



Circumventing Space Charge: Toward Transport-Free Tomography of High-Dimensional Phase Space in High-Intensity Linacs

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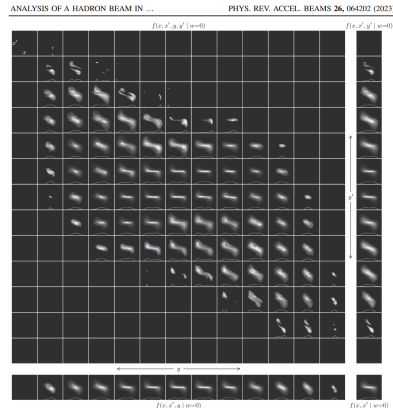


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High-Dimensional Phase Space Tomography

- ▶ Obtaining the complete high-dimensional phase-space beam distribution is key to precise beam manipulation.
- ▶ Due to limitations in measurement techniques and experimental conditions, measurements are often restricted to low-dimensional projections.
- ▶ **Tomography**: reconstructing high-dimensional distribution from low-dimensionsal projections



Dependence of the $x - x'$ distribution on y (columns) and y' (rows) near $w = 0^1$.

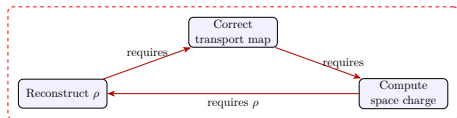
¹A. Hoover, K. Ruisard et al., Analysis of a hadron beam in five-dimensional phase space, Phys. Rev. Accel. Beams, doi: 10.1103/PhysRevAccelBeams.26.064202

Multi-Optics vs. Single-Optics Tomography

Multi-Optics Tomography



- ▶ Reconstructing ρ requires the **correct transport map**, which requires **space charge**, which depends on ρ itself.
- ▶ This circular dependency creates a **self-consistency problem for multi-optics tomography with space charge**.



Circular dependency: not self-consistent

Single-Optics Tomography



Reconstruction point = measurement point

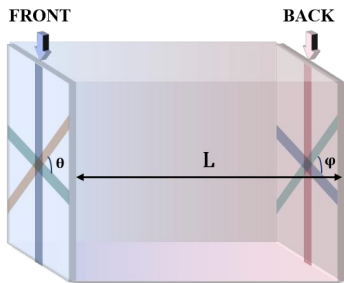
- ▶ **Transport-free**: the measurement point **coincides** with the reconstruction point.
- ▶ This **circumvents the self-consistency problem caused by space charge**.
- ▶ **Examples:**
 - ▶ SOFT: double-rotated slits
 - ▶ VPP: horizontal and vertical slits + screen
 - ▶ ...



How to obtain sufficient information for single-optics tomography?

Measurement Principle of SOFT

SOFT: **S**ingle-**O**ptics **F**our-dimensional phase space **T**omography



*SOFT conducts 2D phase space scans with **two independently rotatable slits** to extract otherwise inaccessible cross-plane information.*

- ▶ **Measured point**: $\mathbf{x} = (x, x', y, y')^T$
- ▶ **Before rotation** (front/back): $\mathbf{x}_1 = (x_f, x_b, y_f, y_b)^T = M_1 \mathbf{x}$
- ▶ **After rotation** (front/back): $\mathbf{x}_2 = (u_f, u_b, v_f, v_b)^T = M_2 \mathbf{x}_1$

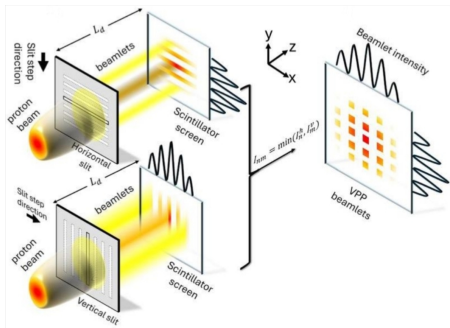
$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & L \end{pmatrix} \quad M_2 = \begin{pmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & \cos \varphi & 0 & -\sin \varphi \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & \sin \varphi & 0 & \cos \varphi \end{pmatrix}$$

- ▶ **3 angles each at front/back** (9 pairs), sufficient for high-dimensional reconstruction.
- ▶ Each pair of rotation angles yields a 2D profile $\rho(u_f, u_b)$

How to choose the optimal angle combination to achieve the best reconstructed distribution?

Measurement Principle of VPP | Collaboration with Prof. Moses Chung's group from POSTECH

VPP: Virtual Pepper-Pot



The VPP setup can be regarded as a 3D measurement using *slits and a screen*².

- ▶ Real input distribution profile measurement:
 - ▶ Horizontal slit scan: $\rho_h(x_f, y_i, y_f)$
 - ▶ Vertical slit scan: $\rho_v(x_i, x_f, y_f)$
- ▶ Reconstruction:

$$\rho_h(x_f, y_i, y_f) + \rho_v(x_i, x_f, y_f) \rightarrow \rho(x_i, y_i, x_f, y_f)$$
- ▶ Under fixed H/V scanning setups, different input distributions yield different ρ_h and ρ_v .
- ▶ Information complementarity between ρ_h and ρ_v directly determines ρ 's reconstruction quality.

What input distribution gives the best reconstruction quality?

²Cosgun, E., Moon, S., Kim, D. et al., Characterization and benchmarking of complete 4D transverse phase space for high-intensity ion beams using novel virtual pepper-pot technique. Sci Rep 15.

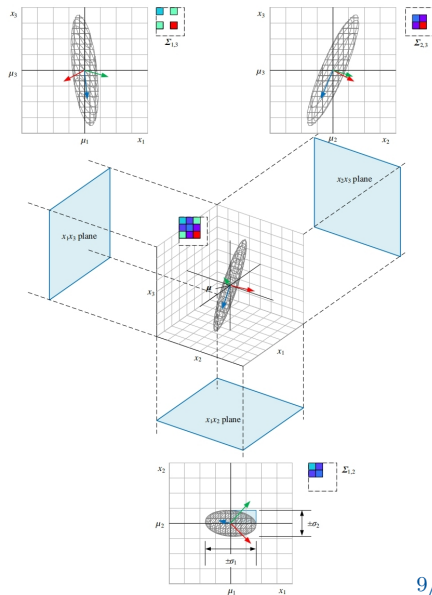


How to select the optimal set of low-dimensional profiles to minimize reconstruction errors?

Geometric Tool for Projections

- ▶ Geometrically, each low-dimensional projection profile corresponds to a projection subspace in the high-dimensional phase space.
- ▶ The similarity between two profiles can be reduced to the similarity between the projection subspaces.
 - ▶ Similar subspaces $\rightarrow$ redundant info $\rightarrow$ worse reconstruction.
 - ▶ Orthogonal subspaces $\rightarrow$ complementary info $\rightarrow$ better reconstruction.

How is the subspace similarity calculated?



Geometric Tool for Projections

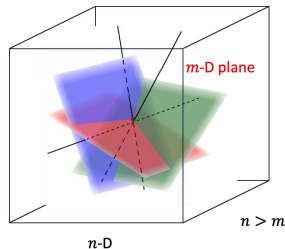
Two definitions of geometric similarity between two projection profiles: ($N = n_p + n_n$):

- Based on an orthonormal basis of the **normal-vector subspace**:

$$\nu_n = \frac{1}{n_n} \sum_{i=1}^{n_n} \sum_{j=1}^{n_n} (\vec{e}_i \cdot \vec{f}_j)^2$$

- Based on an orthonormal basis of the **projection subspace**:

$$\nu_p = \frac{1}{n_p} \sum_{i=n_{n+1}}^N \sum_{j=n_{n+1}}^N (\vec{e}_i \cdot \vec{f}_j)^2$$



- ν_n and ν_p are complementary:

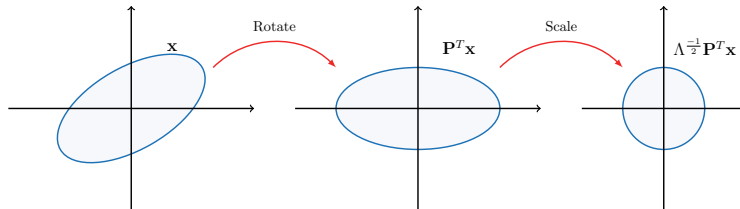
$$\nu_n(1 - \nu_n) = \nu_p(1 - \nu_p)$$

- $\nu_p, \nu_n \in [0, 1]$; for any $\nu \in \{\nu_p, \nu_n\}$:
 - $\nu = 0$: orthogonal $\rightarrow$ Maximum info
 - $\nu = 1$: coincident $\rightarrow$ Minimum info

Spherical Coordinate Transformation

- ▶ Arbitrary beam distributions are anisotropic, so ν depends on the distribution geometry.
- ▶ Transforming the distribution into a spherical one makes all directions equivalent, which is necessary for the geometric tool to fully play its role.

$$\begin{aligned}
 1 &= \mathbf{x}^T \Sigma^{-1} \mathbf{x} = \mathbf{x}^T \mathbf{P} \Lambda^{-1} \mathbf{P}^T \mathbf{x} = \left[\Lambda^{-\frac{1}{2}} \mathbf{P}^T \mathbf{x} \right]^T \Lambda^{-\frac{1}{2}} \mathbf{P}^T \mathbf{x} \\
 &= \mathbf{z}^T \mathbf{z} \Rightarrow \boxed{\mathbf{z} = \Lambda^{-\frac{1}{2}} \mathbf{P}^T \mathbf{x}}
 \end{aligned}$$



SOFT: Optimal Angle Combination Strategy

Fixed Input Distribution: How to Choose the Scanning Angles?

- ▶ **Range:** $\theta, \phi \in [0^\circ, 180^\circ]$; Step: 1°
- ▶ **Brute-force search:** huge computational cost:

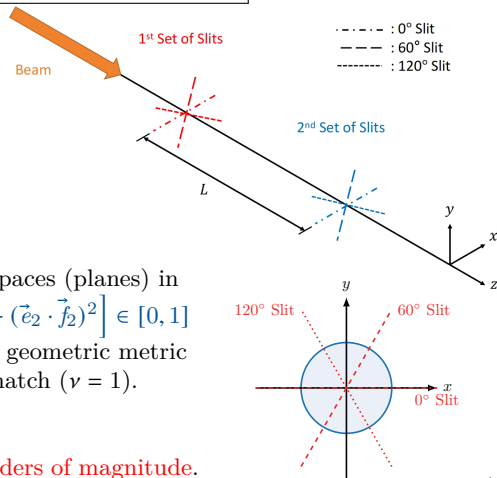
$$C_{180^2}^9 \approx 1.08 \times 10^{35}$$

- ▶ **Geometric optimization:**

- ▶ Coordinate transform: ellipsoid $\rightarrow$ sphere
- ▶ Ideal reference set: $\theta, \phi = 0^\circ, 60^\circ, 120^\circ$ (9 pairs)
- ▶ Similarity metric between two 2D projection subspaces (planes) in 4D space: $\nu = \frac{1}{2} \cdot \left[(\vec{e}_1 \cdot \vec{f}_1)^2 + (\vec{e}_1 \cdot \vec{f}_2)^2 + (\vec{e}_2 \cdot \vec{f}_1)^2 + (\vec{e}_2 \cdot \vec{f}_2)^2 \right] \in [0, 1]$
- ▶ Traverse each candidate (180×180), compute the geometric metric ν with the 9 reference pairs, and select the best match ($\nu = 1$).

$$9 \times 180^2 \approx 2.9 \times 10^5$$

- ▶ The computational cost is **reduced by about 30 orders of magnitude**.

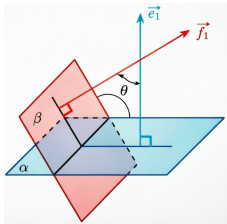


VPP: Choosing the Optimal Input Distribution

Fixed Device: How to Choose the Input Distribution?

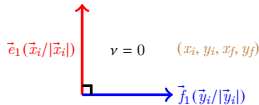
- Similarity metric between two 3D profiles in 4D space:

$$\nu_n = (\vec{e}_1 \cdot \vec{f}_1)^2$$



Similar to the calculation of dihedral angles in 3D space.

- Fixed scanning device in (x_i, x_f, y_i, y_f) , the normal angle between ρ_h and ρ_v is always 90° for all input distributions.



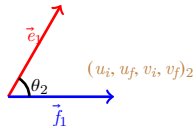
The information content is the same for all different input distributions, which is clearly unreasonable.

- Coordinate transform: ellipsoid $\rightarrow$ sphere
 - The normal angle varies with the input distribution.
 - In the figure below, the input distribution of Case 2 is better than that of Case 1.

Case 1



Case 2





How to reconstruct the high-dimensional distribution from sufficiently good low-dimensional profiles?

Maximum Entropy Tomography (MENT)

Core idea: Reconstruct a high-dimensional distribution from limited low-dimensional projections. Among all solutions satisfying the measurement constraints, **choose the one with maximum information entropy**.

- ▶ 4D distribution $\rho(x, x', y, y')$, information entropy:

$$H[\rho] = - \iiint \rho \ln \rho \, dx \, dx' \, dy \, dy'$$

- ▶ The j -th 2D projection constraint (n total):

$$G_j[\rho] = g_j(u_j, u'_j) - \iint \rho \, dv_j \, dv'_j = 0$$

- ▶ Lagrange multiplier method:

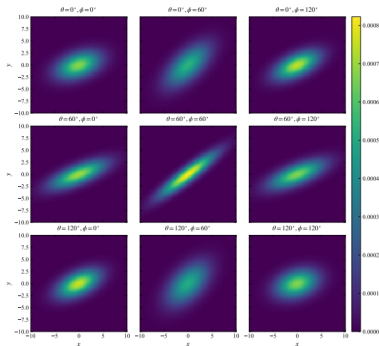
$$K[\rho] = H[\rho] + \sum_{j=1}^n \iint \lambda_j G_j \, du_j \, du'_j$$

- ▶ Vanishing variation $\Rightarrow$ theoretical MENT solution:

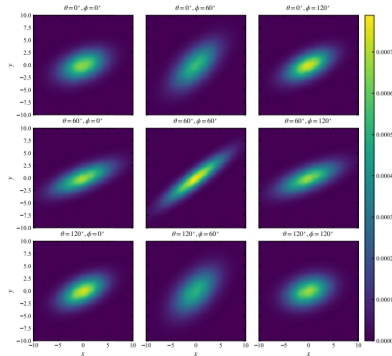
$$\rho = C \prod_{j=1}^n h_j(u_j, u'_j)$$

Reconstruction Feasibility Verification

- ▶ We used the MENT method to reconstruct from the 9 2D projections measured by SOFT.



True distribution: 9 2D projections



Reconstructed distribution: 9 2D projections

- ▶ Metric tool: $D_{TV} = \frac{1}{2} \int |\rho_{true}(\mathbf{x}) - \rho_{reconstructed}(\mathbf{x})| d\mathbf{x} = 0.063 \rightarrow$ The method is feasible.

How to quantify differences between high-dimensional phase space distributions?

- ▶ Assess tomography fidelity
- ▶ Facilitate beam manipulation

f-Divergences

f-Divergences: common methods for measuring distribution differences and capturing comprehensive difference information. Defined as follows:

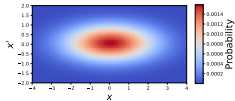
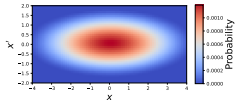
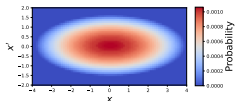
$$D_f[\rho_1(\mathbf{x}) \parallel \rho_2(\mathbf{x})] = \int \rho_2(\mathbf{x}) f\left[\frac{\rho_1(\mathbf{x})}{\rho_2(\mathbf{x})}\right] d\mathbf{x},$$

- ▶ $f(\cdot)$ is a **convex function** and satisfies $f(1) = 0$;
- ▶ Different $f(\cdot)$ correspond to different statistical divergences:

Name	$f(t)$	h	Range
Kullback-Leibler	$t \ln t$	$\rho_1(\mathbf{x}) \ln \left[\frac{\rho_1(\mathbf{x})}{\rho_2(\mathbf{x})} \right]$	$[0, \infty)$
Jensen-Shannon	$\frac{1}{2} \left[\ln \left(\frac{2}{t+1} \right)^{t+1} + t \ln t \right]$	$\frac{1}{2} \left\{ \rho_2(\mathbf{x}) \ln \left[\frac{2\rho_2(\mathbf{x})}{\rho_1(\mathbf{x}) + \rho_2(\mathbf{x})} \right] + \rho_1(\mathbf{x}) \ln \left[\frac{2\rho_1(\mathbf{x})}{\rho_1(\mathbf{x}) + \rho_2(\mathbf{x})} \right] \right\}$	$[0, \ln 2]$
Total Variation	$\frac{1}{2} t - 1 $	$\frac{1}{2} \rho_1(\mathbf{x}) - \rho_2(\mathbf{x}) $	$[0, 1]$
Squared Hellinger	$\frac{1}{2} (\sqrt{t} - 1)^2$	$\frac{1}{2} \left[\sqrt{\rho_1(\mathbf{x})} - \sqrt{\rho_2(\mathbf{x})} \right]^2$	$[0, 1]$

Four forms of f -divergences; $t = \rho_1(\mathbf{x})/\rho_2(\mathbf{x})$; $h = \rho_2(\mathbf{x}) \cdot f[\rho_1(\mathbf{x})/\rho_2(\mathbf{x})]$; $\mathbf{x} \in \mathbb{R}^{2n}$

Several Common 4D Beam Distributions

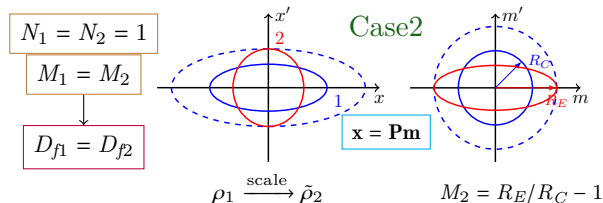
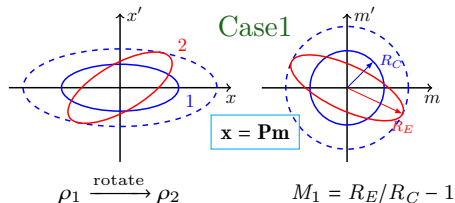
Distribution with Elliptical Symmetry	Definition $\rho(x, x', y, y') = \rho(I)$ $I = \mathbf{x}^T \Sigma^{-1} \mathbf{x}$	Schematic Diagram of 2D Projection (x, x')
Gaussian	$\frac{1}{(\sqrt{2\pi})^4 \Sigma ^{\frac{1}{2}}} \cdot e^{-\frac{1}{2}I}$	
Parabolic	$\frac{6}{(\sqrt{8\pi})^4 \Sigma ^{\frac{1}{2}}} \cdot \left(1 - \frac{I}{8}\right), \quad I < 8$	
Waterbag	$\frac{2}{(\sqrt{6\pi})^4 \Sigma ^{\frac{1}{2}}}, \quad I < 6$	

Quadratic form expression of beam distributions with elliptical symmetry; $\mathbf{x} = (x, x', y, y')^T$;
 Σ is a covariance matrix composed of 10 independent second-order moments.

f-Divergences in Relation to Mismatch Factors

Theorem

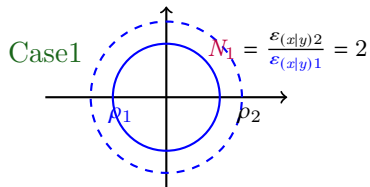
In the $(x_1, x'_1, x_2, x'_2, \dots, x_i, x'_i, \dots, x_n, x'_n)$ phase space $(2n-D)$, given two *elliptically symmetric* and *uncoupled* beam distributions in the $\{x_i | i = 1, 2, \dots, n\}$ motion directions, if their RMS emittances $\{\varepsilon_i | i = 1, 2, \dots, n\}$ in the 2D subspaces $\{(x_i, x'_i) | i = 1, 2, \dots, n\}$ are identical, then the *f-divergence* between them *depends solely on the mismatch factors* $\{M_i | i = 1, 2, \dots, n\}$ in these 2D subspaces.



f-Divergences in Relation to RMS Emittance Scaling Factors

Theorem

In the $(x_1, x'_1, x_2, x'_2, \dots, x_i, x'_i, \dots, x_n, x'_n)$ phase space $(2n-D)$, given two *elliptically symmetric* and *uncoupled* beam distributions in the $\{x_i | i = 1, 2, \dots, n\}$ motion directions, if the mismatch factors $\{M_i | i = 1, 2, \dots, n\}$ in the 2D subspaces represented by $\{(x_i, x'_i) | i = 1, 2, \dots, n\}$ are all zero, then the *f-divergence* between them *depends solely on the RMS emittance scaling factors* $\{N_i | i = 1, 2, \dots, n\}$ in these 2D subspaces.

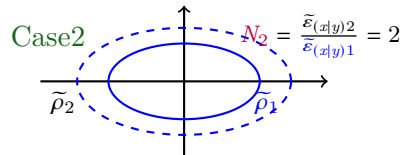


$$M_1 = M_2 = 0$$

$$N_1 = N_2$$

$$\downarrow$$

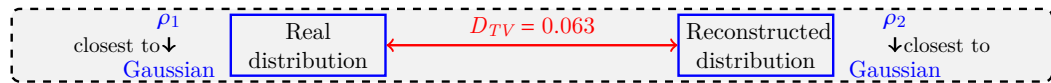
$$D_{f1} = D_{f2}$$



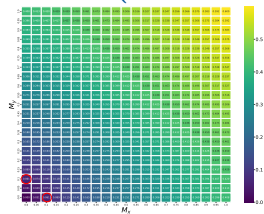
Assessment Standards for f-Divergences

- ▶ We used the previous correspondences and three ideal distributions to establish assessment standards for the f-divergences.
- ▶ This provides a **calibration** for the f-divergence values.

For example: Beam 4D transverse phase space tomography result analysis



Case 1 (Gaussian-Gaussian):

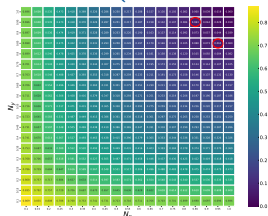


$$N_x = N_y = 1$$

$$M_x = 0 \text{ (or } 0.1)$$

$$M_y = 0.1 \text{ (or } 0)$$

Case 2 (Gaussian-Gaussian):



$$M_x = M_y = 0$$

$$N_x = 0.95 \text{ (or } 0.85)$$

$$N_y = 0.85 \text{ (or } 0.95)$$

This indicates that SOFT enables high-fidelity 4D phase-space reconstruction.



Conclusions

- ▶ Single-optics measurement of low-dimensional profiles is transport-free, thereby circumventing space charge.
- ▶ A geometric metric was proposed to quantify projection subspace similarity and was applied to both SOFT and VPP.
- ▶ The MENT method was shown to effectively reconstruct high-dimensional distributions.
- ▶ f-Divergences were introduced to quantify differences between high-dimensional distributions; their relations to mismatch and emittance scaling factors provided a calibration reference for f-divergence values.



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Thank you for your attention!