Unitarity bounds for form factors in B-meson decays

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BFA 7th, IGFAE, Santiago de Compostela

16 April 2026

Based on arXiv: 2412.04388, 2xxx.xxxxx

UNIVERSITÄT BONN

Emmy Noether-Programm

DFG Deutsche Forschungsgemeinschaft

What's in this talk?

Perform **indirect searches** for NP using **flavour physics**

⇓

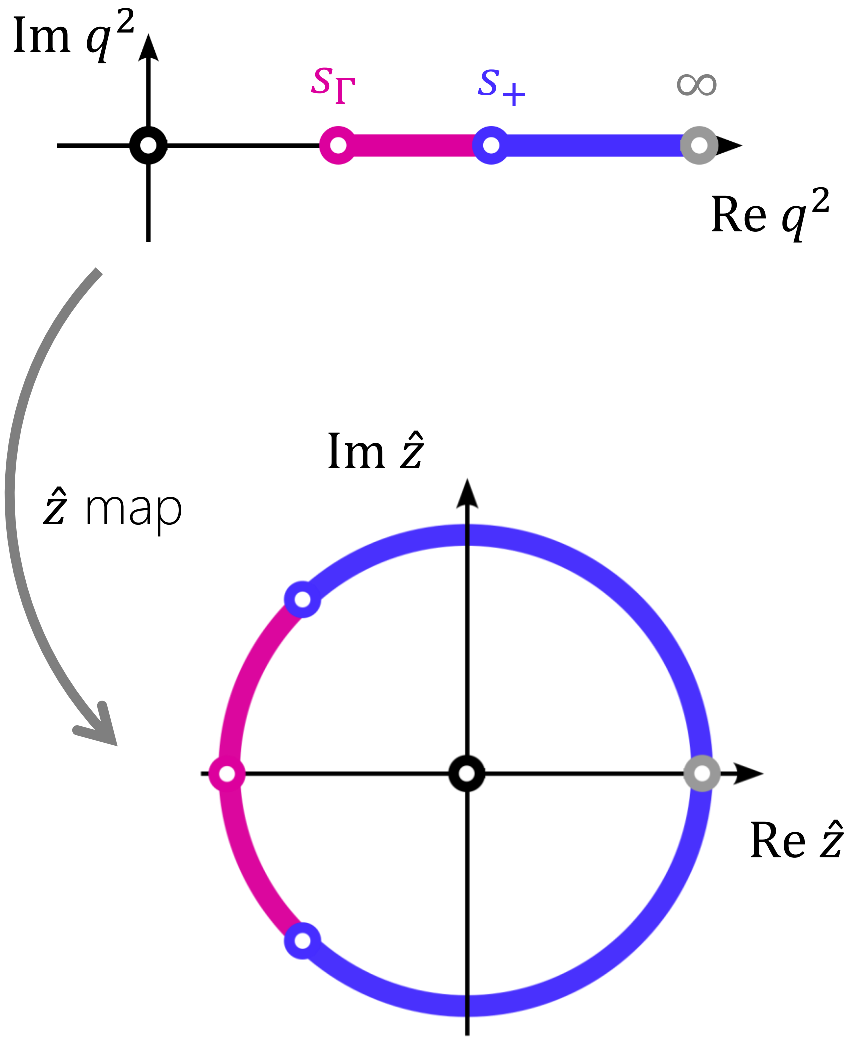
Obtain **precise predictions** for observables
i.e. predictions for hadronic form factors (FFs)

⇓

Combine different FF predictions
i.e. find a suitable **FFs parametrization**

⇓

Overcome the limitations of current parametrizations
by formulating a **new parametrization**



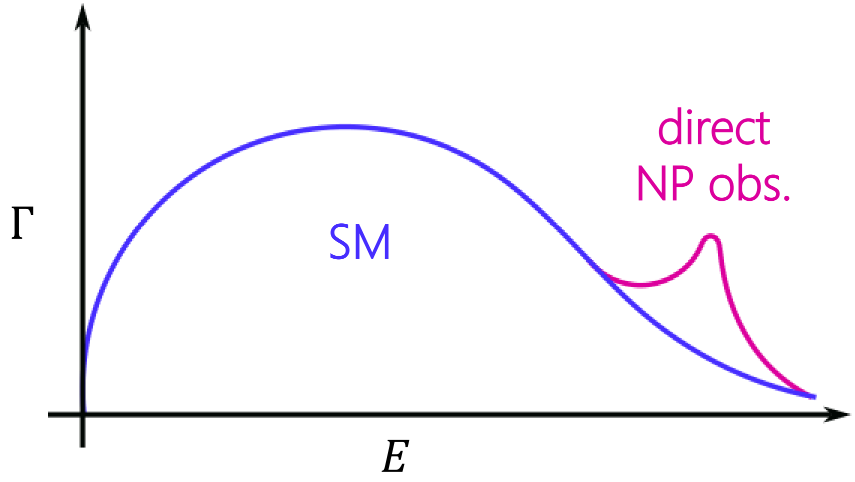
Introduction

Precision flavour tests of the SM

No NP evidence from direct searches so far (too heavy?)

$\Longrightarrow$ LHC has reached its maximum energy

$\Longrightarrow$ Direct NP discovery difficult in coming decades



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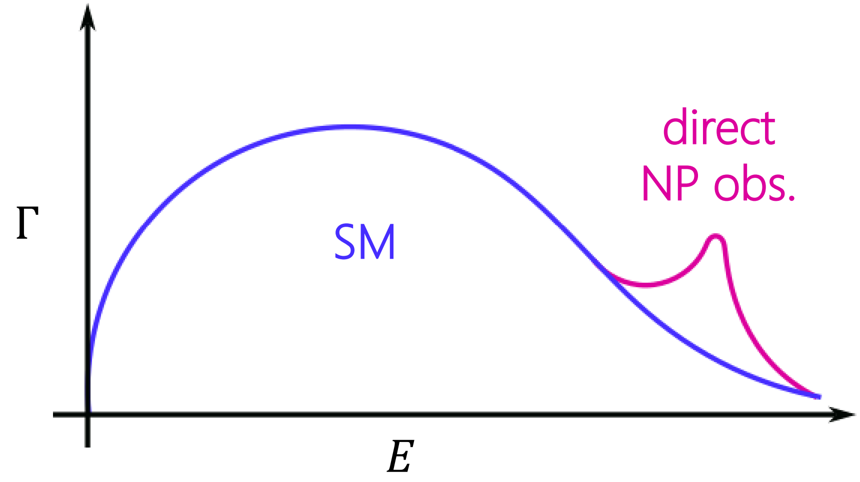


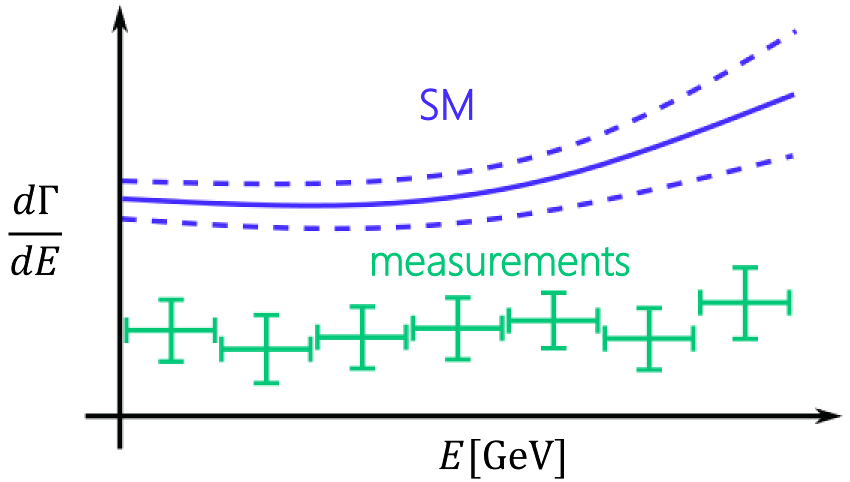
Indirect searches (with flavour)

$\Longrightarrow$ **Compare** precise measurements and calculations

$\Longrightarrow$ Probe the SM at **higher energies** than direct searches

Extraction SM parameters (e.g. $|V_{ub}|$ and $|V_{cb}|$)

$\Longrightarrow$ 13 out of 19 are flavour parameters



Need precise SM predictions

Predictions and form factors

Indirect searches $\Longrightarrow$ **study $\boldsymbol{B}$-meson decays to constrain NP**

Factorise decay amplitude as e.g. (neglecting QED corrections)

$$\langle D\mu\bar{\nu}|\mathcal{O}_{eff}|\bar{B}\rangle = \langle\mu\bar{\nu}|\mathcal{O}_{lep}|0\rangle\langle D|\mathcal{O}_{had}|\bar{B}\rangle$$

Decay amplitudes depend on:

- leptonic matrix elements: perturbative objects, **tiny uncertainties**
- hadronic matrix elements: non-perturbative QCD effects, **moderate uncertainties**

Predictions and form factors

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- leptonic matrix elements: perturbative objects, tiny uncertainties
- hadronic matrix elements: non-perturbative QCD effects, moderate uncertainties

Parametrize hadronic matrix elements in terms of form factors (FFs)

$$\langle D(k)|\mathcal{O}_{had}|\bar{B}(k+q)\rangle = \sum_{\lambda} \mathcal{S}_{\lambda}(k,q)\, \mathcal{F}_{\lambda}(q^2)$$

FFs are functions of q^2 (i.e. the momentum transfer squared)

Form factors predictions

Need to know q^2 dependence of the FFs

Problem: FFs are known (from LCSRs and lattice QCD) at discrete values of q^2 points

To combine theory inputs and obtain FFs for any q^2 value, FFs must be parametrized

Form factors predictions

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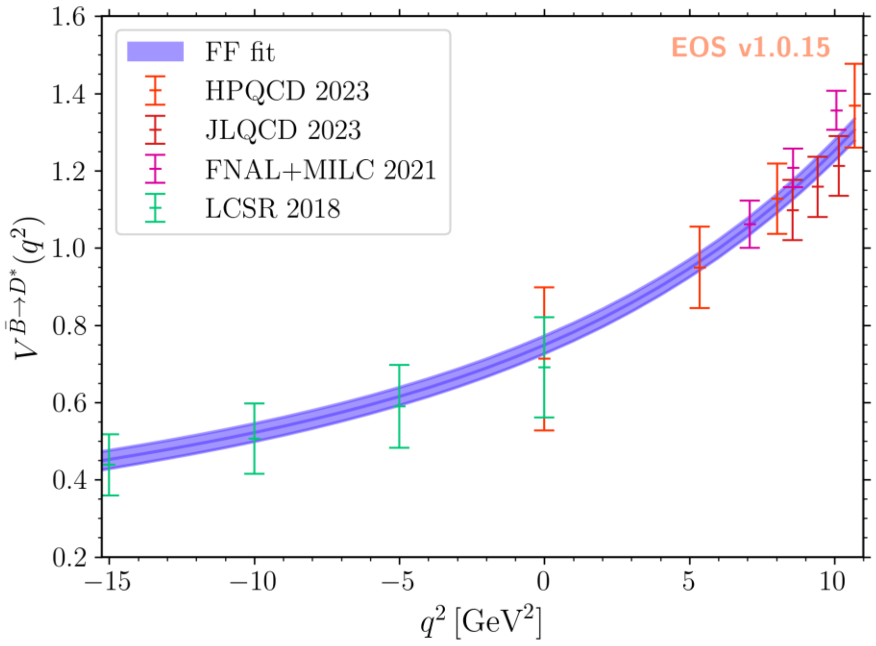
To combine theory inputs and obtain FFs for any q^2 value, FFs must be parametrized

A simple q^2 power expansion is perfectly fine

$$\mathcal{F}(q^2) = \sum_{n=0}^{N} a_n (q^2)^n$$



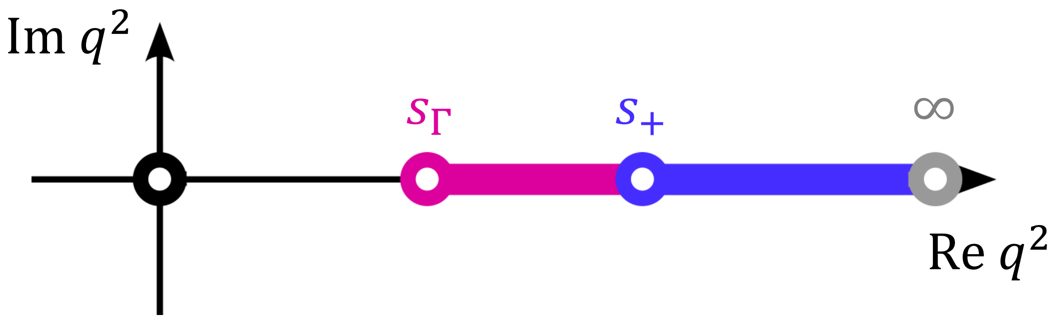
However, the series must be truncated as only a finite number of coefficients a_n can be fitted

Use unitarity bounds to estimate truncation error

FFs parametrization

Analytic properties of FFs

Study FF analytic structure to find a suitable parametrization. Example $B \to K$ FFs



FFs are analytic except for branch cuts (i.e. lines of discontinuity) starting at

$s_+ = (m_B + m_K)^2$, process threshold

$s_\Gamma = (m_{B_s} + m_\pi)^2 < s_+$, subthreshold branch cut

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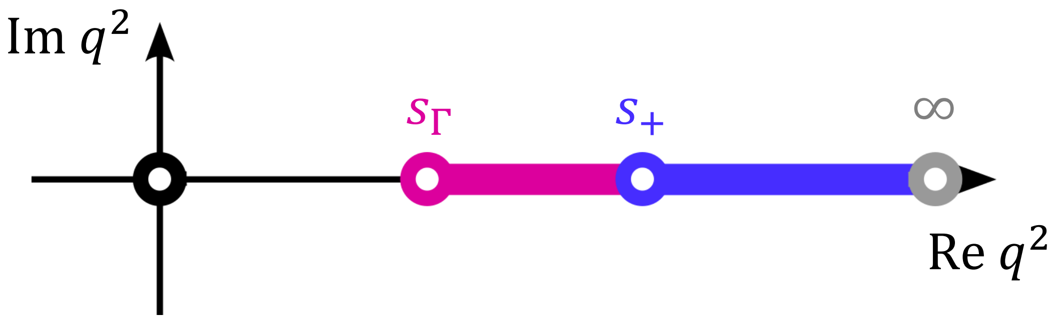

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Obtain a constrain on the FFs using unitarity (see [Okubo 1971])

$$\int_{s_+}^{\infty} dq^2 \,|\det J\,| |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

calculate χ perturbatively, ϕ known function

Traditional approach: BGL

[Boyd/Grinstein/Lebed 1997]

Perform the conformal mapping

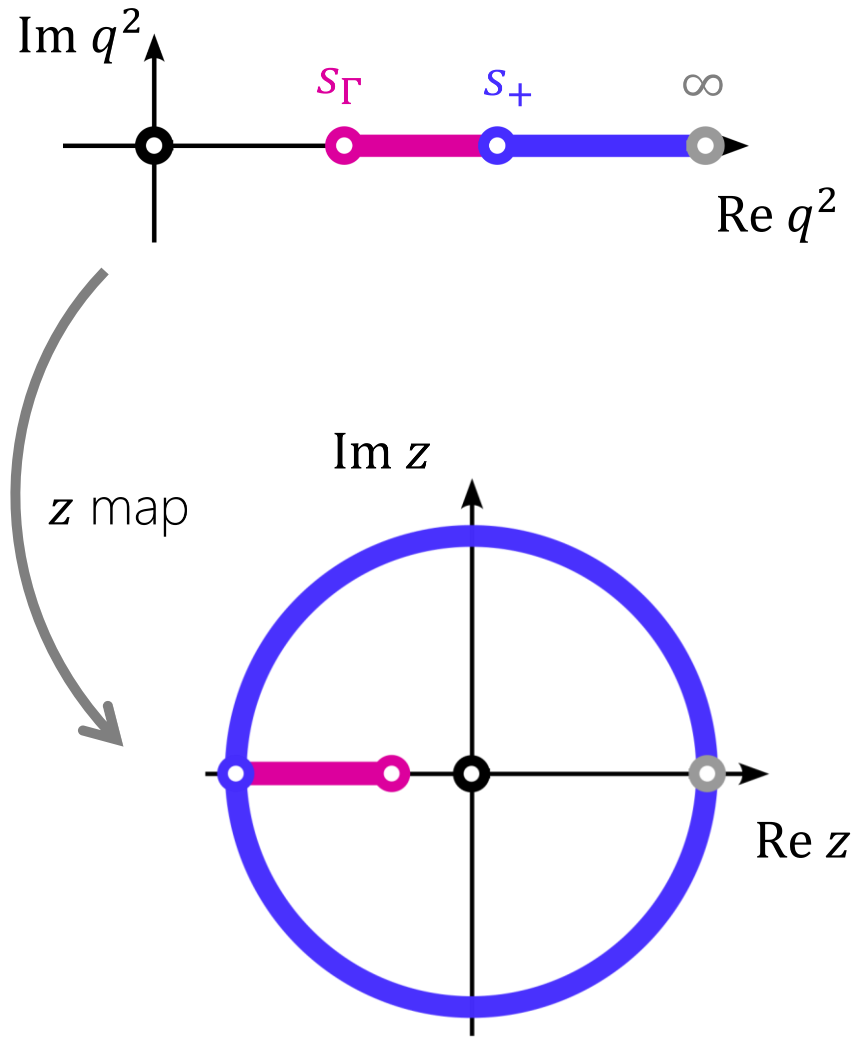
$$z(q^2) = \frac{\sqrt{s_+ - q^2} - \sqrt{s_+}}{\sqrt{s_+ - q^2} + \sqrt{s_+}}$$

expand FFs for $|z| < 1$ as

$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} a_n z(q^2)^n$$

obtain a bound on the coefficients

$$\int_{s_+}^{\infty} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n^2 < \chi$$



BGL truncation error

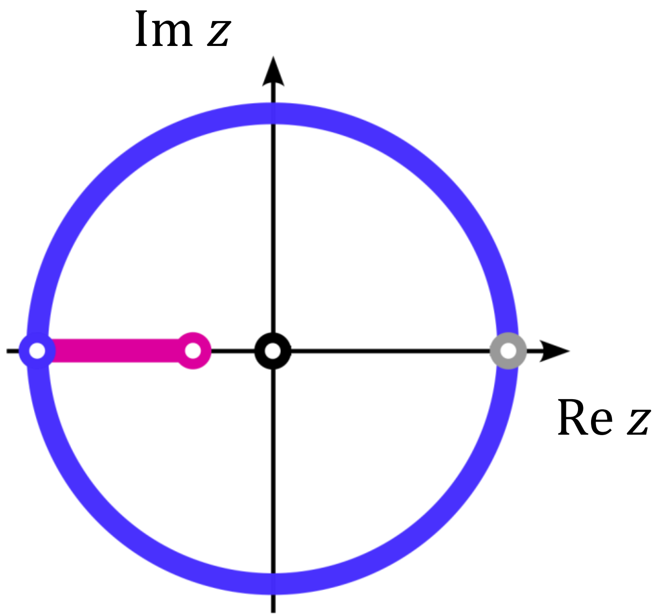
Truncate BGL at order N

$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^{N} a_n z(q^2)^n$$

The truncation error is given by

$$\left| \sum_{n>N}^{\infty} a_n z^n \right| \leq \left(\sum_{n>N}^{\infty} a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n>N}^{\infty} |z|^{2n} \right)^{\frac{1}{2}} \leq \left(\sum_{n>N}^{\infty} |z|^{2n} \right)^{\frac{1}{2}}$$

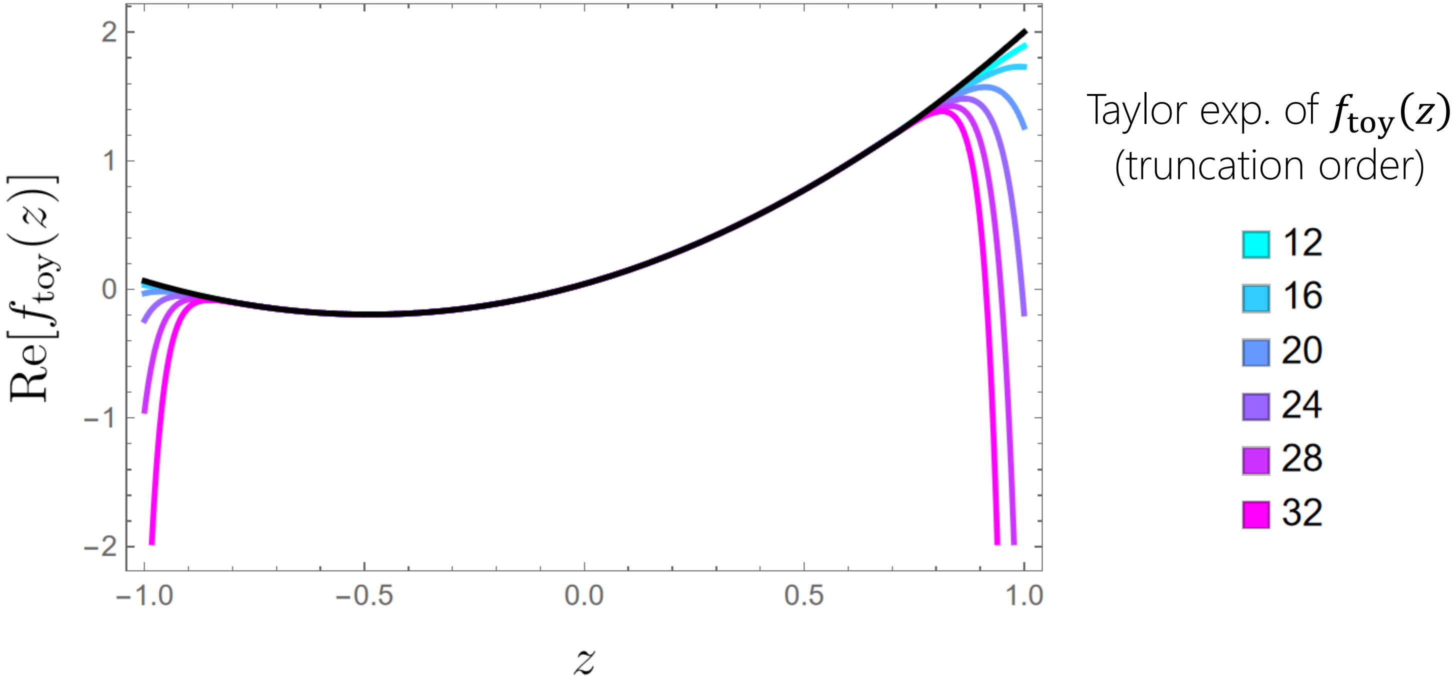
which tends to zero as $N \to \infty$ for any $|z| < 1$ since $\sum_{n>N} |z|^{2n} \to 0$



Problem! series is divergent due to the branch cut in s_Γ

Impact of branch cuts in a Taylor expansion

Toy example: $f_{\text{toy}}(z) = z + z^2 + 0.05\sqrt{0.7 - z}$ $\Longrightarrow$ compare with Taylor expansion



Even if the branch cut is suppressed it generates **divergent coefficients**. Hence:

$$\sum_{n=0}^{\infty} b_n^2 \not< \chi$$

Problems with BGL

Having a branch cut invalidate the expansion for $|z| < 1$

$$\mathcal{F}(q^2) \neq \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} a_n z(q^2)^n \quad \text{for some} \quad |z| < 1$$

same issue appears for FFs in $B \to D^{(*)}$, $\Lambda_b \to \Lambda$, ...

It is crucial to address this issue to accurately estimate uncertainties in b-hadron decays

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It is crucial to address this issue to accurately estimate uncertainties in b-hadron decays

Issue discussed in the literature, but solutions are unsatisfactory:
they do not allow a rigorous estimate of the truncation error (cut modelling, polynomials)

Find a way to recover the unitarity bound:

[Boyd/Grinstein/Lebed 1995]
[Caprini/Neubert 1996]
[NG/van Dyk/Virto 2020]
...

$$\sum_{n=0}^{\infty} a_n^2 < \chi$$

Essential to estimate truncation error! (we can only fit a finite number of a_n)

Dispersive matrix interpolation

[Bourrely/Machet/de Rafael 1981]

Based on the same principles as BGL, i.e. unitarity and analyticity

Not a parametrization in the strict sense but an interpolation method (like Lagrange polynomials)

Assume that $\mathcal{F}$ is known at points $q_1^2, \ldots, q_N^2 \Longrightarrow$ build a matrix

$$\det \begin{pmatrix} \chi & \phi(q^2)\mathcal{F}(q^2) & \phi(q_1^2)\mathcal{F}(q_1^2) & \cdots & \phi(q_N^2)\mathcal{F}(q_N^2) \\ \phi(q^2)\mathcal{F}(q^2) & \dfrac{1}{1-z(q^2)^2} & \dfrac{1}{1-z(q^2)z(q_1^2)} & \cdots & \dfrac{1}{1-z(q^2)z(q_N^2)} \\ \phi(q_1^2)\mathcal{F}(q_1^2) & \dfrac{1}{1-z(q^2)z(q_1^2)} & \dfrac{1}{1-z(q_1^2)^2} & \cdots & \dfrac{1}{1-z(q_1^2)z(q_N^2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(q_N^2)\mathcal{F}(q_N^2) & \dfrac{1}{1-z(q^2)z(q_N^2)} & \dfrac{1}{1-z(q_1^2)z(q_N^2)} & \cdots & \dfrac{1}{1-z(q_N^2)^2} \end{pmatrix} \geq 0$$

By imposing the positivity of this determinant one finds

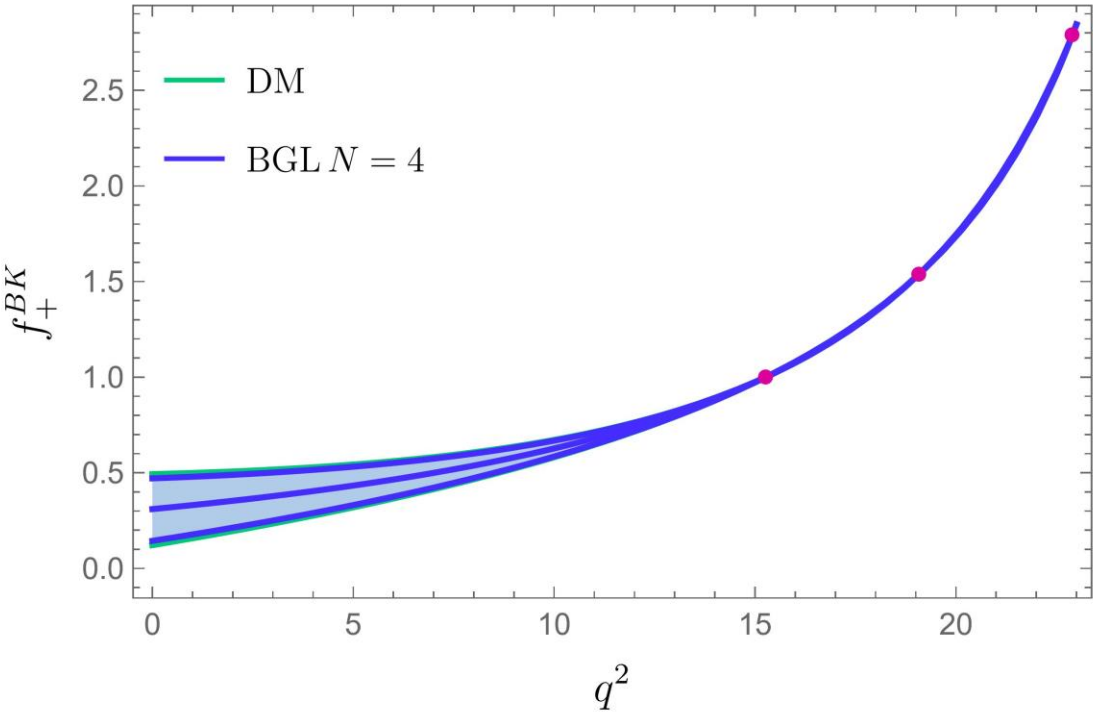
$$\mathcal{F}_{\mathrm{lo}}^{\mathrm{DM}}(q^2) \leq \mathcal{F}(q^2) \leq \mathcal{F}_{\mathrm{up}}^{\mathrm{DM}}(q^2)$$

where $\mathcal{F}_{\mathrm{lo\backslash up}}^{\mathrm{DM}}$ are functions of q^2 and $\mathcal{F}(q_1^2), \ldots, \mathcal{F}(q_N^2)$

Compare BGL and DM

Example: consider the central values of the FF f_+^{BK} from HPQCD 2022 at three high-q^2 points

1. Truncate BGL at $N = 4$
2. Extract the first 3 coefficients analytically (solve a liner system of eqs.)
3. Determine the remaining $N - 3 = 1$ coeffs. by maximizing/minimizing $f_+^{BK}(0)$ and imposing $\sum_{n=0}^{N} a_n^2 < \chi$



Very small difference for $N = 4$

Negligible difference for $N > 4$

DM and BGL lead to equivalent results

Prefer the more convenient method $\Longrightarrow$ DM can be challenging for large data sets

Our approach: GG

[Gopal/Gubernari 2024]

Just a reminder: $s_+ = (m_B + m_K)^2$, $s_\Gamma = \left(m_{B_s} + m_\pi\right)^2$

Modify the conformal mapping ($s_+ \mapsto s_\Gamma$)

$$\hat{z}(q^2) = \frac{\sqrt{s_\Gamma - q^2} - \sqrt{s_\Gamma}}{\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma}}$$

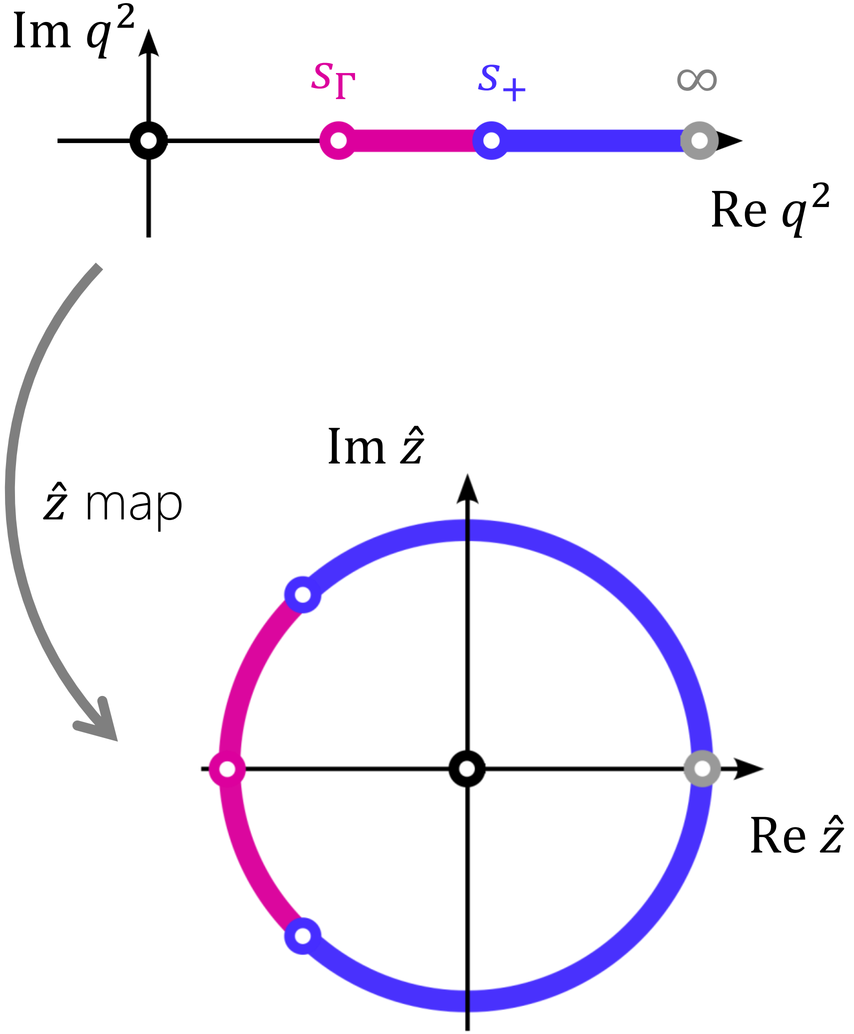
expand FFs for $|\hat{z}| < 1$ (no singularities now!) as

$$\mathcal{F}(q^2) = \frac{1}{\phi(q^2)} \sum_{n=0}^{\infty} b_n \hat{z}(q^2)^n$$

however

$$\int_{s_+}^{\infty} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi \quad \not\Rightarrow \quad \sum_{n=0}^{\infty} b_n^2 < \chi$$

Integral must over the whole circle!



Our derivation of the unitarity bound

Start from

$$\int_{s_+}^{\infty} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

add on both sides

$$\Delta\chi \equiv \int_{s_\Gamma}^{s_+} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2$$

Estimate $\Delta\chi$ using large q^2 scaling behaviour (for $B \to K$ FFs $\frac{\Delta\chi}{\chi} < 1\%$)

Obtain the unitarity bound

$$\int_{s_\Gamma}^{\infty} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi + \Delta\chi \quad \Rightarrow \quad \sum_{n=0}^{\infty} b_n^2 < \chi + \Delta\chi$$

New parametrization for FFs that allows to calculate the truncation error!

$\Delta\chi$ estimate

Approximate FFs using their large q^2 scaling behaviour calculated in perturbative QCD

E.g. for $B \to K$

[Lepage/Brodsky 1980]
[Akhoury et al. 1994]

$$|\mathcal{F}_+(q^2)|^2 \simeq K\left(\frac{s_\Gamma}{q^2}\right)^2$$

According to [Becher/Hill 2005] $K{\sim}1$

Even assuming $K{\sim}100$

$$\frac{\Delta\chi}{\chi} \equiv \frac{1}{\chi}\int_{s_\Gamma}^{s_+} dq^2\, |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2 \simeq 0.005$$

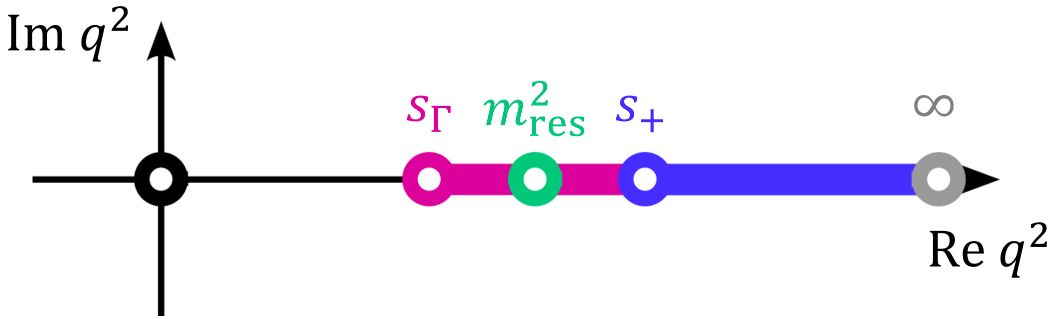
i.e. smaller than the uncertainty on χ

This is due to the fact that $\frac{s_+ - s_\Gamma}{s_\Gamma} \ll 1$ and that χ is an inclusive quantity while $\Delta\chi$ is exclusive

Only upper bound for $\Delta\chi$ is needed, not a precise calculation!

Subthreshold poles

Poles can appear for $s_\Gamma < q^2 < s_+$ making $\Delta\chi$ divergent (happens, e.g., for f_0^{BK})



Modify the dispersion relation (take more subtractions that needed)

$$\Delta\chi \equiv \int_{s_\Gamma}^{s_+} dq^2 |\det J| \frac{(q^2 - m_{\text{res}}^2)^2}{q^2} |\phi(q^2)\mathcal{F}(q^2)|^2 \coloneqq \int_{s_\Gamma}^{s_+} dq^2 |\det J| \left|\tilde{\phi}(q^2)\mathcal{F}(q^2)\right|^2$$

The calculation of χ must also be adjusted accordingly

$$\tilde{\chi} \coloneqq m_{\text{res}}^4 \chi(k+2) - m_{\text{res}}^2 \chi(k+1) + \chi(k)$$

where k is the minimal number of subtractions

Problems with cut modelling and polynomials

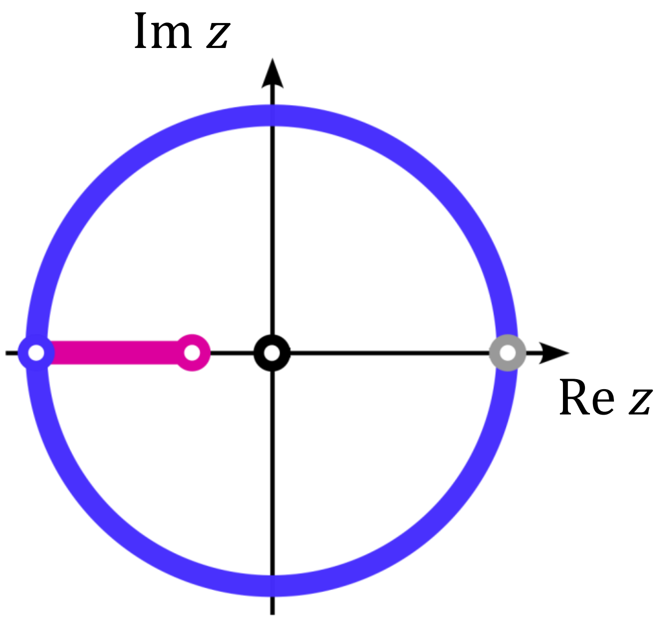
Model the branch cut and subtract it

$$\tilde{\mathcal{F}}(z) \equiv \mathcal{F}(z) - \mathcal{F}_{\mathrm{cut}}(z)$$

expand $\tilde{\mathcal{F}}(z)$ [Boyd/Grinstein/Lebed 1995] [Caprini/Neubert 1996]

Problem: $\mathcal{F}_{\mathrm{cut}}(z)$ is not known

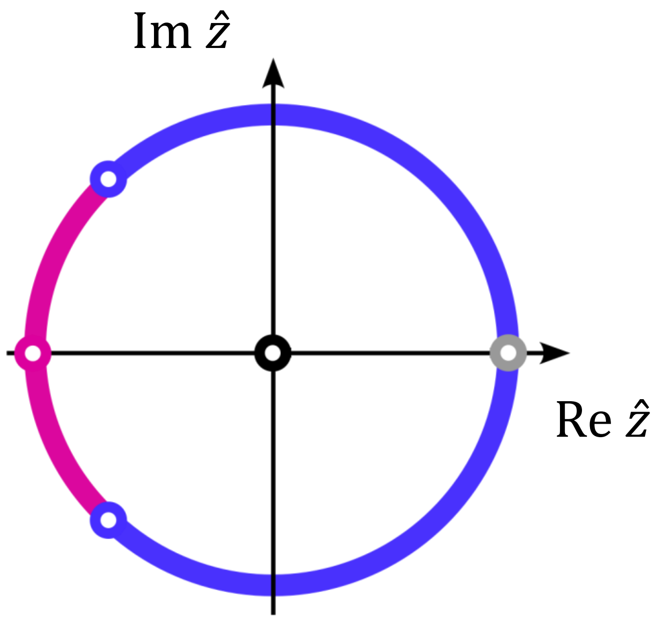
$\Longrightarrow$ cannot rely on exact numerical cancellation of singularities



Expand in polynomials orthogonal on the blue arc [NG/van Dyk/Virto 2020] [Flynn/Jüttner/Tsang 2023]

$$\mathcal{F}(\hat{z}) = \frac{1}{\phi(\hat{z})} \sum_{n=0}^{\infty} b_n p_n(\hat{z})$$

$|\boldsymbol{p_n(\hat{z})}| \to \infty$ for $\boldsymbol{n} \to \infty$ and some $\hat{z}$ in the unit disk

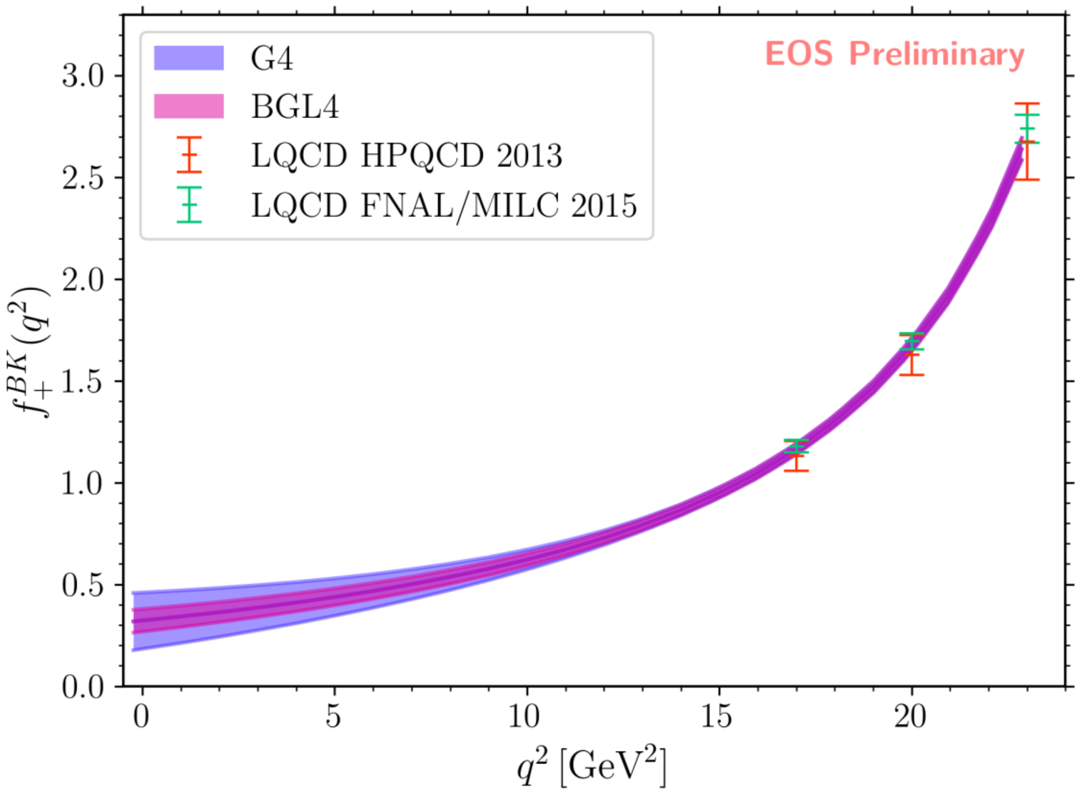


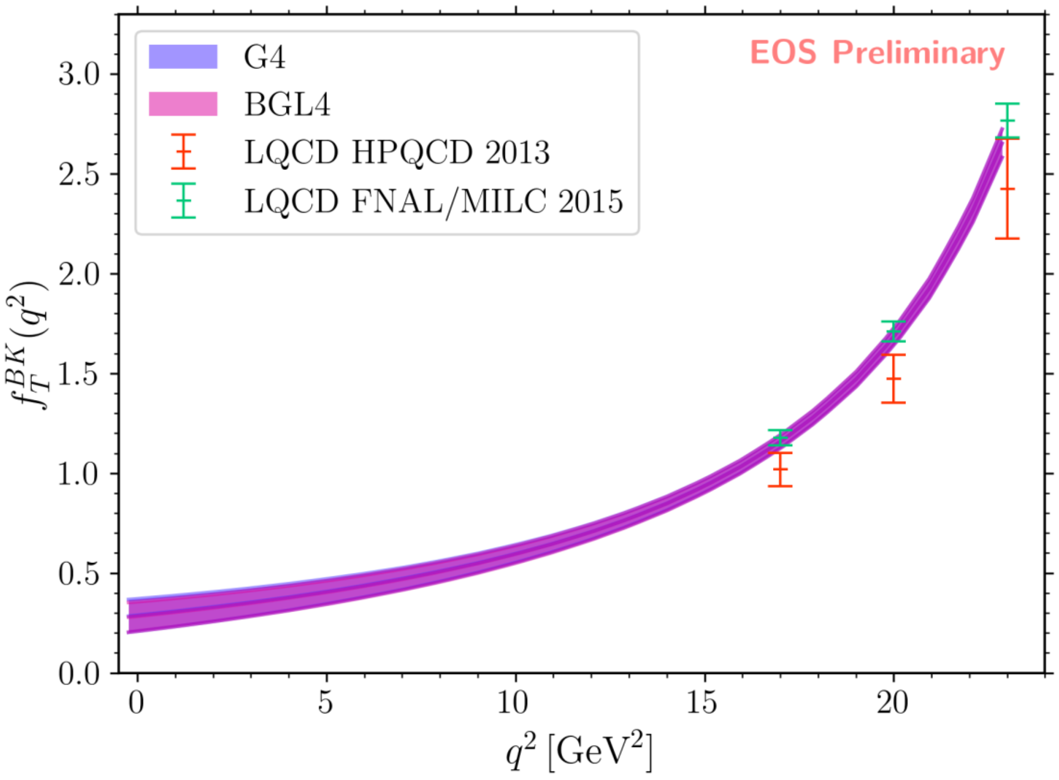
Preliminary results

Preliminary results 1/2

Preliminary Bayesian fit using our new and BGL parametrization for $N = 4$

Example to test the extrapolation (exclude [HPQCD 2022]) $\Longrightarrow$ good agreement for and f_+^{BK} and f_T^{BK}





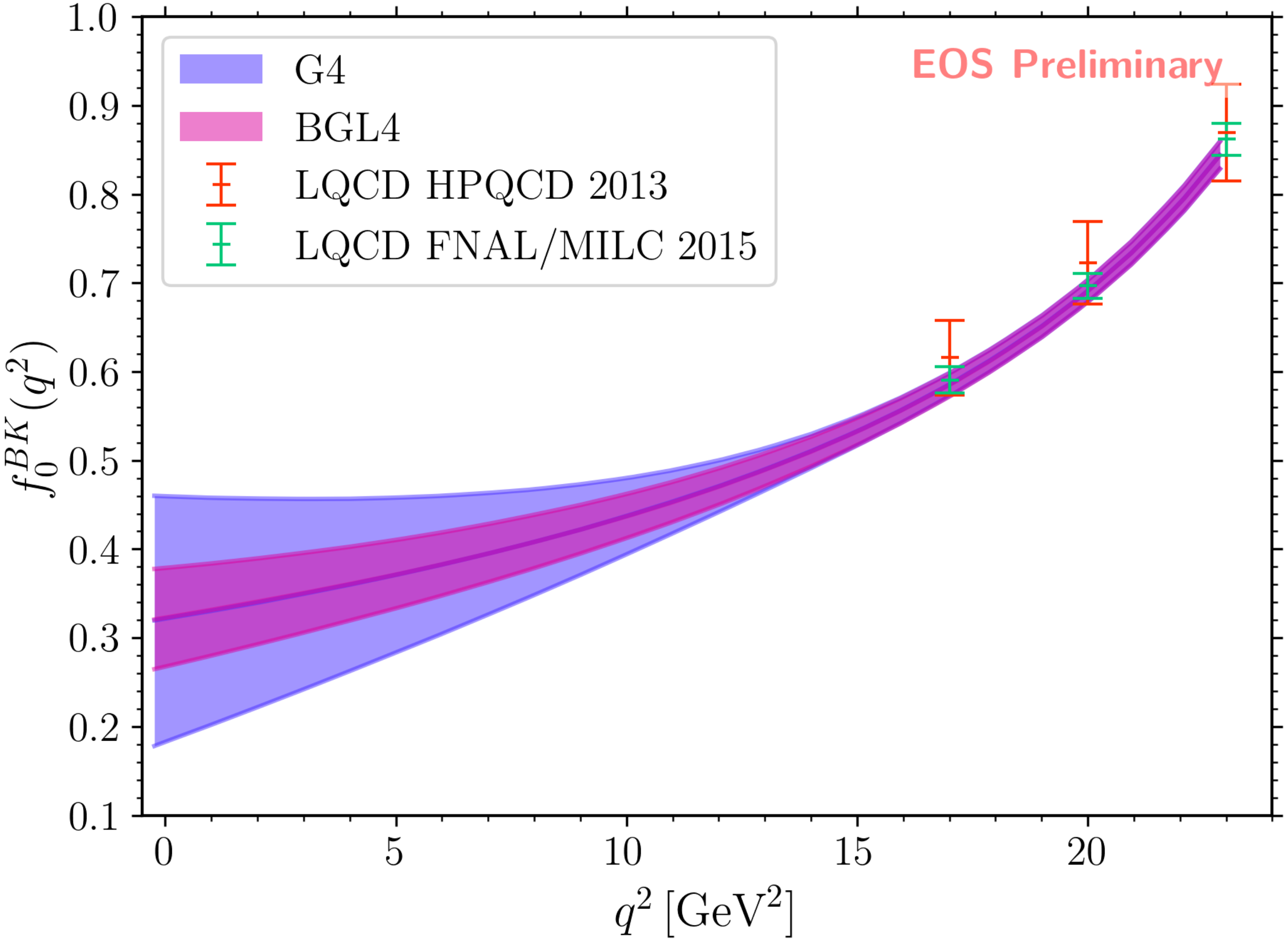
Preliminary results 2/2

Subthreshold poles are present for f_0^{BK}

BGL might underestimate the uncertainties

Unitarity constraints satisfied

N should be high enough to do not underestimate the truncation error



Summary and conclusions

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Predictions of hadronic **form factors (FFs) are crucial** for indirect searches for new physics

Different FFs calculations are combined using a **parametrization**

The **truncation error** can be rigorously calculated using unitarity

However, the traditional (BGL) parametrization neglect subthreshold branch cuts,
leading to hard-to-quantify systematic effects

We propose a new easy to implement parametrization to solve the issue
Our work enables new model independent FF analyses, minimizing systematic uncertainties

Preliminary fits show that subthreshold cuts have a non-negligible impact

Thank you!

Backup slides

Polynomial parametrization

polynomial parametrization ($\hat{z}$ polynomials) [NG/van Dyk/Virto 2020]

$$\mathcal{H}_\lambda(\hat{z}) = \frac{1}{\mathcal{P}(z)\phi(z)} \sum_{n=0}^{\infty} \beta_n p_n(\hat{z}) \qquad \sum_{n=0}^{\infty} |\beta_n|^2 < 1$$

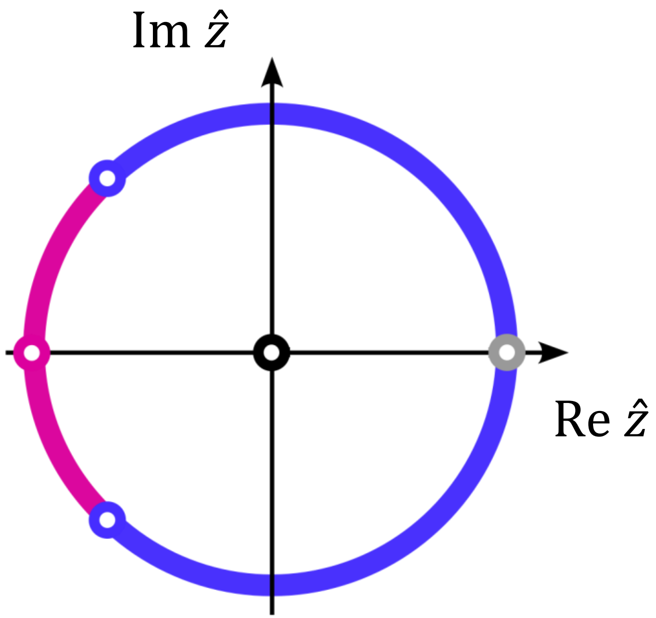
$|p_n(\hat{z})| \to \infty$ for $n \to \infty$ some z in the unit disk

$$p_0^{B\to K}(\hat{z}) = \frac{1}{\sqrt{2\alpha_{BK}}}$$

$$p_1^{B\to K}(\hat{z}) = \left(\hat{z} - \frac{\sin(\alpha_{BK})}{\alpha_{BK}}\right)\sqrt{\frac{\alpha_{BK}}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}}$$

$$p_2^{B\to K}(\hat{z}) = \left(\hat{z}^2 + \frac{\sin(\alpha_{BK})(\sin(2\alpha_{BK})-2\alpha_{BK})}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}\hat{z} + \frac{2\sin(\alpha_{BK})(\sin(\alpha_{BK})-2\alpha_{BK})}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}\right)$$

$$p_3^{B\to K}(\hat{z}) = \cdots$$



Anomalous branch cuts

Rare decays (e.g. $B \to K\mu^+\mu^-$) also depend on non-local FFs

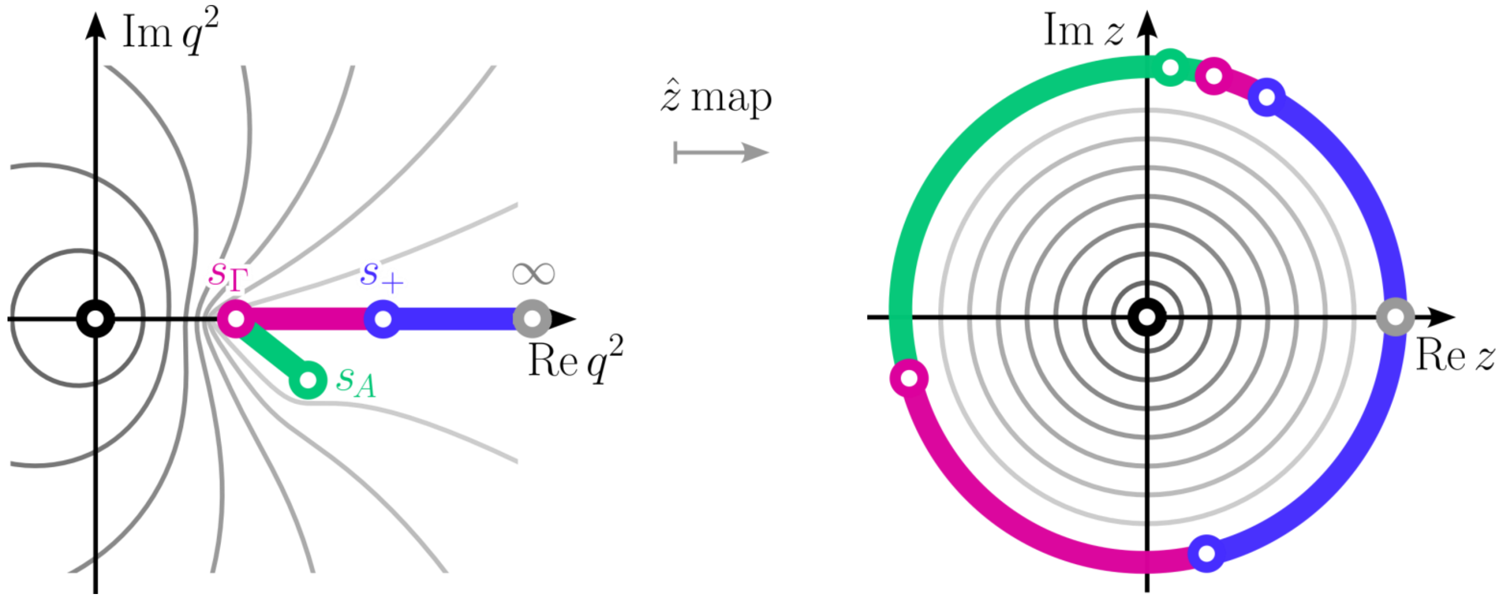
$$\langle K\mu^+\mu^-|\mathcal{O}_{eff}|B\rangle = \langle \mu^+\mu^-|\mathcal{O}_{lep}|0\rangle\langle K|\mathcal{O}_{had}|B\rangle + \text{non local}$$

Rescattering, e.g. $B \to DD_s^* \to K\mu^+\mu^-$, generates anomalous branch cuts in the complex plane

[Ciuchini et al. 2022]
[Mutke et al. 2024]

Apply the same procedure as for the subthreshold branch cuts, but:

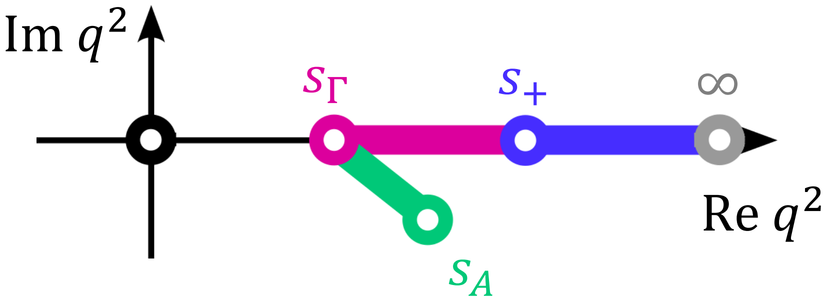
- $\hat{z}$ map is very hard to obtain
- $\Delta\chi$ calculation extremely challenging



Schwarz–Christoffel formula

Map the unit disk $\boldsymbol{q^2}$ plane (existence guaranteed by the Riemann Mapping Theorem):

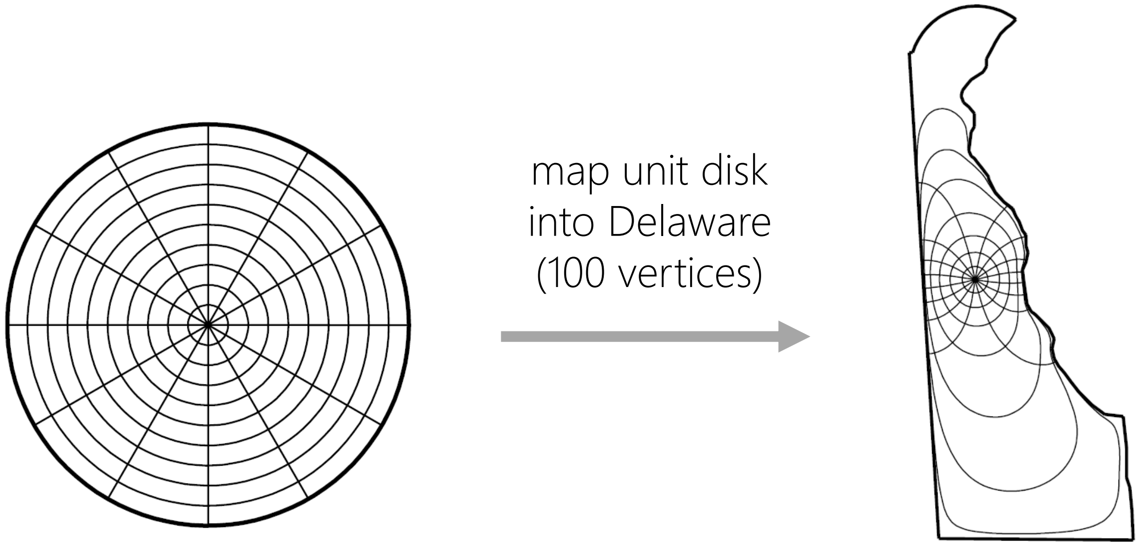
use **Schwarz–Christoffel formula**



$$g(z) = A + C \int_0^z d\,\zeta \prod_{k=1}^{4} \left(1 - \frac{\zeta}{z_k}\right)^{\frac{\phi_k}{\pi} - 1}$$

Analytic formula but $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{z_k}$ have to be determined numerically

Extremely powerful method!



[Driscoll/Trefethen 2002]