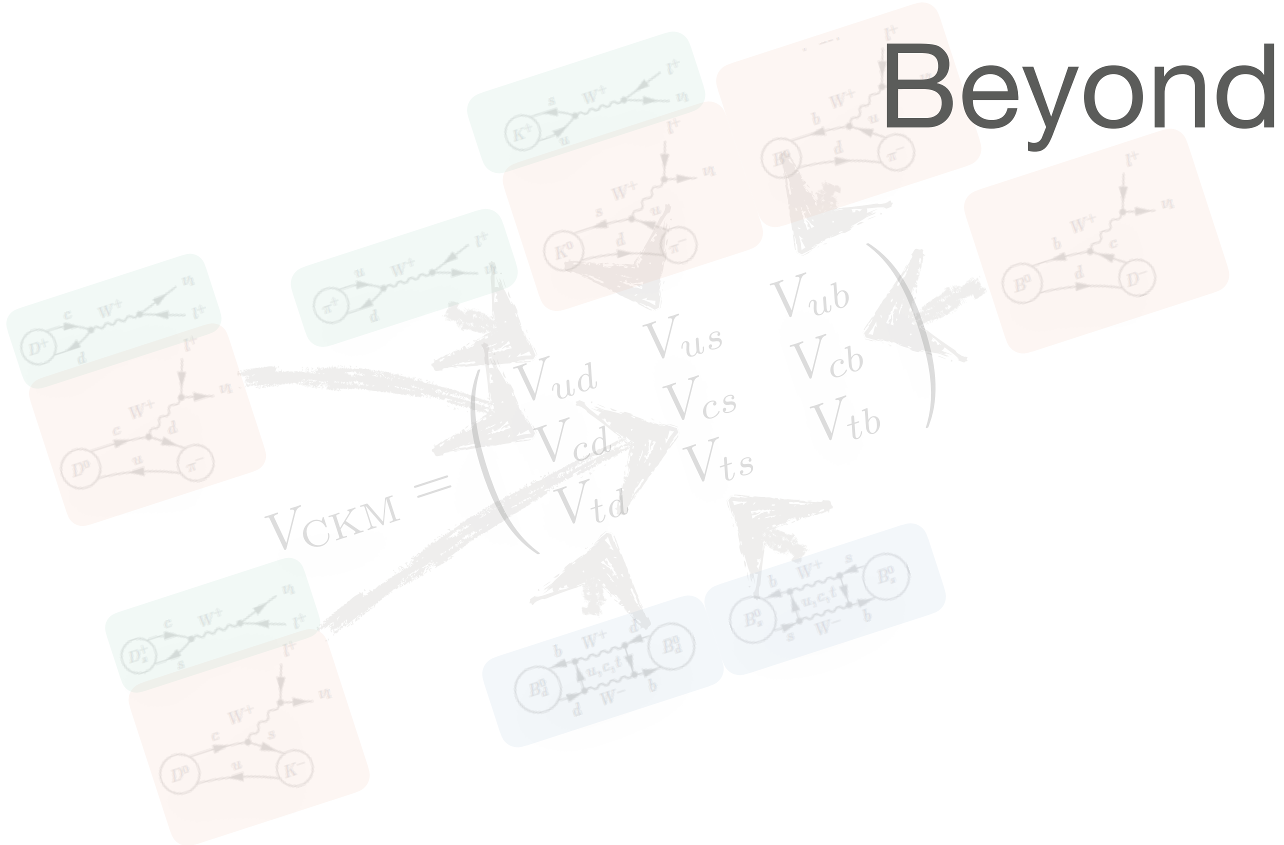
Inclusive meson decay on the lattice



Beyond the Flavour Anomalies Workshop 2026

INSTITUTO GALEGO DE FÍSICA DE ALTAS ENERXÍAS IGFAE

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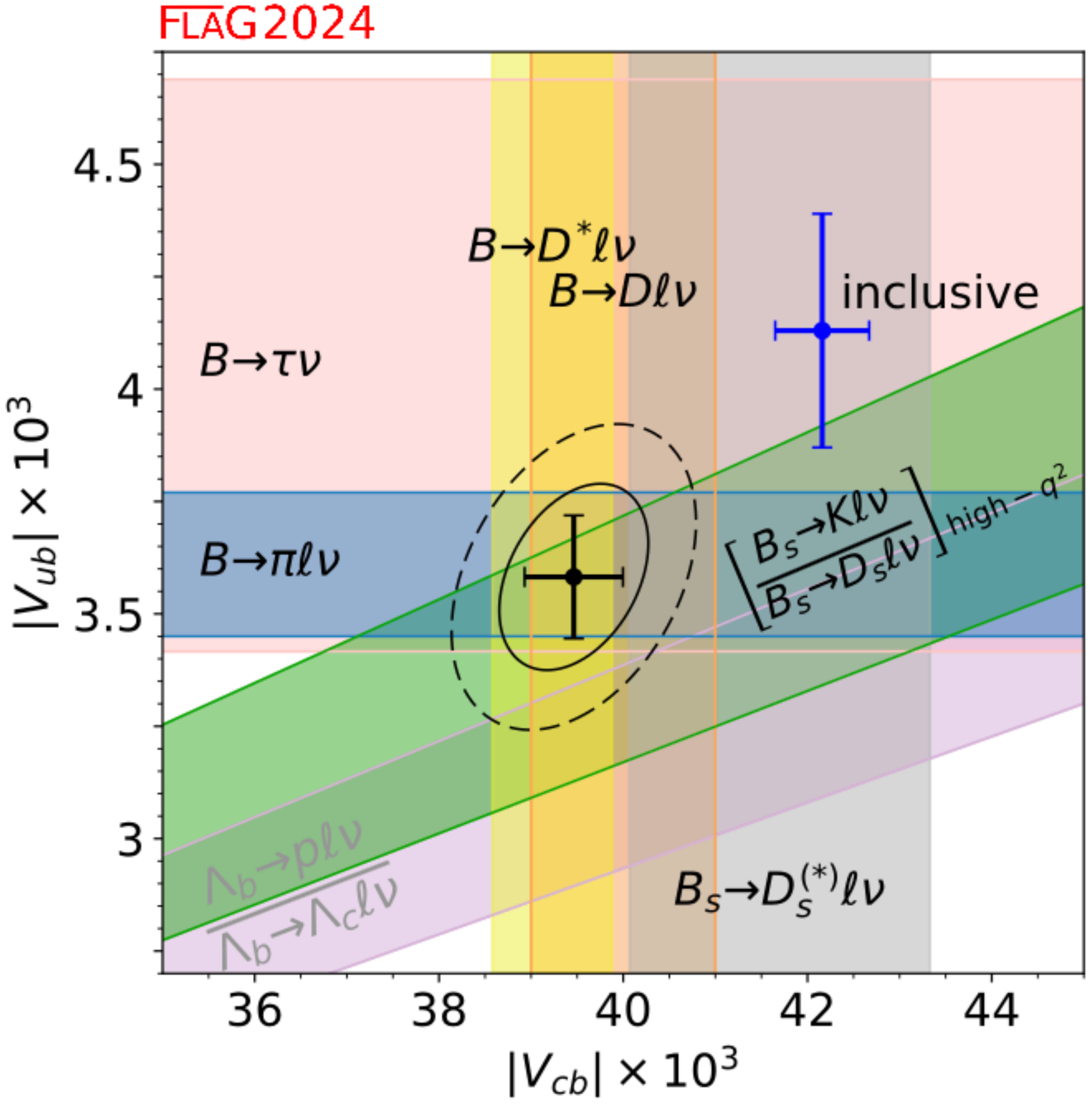
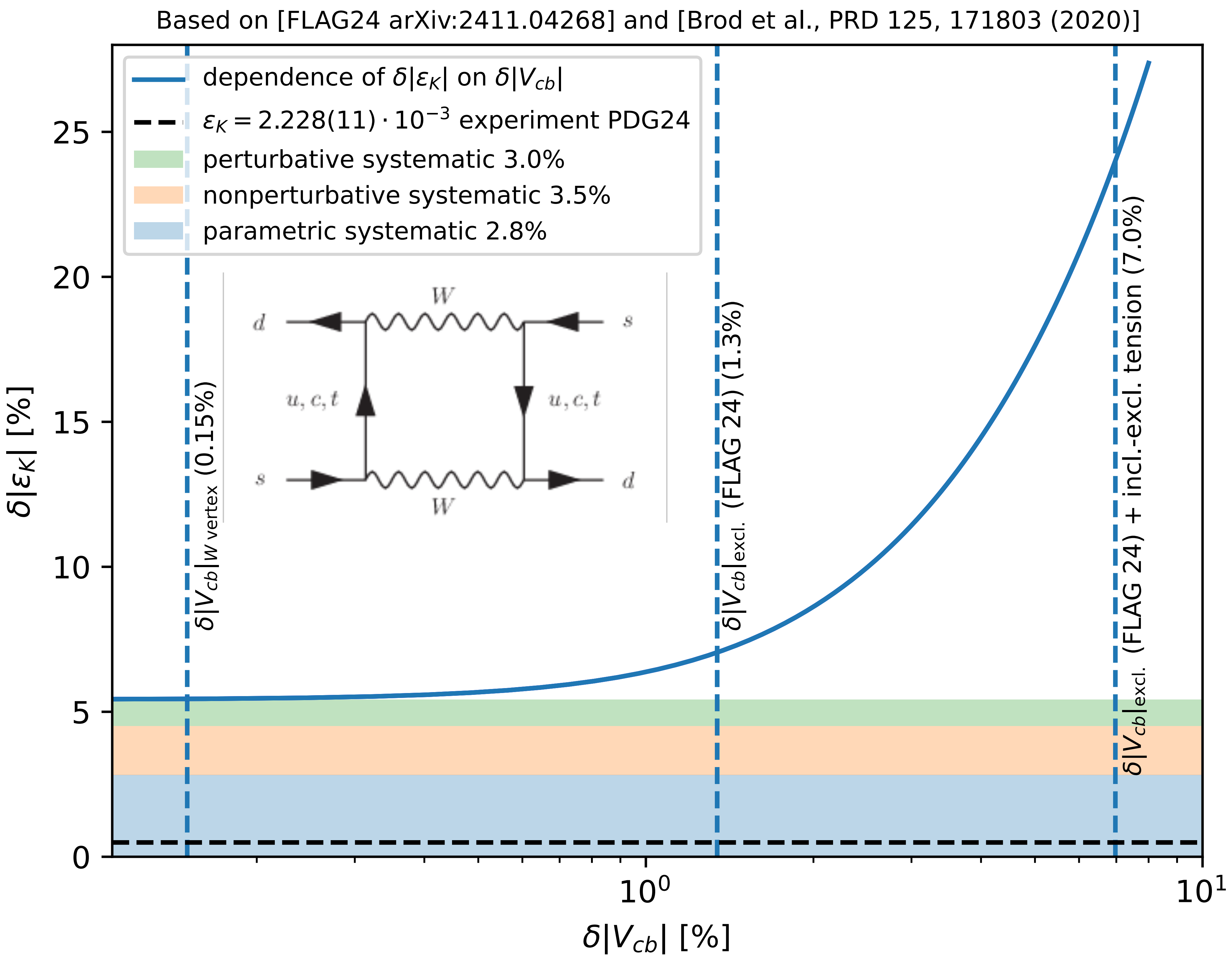
PRD 112 (2025) 1

Puzzle and implications

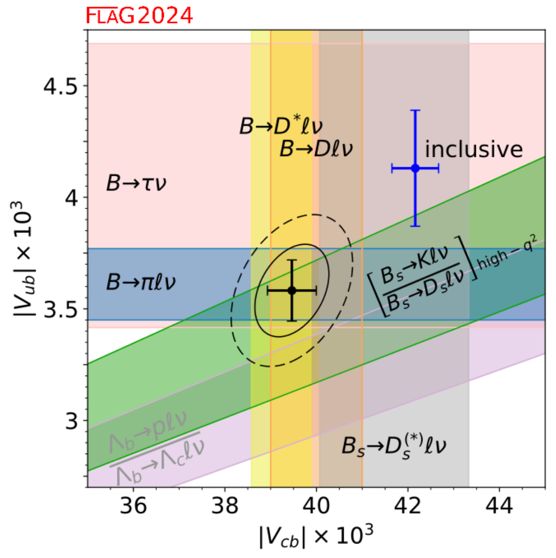


ϵ_K of kaon mixing depends on $|V_{cb}|$



- Consistent picture for exclusive decay
- Consistent picture for inclusive decay
- **Inconsistent** picture between both → **puzzle**

- V_{cb} crucial input to highly sensitive SM probes
- relevant question even on time scale of FCC-*ee*



What's the problem?

Exclusive decay

- $d\Gamma/dq^2 \sim |V_{Qq}|^2 \mathcal{F}(f_+(q^2), f_0(q^2), \dots)$
- Many Lattice-QCD results for exclusive decay (single decay channel)

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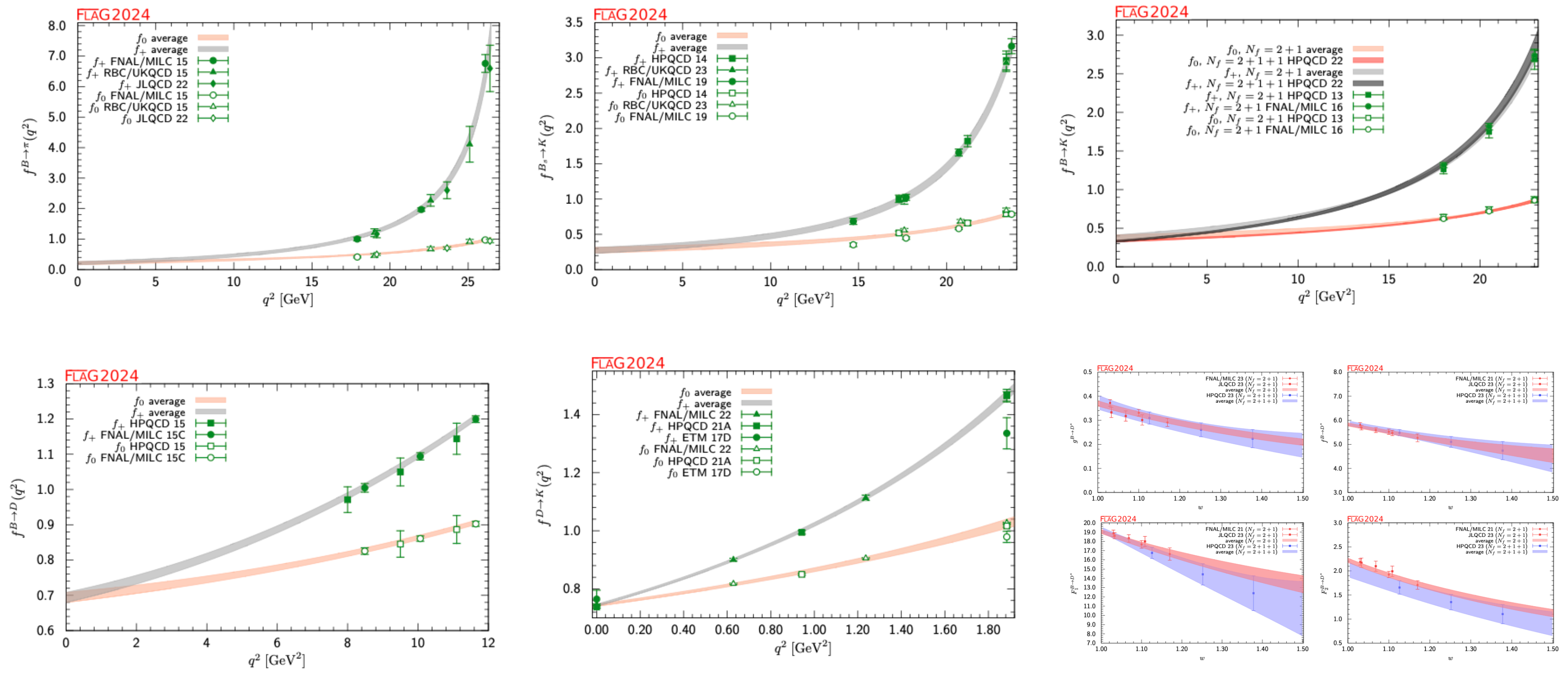
Decay	form factor	fit	Sec.	Fig.	Tab. $N_f = 2+1+1$	Refs.	Tab. $N_f = 2+1$	Refs.
$D \to K\ell\nu$	f_+, f_0	lat	7.2	18	30	[63, 65, 123]		
$D \to K\ell\nu$	f_+, f_0	lat+exp	7.5	19	32	[63, 65, 123]		
$B \to \pi\ell\nu$	f_+, f_0	lat	8.3.1	25			38	[124–126]
$B_s \to K\ell\nu$	f_+, f_0	lat	8.3.3	26			40	[127, 128, 128]
$B \to \pi\ell^-\ell^+$	f_T	lat	8.3.4				42	[129]
$B \to K\ell^+\ell^-(\nu\bar{\nu})$	f_+, f_0, f_T	lat	8.3.4	27			43	[130, 131]
$B \to D\ell\nu$	f_+, f_0	lat	8.4.1	28			45	[132, 133]
$B_s \to D_s\ell\nu$	f_+, f_0	lat	8.4.1		46	[134]		
$B \to D^*\ell\nu$	g, f, F_1, F_2	lat	8.4.2	29	47	[135]	47	[136, 137]
$B_s \to D_s^*\ell\nu$	g, f, F_1, F_2	lat	8.4.2	30		[135]		
$B \to \pi\ell\nu$	f_+, f_0	lat+exp	8.8	31			50	[124–126, 138–141]
$B \to D\ell\nu$	f_+, f_0	lat+exp	8.9	32			52	[132, 133, 142, 143]
$B \to D^*\ell\nu$	g, f, F_1, F_2	lat+exp	8.9	33, 34	53	[135, 144–148]	53	[136, 137, 144–148]



Inclusive decay

- all results to date from OPE/PT
- Until recently nothing on inclusive decay in Lattice QCD (integration over allowed final states)

PDGlive

$B^0 \to X_c \ell^+ \nu_l$

Semileptonic and leptonic modes

Γ_1	$\ell^+\nu_\ell X$	[1]	$(10.99 \pm 0.28)\%$		
Γ_2	$e^+\nu_e X_c$		$(10.8 \pm 0.4)\%$		
Γ_3	$D\ell^+\nu_\ell X$		$(9.7 \pm 0.7)\%$		
Γ_4	$\overline{D}^0\ell^+\nu_\ell$	[1]	$(2.35 \pm 0.09)\%$		2310
Γ_5	$\overline{D}^0\tau^+\nu_\tau$		$(7.7 \pm 2.5) \times 10^{-3}$		1911
Γ_6	$\overline{D}^*(2007)^0\ell^+\nu_\ell$	[1]	$(5.66 \pm 0.22)\%$		2258
Γ_7	$\overline{D}^*(2007)^0\tau^+\nu_\tau$		$(1.88 \pm 0.20)\%$	S=1.0	1839
Γ_8	$D^-\pi^+\ell^+\nu_\ell$		$(4.4 \pm 0.4) \times 10^{-3}$		2306
Γ_9	$\overline{D}_0^*(2420)^0\ell^+\nu_\ell$, $\overline{D}_0^{*0} \to D^-\pi^+$		$(2.5 \pm 0.5) \times 10^{-3}$		
Γ_{10}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^-\pi^+$		$(1.53 \pm 0.16) \times 10^{-3}$	S=1.0	2065
Γ_{11}	$D^{(*)}$ n $\pi\ell^+\nu_\ell$ (n $\geq$ 1)		$(1.88 \pm 0.25)\%$		
Γ_{12}	$D^{*-}\pi^+\ell^+\nu_\ell$		$(6.0 \pm 0.4) \times 10^{-3}$		2254
Γ_{13}	$\overline{D}_1(2420)^0\ell^+\nu_\ell$, $\overline{D}_1^0 \to D^{*-}\pi^+$		$(3.03 \pm 0.20) \times 10^{-3}$		2084
Γ_{14}	$\overline{D}_1'(2430)^0\ell^+\nu_\ell$, $\overline{D}_1'^0 \to D^{*-}\pi^+$		$(2.7 \pm 0.6) \times 10^{-3}$		
Γ_{15}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^{*-}\pi^+$		$(1.01 \pm 0.24) \times 10^{-3}$	S=2.0	2065
Γ_{16}	$\overline{D}^0\pi^+\pi^-\ell^+\nu_\ell$		$(1.7 \pm 0.4) \times 10^{-3}$		2301
Γ_{17}	$\overline{D}^{*0}\pi^+\pi^-\ell^+\nu_\ell$		$(8 \pm 5) \times 10^{-4}$		2248
Γ_{18}	$D_s^{(*)-}K^+\ell^+\nu_\ell$		$(6.1 \pm 1.0) \times 10^{-4}$		
Γ_{19}	$D_s^-K^+\ell^+\nu_\ell$		$(3.0^{+1.4}_{-1.2}) \times 10^{-4}$		2242
Γ_{20}	$D_s^{*-}K^+\ell^+\nu_\ell$		$(2.9 \pm 1.9) \times 10^{-4}$		2185
Γ_{21}	$\pi^0\ell^+\nu_\ell$		$(7.80 \pm 0.27) \times 10^{-5}$		2638
Γ_{22}	$\pi^0 e^+\nu_e$				2638
Γ_{23}	$\eta\ell^+\nu_\ell$		$(3.9 \pm 0.5) \times 10^{-5}$		2611
Γ_{24}	$\eta'\ell^+\nu_\ell$		$(2.3 \pm 0.8) \times 10^{-5}$		2553
Γ_{25}	$\omega\ell^+\nu_\ell$	[1]	$(1.19 \pm 0.09) \times 10^{-4}$		2582
Γ_{26}	$\omega\mu^+\nu_\mu$				2581
Γ_{27}	$\rho^0\ell^+\nu_\ell$	[1]	$(1.58 \pm 0.11) \times 10^{-4}$		2583
Γ_{28}	$p\bar{p}\ell^+\nu_\ell$		$(5.8^{+2.6}_{-2.3}) \times 10^{-6}$		2467
Γ_{29}	$p\bar{p}\mu^+\nu_\mu$		$< 8.5 \times 10^{-6}$	CL=90%	2446
Γ_{30}	$p\bar{p}e^+\nu_e$		$(8.2^{+4.0}_{-3.3}) \times 10^{-6}$		2467
Γ_{31}	$e^+\nu_e$		$< 9.8 \times 10^{-7}$	CL=90%	2640
Γ_{32}	$\mu^+\nu_\mu$		2.90E-07 to 1.07E-06	CL=90%	2639
Γ_{33}	$\tau^+\nu_\tau$		$(1.09 \pm 0.24) \times 10^{-4}$	S=1.2	2341
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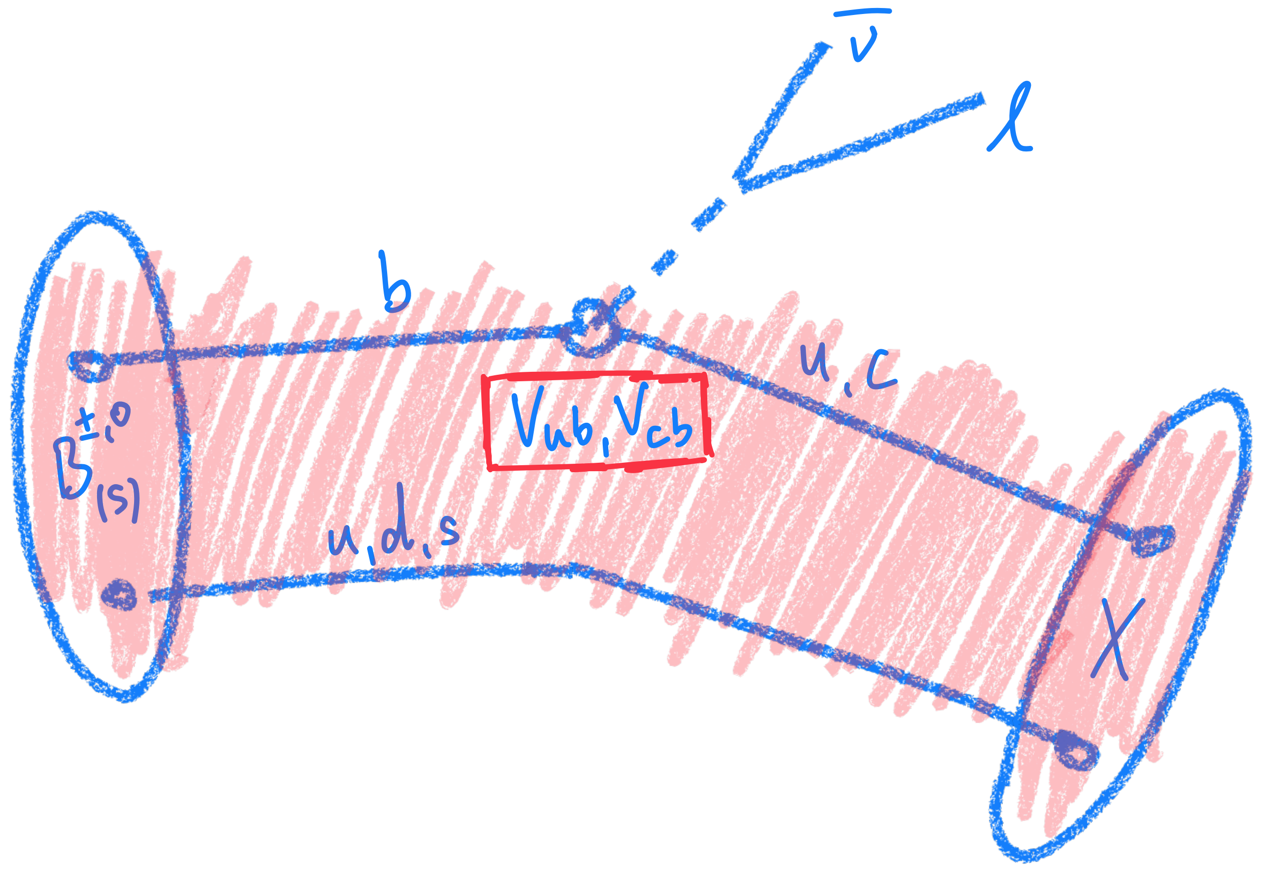
Γ_{37}	$\mu^+\mu^-\mu^+\nu_\mu$		$< 1.6 \times 10^{-8}$	CL=95%	2634

Inclusive SL decay in the SM

$$\frac{d\Gamma}{dq^2 dq^0 dE_l} = \frac{G_F^2 |V_{qQ}|^2}{8\pi} L^{\mu\nu} W_{\mu\nu}$$

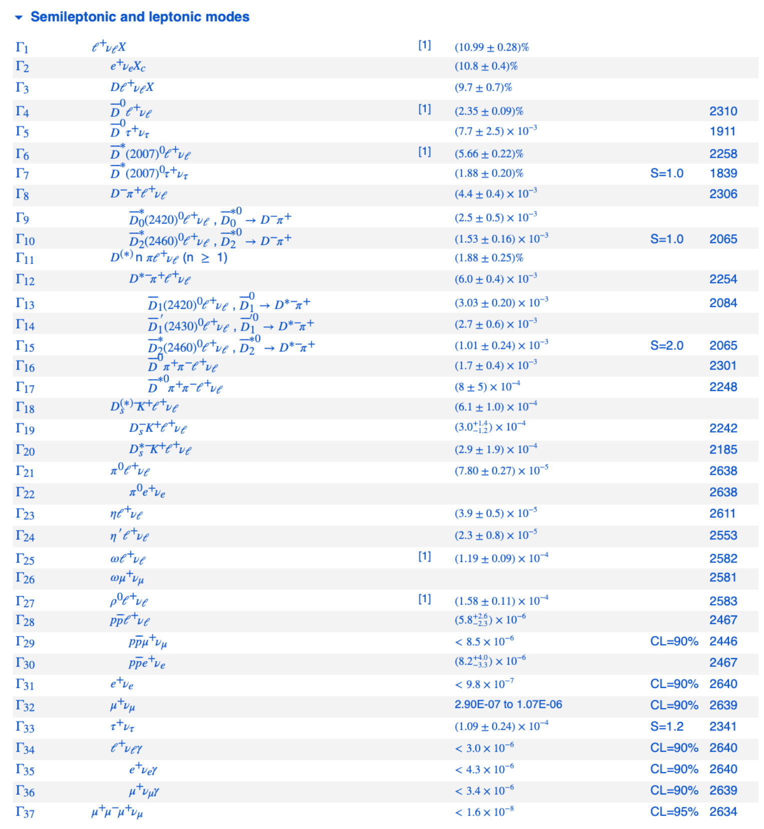
hadronic tensor

leptonic tensor



$B \to X_c \ell \nu$ in the B rest frame:

$$W^{\mu\nu}(p_B, q) = \frac{1}{2E_B} \sum_{X_c} (2\pi)^3 \delta^{(4)}(p_B - q - p_{X_c}) \langle B(\mathbf{0}) | (\tilde{J}^\mu(q^2))^\dagger | X_c(p_{X_c}) \rangle \langle X_c(p_{X_c}) | \tilde{J}^\nu(q^2) | B(\mathbf{0}) \rangle$$



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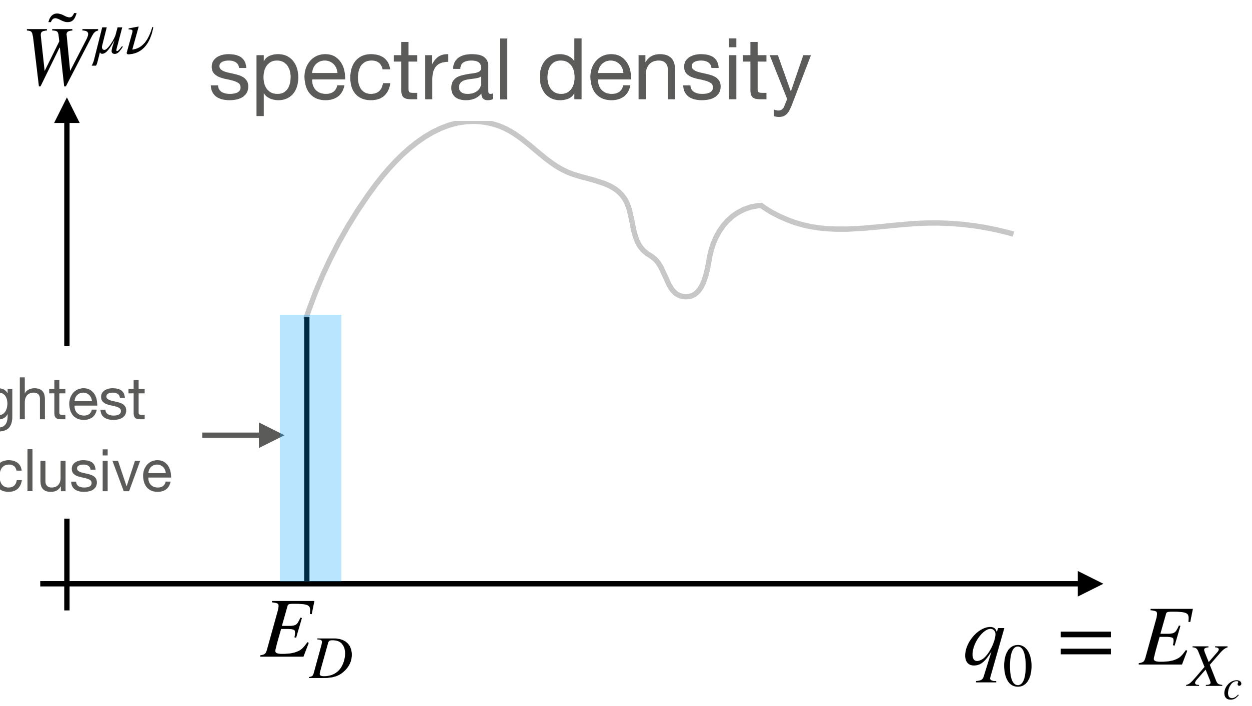


- $W_{\mu\nu}$ is **spectral density**
- continuous distribution in infinite volume

Inclusive SL decay in the SM

$$\frac{d\Gamma(B \to X_c \ell \nu)}{dq^2 dq^0 dE_l} = \frac{G_F^2 |V_{cb}|^2}{8\pi} L^{\mu\nu} W_{\mu\nu}$$

hadronic tensor

leptonic tensor

kinematics:

$$\omega_{\min} = \sqrt{M_D^2 + \mathbf{q}^2}$$

$$\omega_{\max} = M_B - \sqrt{\mathbf{q}^2}$$

$$\mathbf{q}^2_{\max} = \frac{(M_B^2 - M_D^2)^2}{4M_B^2}$$

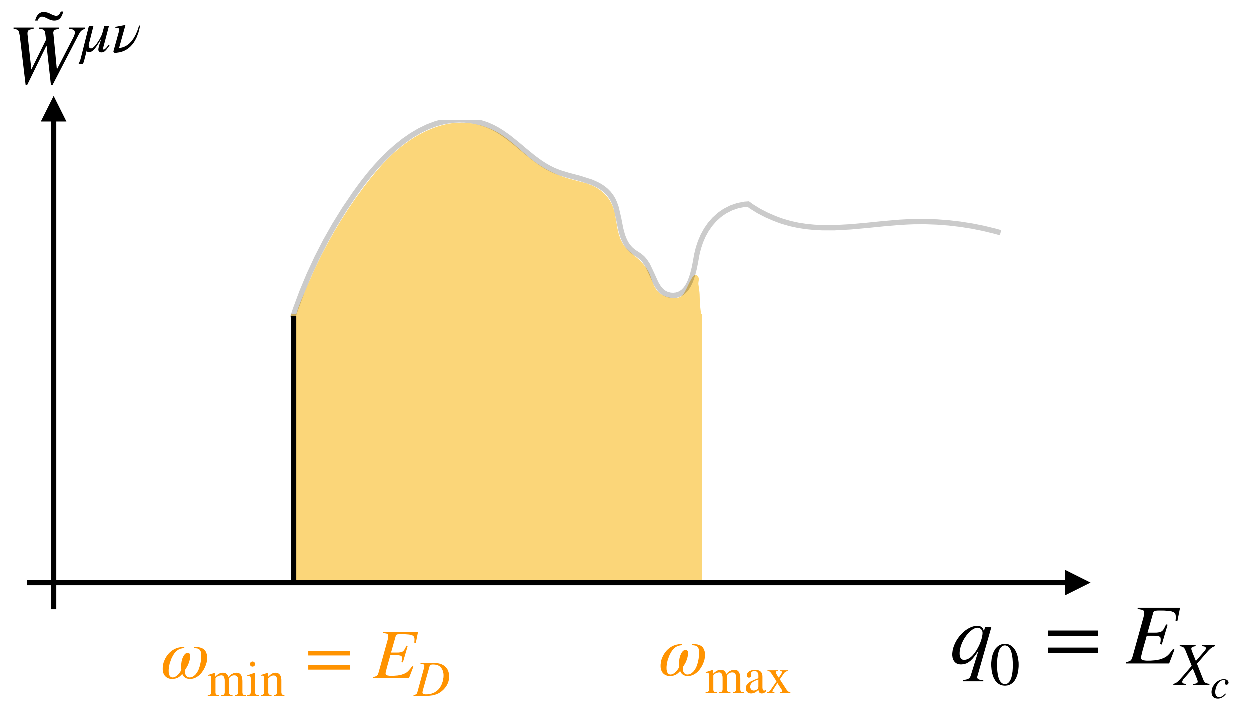
Integrate phase space:

$$\Gamma(B \to X_c l \nu) = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\mathbf{q}^2_{\max}} d\mathbf{q}^2 \sqrt{\mathbf{q}^2}\, \bar{X}(\mathbf{q}^2)$$

where

$$\bar{X}(\mathbf{q}^2) = \int_{\omega_{\min}}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q}) K^{\mu\nu}(\omega, \mathbf{q})$$

- integration over lepton energy done analytically: $L^{\mu\nu} \to K^{\mu\nu}$
- $\mathbf{q}$ is three-momentum transfer
- ω is energy of intermediate state X_c

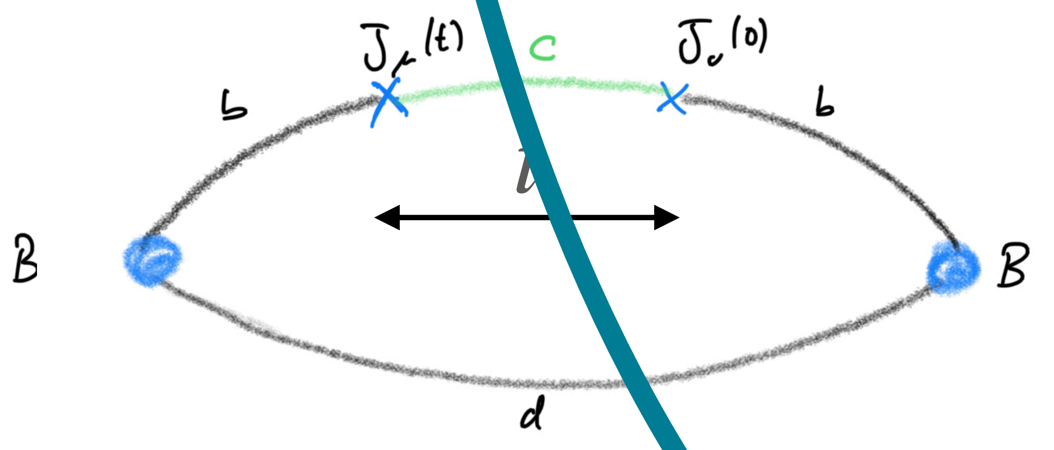


Inclusive SL decay on the lattice

$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega \, W_{\mu\nu}(\omega, \mathbf{q}) \, K^{\mu\nu}(\omega, \mathbf{q})$$

$$W_{\mu\nu}(\omega, \mathbf{q}) = \frac{i}{2M_B} \int d^4x \, e^{iq\cdot x} \langle B(\mathbf{0}) \,|\, J_\mu^\dagger(x) J_\nu(0) \,|\, B(\mathbf{0}) \rangle$$

standard lattice computation:



$$C_{\mu\nu}(t, \mathbf{q}) = \frac{C_{4\text{pt}}^{\mu\nu}(t_2, t_1)}{C_{2\text{pt}}(t_2) C_{2\text{pt}}(t_1)} \to \sum_{\mathbf{x}} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{2M_B} \langle B \,|\, J_\mu^\dagger(\mathbf{x}, t) \, J_\nu(\mathbf{0}, 0) \,|\, B \rangle \qquad (t = t_2 - t_1 \geq 0)$$

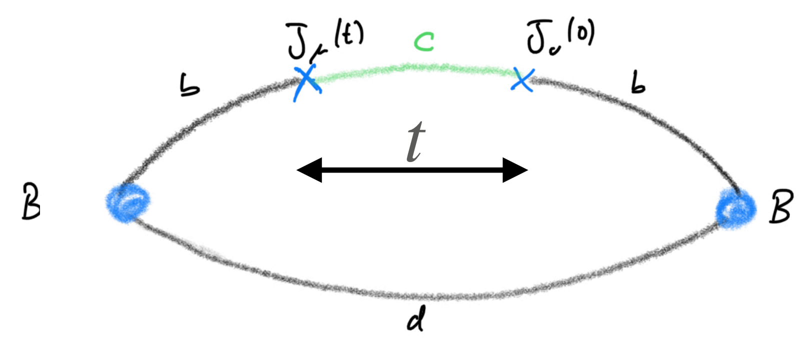
$$= \int_{\omega_{\min}}^{\infty} d\omega \frac{1}{2M_B} \langle B \,|\, (\tilde{J}_\mu(\mathbf{q}, 0)^\dagger \delta(\hat{H} - \omega) \tilde{J}_\nu(\mathbf{q}, 0) \,|\, B \rangle \, e^{-t\omega}$$

?

$$C_{\mu\nu}(t, \mathbf{q}) = \int_{\omega_{\min}}^{\infty} d\omega \, W_{\mu\nu}(\omega, \mathbf{q}) \, e^{-\omega t}$$

Euclidean 4pt function (computable on the lattice) is Laplace transform of hadronic tensor

$\bar{X}(\mathbf{q})$ reconstruction



$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega,\mathbf{q})\, K^{\mu\nu}(\mathbf{q},\omega) \quad \longleftrightarrow \quad C_{\mu\nu}(t,\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega,\mathbf{q})\, e^{-\omega t}$$

Expand the kernel K (analytically known) in powers of $e^{-a\omega}$: $K^{\mu\nu}(\mathbf{q},\omega) = \sum_k c_k^{\mu\nu}(\mathbf{q})\left(e^{-a\omega}\right)^k$

$$\bar{X}(\mathbf{q}) \approx c_0^{\mu\nu}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q}) \quad + c_1^{\mu\nu}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-a\omega} \quad + c_2^{\mu\nu}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-2a\omega} \quad + \ldots$$

$$= c_0^{\mu\nu}(\mathbf{q})\;\; C_{\mu\nu}(0,\mathbf{q}) \quad + c_1^{\mu\nu}(\mathbf{q})\;\; C_{\mu\nu}(a,\mathbf{q}) \quad + c_2^{\mu\nu}(\mathbf{q})\;\; C_{\mu\nu}(2a,\mathbf{q}) \quad + \ldots$$

linear transformation (basis change) to (shifted) Chebyshev basis

$$= \tilde{c}_0^{\mu\nu}(\mathbf{q})\;\; \langle \tilde{T}_0 \rangle_{\mu\nu}(\mathbf{q}) \quad + \tilde{c}_1^{\mu\nu}(\mathbf{q})\;\; \langle \tilde{T}_1 \rangle_{\mu\nu}(\mathbf{q}) \quad + \tilde{c}_2^{\mu\nu}(\mathbf{q})\;\; \langle \tilde{T}_2 \rangle_{\mu\nu}(\mathbf{q}) \quad + \ldots$$

$\langle \tilde{T}_k \rangle_{\mu\nu}(\mathbf{q})$ *Chebyshev Matrix Elements* computed on the lattice

$\tilde{c}_k^{\mu\nu}(\mathbf{q})$ coefficients computed *analytically* (no lattice input required)

$\bar{X}(\mathbf{q})$ reconstruction

$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, K^{\mu\nu}(\mathbf{q}, \omega) \quad \longleftrightarrow \quad C_{\mu\nu}(t, \mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega t}$$

$$\begin{aligned} \bar{X}(\mathbf{q}) &= \sum_k c_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega k} \\ &= \sum_k c_{\mu\nu,k}(\mathbf{q})\, C_{\mu\nu}(ak, \mathbf{q}) \\ &= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q}) \langle \tilde{T}_k \rangle_{\mu\nu} \end{aligned}$$

- fully determined linear system
- coefficients $\tilde{c}_{\mu\nu,k}(\mathrm{q})$ known

required nonperturbative input:

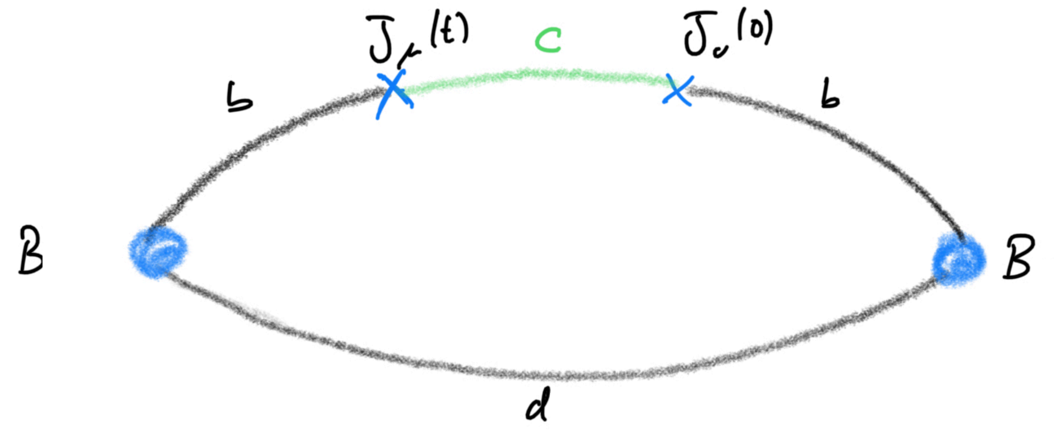
- $C_{\mu\nu}(t, \mathbf{q})$ computed in Lattice QCD
- $\langle \tilde{T}_k \rangle_{\mu\nu}$ linear combination of $C_{\mu\nu}(t, \mathbf{q})$

$\bar{X}$ can be *reconstructed* from a lattice computation of the Euclidean correlator $C_{\mu\nu}(t, \mathbf{q})$

$\bar{X}(\mathbf{q})$ reconstruction

Barata, Fredenhagen, CMP 138 (1991)
Hashimoto PTEP 53-56 (2017)
Hansent et al. PRD 99 (2019)
Bailas et al. PTEP 43-50 (2020)
Gambino and Hashimoto PRL 125 32001 (2020)
Gambino et al. JHEP 07 (2022) 083
Barone et al. JHEP 07 (2023) 145
Barone et al. PRD 112 (2025) 1
De Santis et al. PRD 112 (2025) 5
De Santis et al. PRL 135 (2025) 12
Tantalo, Patella, JHEP 01 (2025) 091

$$\bar{X}(\mathbf{q}) = \sum_k c_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega k}$$
$$= \sum_k c_{\mu\nu,k}(\mathbf{q})\, C_{\mu\nu}(ak, \mathbf{q})$$
$$= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q})\, \langle \tilde{T}_k \rangle_{\mu\nu}$$



In practice:

- signal-to-noise for Euclidean correlators in lattice QCD deteriorates with t
  → cannot extract meaningful signal

Mitigation:

- use that $C_{\mu\nu}(t)$ monotonously decreasing with t
- **translates into $|\langle \tilde{T}_k \rangle_{\mu\nu}| \leq 1$ as regulator implemented as uniform Bayesian prior**

alternative mitigation strategy:
balance resolution power of kernel approximation against stat. noise (see work by De Santis et al.)

$\bar{X}(\mathbf{q})$ reconstruction

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De Santis et al. PRL 135 (2025) 12
Tantalo, Patella, JHEP 01 (2025) 091

$$\begin{aligned}\bar{X}(\mathbf{q}) &= \sum_k c_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega k} \\ &= \sum_k c_{\mu\nu,k}(\mathbf{q})\, C_{\mu\nu}(ak, \mathbf{q}) \\ &= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q})\, \langle \tilde{T}_k \rangle_{\mu\nu}\end{aligned}$$

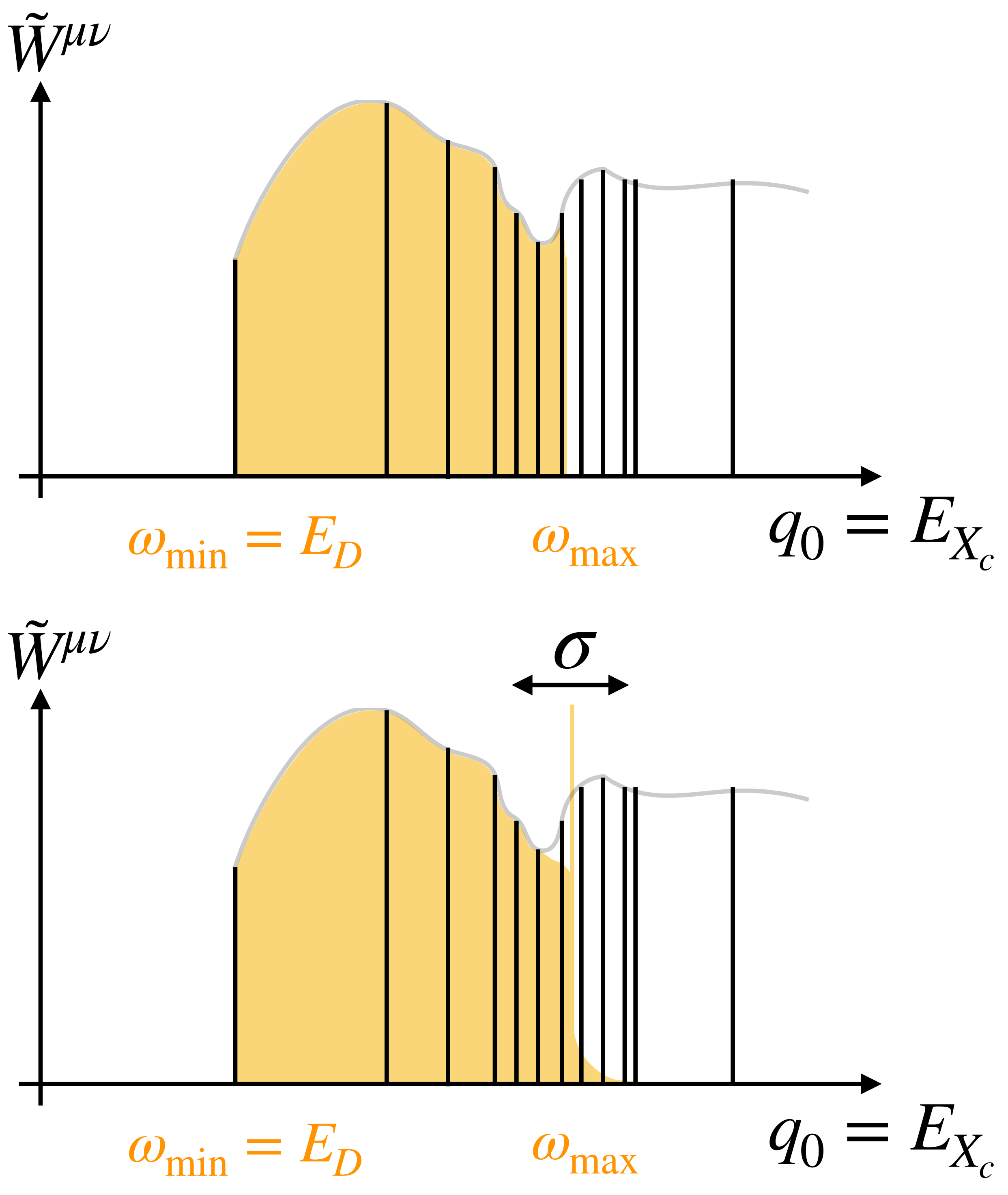
I will now discuss

- Chebyshev approximation of the leptonic kernel $K_{\mu\nu}$
- The determination of the Chebyshev matrix elements $\langle \tilde{T}_k \rangle_{\mu\nu}$

A bit of smearing

$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, K^{\mu\nu}(\mathbf{q}, \omega) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, \tilde{K}^{\mu\nu}(\mathbf{q}, \omega)\theta_\sigma(\omega_{\max} - \omega)$$

where $\omega_{\max} = M_B - \sqrt{\mathbf{q}^2}$

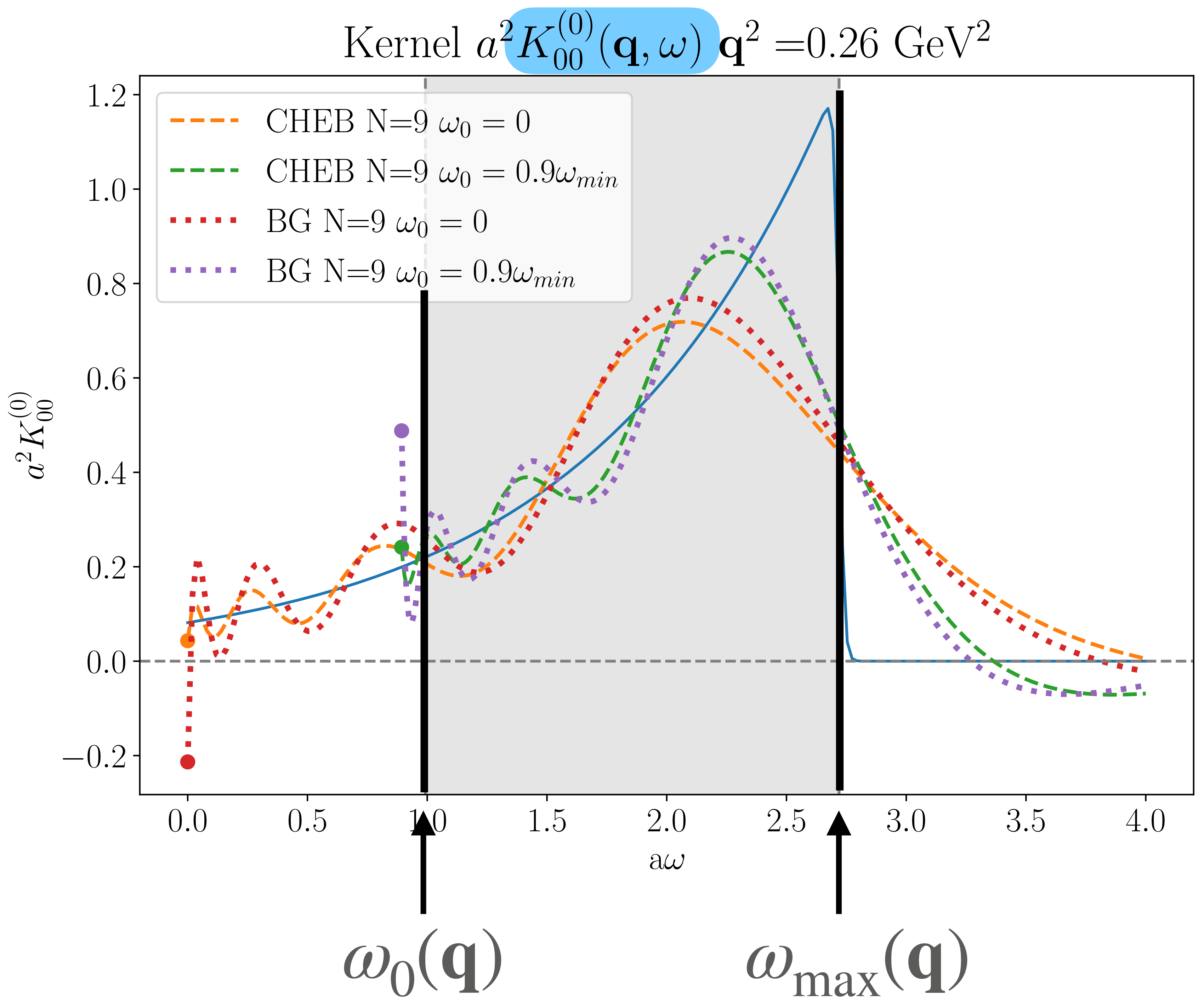


- Spectral density in finite volume is discrete
- Integration with the leptonic kernel *smears out* discrete nature ("*nature-given* smearing")
- In addition, **we smear the Heaviside step function** (sigmoid) over a width σ
  - this allows for well-defined convergence of the Chebyshev approximation
  - it also smoothes out finite-volume effects
  - smearing needs to be removed eventually*

Chebyshev expansion of the leptonic kernel

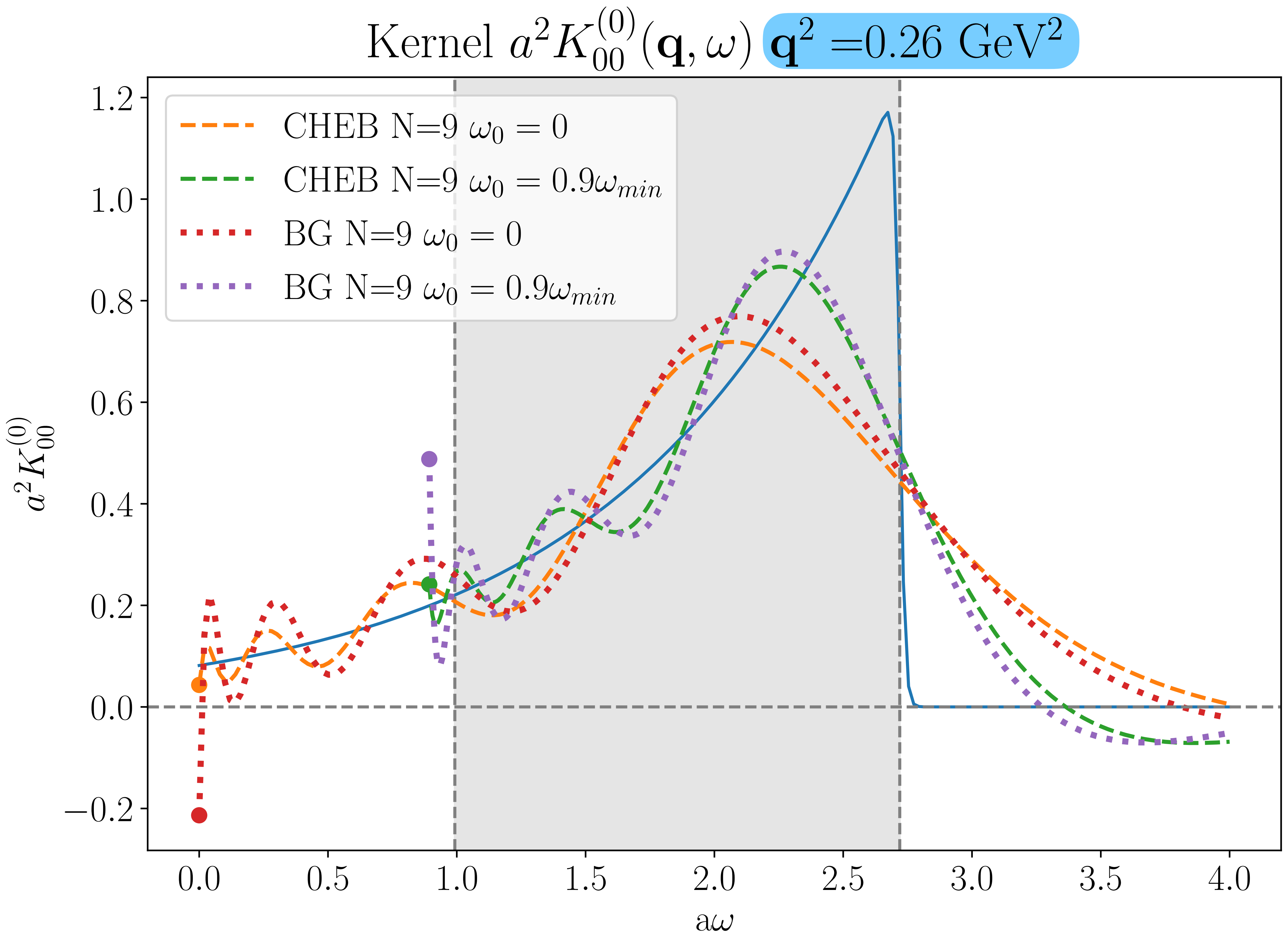
$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, K^{\mu\nu}(\mathbf{q}, \omega)$$

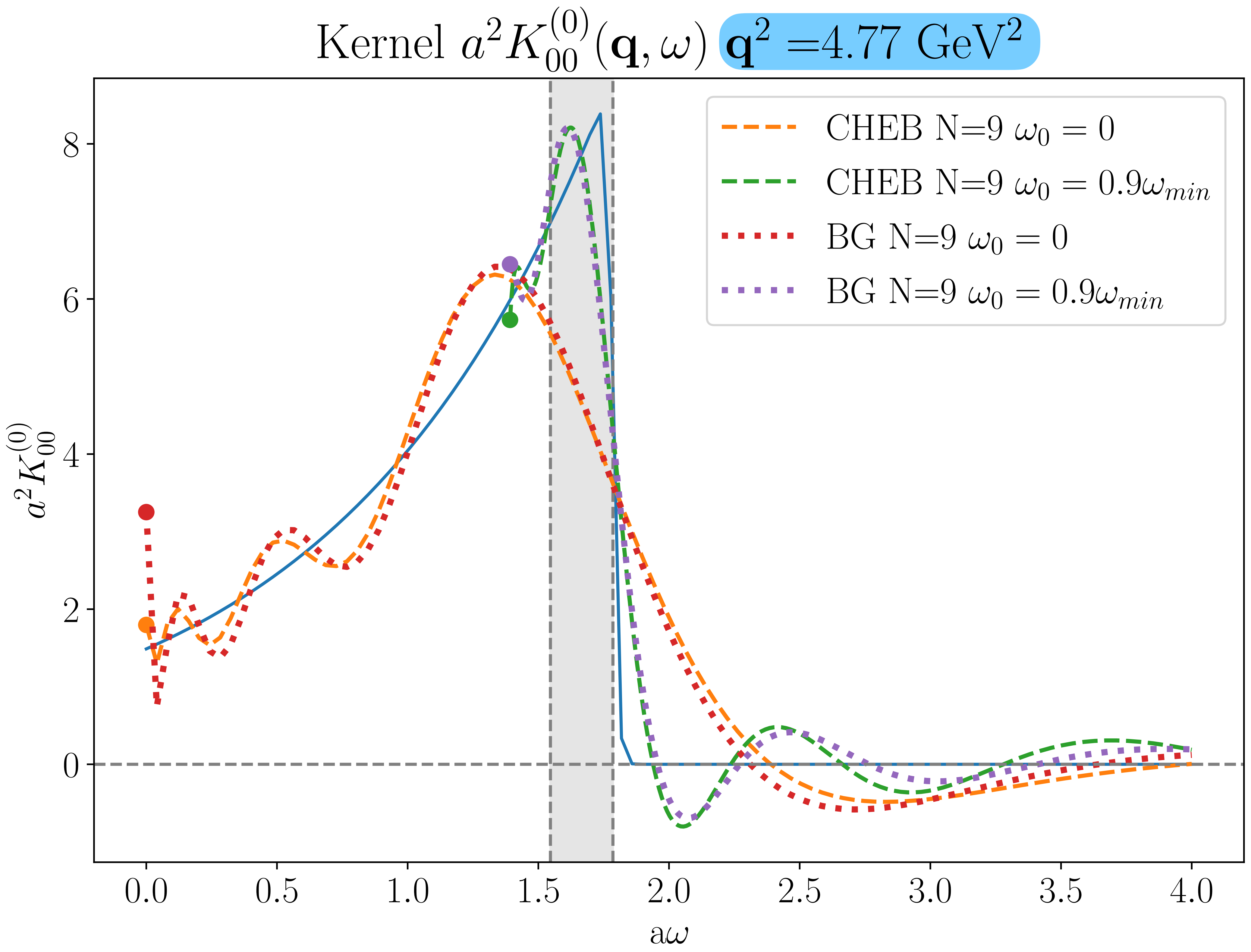
massless-lepton kernel $K^{\mu\nu}(q) \sim \frac{|\mathbf{q}^2| q^2}{3} \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right) \approx \frac{\tilde{c}_{\mu\nu,0}}{2} + \sum_{k=1}^{N} \tilde{c}_{\mu\nu,k} \tilde{T}_k(\omega)$



- this analysis stage **independent of data**
- order of approximation limit by lattice calculation $N \leftrightarrow C_{\mu\nu}(t)$ (noise/signal of lattice data)
- suggests relation $\sigma N = \text{const}$.
- $\omega_0 \lesssim E_D$

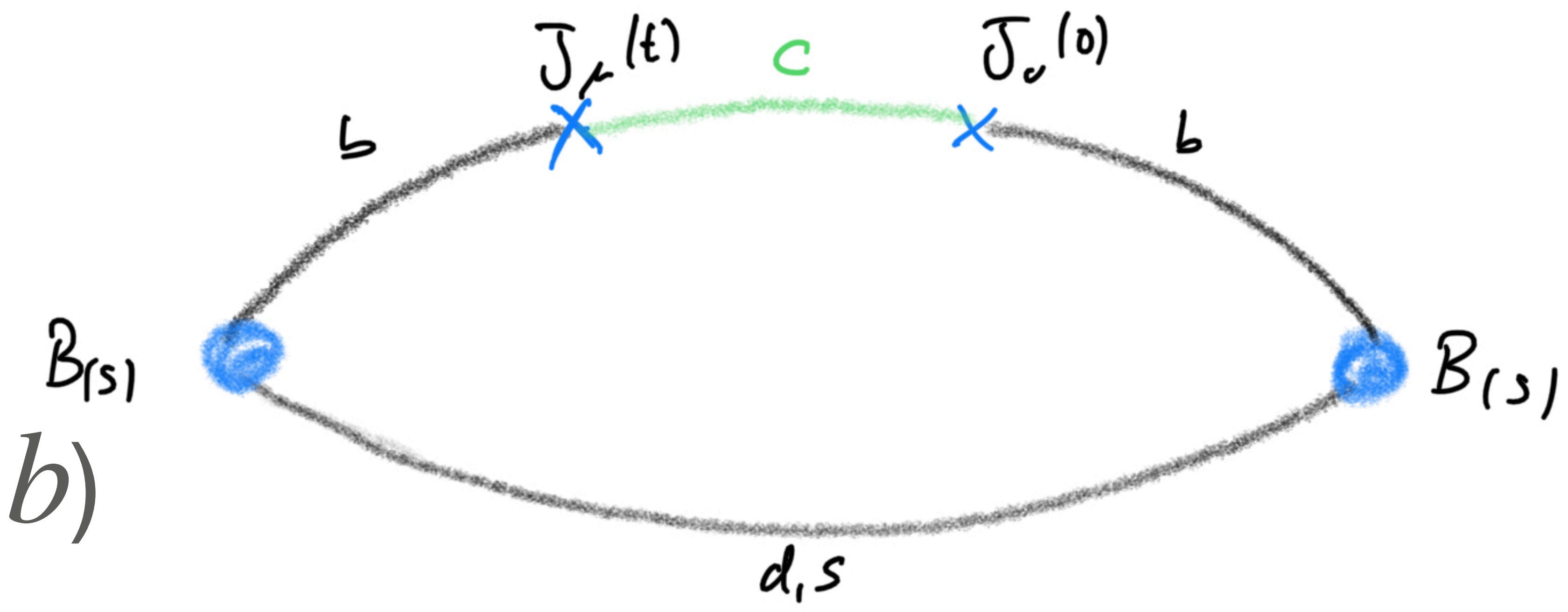
Chebyshev expansion of the leptonic kernel



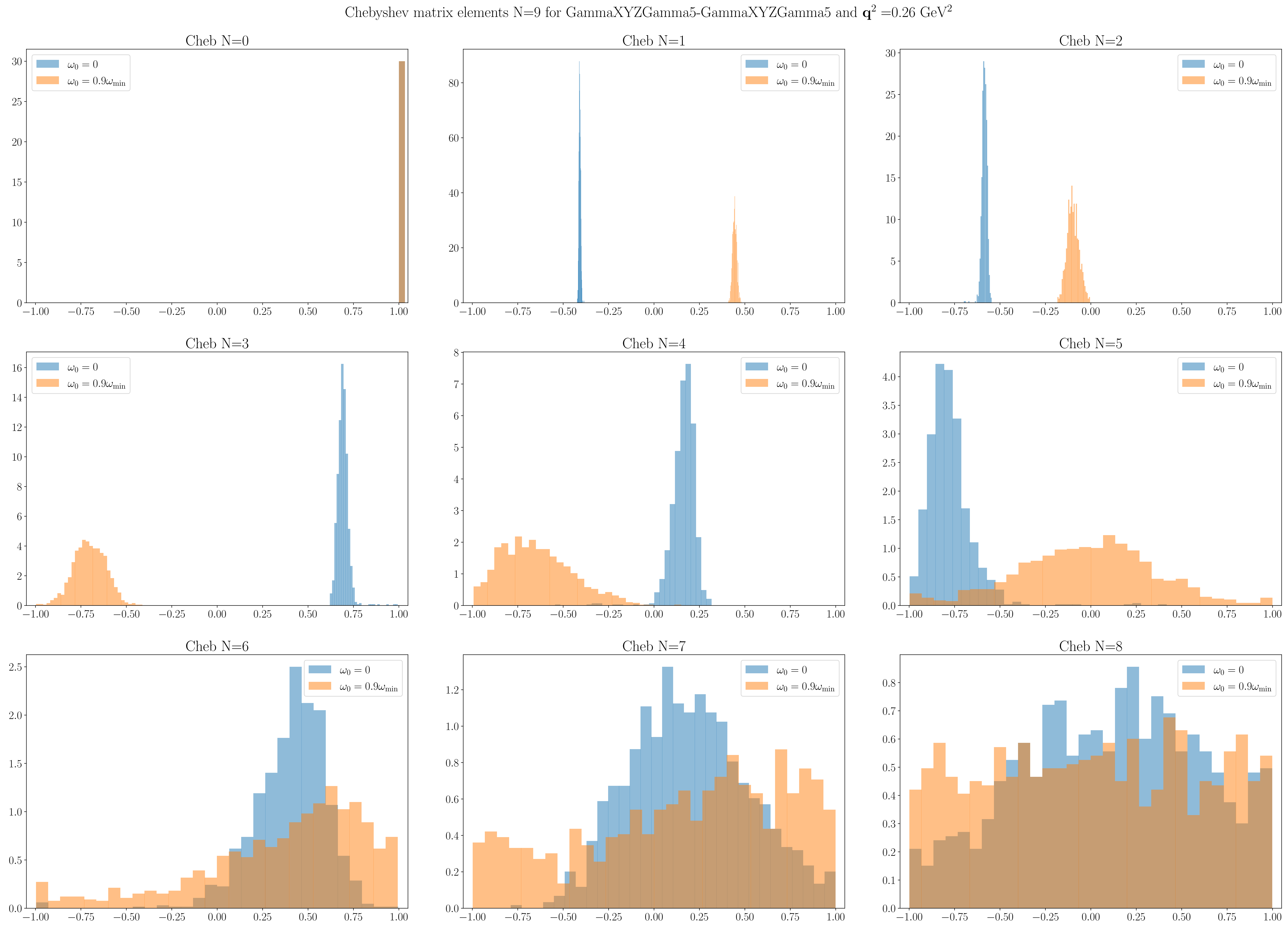


Exploratory study

- $B_s \to X_c \ell \nu$
- lattice study on $24^3 \times 64$ RBC/UKQCD DWF ensemble ($M_\pi^{\mathrm{sea}} \approx 330\,\mathrm{MeV}$)
- physical m_s- and m_b-quark masses (RHQ action for b) near-physical m_c (domain-wall)
- implemented in Grid/Hadrons
- run on DiRAC Extreme-scaling service Tursa (A100-40 nodes)
- 120 gauge configs, 8 Z_2 noise-source planes



Results for $\langle \tilde{T}_k \rangle_{\mu\nu}$

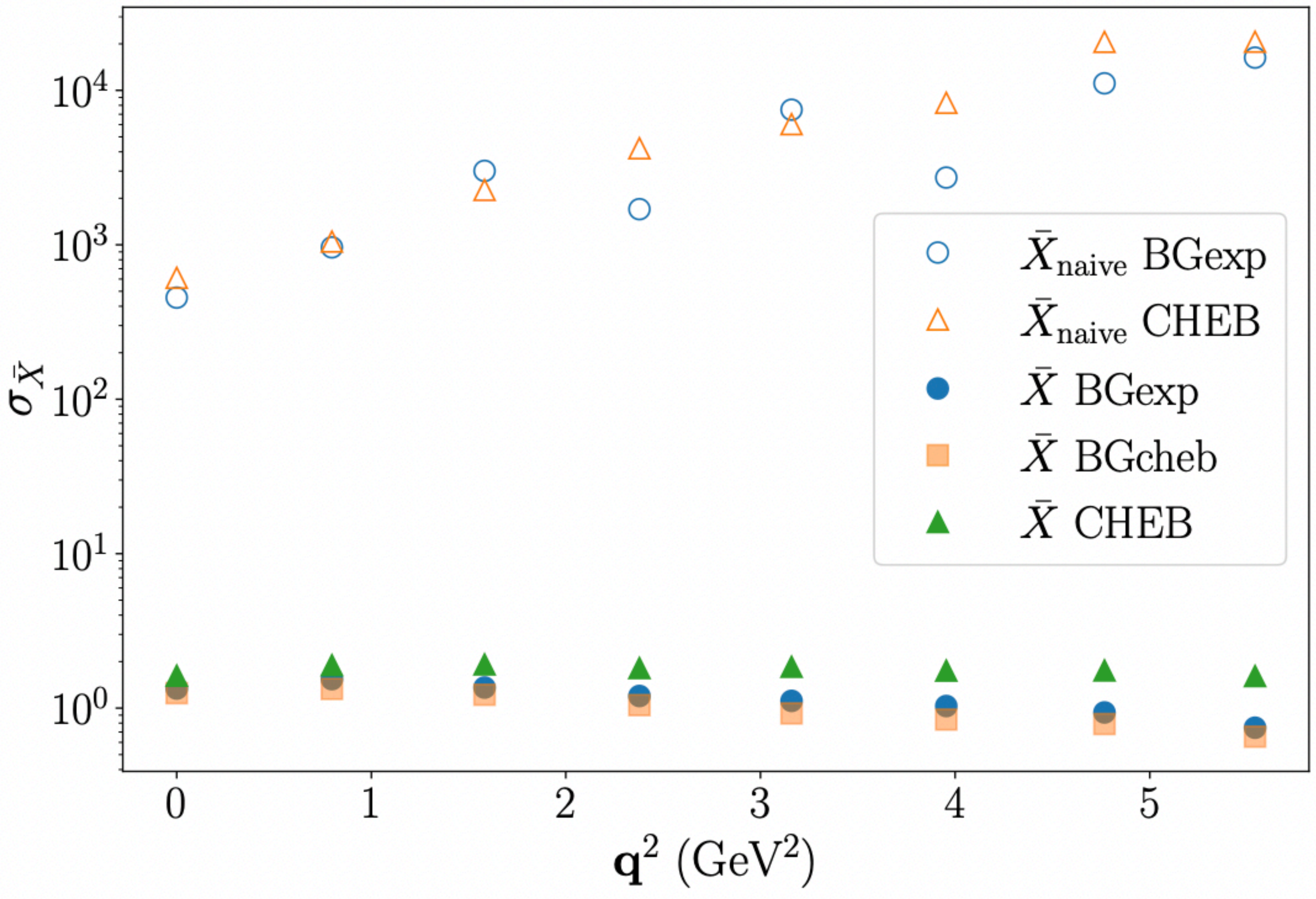


$$\bar{X}(\mathbf{q}) = \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q}) \langle \tilde{T}_k \rangle_{\mu\nu}$$

where $\langle \tilde{T}_k \rangle_{\mu\nu}(\mathbf{q}) = \sum_j \tilde{t}_j^{(k)} \tilde{C}_{\mu\nu}(j, \mathbf{q})$

- $\langle \tilde{T}_k \rangle_{\mu\nu}$ from constrained fit to lattice data for $C_{\mu\nu}(ak, \mathbf{q})$
- $|\langle \tilde{T}_k \rangle_{\mu\nu}| \leq 1$ Bayesian constraint
- higher orders affected by noise - regulator kicks in

Impact of regulator

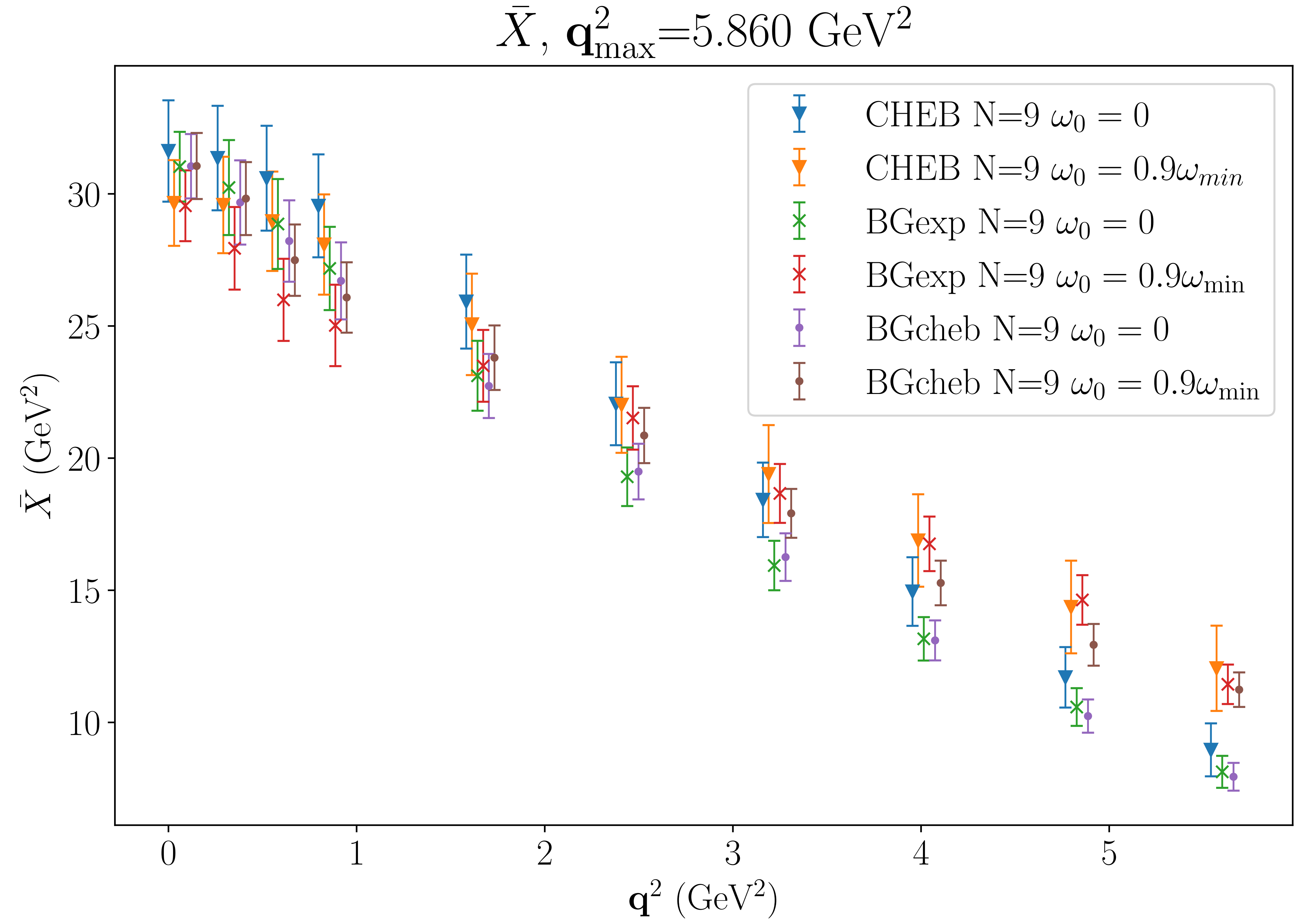


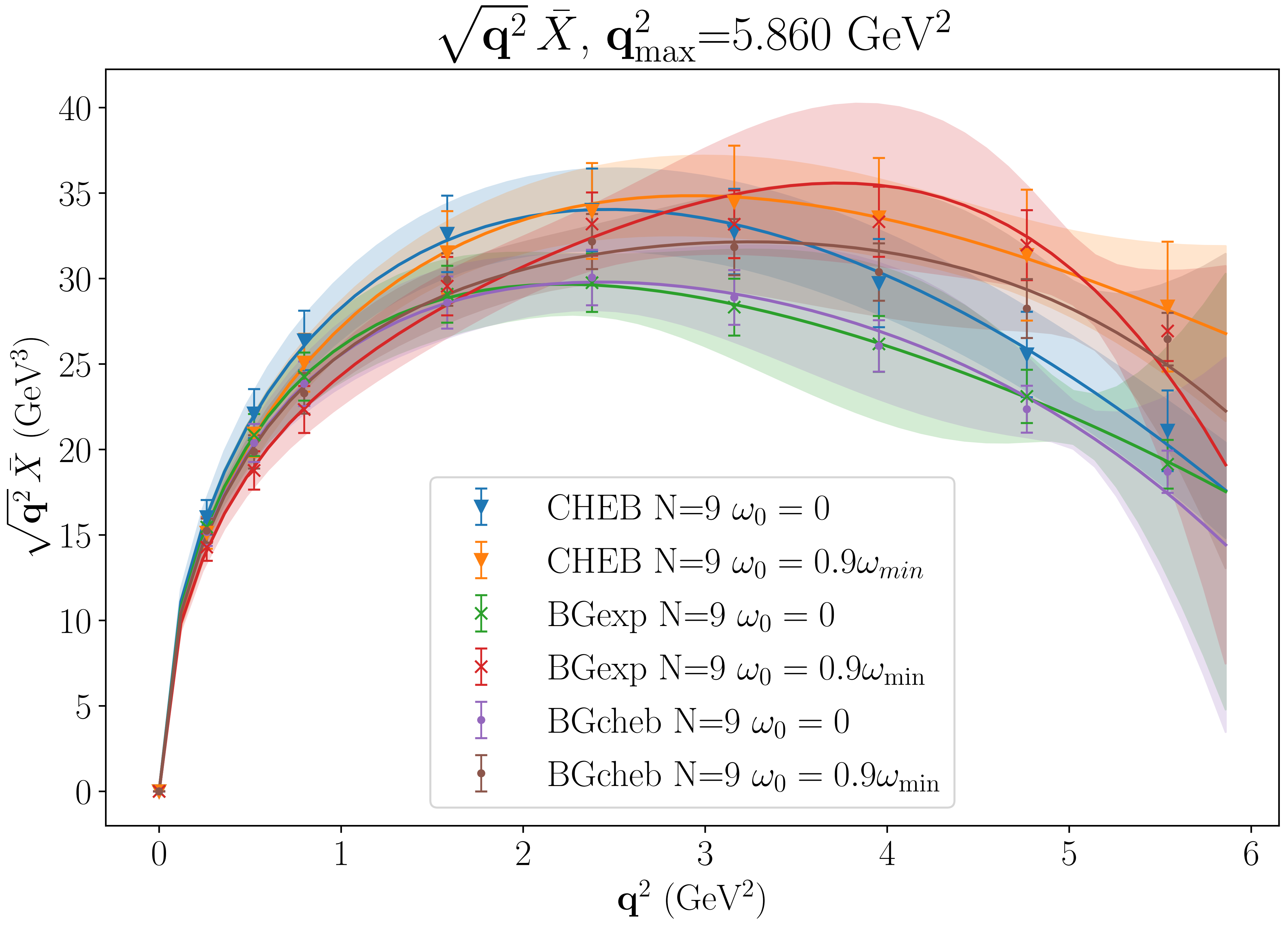
Noise reduction due to regulator term absolutely essential!

- "CHEB" is analysis as discussed
- "BGexp" is Backus-Gilbert-inspired Hansen-Lupo-Tantalo approach

Hansent et al. PRD 99 (2019),
De Santis et al. PRD 112 (2025) 5, PRL 135 (2025) 12

Results for $\bar{X}(q)$



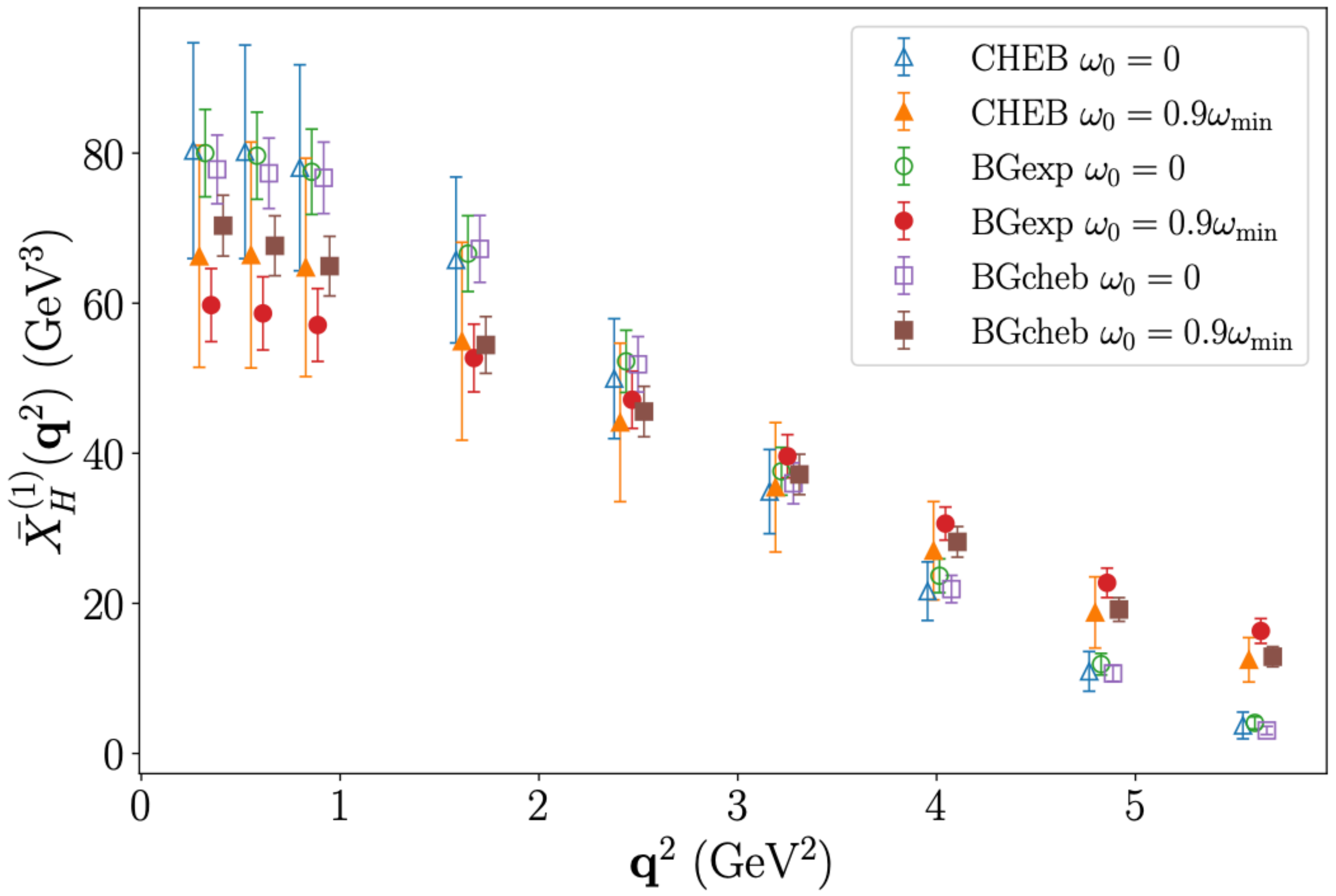


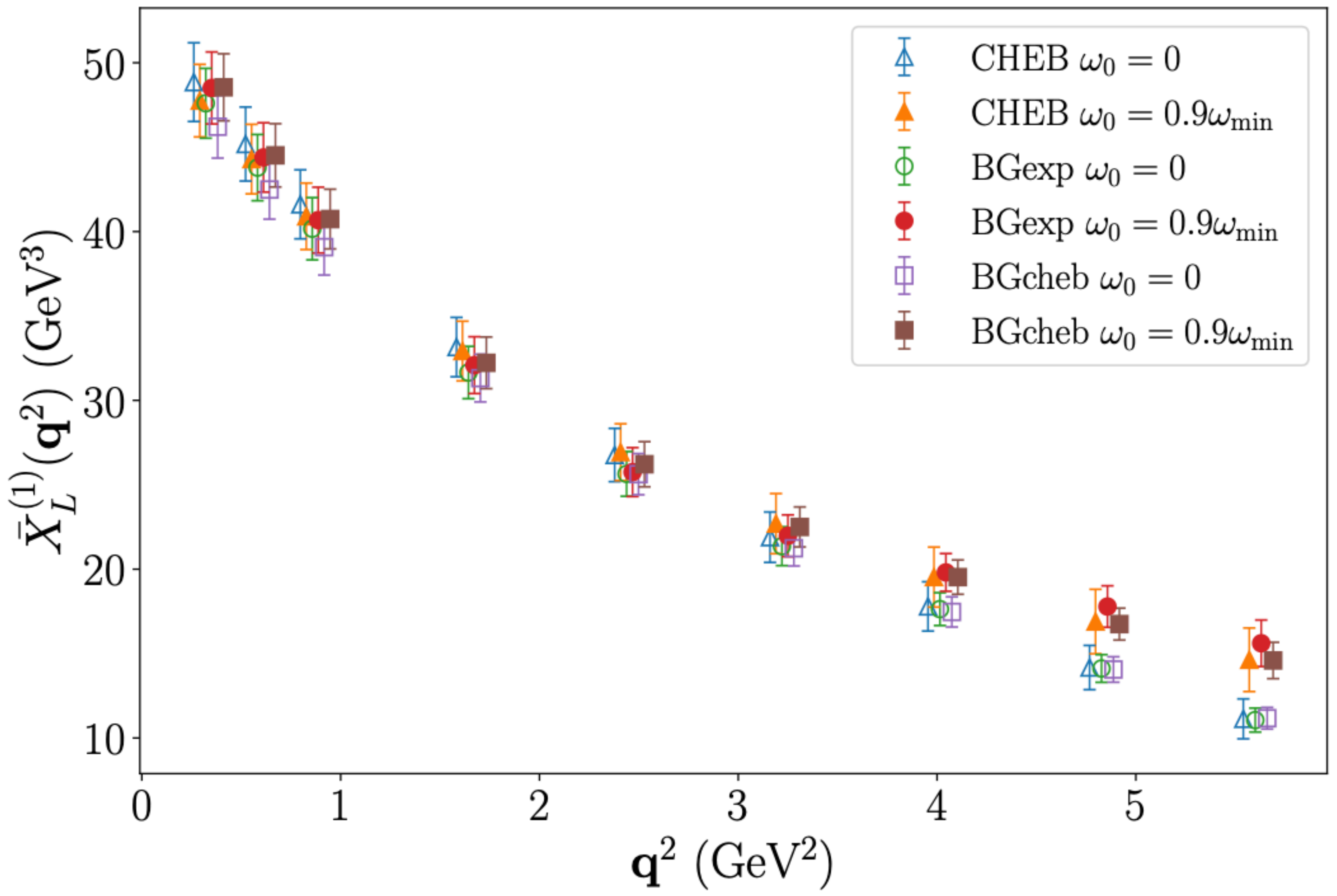
variations of analysis techniques largely consistent — tension at larger $\mathbf{q}^2$ visible

Integral of $\sqrt{\mathbf{q}^2}\,\bar{X}(\mathbf{q}^2)$ proportional to Γ;

Moments

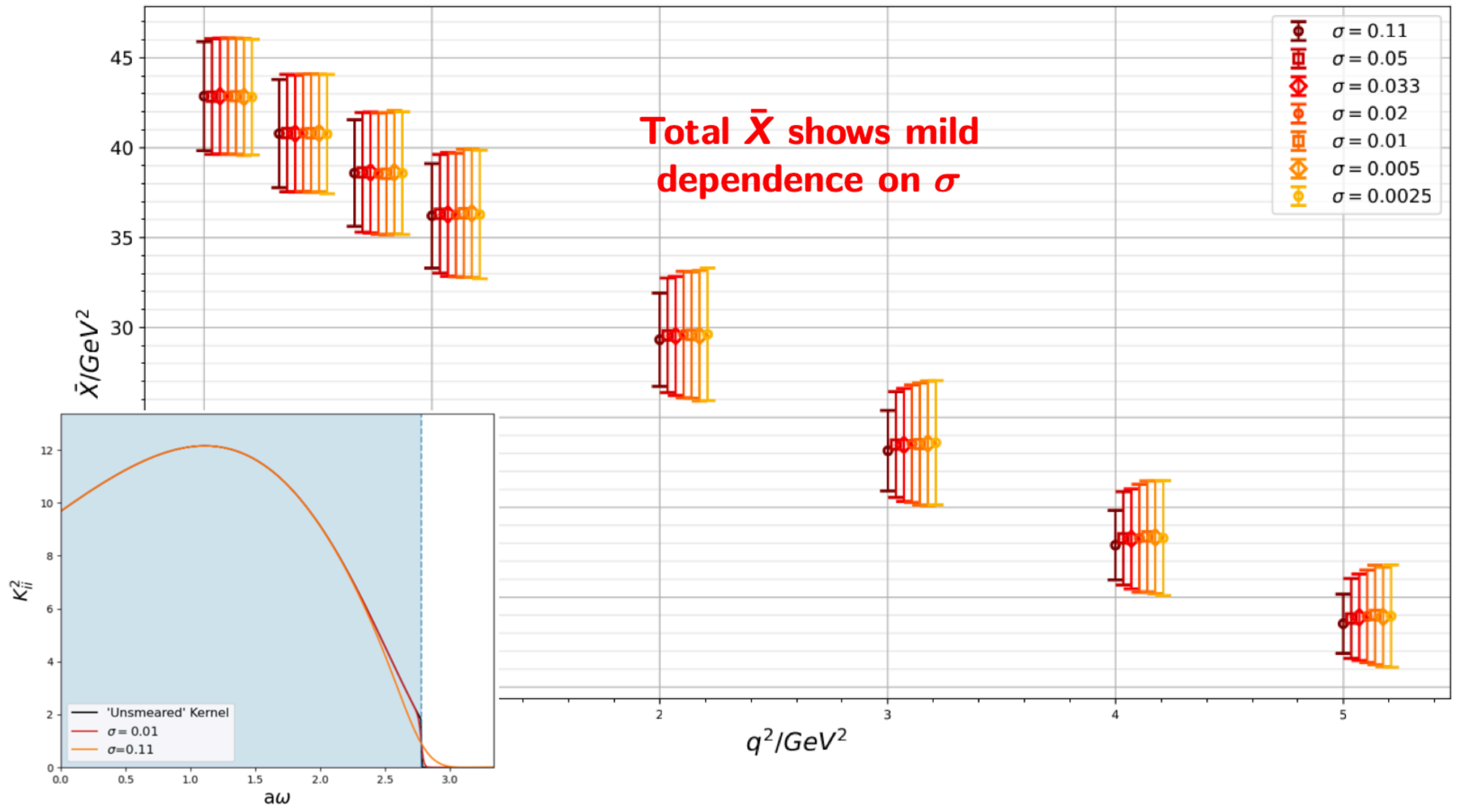
- hadronic or leptonic moments are essential building block of OPE analysis of inclusive decays
- they can be computed from the lattice data and allow for mutually scrutinising continuum and lattice computations



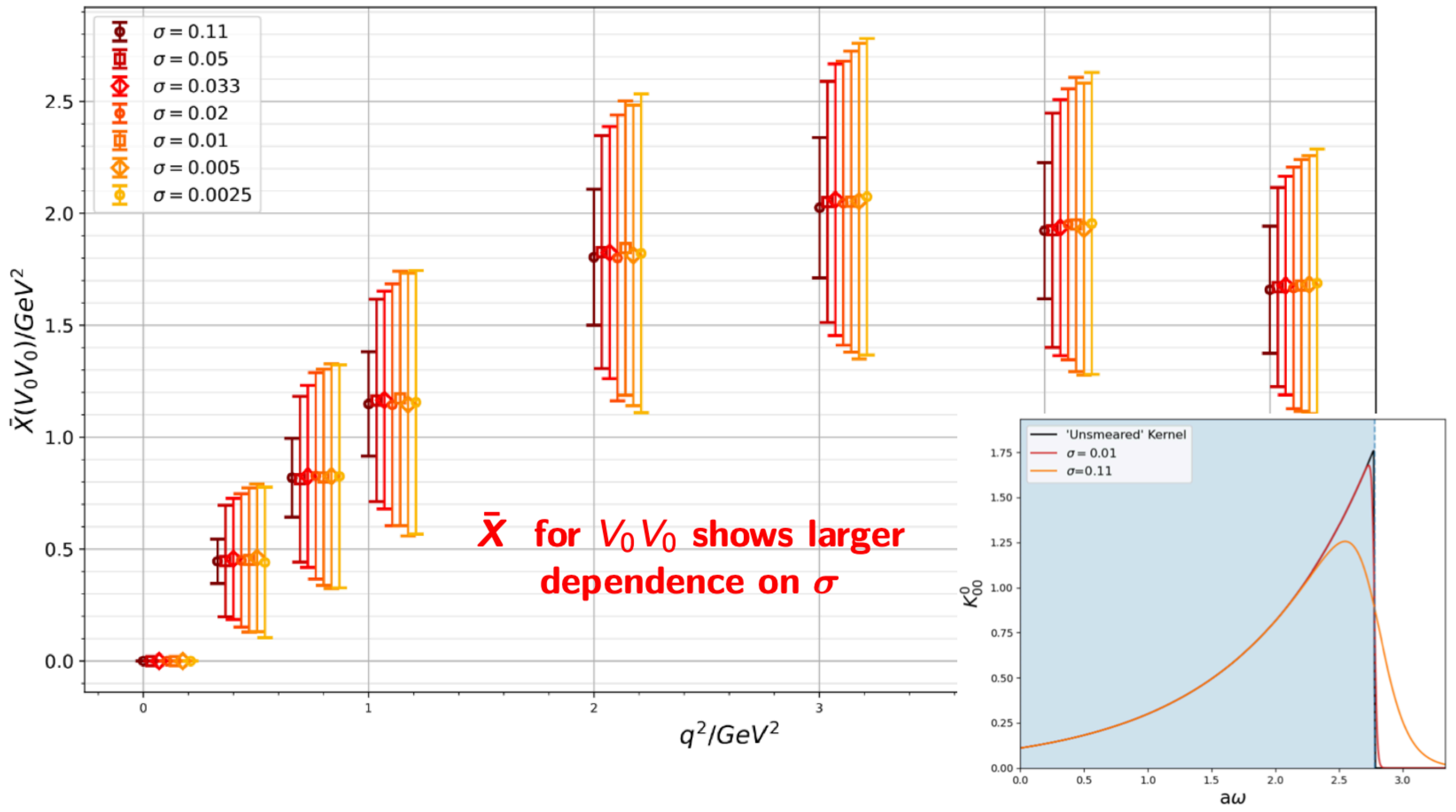


[Barone@Lattice2023]

Systematics — σ



Ahmed's Lattice 2025 talk



Convergence of Chebyshev approximation depending on the kernel and kinematics

Ground-state limit

What is the ground-state contribution to inclusive decay?

We restrict the analysis to the ground-state $B_s \to D_s$ decay: $C_{\mu\nu}(t) = C_{\mu\nu}^{\mathrm{GS}}(t) + C_{\mu\nu}^{\mathrm{ES}}(t)$

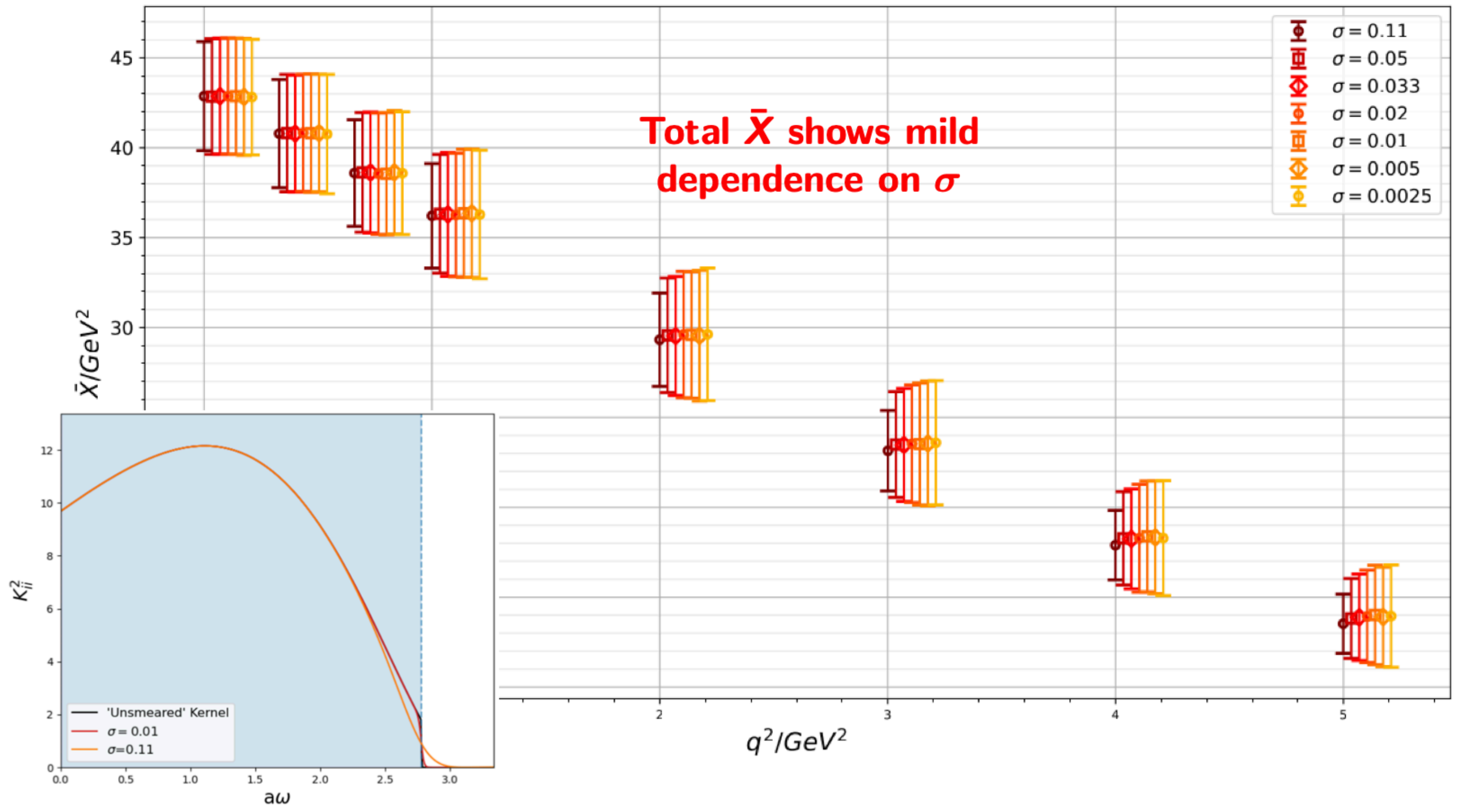
$$W_{\mu\nu} \to \delta(\omega - E_{D_s}) \frac{1}{4M_{B_s}E_{D_s}} \langle B_s | J_\mu^\dagger | D_s \rangle \langle D_s | J_\nu | B_s \rangle$$

The corresponding data is generated on the lattice (analysis of $B_s \to D_s$ 3pt/4pt correlators):

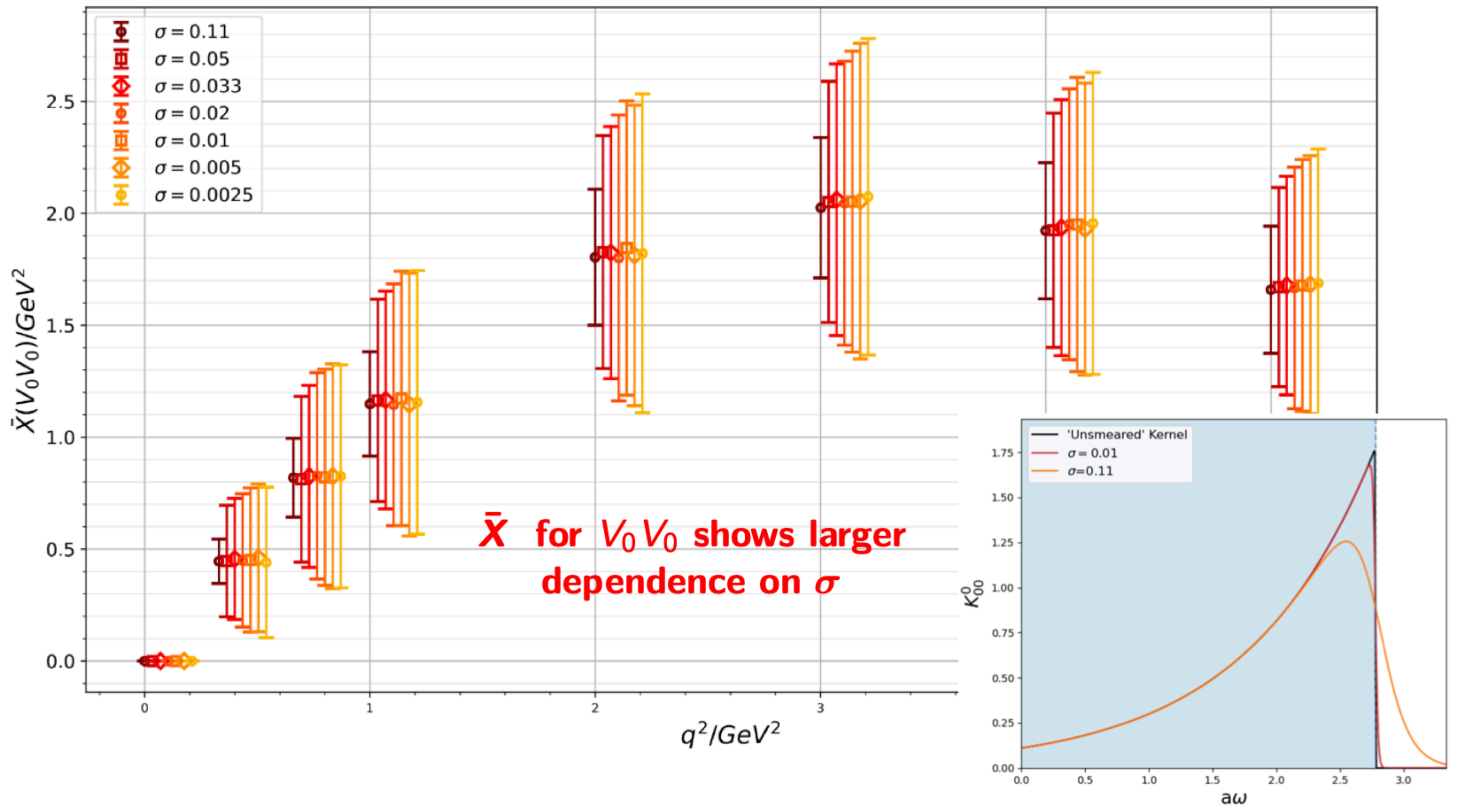
$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

$$\bar{X}_{VV}^{\parallel} \to \frac{M_{B_s}}{E_{D_s}} \mathbf{q}^2 \, | f_+(q^2) |^2$$

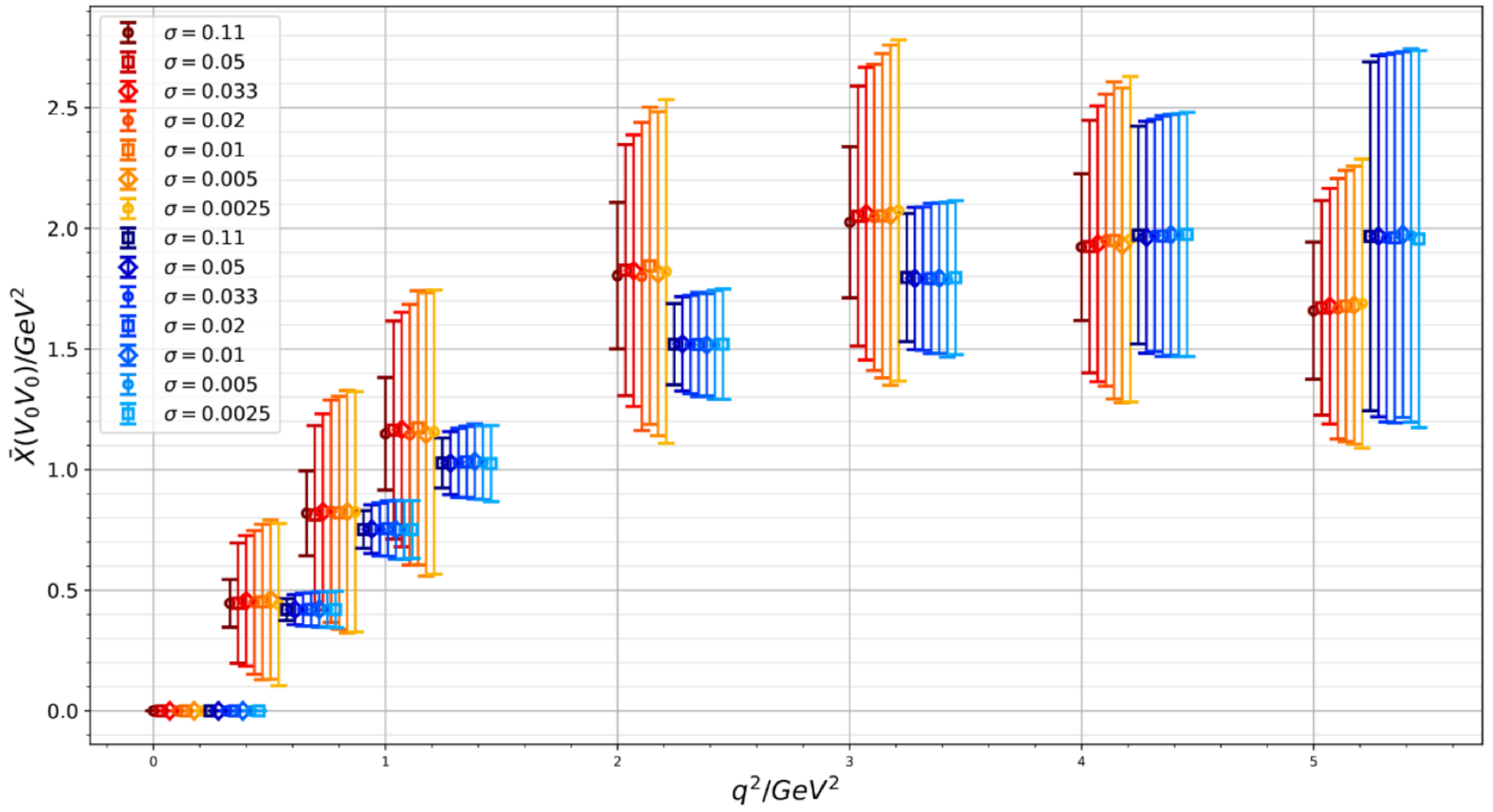
Reducing smearing systematics



[Ahmed's Lattice 2025 talk]

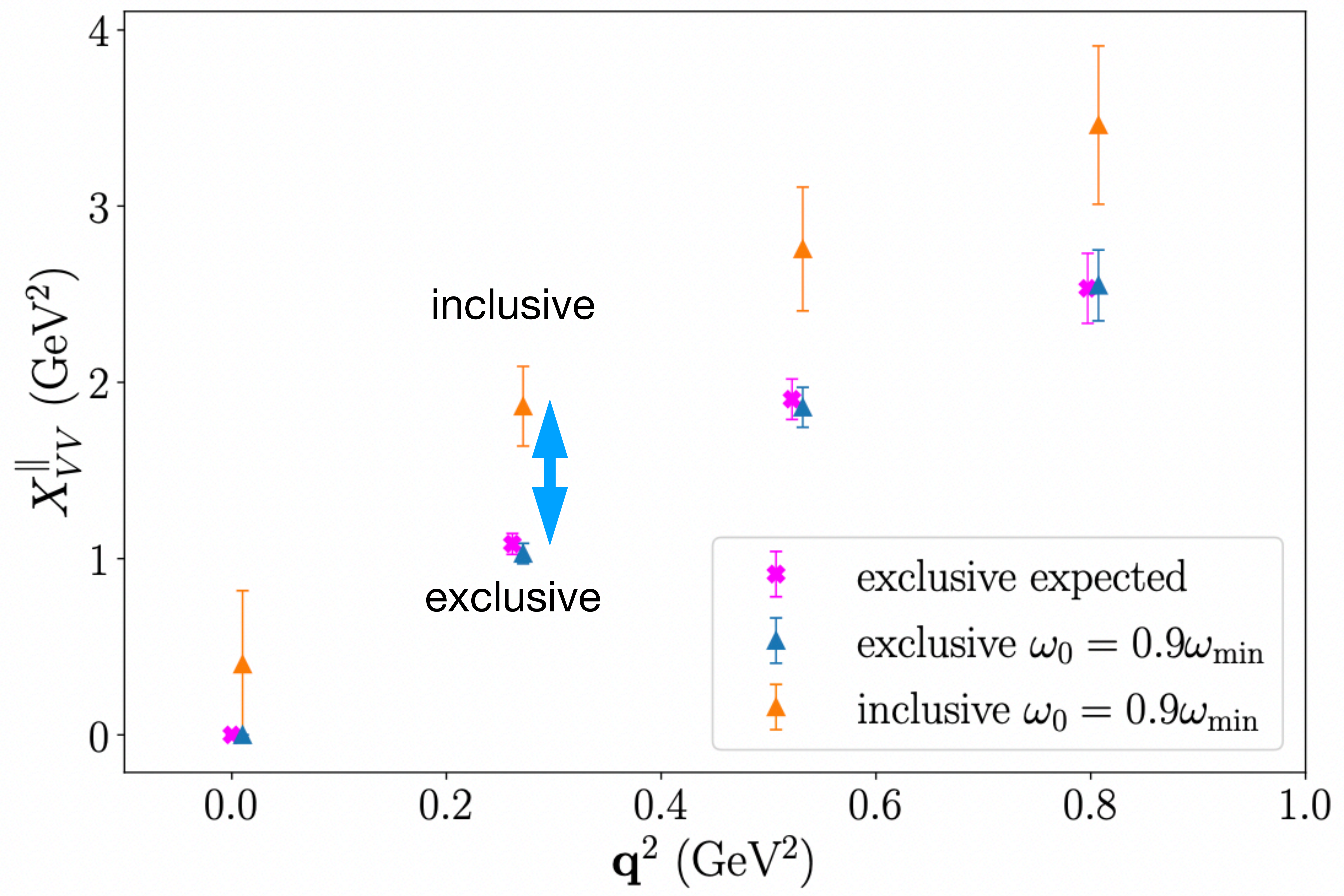


1. compute ground-state contribution with conventional lattice methods
2. subtract it from inclusive data
3. reconstruct only the excited states
4. combine



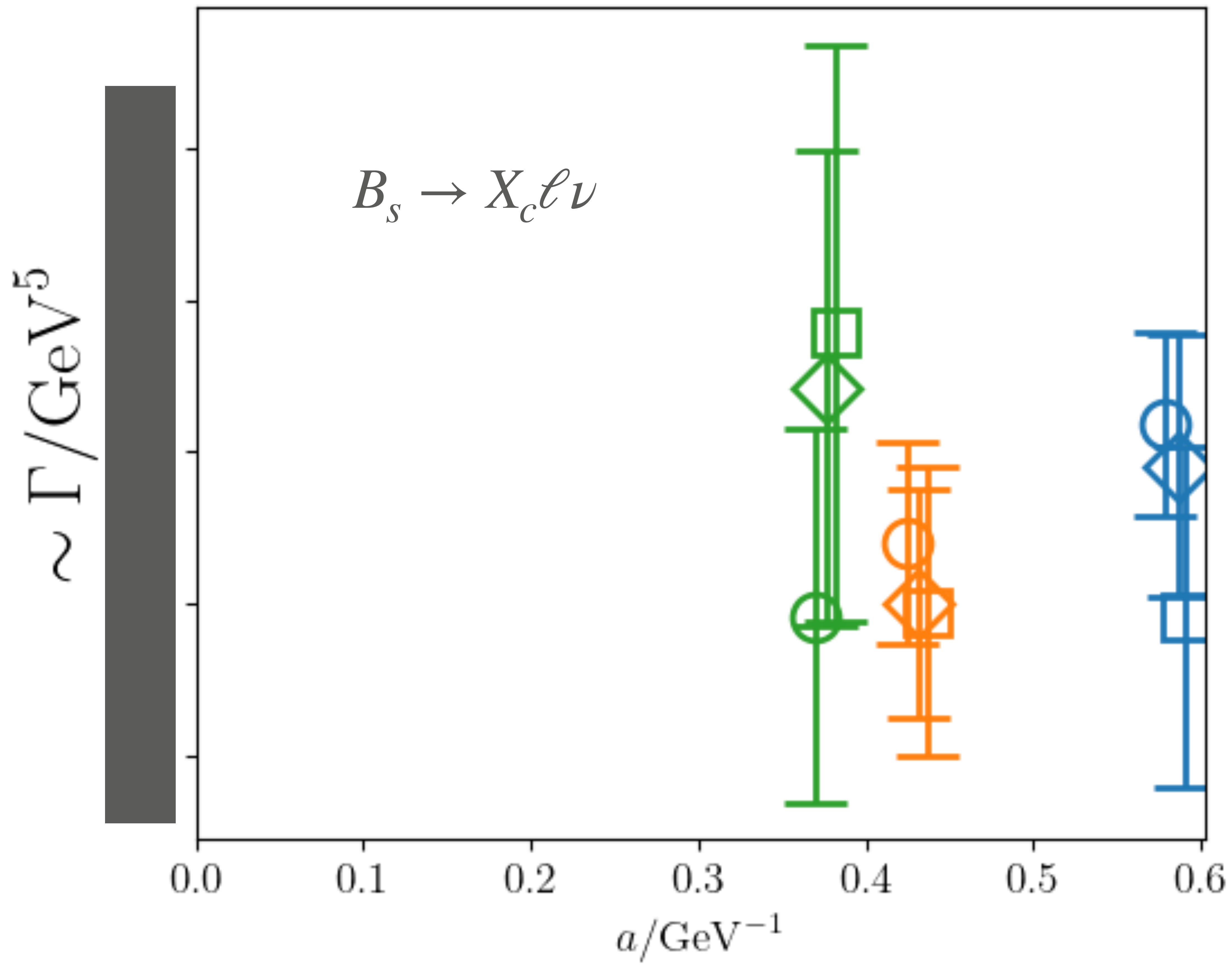
Depending on kinematics and the shape of the smearing kernel, subtracting the GS prior to reconstruction can substantially reduce systematic effects

Ground-state limit



- Results for exclusive channel agree for both ways of data analysis (standard 3pt vs. Chebychev)
- clear distinction between ground-state and full inclusive determination

First glimpse on continuum limit

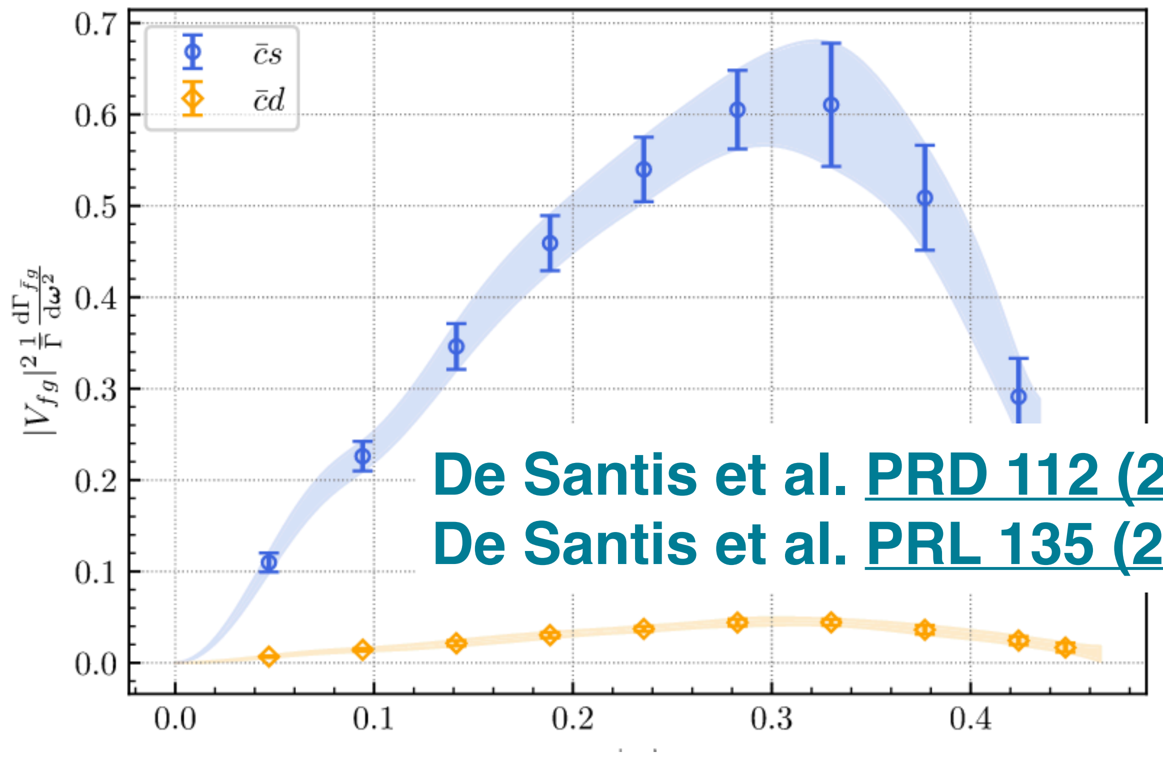


- new data for three lattice spacings, three volumes and variations of correlation function data → comprehensive systematics study
- data shown without $O(a)$ improvement (data on disk for that) and with partial statistics
- different symbols for a given colour correspond to different systematics
- we have worked on variance-reduction techniques that will allow to bring down stat. error substantially

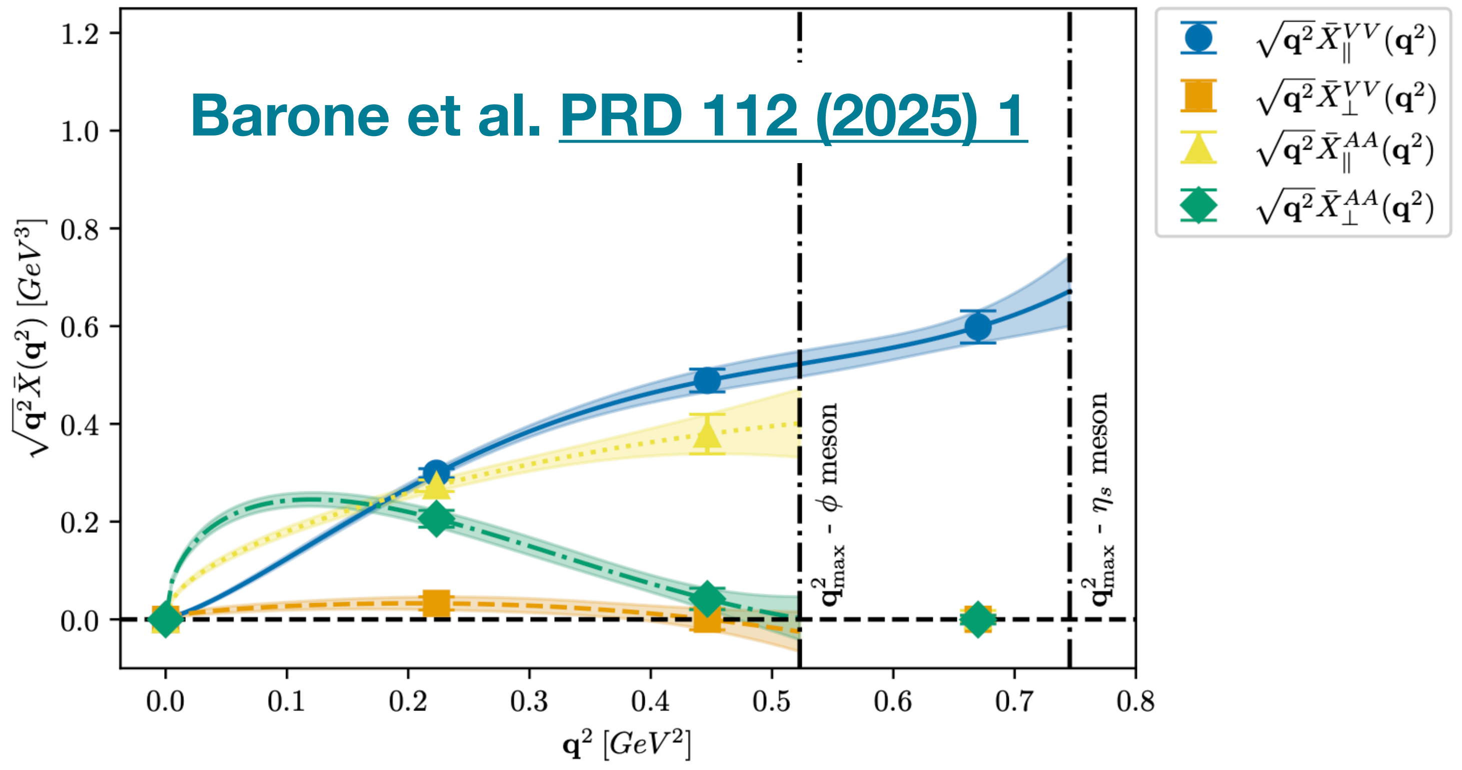
Outlook: Inclusive D_s decay

First lattice study with full systematic error budget for D_s inclusive decay

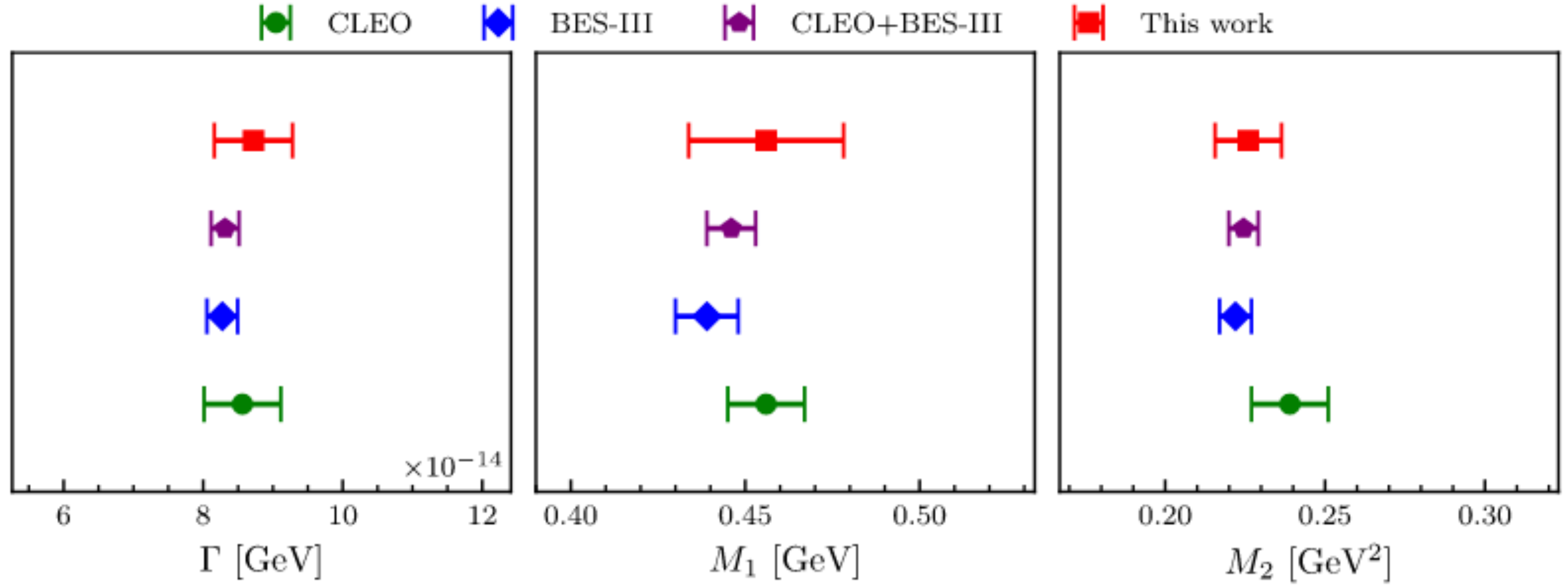
inclusive differential decay rates



De Santis et al. PRD 112 (2025) 5
De Santis et al. PRL 135 (2025) 12



Barone et al. PRD 112 (2025) 1



leptonic moments M_1 and M_2

- Rome: Impressive demonstration for D_s — most advanced
- CERN/Mainz/KEK/Soton Results for $D_s \to X_x \ell \nu$ in preparation
- as always $B_{(s)}$ more challenging on the lattice …

Slightly off the beaten track*

Standard Model tests with smeared experiment and theory

[AJ arXiv:2603.15487]

$$\frac{d\Gamma}{d\mathbf{q}^2}/|V_{qQ}|^2$$

“Standard analysis”:

Lattice QCD can make parameter-free predictions (modulo the CKM ME) for

- bins for the differential decay rate
- the same moments that are also input to the OPE analysis

$$\left\langle \frac{d^2\Gamma}{d\mathbf{q}^2 dq_0}/|V_{qQ}|^2 \right\rangle_\epsilon$$

Nobody forces us to use *standard* Standard Model quantities for SM tests!

So why not consider also new observables if they promise

- smaller systematic and statistical errors for experiment
- smaller systematic and statistical errors for theory

Slightly off the beaten track

Standard Model tests with smeared experiment and theory

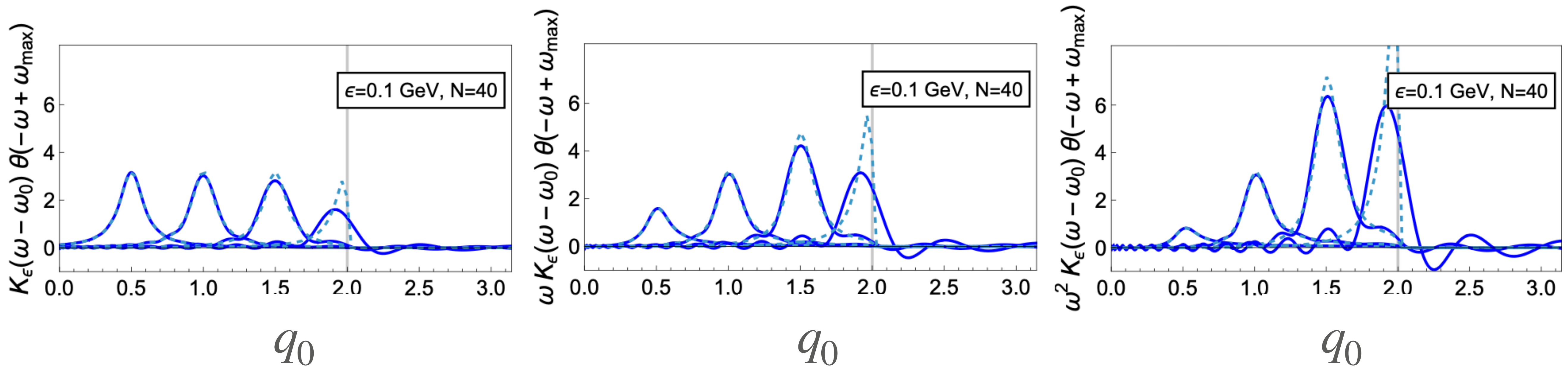
[AJ arXiv:2603.15487]

Same smeared observables accessible to both experiment and theory (not always!)

smearing kernel

$$\left\langle \frac{d^2\Gamma}{d\mathbf{q}^2 dq_0} / |V_{qQ}|^2 \right\rangle_\epsilon = \int dq_0' \frac{d^2\Gamma}{d\mathbf{q}^2 dq_0'}(q_0') / |V_{qQ}|^2 \, K_\epsilon(q_0' - q_0)$$

$$K_\epsilon(q_0 - q_0') = \frac{1}{\pi} \frac{\epsilon}{(q_0 - q_0')^2 + \epsilon^2}$$



Smeared analysis

- determine $|V_{cb}|$ at finite smearing!
- $\epsilon > 0$**:** reduced statistical and systematic errors
- can also be obtained for experimental data
- flexibility in choosing a kernel

Idea more generally applicable

- same idea applies to exclusive rare decay $B \to K^{(*)}\ell\ell, D \to \pi\ell\ell$ long-distance contributions around resonance peaks
- let's discuss

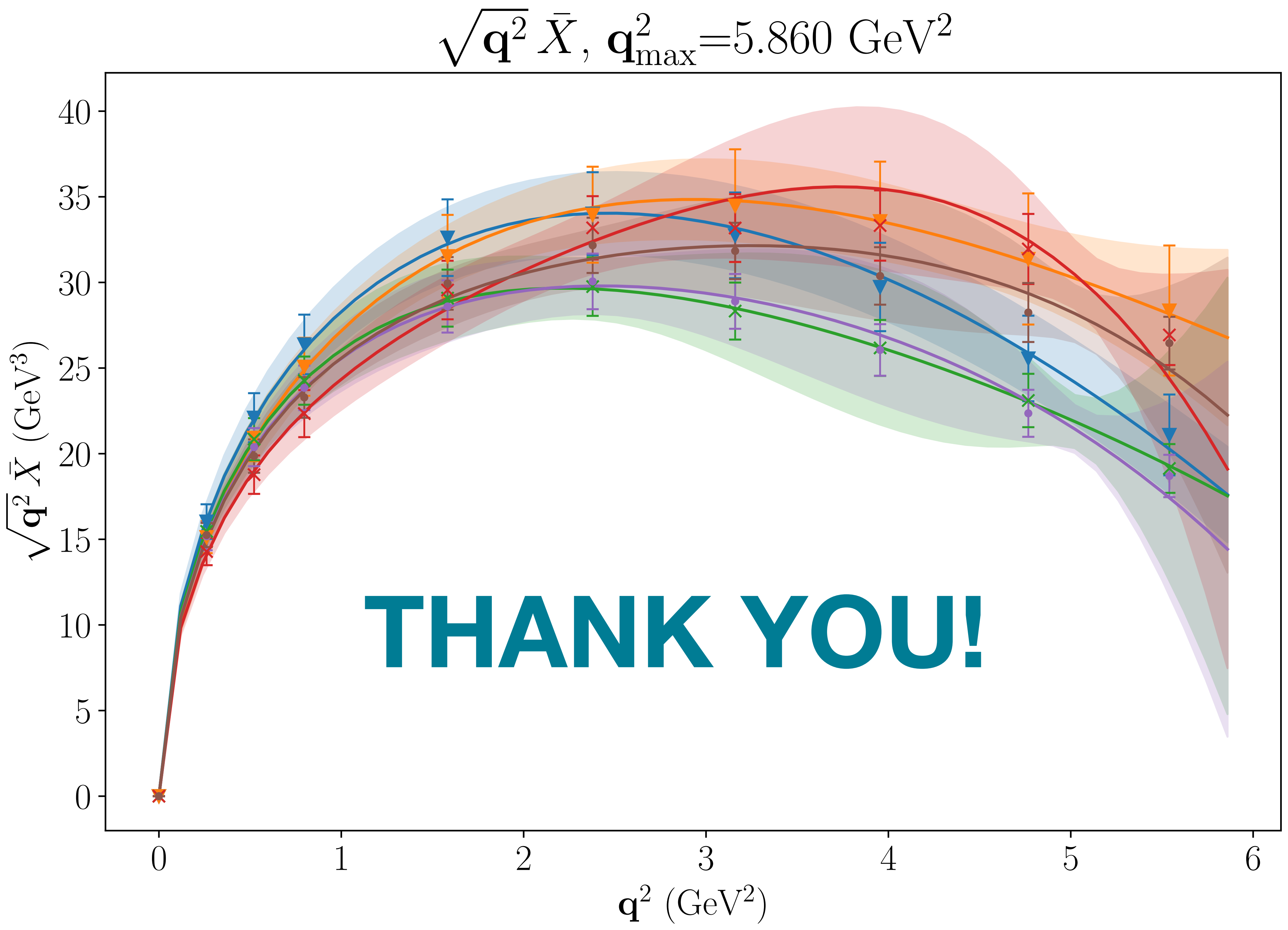
see also Luca's talk tmw.

Conclusions

- A first lattice-QCD calculation of $|V_{ub}|$ or $|V_{cb}|$ from inclusive decay is on its way
- Recently new results on inclusive D_s decay [Barone et al. PRD 112 (2025) 1], [De Santis et al. PRD 112 (2025) 5, PRL 135 (2025) 12]
- $B_s \to X_c \ell \bar{\nu}$ in preparation by CERN/KEK/Mainz/Soton
- Going forward, the focus will be on reducing stat. errors and systematic effects

- It's worthwhile considering novel observables for SM tests with reduced stat. and syst. errors [AJ arXiv:2603.15487]
- It's also worthwhile thinking about inclusive rare decays!

Resolving the V_{cb} tension remains a formidable but extremely important challenge with wide-ranging implications for Standard Model tests


$\sqrt{\mathbf{q}^2}\,\bar{X}$, $\mathbf{q}^2_{\mathrm{max}}$=5.860 GeV^2
$\sqrt{\mathbf{q}^2}\,\bar{X}$ (GeV^3)
$\mathbf{q}^2$ (GeV^2)
0
5
10
15
20
25
30
35
40
0
1
2
3
4
5
6

THANK YOU!