Reconciling hadronic and partonic analyticity in $b \to s\ell\ell$ transitions

Simon Mutke

Helmholtz-Institut für Strahlen- und Kernphysik,

Bethe Center for Theoretical Physics, Universität Bonn

April 16, 2026

Beyond the Flavour Anomalies 2026

Santiago de Compostela

based on arXiv:2604.01284

in collaboration with Martin Hoferichter and Bastian Kubis

Non-local form factors in $B \to K^{(*)}\ell\ell$

- Hadronic matrix element for $B \to K^{(*)}\ell\ell$ in Weak Effective Theory

$$\mathcal{A}(B \to K^{(*)}\ell\ell) \sim \mathcal{N}\left[\left(C_9 L_V^\mu + C_{10} L_A^\mu\right)\boldsymbol{\mathcal{F}_\mu} - \frac{L_V^\mu}{q^2}\left(C_7 \boldsymbol{\mathcal{F}_{T,\mu}} + \boldsymbol{\mathcal{H}_\mu}\right)\right]$$

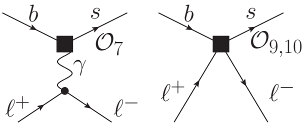
conventions from Asatrian, Greub, Virto 2019

- **Local form factors** (FFs) $\mathcal{F}_\mu$ and $\mathcal{F}_{T,\mu}$

  ↪ calculated with Lattice QCD,

  Light-Cone Sum Rules, . . .

  ↪ known with good precision



- **Non-local FFs** $\mathcal{H}_\mu$

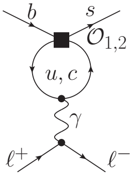
  ↪ charm-loop effects important

  ↪ calculated with Operator Product Expansion,

  QCD factorization, . . .

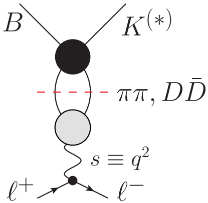
  ↪ still with large uncertainties

  ↪ complicated **analytic structure!**



Analytic structure of hadronic form factors

- Fundamental principles:
  - **Analyticity** (causality)
  - **Unitarity** (probability conservation)
- Start with **analyticity**: amplitudes are analytic in all **kinematic invariants**
  - **Meson masses** $p^2 = M_B^2$, $(p-q)^2 = M_{K^{(*)}}^2$
    $\hookrightarrow$ only defined on-shell
  - **Photon virtuality** q^2
    $\hookrightarrow$ can define analytic continuation for arbitrary $s \equiv q^2$ in the complex plane
- **Singularities in q^2**
  - **Poles**: (infinitely) narrow bound states
    $\hookrightarrow q^2 = \{M_{J/\psi}^2, M_{\psi(2S)}^2\}$
  - **Thresholds**: branch points of $\gamma^* \to \{\pi^+\pi^-, D\bar{D}, \ldots\}$ cuts
    $\hookrightarrow q^2 = \{4M_\pi^2, 4M_D^2, \ldots\}$
  - **Anomalous thresholds**: anomalous branch points
    $\hookrightarrow$ kinematic singularities, e.g., of the triangle diagram
    $\hookrightarrow$ position depends on all the masses



Dispersion relations

- Next up: **unitarity** of the S-matrix implies **unitarity relation**

$$\mathrm{disc}\,\mathcal{M}_{if}(s) \equiv \lim_{\varepsilon\to 0}\Big[\mathcal{M}_{if}(s+\mathrm{i}\varepsilon)-\mathcal{M}_{if}(s-\mathrm{i}\varepsilon)\Big] = \mathrm{i}\sum_n \mathcal{M}_{fn}^*\mathcal{M}_{in}$$

  ↪ summing over intermediate states $n \in \{\pi^+\pi^-, D\bar{D}, \ldots\}$

- Amplitudes are analytic with **branch cuts** along real axis

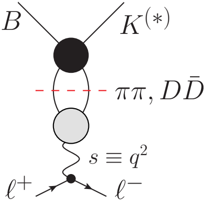
  ↪ starting at thresholds $s_{\mathrm{thr}} = \{4M_\pi^2, 4M_D^2, \ldots\}$

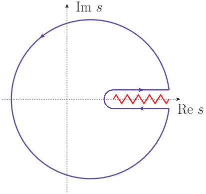
- Know **discontinuity** along cuts from **unitarity relation**

- Reconstruct from discontinuity via **dispersion relation**:

$$\mathcal{M}_{if}(s) = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}s'\,\frac{\mathcal{M}_{if}(s')}{s'-s} = \frac{1}{2\pi\mathrm{i}}\int_{s_{\mathrm{thr}}}^{\infty}\mathrm{d}s'\,\frac{\mathrm{disc}\,\mathcal{M}_{if}(s')}{s'-s}$$

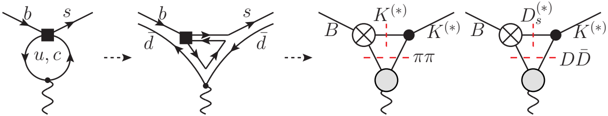
  ↪ using Cauchy's theorem





Dispersion relations with anomalous thresholds

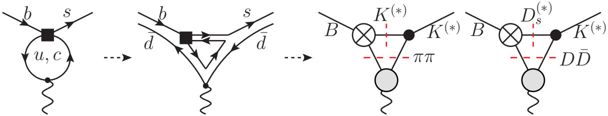
- Light-quark / charm loops hadronize into triangle topology:



$\hookrightarrow$ all three particles can go on-shell simultaneously

Dispersion relations with anomalous thresholds

- Light-quark / charm loops hadronize into triangle topology:

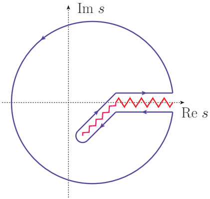


$\hookrightarrow$ all three particles can go on-shell simultaneously

- Additional **anomalous threshold** (“triangle singularity”)
- **Additional cut**: need to modify dispersion relation

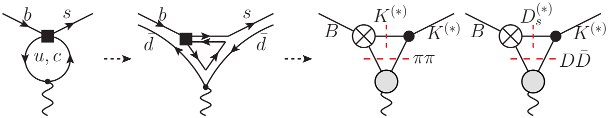
$$\mathcal{M}_{if}(s) = \frac{1}{2\pi i}\int_{s_{thr}}^{\infty} ds' \, \frac{\text{disc}\,\mathcal{M}_{if}(s')}{s'-s} + \frac{1}{2\pi i}\int_0^1 dx \, \frac{\partial s_x}{\partial x}\,\frac{\text{disc}_{an}\,\mathcal{M}_{if}(s_x)}{s_x - s}$$

$\hookrightarrow$ with integration contour $s_x = x\, s_{thr} + (1-x)\, s_+$

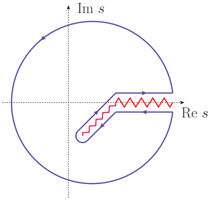


Dispersion relations with anomalous thresholds

- Light-quark / charm loops hadronize into triangle topology:



$\hookrightarrow$ all three particles can go on-shell simultaneously

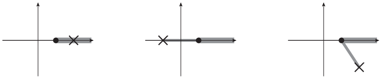


- Additional **anomalous threshold** ("triangle singularity")
- **Additional cut**: need to modify dispersion relation

$$\mathcal{M}_{if}(s) = \frac{1}{2\pi i}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{disc}\,\mathcal{M}_{if}(s')}{s'-s} + \frac{1}{2\pi i}\int_0^1 dx\, \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\,\mathcal{M}_{if}(s_x)}{s_x - s}$$

$\hookrightarrow$ with integration contour $s_x = x\, s_{\text{thr}} + (1-x)\, s_+$

- Three cases $\rightarrow$ all can occur!



- For light-quark loops ($\widehat{=}\,\pi\pi$): anomalous effects $\gtrsim \mathcal{O}(10\%)$

SM, Hoferichter, Kubis 2024

$\hookrightarrow$ better not neglect them!

Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{\mathrm{i}}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{i}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

- **Leading singularities** behave as (scalar case)

$$\sim \begin{cases} (s - s_0)^{\frac{4L-n-1}{2}} \log(s - s_0), & \text{if } 4L - n - 1 \text{ is even and nonnegative,} \\ (s - s_0)^{\frac{4L-n-1}{2}}, & \text{else.} \end{cases}$$

Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{\mathrm{i}}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

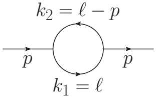
$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

- **Leading singularities** behave as (scalar case)

$$\sim \begin{cases} (s - s_0)^{\frac{4L-n-1}{2}} \log(s - s_0), & \text{if } 4L - n - 1 \text{ is even and nonnegative,} \\ (s - s_0)^{\frac{4L-n-1}{2}}, & \text{else.} \end{cases}$$

- **Two-particle threshold**: $L = 1, n = 2 \rightarrow$ behaves as $\sim (s - s_0)^{1/2}$

$$\alpha_i (k_i^2 - m_i^2) = 0, \qquad \alpha_1 k_1 + \alpha_2 k_2 = 0 \qquad \Rightarrow \qquad t = p^2 = (m_1 \pm m_2)^2$$

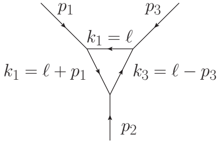


$\hookrightarrow$ Two-particle threshold at $s_{\text{thr}} = (m_1 + m_2)^2$

$\hookrightarrow$ Pseudothreshold $(m_1 - m_2)^2 \rightarrow$ left-hand cut on 2nd sheet

Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

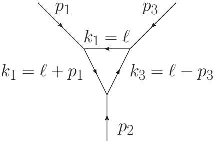


$$\alpha_i(k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$

Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

$$\alpha_i(k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$



- **Normal thresholds**: subleading singularities

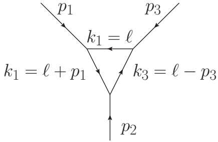
  $\hookrightarrow$ e.g. $\alpha_1 = 0 \Rightarrow s \equiv p_2^2 = (m_2 \pm m_3)^2$

  $\hookrightarrow$ inherits $\sim (s - s_0)^{1/2}$ behavior

Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

$$\alpha_i(k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$



- **Normal thresholds**: subleading singularities

$\hookrightarrow$ e.g. $\alpha_1 = 0 \Rightarrow s \equiv p_2^2 = (m_2 \pm m_3)^2$

$\hookrightarrow$ inherits $\sim (s - s_0)^{1/2}$ behavior

- **Anomalous threshold**: all $\alpha_i \neq 0 \rightarrow$ behaves as $\sim \log(s - s_0)$

$\hookrightarrow$ $s_\pm \equiv p_1^2 \frac{m_1^2+m_3^2}{2m_1^2} + p_3^2 \frac{m_1^2+m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2-m_2^2)(m_1^2-m_3^2)}{2m_1^2} \pm \frac{1}{2m_1^2}\sqrt{\lambda(p_1^2, m_1^2, m_2^2)\lambda(p_3^2, m_1^2, m_3^2)}$

$\hookrightarrow$ can be complex-valued

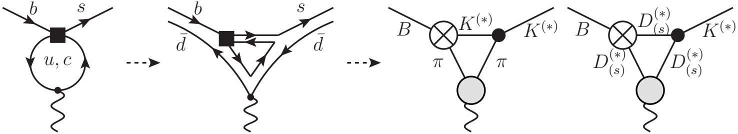
$\hookrightarrow$ s_- always on 2nd sheet

$\hookrightarrow$ s_+ can move to 1st sheet, if external masses big enough (B decays!),

$$m_3 p_1^2 + m_2 p_3^2 - (m_2 + m_3)(m_1^2 + m_2 m_3) > 0$$

Triangle loops in non-local $B \to K^{(*)}\gamma^*$ form factors

- **Triangle loop** contributions to non-local FFs:



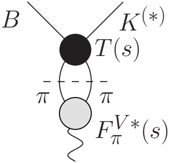
- How to assess size of **anomalous contributions**?
- Start with ***u*-quark loop** and **$\pi\pi$ intermediate states**:
  - CKM-suppressed $\sim \lambda^4$ compared to ***c*-quark loop** $\sim \lambda^2$
  - Input (form factors, branching ratios, polarization fractions . . .) well known
  - Sizable energy gap to next state $\pi\omega$
    $\hookrightarrow$ cf. various $D^{(*)}_{(s)}\bar{D}^{(*)}_{(s)}$ for hadronization of charm loop within close proximity
- Build **dispersive framework**

Form factor dispersion relation

- **Unitarity relation** for $B \to K^{(*)}\gamma^*$ FF with intermediate $\pi\pi$

$$\mathrm{disc}\,\Pi(s) = 2i\, s\, \sigma_\pi(s)^3\, T(s)\, F_\pi^{V*}(s)$$

$\hookrightarrow$ pion vector FF $F_\pi^V(s)$, $B \to K^{(*)}\pi\pi$ P-wave amplitude $T(s)$



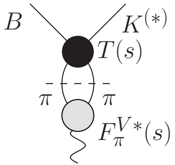
- FF **dispersion relation**

$$\Pi(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{s'\,\sigma_\pi(s')^3\, T(s')\, F_\pi^{V*}(s')}{s'-s}$$

Form factor dispersion relation

- **Unitarity relation** for $B \to K^{(*)}\gamma^*$ FF with intermediate $\pi\pi$

$$\mathrm{disc}\,\Pi(s) = 2i\, s\, \sigma_\pi(s)^3\, T(s)\, F_\pi^{V*}(s)$$



$\hookrightarrow$ pion vector FF $F_\pi^V(s)$, $B \to K^{(*)}\pi\pi$ P-wave amplitude $T(s)$

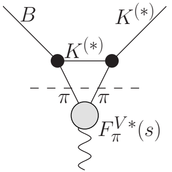
- FF **dispersion relation**

$$\Pi(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s'\, \frac{s'\,\sigma_\pi(s')^3\, T(s')\, F_\pi^{V*}(s')}{s'-s}$$

- Left-hand cut from crossed-channel $K^{(*)}$-exchange in $T(s)$

$\hookrightarrow$ leads to **triangle topology**

$\hookrightarrow$ add **anomalous cut**, $\Pi(s) = \Pi^{\mathrm{norm}}(s) + \Pi^{\mathrm{anom}}(s)$

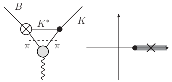


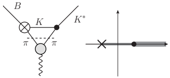
- Simple Born amplitude violates unitarity (Watson's theorem)

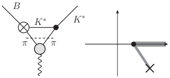
$\hookrightarrow$ need to include **$\pi\pi$ rescattering**

$\hookrightarrow$ unitarize via Muskhelishvili–Omnès representation

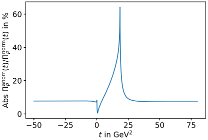
Anomalous fractions for $B^0 \to K^{(*)0}\gamma^*$

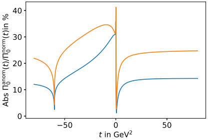


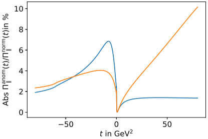


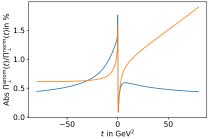


s_+ [GeV^2] $= 18.6$ (case 1)	-57.8 (case 2)	$0.5 - 4.2i$ (case 3)









SM, Hoferichter, Kubis 2024

	$\lambda = 0$	$\lambda = \parallel$	$\lambda = \perp$

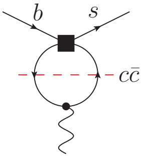
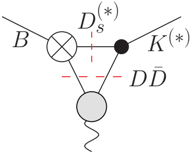
- How important are **anom. contributions**? $\to$ anom. fraction $|\Pi^{\text{anom}}(s)/\Pi^{\text{norm}}(s)|$
- All parameters fixed from **data**!

  $\hookrightarrow$ one ambiguity left due to lack of Dalitz plot data (blue and orange curve)
- **Anomalous contributions** can be $\gtrsim$ **10%** away from thresholds, resonances

- Established analytic structures on the **hadronic** side
- Non-local FFs $\mathcal{H}_\mu$ computed via Operator Product Expansion

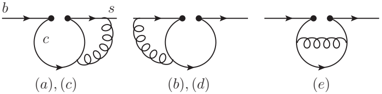
  $\hookrightarrow$ applicable for $s \ll 4m_c^2$, but rely on **partonic** calculations

  $\hookrightarrow$ how does partonic analyticity compare?

Partonic picture	**Hadronic picture**
	
$s_{\text{thr}} = 4m_c^2 \simeq 6.5\,\text{GeV}^2$	$s_{\text{thr}} = 4M_D^2 \simeq 14\,\text{GeV}^2$
?	$\mathbf{s_+} \simeq (24 - 8i)\,\text{GeV}^2$

- Are **anomalous contributions** missed by the partonic calculations?

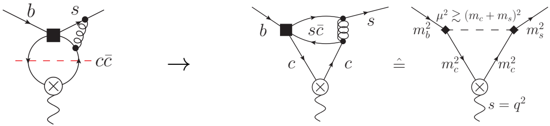
- Consider $b \to s\ell\ell$ at two-loop order
- Five gauge-invariant classes of functions $F^{(7)}_{2,(i)}(s)$, $i \in \{a, b, c, d, e\}$



Asatrian, Greub, Virto 2019

$\hookrightarrow$ with photon inserted at all possible points

- Focus on diagram (c) for now:

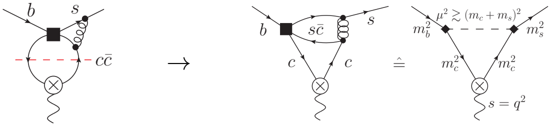


$\hookrightarrow$ bring into "triangle shape"

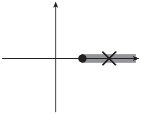
$\hookrightarrow$ can do the same for (a)–(d); only (e) does not yield triangle topology

$b \to s\ell\ell$ at partonic two-loop order

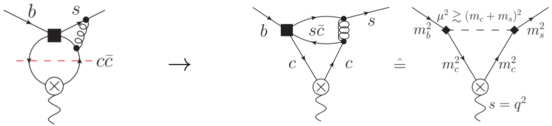
- Diagram (c):



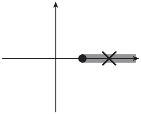
- Which singularities in s do we expect? ($m_c^2 = 0.1\, m_b^2$ and $m_s^2 = 0$)
  - Two-particle $c\bar{c}$ threshold at $s_{\text{thr}} = 4m_c^2 = 0.4\, m_b^2 \to$ unitarity cut
  - **Anomalous threshold $s_+ \sim m_b^2$** (for $\mu^2 \sim (m_c + m_s)^2$)



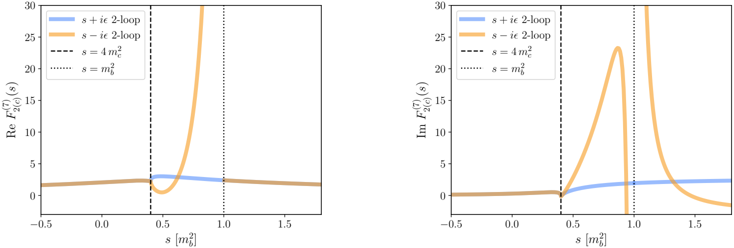
- Diagram (c):



- Which singularities in s do we expect? ($m_c^2 = 0.1\, m_b^2$ and $m_s^2 = 0$)
  - Two-particle $c\bar{c}$ threshold at $s_{\text{thr}} = 4m_c^2 = 0.4\, m_b^2 \to$ unitarity cut
  - **Anomalous threshold $s_+ \sim m_b^2$** (for $\mu^2 \sim (m_c + m_s)^2$)



- Matches perturbative two-loop calculation from Asatrian, Greub, Virto 2019



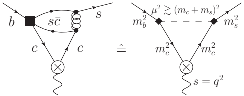
Dispersive representation

- Want **dispersion relation** ("DR") to confirm suspected **analytic structures**

$$F_{2,(c)}^{(7)}(s) = F_{2,(c)}^{(7)}(0) + \frac{s}{2\pi i} \int_{s_{\text{thr}}=4m_c^2}^{\infty} \mathrm{d}s' \, \frac{\text{disc}\, F_{2,(c)}^{(7)}(s')}{s'(s'-s)}$$

- Discontinuity determined by **triangle diagram**

$\hookrightarrow$ allow for $\mu^2 \geq \mu_{\text{thr}}^2 = (m_c + m_s)^2$ via spectral function



$$\text{disc}\, F_{2,(c)}^{(7)}(s) = \int_{\mu_{\text{thr}}^2}^{\infty} \mathrm{d}\mu^2 \, \rho(\mu^2) \, \text{disc}\, F_{2,(c)}^{(7)}(s; \mu^2)$$

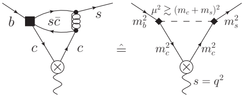
$\hookrightarrow$ use conformal polynomial as ansatz for $\rho(\mu^2)$

- Want **dispersion relation** ("DR") to confirm suspected **analytic structures**

$$F^{(7)}_{2,(c)}(s) = F^{(7)}_{2,(c)}(0) + \frac{s}{2\pi i}\int_{s_{\rm thr}=4m_c^2}^{\infty} \mathrm{d}s' \, \frac{\mathrm{disc}\, F^{(7)}_{2,(c)}(s')}{s'(s'-s)}$$

- Discontinuity determined by **triangle diagram**

  $\hookrightarrow$ allow for $\mu^2 \geq \mu^2_{\rm thr} = (m_c + m_s)^2$ via spectral function

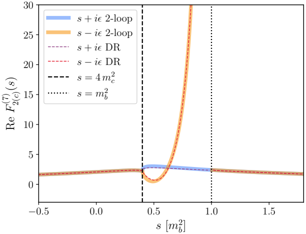


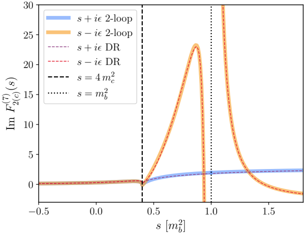
$$\mathrm{disc}\, F^{(7)}_{2,(c)}(s) = \int_{\mu^2_{\rm thr}}^{\infty} \mathrm{d}\mu^2 \, \rho(\mu^2)\, \mathrm{disc}\, F^{(7)}_{2,(c)}(s;\mu^2)$$

  $\hookrightarrow$ use conformal polynomial as ansatz for $\rho(\mu^2)$

- Fit $\rho(\mu^2)$ to match discontinuity of 2-loop results from Asatrian, Greub, Virto 2019

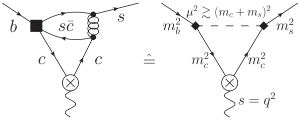
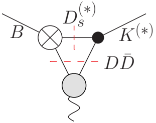
  $\hookrightarrow$ expected structures confirmed! ($\rightarrow s_{\rm thr} = 4\, m_c^2$, $s_+ \sim m_b^2$)



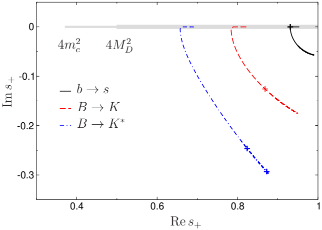


Reconciling partonic and hadronic analyticity

- Compare the pictures again

Partonic picture	**Hadronic picture**
	
$s_{\text{thr}} = 4m_c^2 \simeq 0.4\, m_b^2$	$s_{\text{thr}} = 4M_D^2 \simeq 0.5\, M_B^2$
$s_+ \simeq m_b^2$	$s_+ \simeq (0.85 - 0.3i)\, M_B^2$

- Both share the **same analytic structures!**
- Hadronic singularities in complex plane since $D_s^{(*)} \to DK^{(*)}$ kinematically forbidden
  $\hookrightarrow$ move back to real axis for higher $M_{D_s^{(*)}}$
  $\hookrightarrow$ differences: mass effects & hadronization



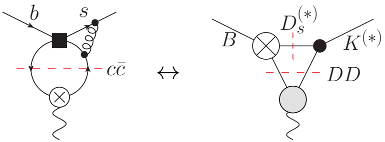
Summary and Outlook

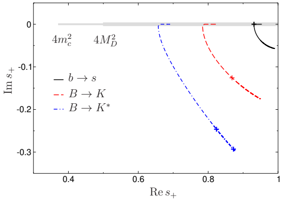
- **Analytic structure** of non-local FFs richer than previously thought

  ↪ **anomalous thresholds**!

  ↪ essential for detailed understanding both on **hadronic** and **partonic level**

- Both pictures match each other!





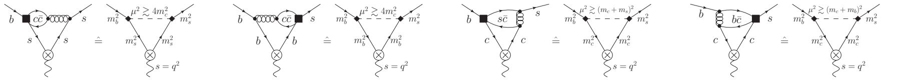
  ↪ justifies using partonic calculations in non-local FFs

- More **robust interpretation** of $b \to s\ell\ell$ flavor anomalies!

for details: Hoferichter, Kubis, SM 2026 [arXiv:2604.01284]

Spare slides

Partonic analyticity at two-loop order



(a)	(b)	(c)	(d)
$s_{\text{thr}} = 0$	$4m_b^2$	$4m_c^2$	$4m_c^2$
$\mu^2_{\text{thr}} = 4m_c^2$	$4m_c^2$	$(m_c + m_s)^2$	$(m_c + m_b)^2$
$s_+ \sim 0.6\, m_b^2$		$\sim m_b^2$