Local correction of nonlinear space-charge effects by ring lattice design

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Conventional High-Intensity Ring Design

The performance of high-intensity hadron rings is limited by beam loss.

Beam loss is caused by resonance crossing due to detuning.

SC induces both detuning and resonance excitation.

Mitigation methods for the SC tune shift

- Higher injection energy
- Larger aperture
- Longitudinal bunch lengthening
- Electron-lens compensation?

The first three are well-established and effective approaches.

Conventional High-Intensity Ring Design

Now, let's design a high-intensity ring!

Mitigation methods for the SC tune shift

- Higher injection energy → as high as possible
- Larger aperture → limited by budget
- Longitudinal bunch lengthening → 2nd(3rd) harmonic RF
- Electron-lens compensation? → if you like

But how should we design the beam optics?

Key parameters in beam optics design

- **Working point (ν_x, ν_y)**
  - Away from major resonances, considering the tune spread
- **Dispersion η_x**
  - Zero in straight sections
- **Momentum compaction factor α_c**
  - Avoid transition crossing
  - Control the slip factor to mitigate beam instabilities

These are certainly important conditions.

However, these are general requirements for ring design, rather than specific measures against space charge.

Conventional High-Intensity Ring Design

What can we do if SC-induced resonances become the performance limit?

Mitigation methods for the SC tune shift

- Higher injection energy
- Larger aperture
- Longitudinal bunch lengthening
- Electron-lens compensation?

Or using octupoles? Nonlinear integrable optics?

Can the lattice itself be designed to suppress SC-induced resonances?

Hint for resonance suppression

The conventional mitigation methods for the SC tune shift mainly address the linear effect.

For nonlinear effects, such as resonances, the phase advance provides another control knob.

$$U_{m_x,m_y} = \frac{1}{(2\pi)^2}\iint \mathrm{d}\psi_x \mathrm{d}\psi_y U e^{-i[m_x\psi_x + m_y\psi_y]}$$

Tune shift:
$$\Delta\nu_u = \frac{1}{2\pi}\int_0^C \mathrm{d}s \frac{\partial U_{0,0}}{\partial J_u} \propto \frac{\lambda r_0}{\beta_{\mathrm{rel}}^2\gamma_{\mathrm{rel}}^3} \equiv K_{\mathrm{sc}}$$

Resonance potential:
$$u_{m_x,m_y,n} = \frac{1}{2\pi}\int_0^C \mathrm{d}s U_{m_x,m_y} e^{i[m_x\chi_x + m_y\chi_y - \delta\frac{2\pi n s}{C}]}$$

$$\delta = m_x\nu_x + m_y\nu_y - n$$

Hint for resonance suppression

The conventional mitigation methods for the SC tune shift mainly address the linear effect.

For nonlinear effects, such as resonances, the phase advance provides another control knob.

SC-induced nonlinear effects are difficult to control because
- they depend on beam intensity
- they act throughout the entire ring.

Nevertheless, **nonlinear SC effects are largely determined at the ring-design stage**.

So, how should we design the beam optics?

Theory of local correction

Consider a ring with N-fold symmetry. We assume

$$U(s + C/N) = U(s)\,.$$

The potential for the structure resonance becomes

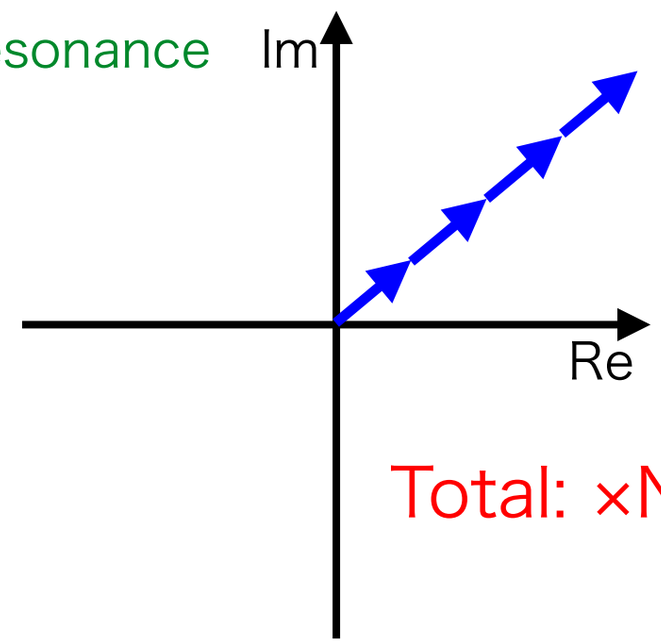
$$u_{m_x,m_y,n} = N\frac{1}{2\pi}\int_0^{C/N} \mathrm{d}s U_{m_x,m_y} e^{i[m_x\chi_x + m_y\chi_y - \delta\frac{2\pi n s}{C}]}.$$

contribution from a single superperiod

Example: $N = 4$

structure resonance

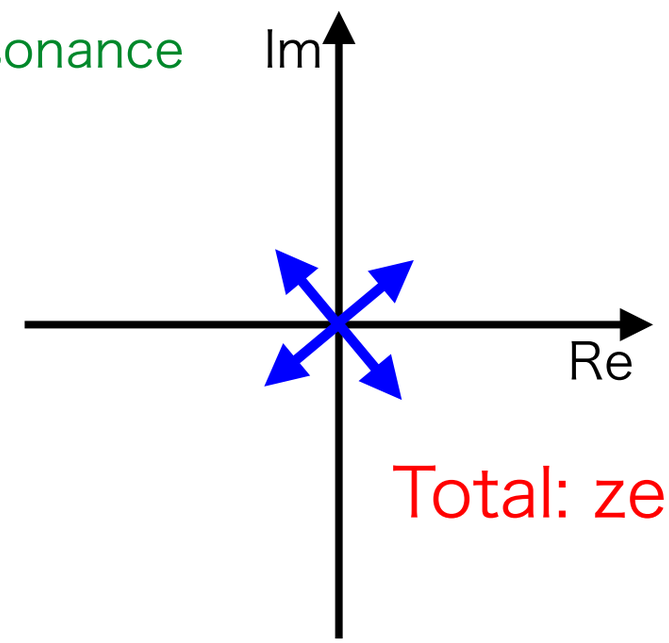
$n \bmod N = 0$



Total: ×N

nonstructure resonance

$n \bmod N \neq 0$



Total: zero

Theory of local correction

Consider a ring with N-fold symmetry. We assume

$$U(s + C/N) = U(s)\,.$$

The potential for the structure resonance becomes

$$u_{m_x,m_y,n} = N\frac{1}{2\pi}\int_0^{C/N} \mathrm{d}s U_{m_x,m_y} e^{i[m_x\chi_x + m_y\chi_y - \delta\frac{2\pi n s}{C}]}$$
$$= N[u_{m_x,m_y,n,\mathrm{arc}} + u_{m_x,m_y,n,\mathrm{straight}}]$$

Correct this　　Leave this uncorrected

Straight sections lack a periodic cell structure and are typically shorter than arc sections.

→ Even local correction can provide a significant effect.

Theory of local correction

Consider an arc section comprising M cells.
We impose a stronger periodic condition within the arc:

$$U(s + L_{\mathrm{cell}}) = U(s).$$

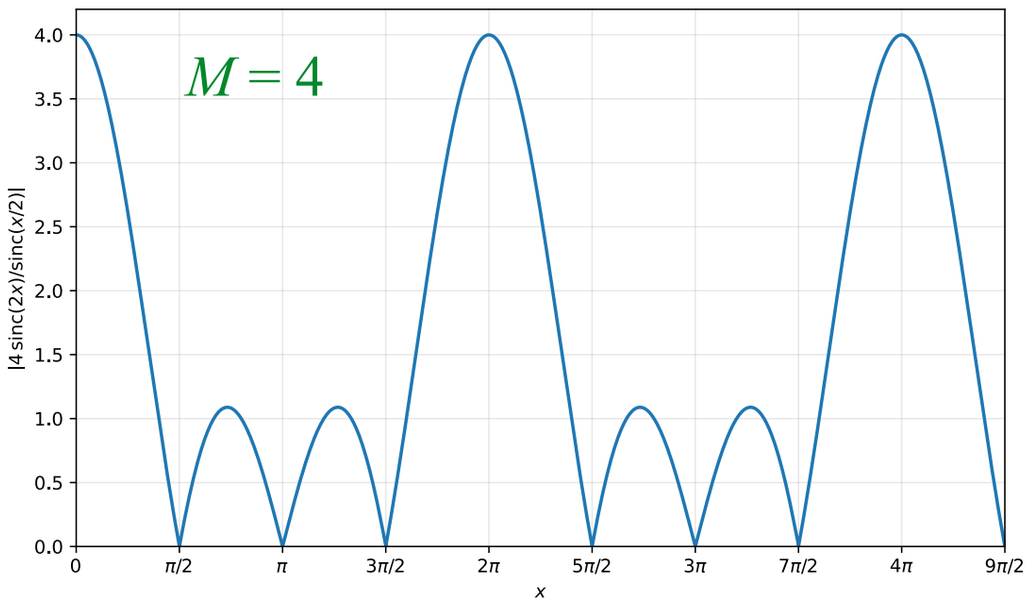
The arc-section contribution becomes

$$u_{m_x,m_y,n,\mathrm{arc}} = S_M(m_x\mu_x + m_y\mu_y - \delta\frac{2\pi L_{\mathrm{cell}}}{C})u_{m_x,m_y,n,\mathrm{cell}},$$

where $\mu_u = \chi_u(s + L_{\mathrm{cell}}) - \chi_u(s)$ is cell phase advance

$$S_M(x) = \sum_{k=0}^{M-1} e^{ikx} = \frac{\sin(Mx/2)}{\sin(x/2)} e^{i\frac{(M-1)x}{2}}$$

$|S_M(x)|$ reaches its maximum at $x = 2\pi k$
and vanishes at $x = 2\pi k / M \quad (k \bmod M \neq 0)$.



Local correction condition

At exact resonance ($\delta = 0$), the conditions for the maximum and null arc contribution $u_{mx,my,n,\boldsymbol{arc}}$ are:

Maximum condition:

$$m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$$

Null condition:

$$m_x\mu_x + m_y\mu_y = \frac{2\pi k}{M} \quad (k \bmod M \neq 0)$$

The same reasoning also applies to the non-resonant case, leading to the same maximum and null conditions.

Strengths of local correction

Correction is largely independent of the straight-section optics.

→ Flexibility in the straight sections

→ The tune can be varied by modifying only the straight sections, while preserving the local correction.

The maximum/null conditions are identical for the resonant and non-resonant cases.

→ The cell phase advance can be chosen simply to avoid low-order nonlinear effects,
without considering the distance from resonances.

Extensibility

Only constraints imposed on the potential were

$$U(s + C/N) = U(s) \ , \quad U(s + L_{\rm cell}) = U(s)$$

$\rightarrow U(s)$ need not be space-charge-driven (e.g., sextupole-driven).

Case 1: only sextupoles in arcs, no space charge

if $u_{mx,my,n,\rm straight} = 0$, the resonance can be completely corrected!

T. Yasui et al., Prog. Theor. Exp. Phys. 2022, 013G01.

Case 2: space charge

The beam sizes $\sigma_u(s)$ need to satisfy the periodicity condition.

Since $\sigma_u^2 \simeq \beta_u \varepsilon_u + \sigma_\delta^2 \eta_u^2$, the periodicity of betas is important.

When the momentum spread is large,
the dispersion must also be periodic.

Supplement:
A more rigorous discussion

The maximum/null conditions can shift because

- The beam size is determined by the envelope equation
  and may deviate from the periodicity condition.

- The cell phase advance μ is shifted by SC effects.

- Beta-beat in the actual machine

There is no need to insist on the exact null condition.
(For example, $|u_{mx,my,n,\mathrm{arc}}| < |u_{mx,my,n,\mathrm{straight}}|$ can be sufficient.)

Under the maximum condition, even if there is a slight deviation,
$|u_{mx,my,n,\mathrm{arc}}| \gg |u_{mx,my,n,\mathrm{straight}}|$, so such a design should be avoided.

Insights
from the local correction scheme

Pitfall in the simple achromat lattices

Simple achromat condition: $M\mu_x = 2\pi k \quad (k \text{ is an integer})$

Maximum condition: $m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$

The maximum condition may be inadvertently satisfied!

Excitation of Montague resonance

If $\mu_x = \mu_y$, all terms with $m_x = -m_y$ satisfy the maximum condition.

→ Montague resonance $2\nu_x - 2\nu_y = 0$ can be strongly excited!

Example in J-PARC Main Ring

Three-fold symmetry ($N = 3$), arc comprises eight cells ($\boldsymbol{M = 8}$).

Present optics for Neutrino operation (Nov. 2025-)

$(\mu_x, \mu_y) = 2\pi \times (3/4, 11/16)$

Maximum condition:

$$m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$$

Null condition:

$$m_x\mu_x + m_y\mu_y = \frac{2\pi k}{M} \quad (k \bmod M \neq 0)$$

Achromat condition:

$$M\mu_x = 2\pi k \quad (k \text{ is an integer})$$

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$(\mu_x, \mu_y) = 2\pi \times (3/4, 11/16)$

$(m_x, m_y) = (2, 0)$ → null
$(m_x, m_y) = (0, 2)$ → null
$(m_x, m_y) = (4, 0)$ → max
$(m_x, m_y) = (0, 4)$ → null
$(m_x, m_y) = (2, -2)$ → null
$(m_x, m_y) = (2, 2)$ → null

} SC

$(m_x, m_y) = (3, 0)$ → null
$(m_x, m_y) = (1, 2)$ → null
$(m_x, m_y) = (1, -2)$ → null

} Sext.

Maximum condition:

$$m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$$

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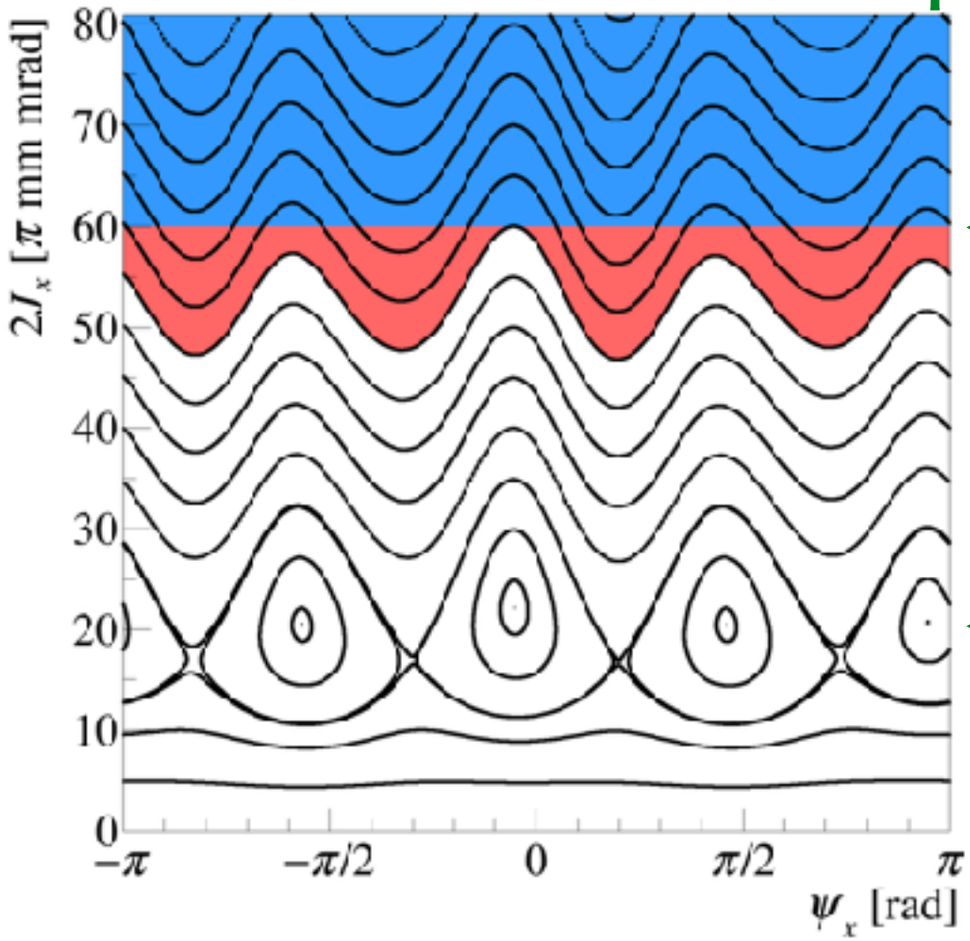
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Horizontal Poincaré map



← collimator

← resonance
$4\nu_x = 84$

$(\nu_x, \nu_y) = (21.35, 21.39)$, $J_y = 0$, $\lambda = 1.1\times10^{12}$ protons/m,
2D Gauss (Basetti-Erskine), (σ_x, σ_y) ⋯ from envelope eq.

Suppression of 4th order

To suppress 4th-order nonlinear effects, μ_x must be changed.
At the same time, we want to preserve the achromat condition.

→ Let's consider $(\mu_x, \mu_y) = 2\pi \times (5/8, 9/16)$

Maximum condition:

$$m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$$

Null condition:

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Achromat condition:

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$M = 8$ in J-PARC MR

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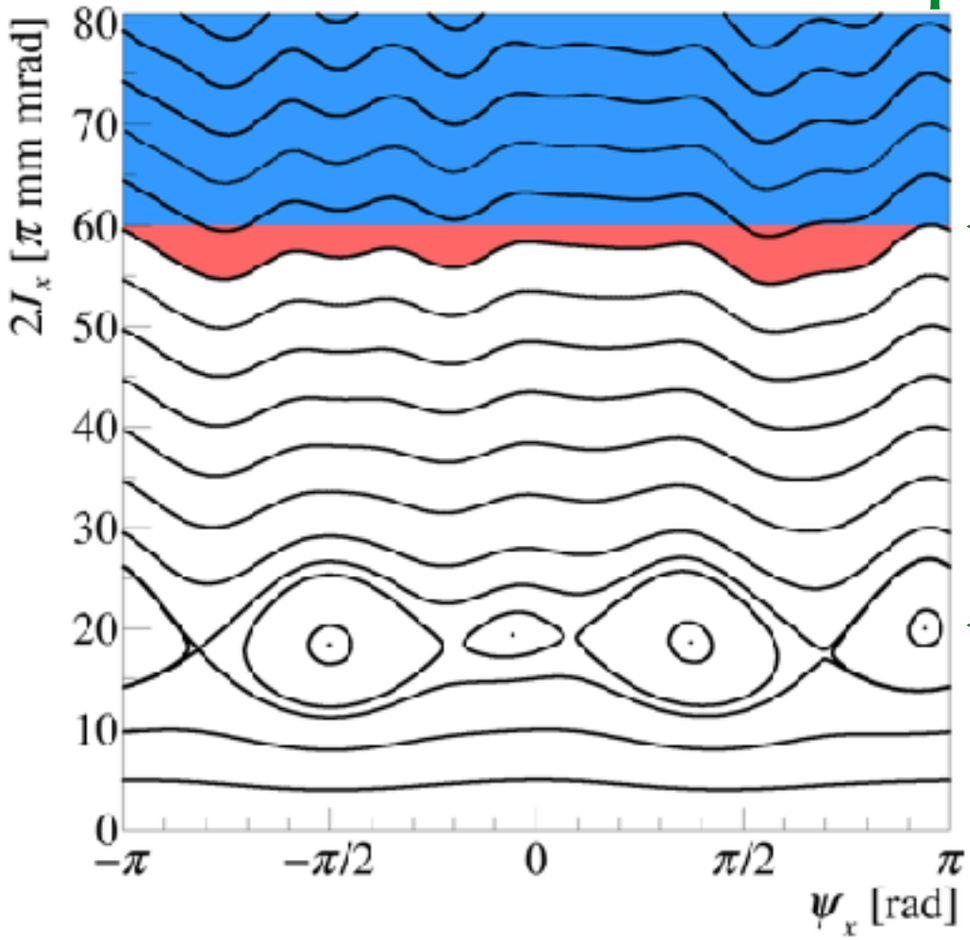
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Horizontal Poincaré map

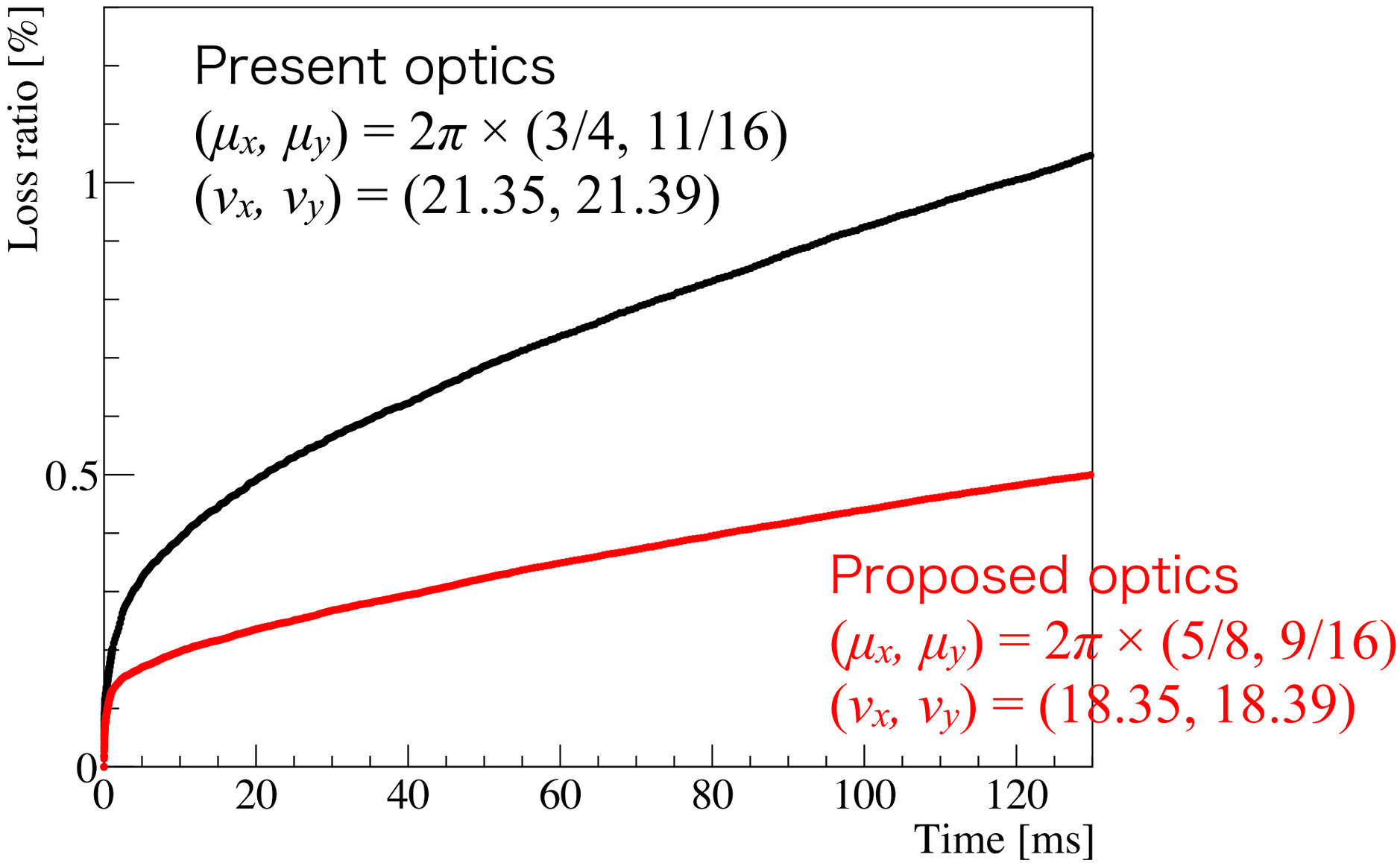


← collimator

← resonance
$4\nu_x = 72$

$(\nu_x, \nu_y) = (18.35, 18.39)$, $J_y = 0$, $\lambda = 1.1\times10^{12}$ protons/m,
2D Gauss (Basetti-Erskine), (σ_x, σ_y) ⋯ from envelope eq.

Beam loss simulation



4.1×10^{13} protons, $(\varepsilon_x, \varepsilon_y) = (4.1\pi, 4.3\pi)$ mm mrad, macro pnum = 2^{20} (particle in cell),
Transv. distr. ⋯ Gaussian, Init. cut ⋯ $(2J_x, 2J_y) = (59\pi, 59\pi)$ mm mrad,
Ring collimator ⋯ $(2J_x, 2J_y) = (60\pi, 60\pi)$ mm mrad, longitudinal matched,
Chromaticity ⋯ $(\xi_x, \xi_y) = (-7, -7)$, no magnetic imperfections.

Summary

We propose a local correction scheme for nonlinear SC effects in the arc section.

Maximum condition:

$$m_x\mu_x + m_y\mu_y = 2\pi k \quad (k \text{ is an integer})$$

Null condition:

$$m_x\mu_x + m_y\mu_y = \frac{2\pi k}{M} \quad (k \bmod M \neq 0)$$

Care is needed when designing a simple achromat lattice:

$$M\mu_x = 2\pi k \quad (k \text{ is an integer})$$

Using the local correction scheme, we suppress the 4th-order SC effect in the J-PARC MR.