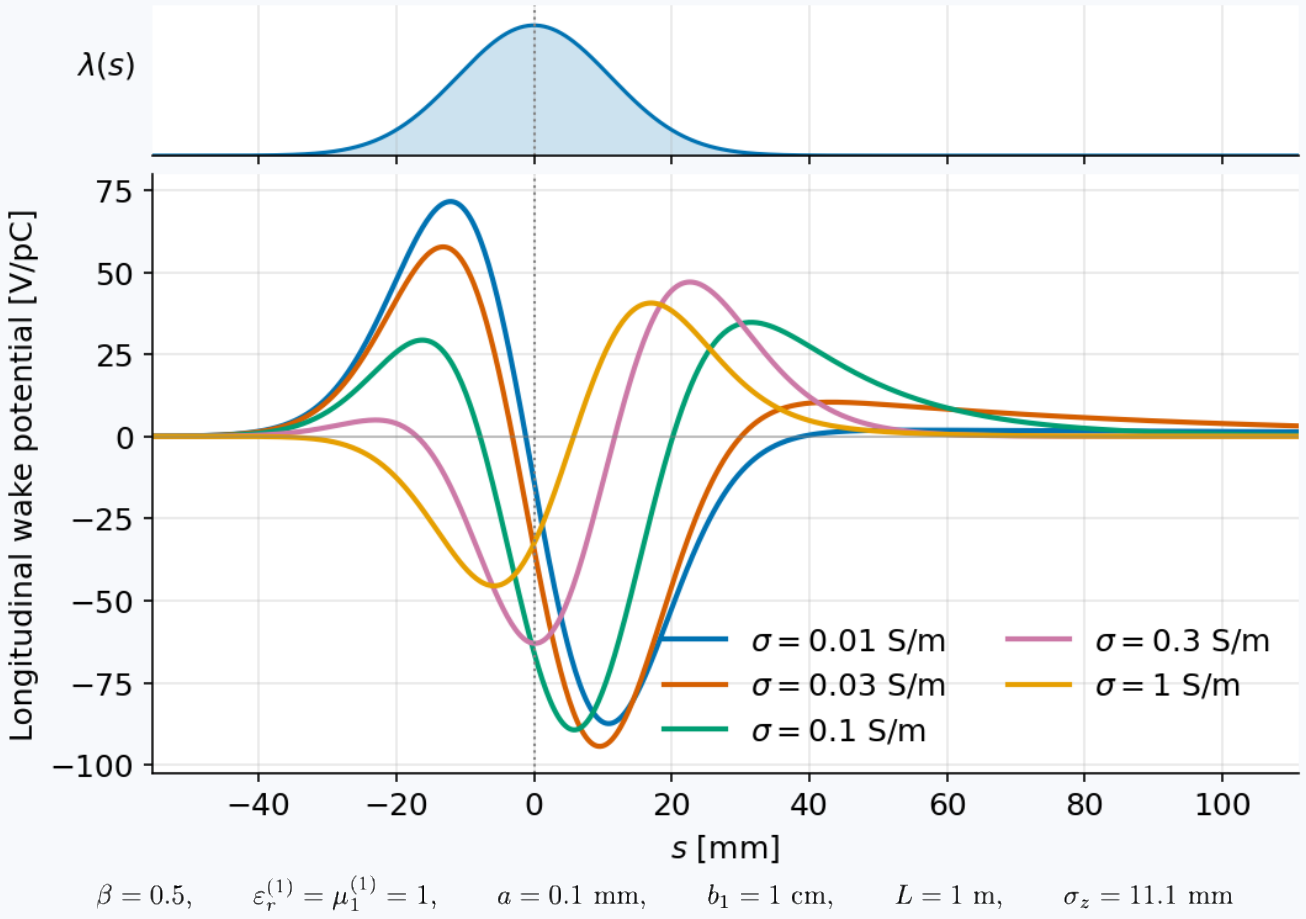
Transverse Impedance →

Space Charge in Matter

Wake Potential and Electromagnetic benchmark

Conclusion and Outlook

From impedance to wake potential



$\beta = 0.5, \quad \varepsilon_r^{(1)} = \mu_1^{(1)} = 1, \quad a = 0.1 \text{ mm}, \quad b_1 = 1 \text{ cm}, \quad L = 1 \text{ m}, \quad \sigma_z = 11.1 \text{ mm}$

$$W_\parallel(s) = \int_{-\infty}^{\infty} w_\parallel(s - s')\, \lambda(s')\, \mathrm{d}s'$$

$$W_\parallel(s) = \mathcal{F}^{-1}\left[Z_\parallel(\omega)\, \tilde{\lambda}(\omega) \right]$$

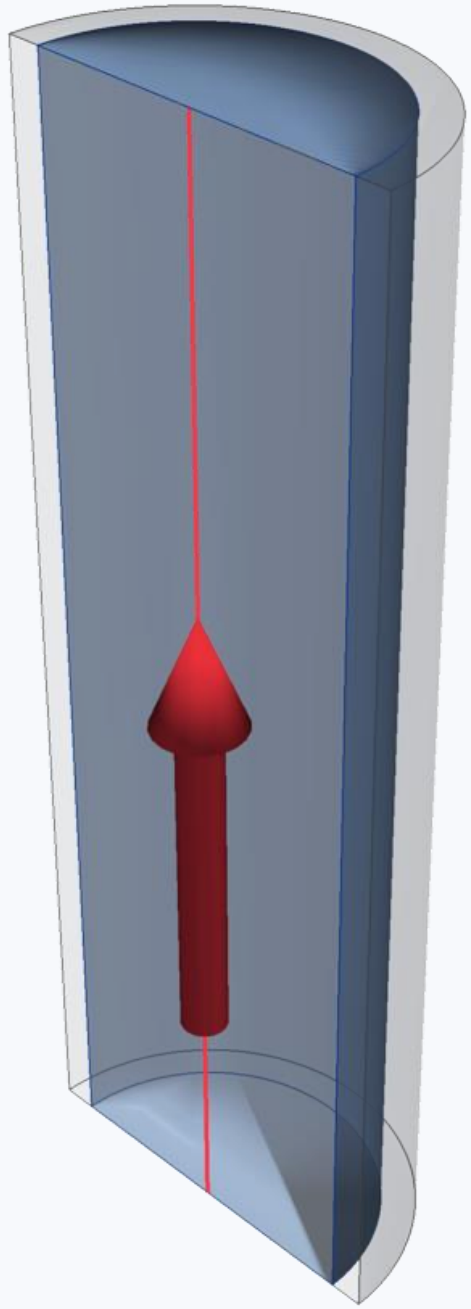
- A finite bunch samples the impedance through its **frequency spectrum**
- Material properties modify the shape and amplitude of the wake potential
- The **total longitudinal and transverse wake potentials** are compared directly with WAKIS [3]

Electromagnetic benchmark with WAKIS

- **Time-domain electromagnetic simulation** of a bunch propagating through the material-filled beam region
- The **total longitudinal and transverse wake potentials and impedances** are obtained from the simulated fields
- Results are compared with the **corresponding analytical prediction for each simulation case**

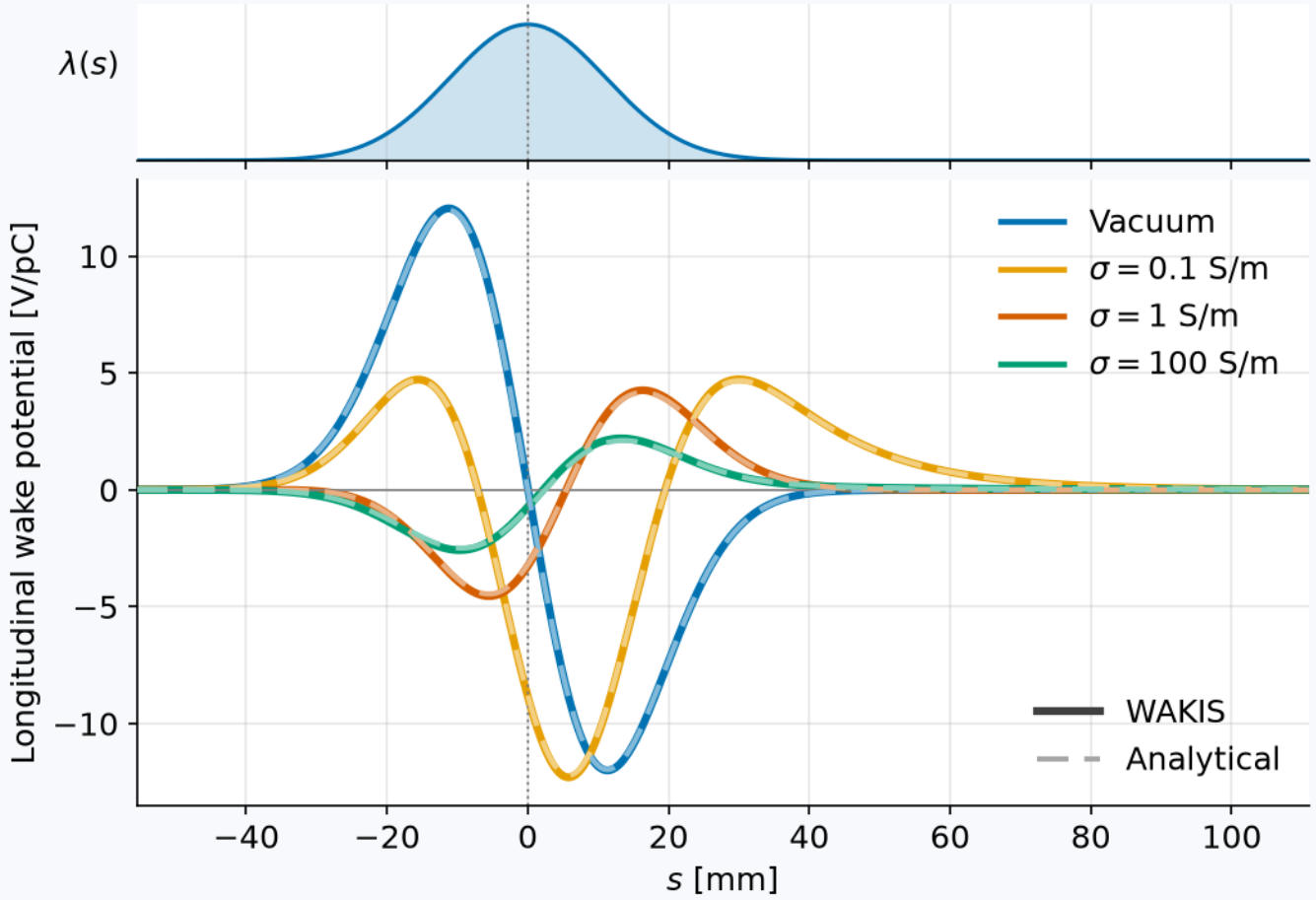
Analytical model $\longleftrightarrow$ WAKIS time-domain simulation

Resistive case Dielectric case



WAKIS benchmark: longitudinal wake potential – resistive beam region

- **Very good agreement** between WAKIS and the analytical prediction over the full wake shape
- WAKIS reproduces the **conductivity-dependent modification** of the longitudinal wake potential



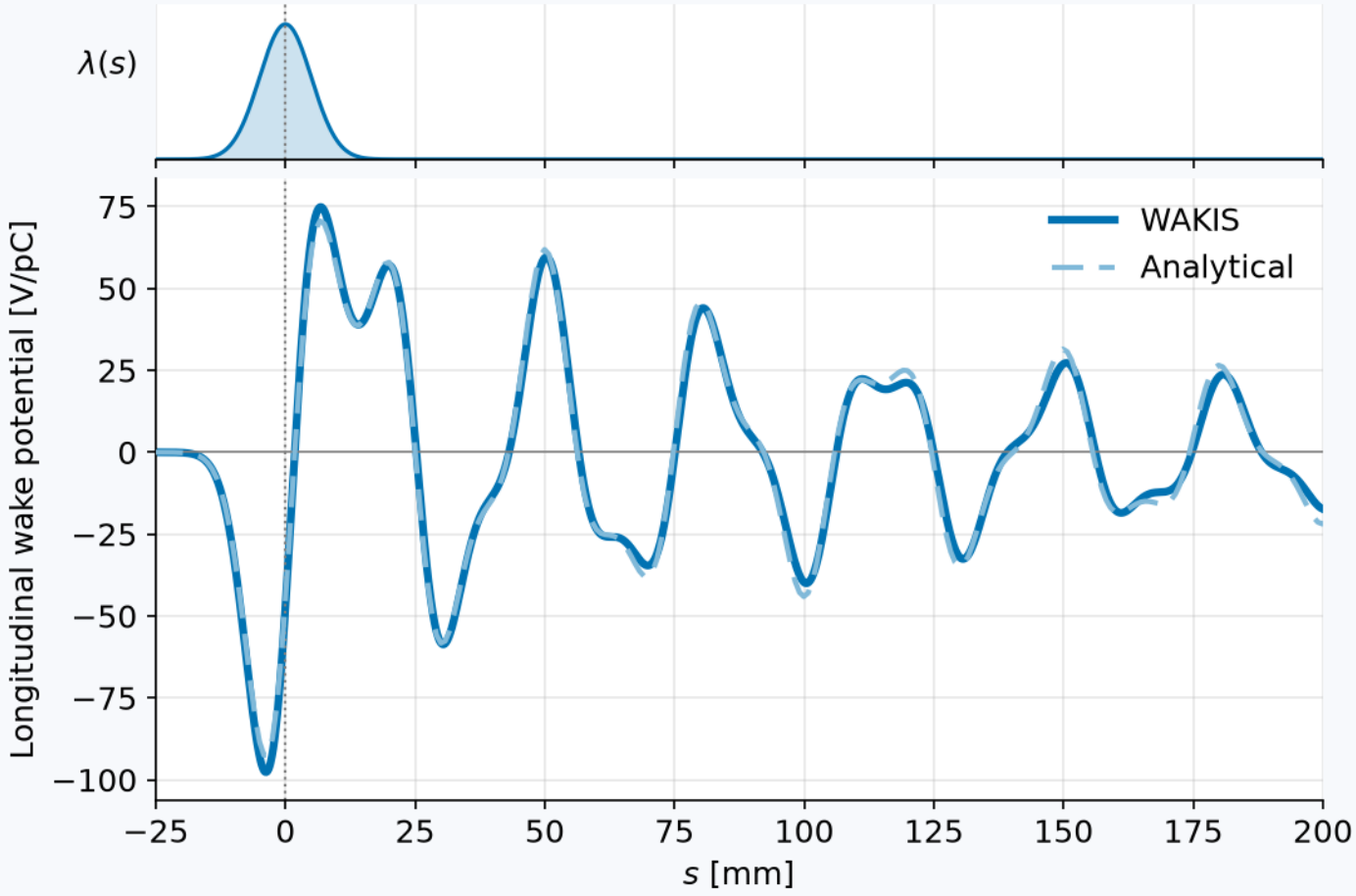
Simulation parameters

$$\beta = 0.5, \qquad \sigma_z = 11.1 \text{ mm}, \qquad b_1 = 10 \text{ mm},$$
$$L = 100 \text{ mm}, \qquad N_x = N_y = 101, \qquad N_z = 150$$

Impedance plot →

WAKIS benchmark: longitudinal wake potential – dielectric beam region

- **Very good agreement** between WAKIS and the analytical prediction in the dielectric regime
- WAKIS reproduces the **predicted resonant structure**



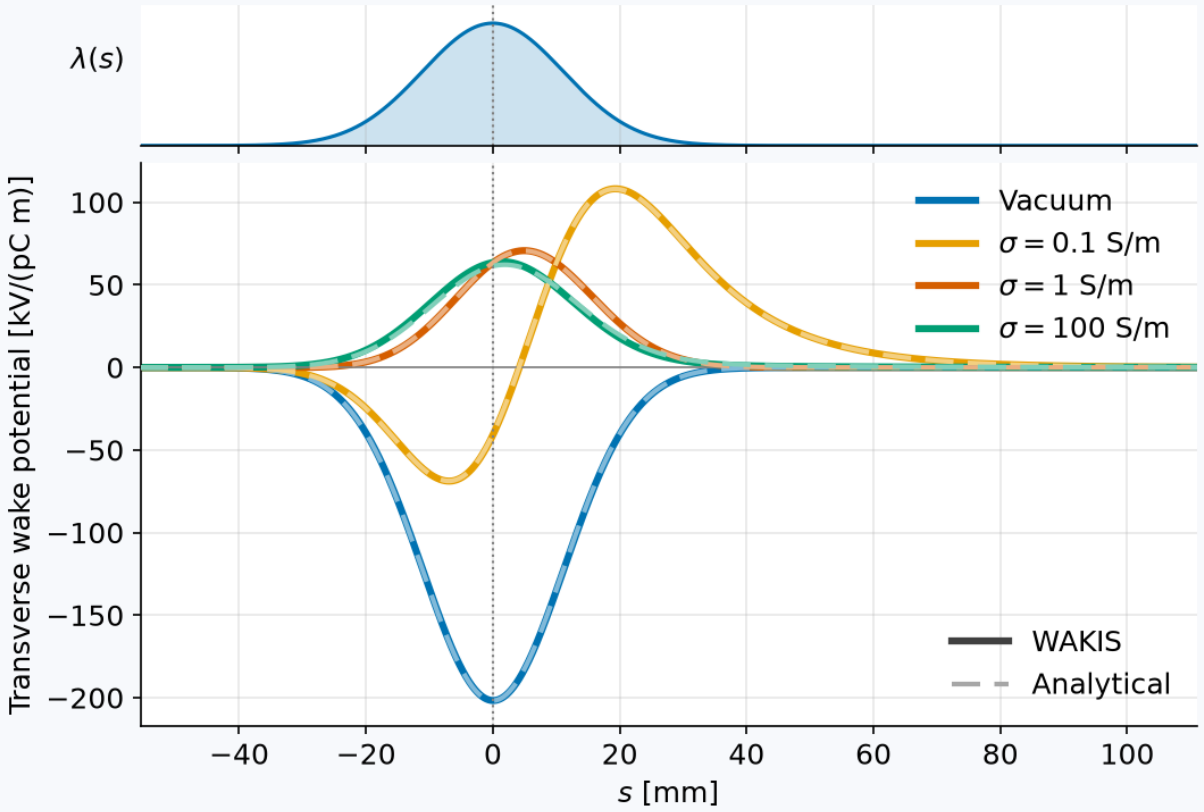
Simulation parameters

$$\beta = 0.5, \quad \varepsilon_r^{(1)} = 10, \quad \mu_r^{(1)} = 1, \quad \sigma^{(1)} = 0.1\ \mathrm{S/m},$$
$$\sigma_z = 5\ \mathrm{mm}, \quad b_1 = 10\ \mathrm{mm}, \quad L = 800\ \mathrm{mm},$$
$$N_x = N_y = 51, \quad N_z = 1200$$

Impedance plot →

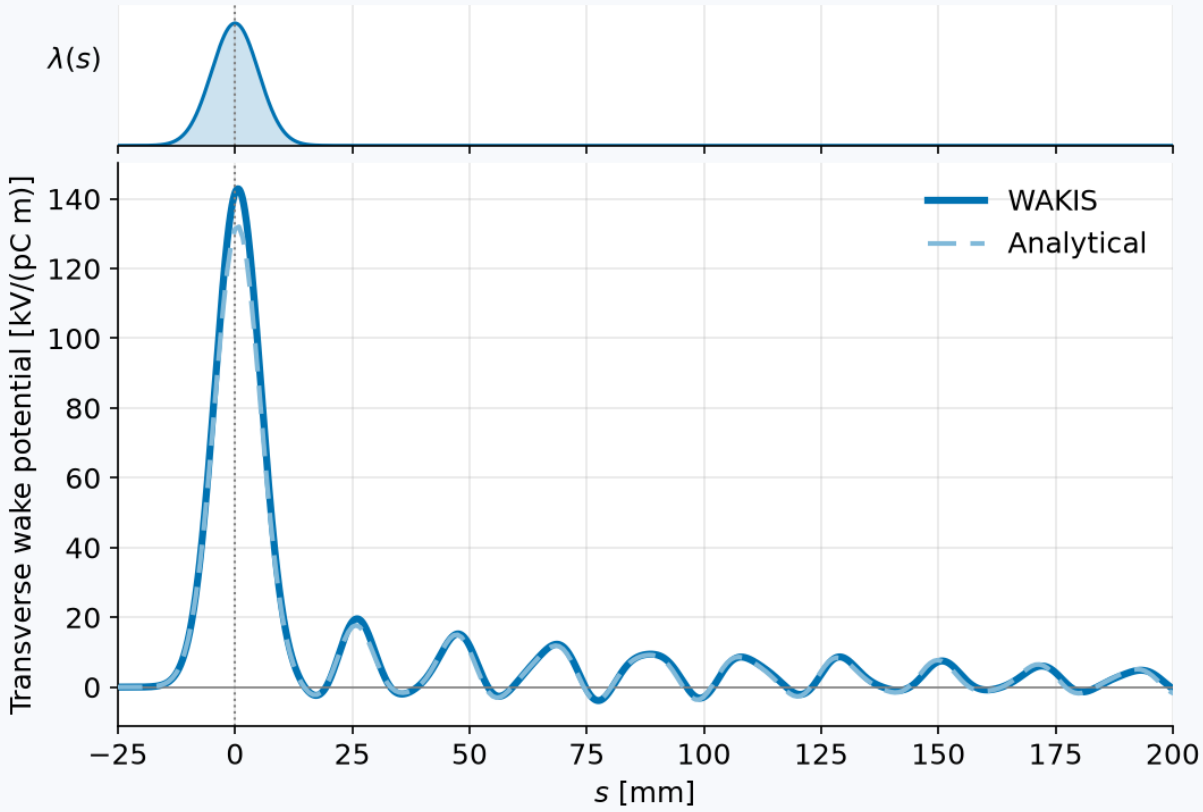
WAKIS benchmark: transverse wake potential

Resistive beam region



$\beta = 0.5$, $\sigma_z = 11.1$ mm, $b_1 = 10$ mm,
$L = 100$ mm, $\varepsilon_r = 1$, $\mu_r = 1$,
$\sigma = 0,\ 0.1,\ 1,\ 100$ S/m, $N_x = N_y = 101$, $N_z = 150$.

Dielectric beam region



$\beta = 0.5$, $\sigma_z = 5$ mm, $b_1 = 10$ mm,
$L = 800$ mm, $\varepsilon_r = 10$, $\mu_r = 1$,
$\sigma = 0.1$ S/m, $N_x = N_y = 51$, $N_z = 1200$.

Space Charge in Matter

Wake Potential and Electromagnetic benchmark

Conclusion and Outlook



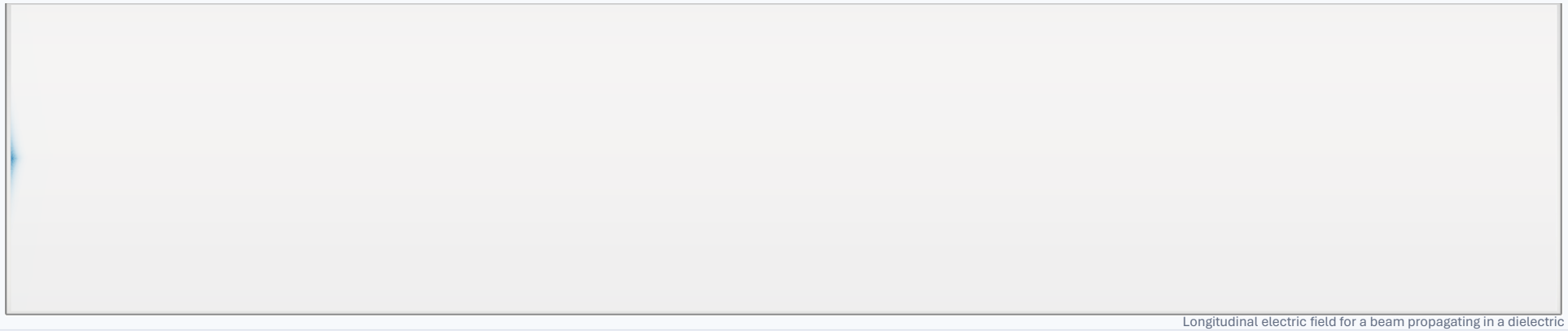
Conclusion and Outlook

Conclusions

- Space-charge impedance can change **qualitatively** when the beam region is filled with matter
- Conductive media can significantly modify the **magnitude, phase and sign of the space-charge impedance**, while dielectric media can introduce **resonant behaviour**
- Analytical predictions are reproduced with **very good agreement by WAKIS**

Outlook

- Extend the benchmark to **realistic finite-length absorbers, transitions, materials and geometries**
- Quantify the resulting effects for **representative ionization-cooling stages**
- Include the resulting impedances/wakes in **beam-dynamics simulations** to quantify their impact on the cooling channel.



Longitudinal electric field for a beam propagating in a dielectric

Thank you for your attention!



Funded by the European Union (EU). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the EU or European Research Executive Agency (REA). Neither the EU nor the REA can be held responsible for them.

References

[1] R. L. Gluckstern, Analytic Methods for Calculating Coupling Impedances, CERN-2000-011 (2000).

[2] E. Kvikne, E. Métral and E. Adli, “Impedance of multilayer cylindrical structures with a material-filled beam region,” arXiv:2606.16761 (2026).

[3] E. de la Fuente Garcia et al., WAKIS, Zenodo (2025), doi:10.5281/zenodo.15527405.

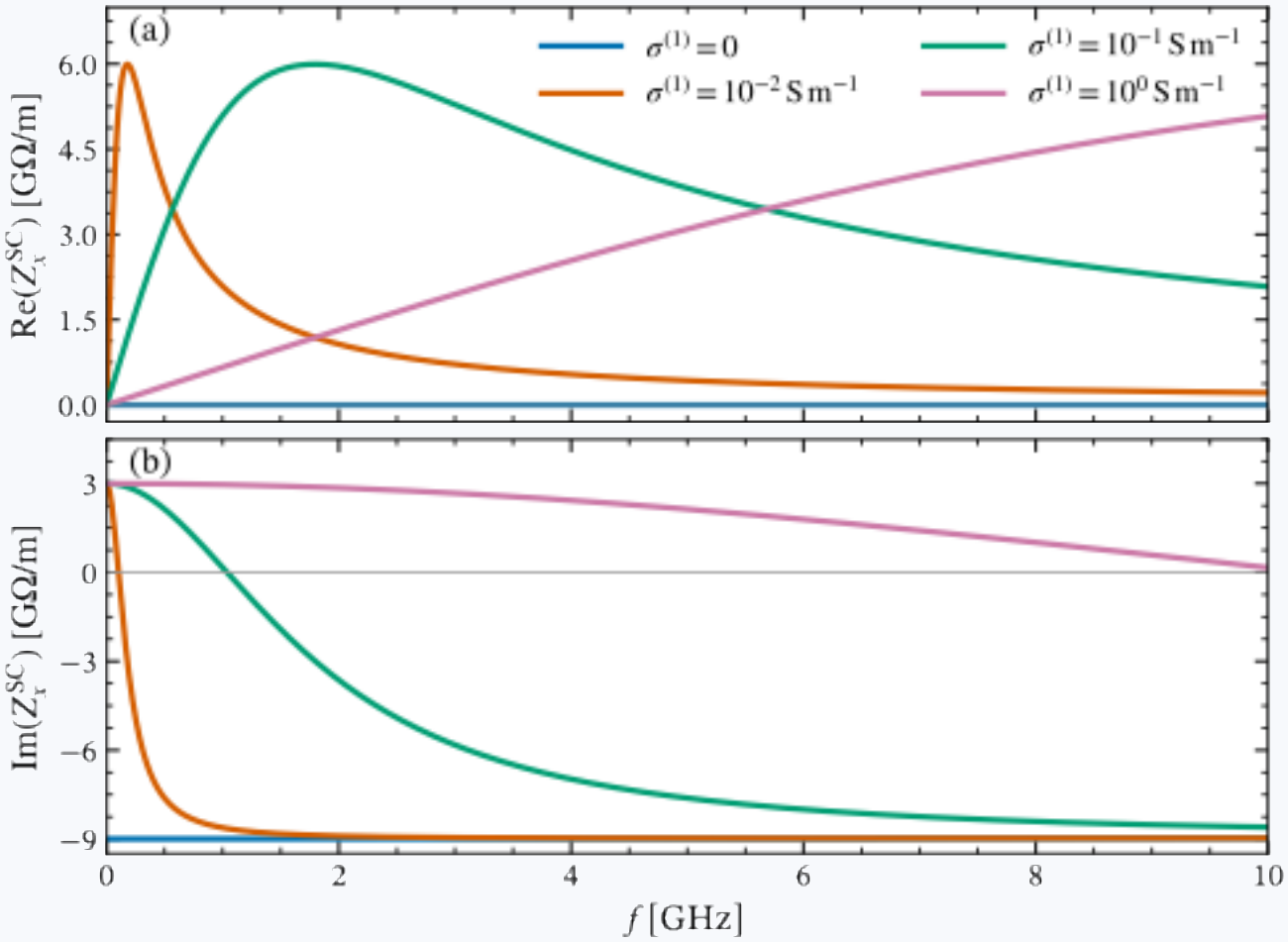
Backup



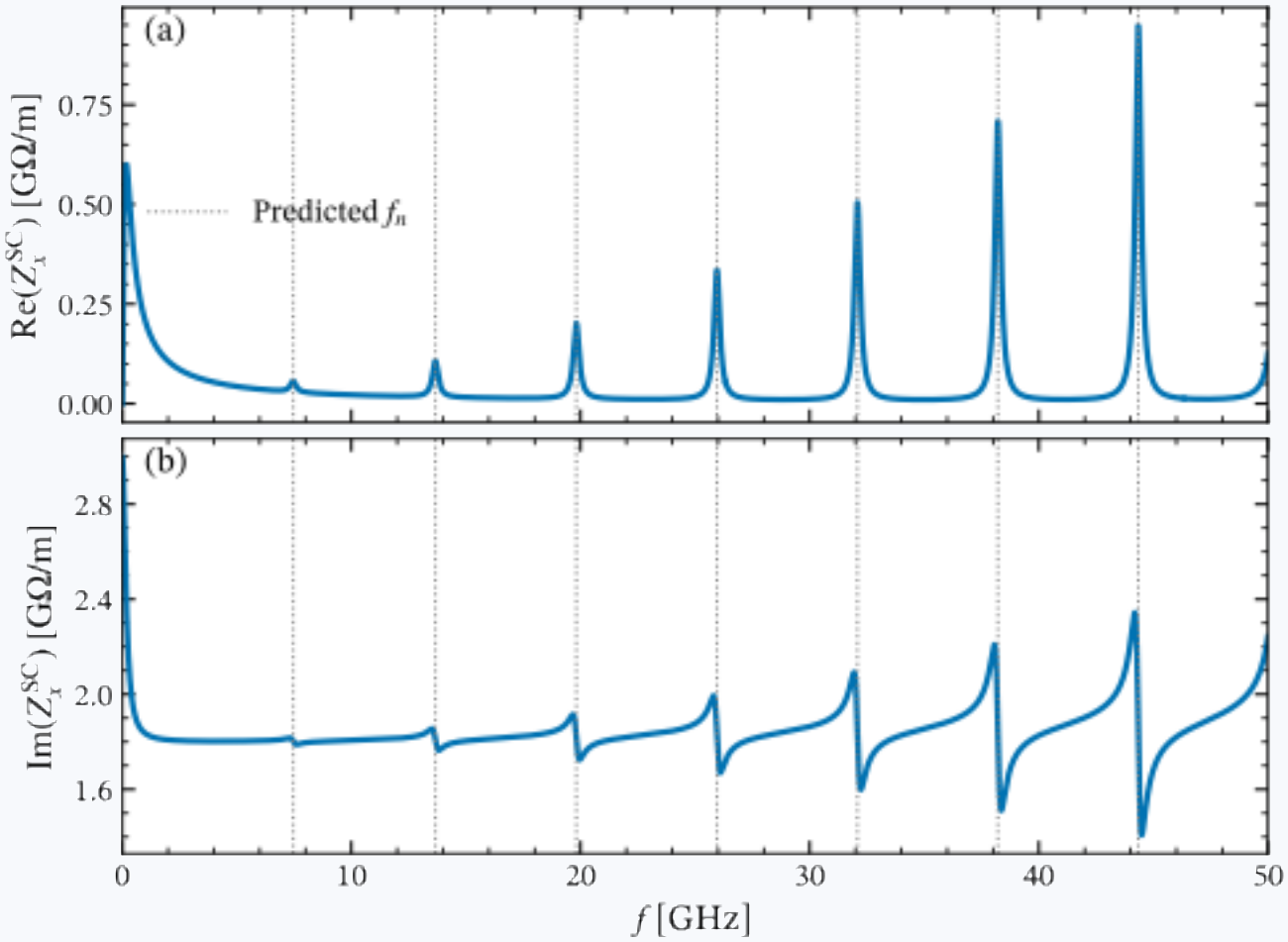
Definitions

$\beta = v/c, \quad \gamma = (1-\beta^2)^{-1/2}$	normalized beam velocity and Lorentz factor
$\omega = 2\pi f, \quad k = \dfrac{\omega}{\beta c}$	angular frequency and longitudinal wavenumber
$Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$	vacuum impedance
$\varepsilon_0,\ \mu_0$	vacuum permittivity and permeability
$j = \sqrt{-1}$	imaginary unit
$I_m,\ K_m$	modified Bessel functions of order m
J_m	Bessel function of the first kind of order m
$j_{m,n}$	the nth zero of J_m
$Z_{\parallel},\ Z_x$	longitudinal and transverse beam-coupling impedance
$\lambda(s),\ \tilde{\lambda}(f)$	longitudinal bunch distribution and its frequency spectrum

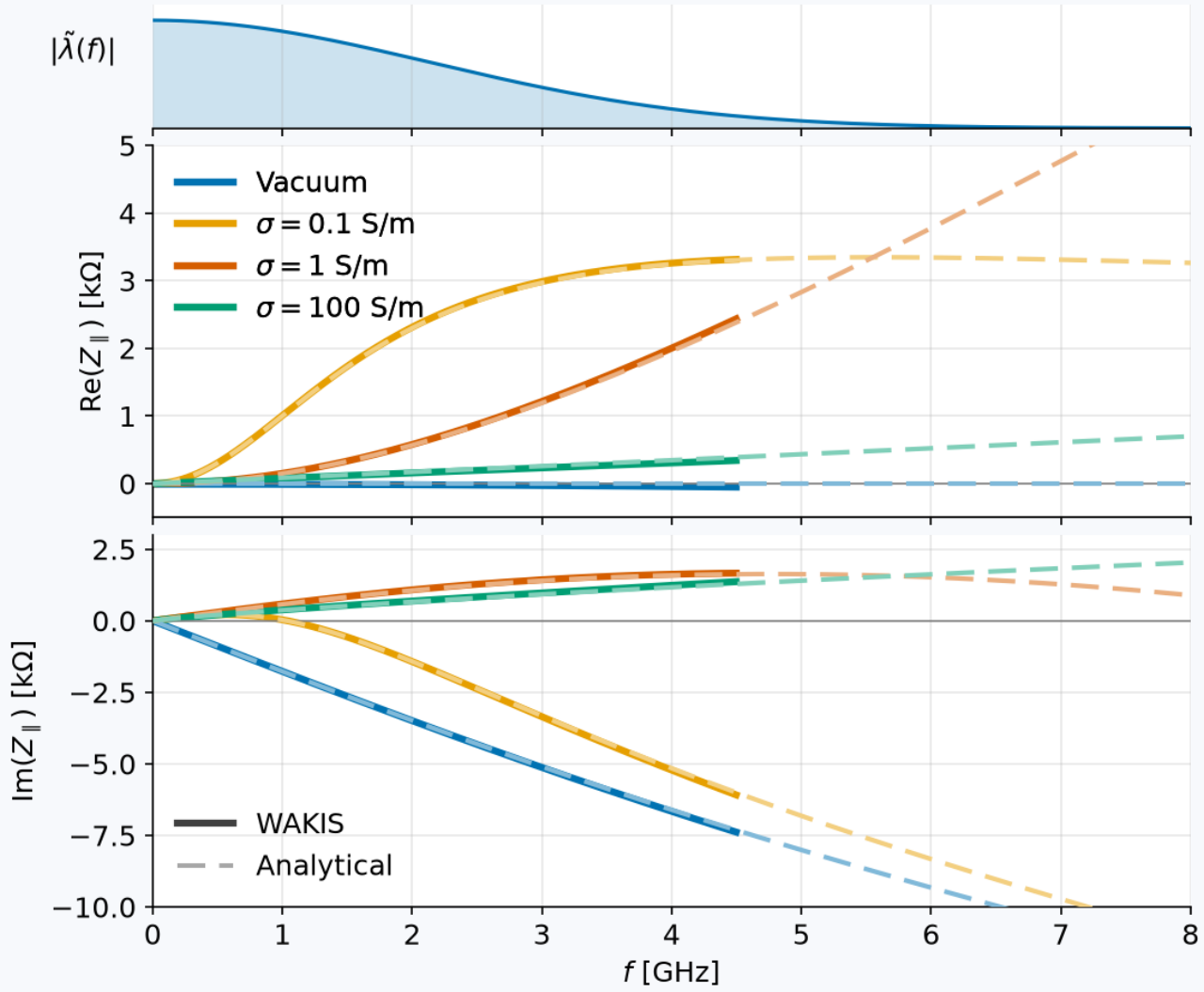
Transverse impedance: resistive beam region



Transverse impedance: dielectric beam region



WAKIS longitudinal impedance benchmark: resistive beam region



WAKIS longitudinal impedance benchmark: dielectric beam region

