Some more progress on multi-denominator angular integrals

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Juliane Haug, Vladimir A. Smirnov, and FW,
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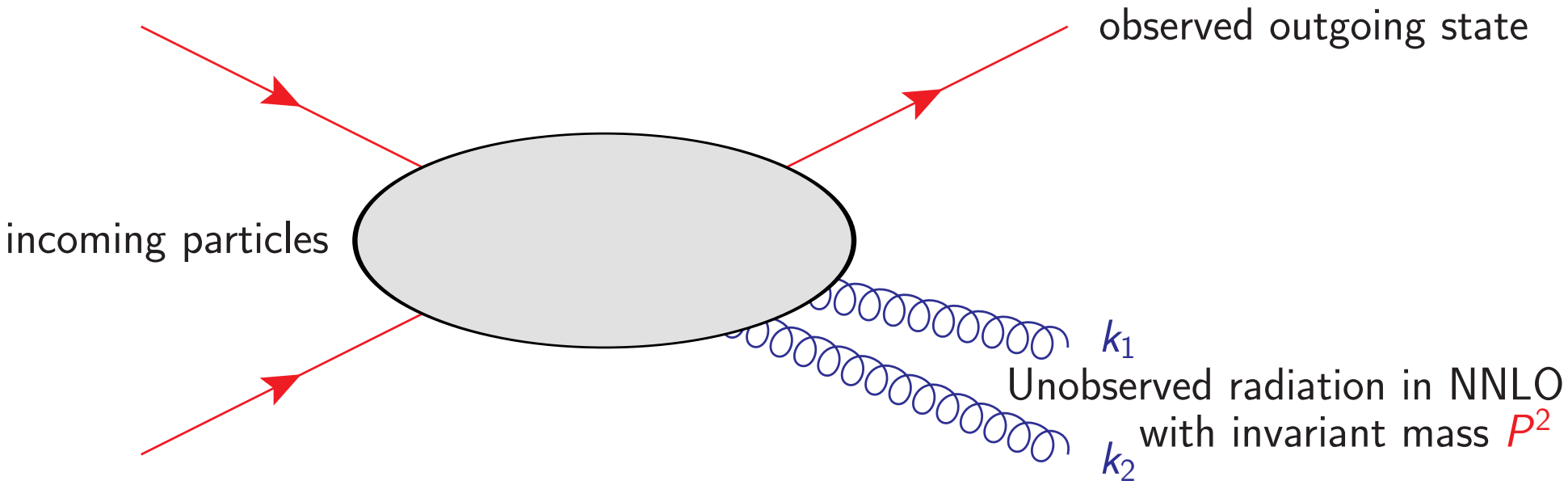
[Note parallel work by Regensburg group: arXiv:2508.15952]

Outline

1. Recap: Angular integrals
2. Why angular integrals again?
3. IBP without Laporta
4. Scale reduction via "branch integrals"
5. Conclusion

1. Recap: Angular integrals

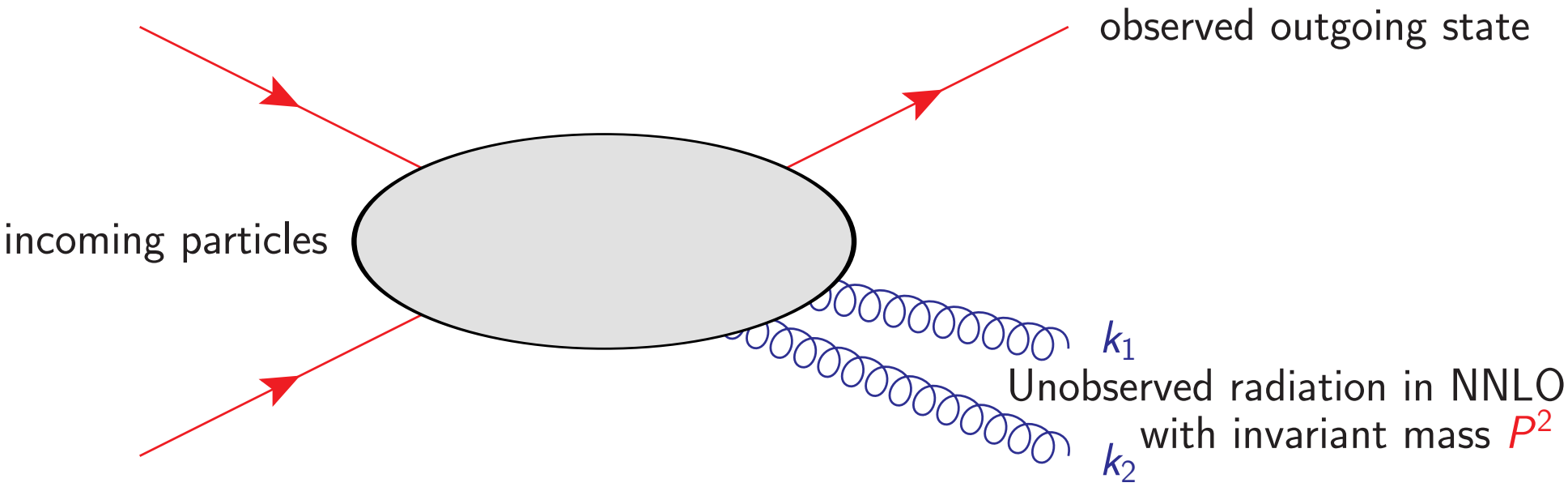
Here we go again, again: Angular integrals



Two particle phase space ($d = 4 - 2\varepsilon$):

$$\int \mathrm{dPS}_{2,P} = \int \frac{\mathrm{d}^{d-1} k_1}{(2\pi)^{d-1} 2k_1^0} \int \frac{\mathrm{d}^{d-1} k_2}{(2\pi)^{d-1} 2k_2^0} (2\pi)^d \delta^d (P - k_1 - k_2)$$

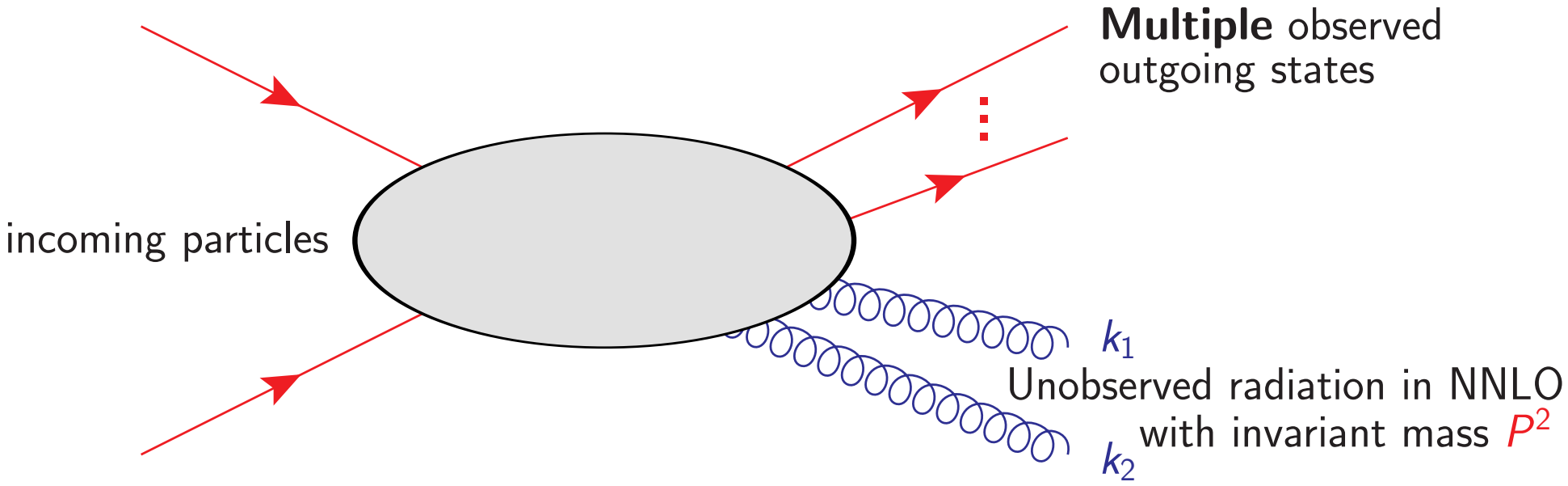
Here we go again, again: Angular integrals



Two particle phase space ($d = 4 - 2\varepsilon$):

$$\int \mathrm{dPS}_{2,P} = \frac{\left(P^2\right)^{-\varepsilon}}{2(4\pi)^{2-2\varepsilon}} \int \mathrm{d}\Omega_{3-2\varepsilon}$$

Here we go again, again: Angular integrals



Two particle phase space ($d = 4 - 2\varepsilon$):

$$\int \mathrm{dPS}_{2,P} = \frac{\left(P^2\right)^{-\varepsilon}}{2(4\pi)^{2-2\varepsilon}} \int \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2 \ldots$$

$$\times \ldots \underbrace{\mathrm{d}\theta_{n+1} \sin^{1-n-2\varepsilon}\theta_{n+1}}_{\text{explicit } \theta \text{ dependence}} \underbrace{\mathrm{d}\Omega_{2-n-2\varepsilon}}_{\text{spher. sym.}}$$

Definition of Angular Integrals [Somogyi 2011]

$$\Omega_{j_1,j_2,\ldots,j_n}(v_1,\ldots,v_n;d) \equiv \int \mathrm{d}\Omega_{d-1}(k) \prod_{i=1}^{n} \frac{1}{(v_i\cdot k)^{j_i}}$$

with normalized d-vectors

$$v_i = (1,\mathbf{v}_i)\,,$$

$$k = (1,\mathbf{k}) = (1,\ldots,\cos\theta_n \prod_{i=1}^{n-1}\sin\theta_i,\ldots,\cos\theta_2\sin\theta_1,\cos\theta_1)\,.$$

Common integral normalization ($v_{ij} = v_i\cdot v_j$ with $i\leq j$):

$$I^{(\#\text{non-zero masses})}_{j_1,\ldots,j_n}(v_{ij};\varepsilon) \equiv \frac{\Omega_{j_1,j_2,\ldots,j_n}(v_1,\ldots,v_n;4-2\varepsilon)}{\Omega_{1-2\varepsilon}}$$

State-of-the-art (before our paper)

Known orders in the ε-expansion:

# denom. \ # masses	0	1	2	3	4
0	∞^+ [Ellis '80]	–	–	–	–
1	∞^+ [Ellis '80]	∞^+ [Beenakker '88]	–	–	–
2	∞^+ [Neerven '85]	∞^+ [Somogyi '11]	∞^+ [Lyubovitskij '21]	–	–
3	2 [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	–
4	-1 [Smirnov '24]	0 [Salvatori '24]	-1 [Smirnov '24]	-1 [Smirnov '24]	-1 [Smirnov '24]
⋮	⋮	⋮	⋮	⋮	⋮
n	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]

This talk

Known orders in the ε-expansion:

# denom. \ # masses	0	1	2	3	4
0	∞^+ [Ellis '80]	–	–	–	–
1	∞^+ [Ellis '80]	∞^+ [Beenakker '88]	–	–	–
2	∞^+ [Neerven '85]	∞^+ [Somogyi '11]	∞^+ [Lyubovitskij '21]	–	–
3	$2 \to \infty^+$ [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	1 [Haug '24,Ahmed '24]	–
4	$-1 \to 0$ [Smirnov '24]	$0 \to 0$ [Salvatori '24]	$-1 \to 0$ [Smirnov '24]	$-1 \to 0$ [Smirnov '24]	$-1 \to 0$ [Smirnov '24]
⋮	⋮	⋮	⋮	⋮	⋮
n	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]	-1^* [Smirnov '24]

2. Why angular integrals again?

Last year's talk in Regensburg:

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Expansion by regions meets angular integrals

Vladimir A. Smirnov [a,b] and Fabian Wunder [c]

Angular integrals: $\Omega_{j_1,j_2,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(k) \prod_{i=1}^{n} \frac{1}{(v_i \cdot k)^{j_i}}$

Euler integrals: $I = \int_0^\infty \mathrm{d}t_1 \cdots \int_0^\infty \mathrm{d}t_n \prod_{i=1}^{N} P_i^{a_i+\varepsilon b_i}(t_1, \ldots, t_n)$

Salvatori's subtraction scheme (arXiv: 2406.14606)

The Tropical Geometry of Subtraction Schemes

Giulio Salvatori[a]

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- uses four denominator angular integral as example:

7.7 An angular integral

We provide another example beyond the case of Feynman integrals, a class of phase-space integrals known as *angular integrals*. In [11] it was introduced a parametric formulation which allows to apply our master formula to this case as well. In the language of [11], we consider the four denominators, one-mass integral, given by

$$I(\mathbf{v};\epsilon) = \mathcal{N} \int_{\mathbb{R}^4_+} \frac{dt}{t} t^{(1,1,1,5/2)} (1+t_2)(1+t_3)^2 (1+t_4)^{\epsilon+1/2} \mathcal{Q}^{-2}, \tag{7.25}$$

Why look at angular integrals again?

- Salvatori calculated angular integral $I^{(1)}_{1111}$ to $\mathcal{O}(\varepsilon)$ from Euler representation of [Smirnov, FW 2024]
- correspondence about result, check with `FIESTA`, mistake in initial version fixed
- **Idea:** Use mass-reduction to calculate all other four-denominator integrals from it ($m = 0, 2, 3, 4$)
- Salvatori's result: Non-symmetric, complicated coefficients and square roots $\longrightarrow$ recalculation by differential equation method as in [Haug, FW 2024]

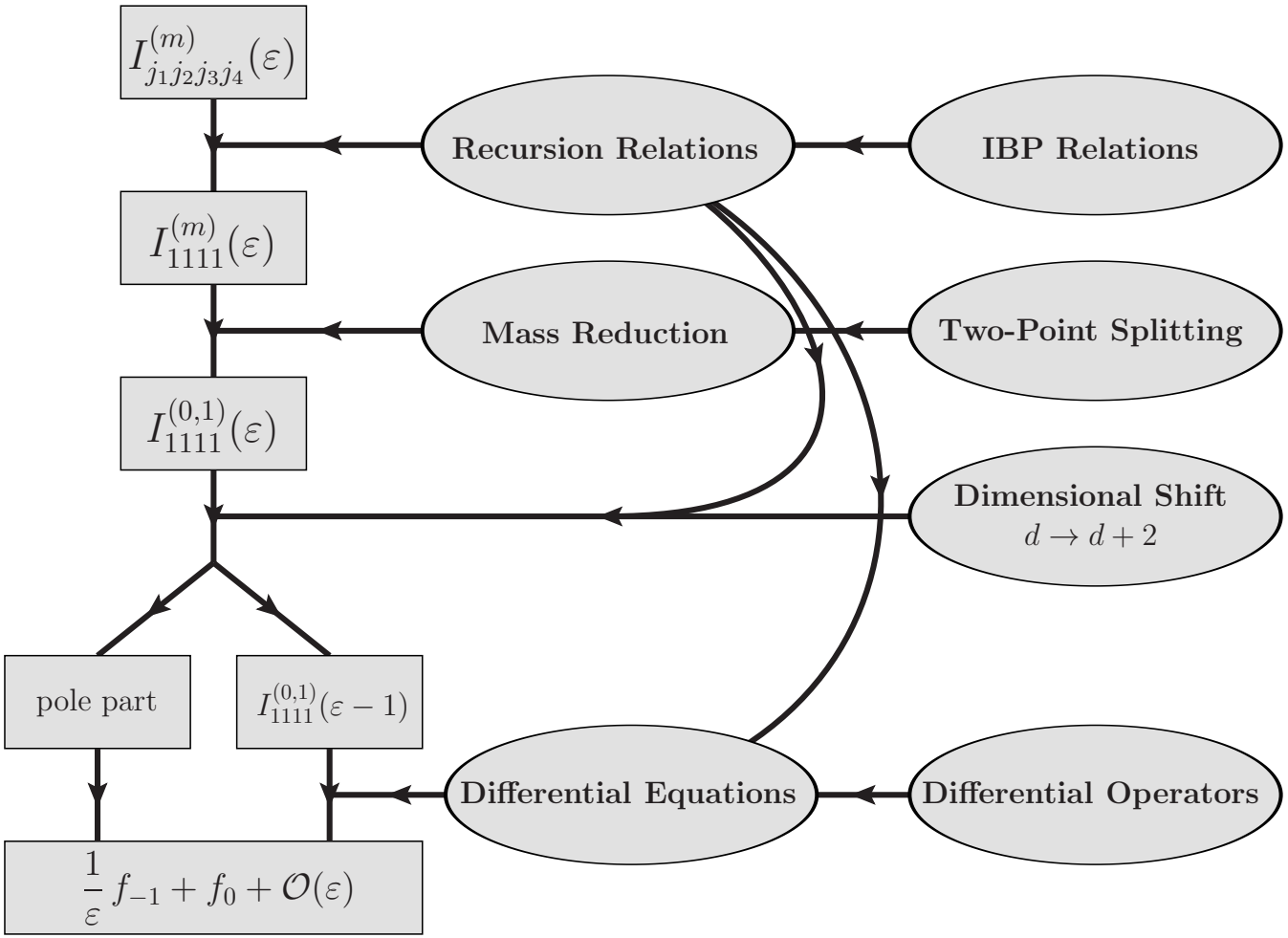
Salvatori's result (organized)

$$\underbrace{I^{(1)}_{1111}(v_4, v_1, v_2, v_3)}_{\text{4 denom. integral}} = \underbrace{\frac{1}{X_4}\Big[X_{4,4} I^{(0)}_{111}(v_1, v_2, v_3) + X_{4,3} I^{(1)}_{111}(v_4, v_1, v_2) + X_{4,2} I^{(1)}_{111}(v_4, v_1, v_3) + X_{4,1} I^{(1)}_{111}(v_4, v_2, v_3)\Big]}_{\text{"partial fractioning" part}} + \underbrace{\frac{\pi\, Y_4}{X_4^{3/2}}\, C_2}_{\text{genuine 4 denom.}} + \mathcal{O}(\varepsilon)$$

- X_4 is the Gram determinant $\det(v_{ij})$, $X_{4,i}$ the GD where the i-th column is replaced with 1s, Y_4 the Euclidean GD $\det(1 - v_{ij})$
- C_2 is a sum of 12 weight-2 GPLs, not explicitly symmetric in $1 \leftrightarrow 2 \leftrightarrow 3$.

Four denominator integral: Calculation

- As for 3 denom. in [Haug, FW 2024], even same integrals:



Four denominator integral: Result

- Identification of $\pi C_2 = \sqrt{X_4} I^{(m),d=6}_{1111} \equiv J^{(m),d=6}_{1111}$.
- Massless case:

$$J^{(0),d=6}_{1111} = 2\pi \left[\mathrm{Cl}_2\left(2\theta^{(0)}_{23}\right) + \mathrm{Cl}_2\left(2\theta^{(0)}_{24}\right) + \mathrm{Cl}_2\left(2\theta^{(0)}_{34}\right) \right]$$

where $\theta^{(m)}_{ij} = \arctan\left(\frac{\sqrt{X^{(m)}_4}}{v_{il}\, v_{jk} + v_{ik}\, v_{jl} - v_{ij}\, v_{lk}} \right)$.

- Single-massive case:

$$\begin{aligned} J^{(1),d=6}_{1111} = \pi \Big[& 2\,\mathrm{Cl}_2\left(2\theta^{(1)}_{23}\right) + 2\,\mathrm{Cl}_2\left(2\theta^{(1)}_{24}\right) + 2\,\mathrm{Cl}_2\left(2\theta^{(1)}_{34}\right) \\ & + \mathrm{Cl}_2\left(2\theta^{(1)}_{23} + 2\theta^{(0)}_{23}\right) + \mathrm{Cl}_2\left(2\theta^{(1)}_{23} - 2\theta^{(0)}_{23}\right) - \mathrm{Cl}_2\left(4\theta^{(1)}_{23}\right) \\ & + \mathrm{Cl}_2\left(2\theta^{(1)}_{24} + 2\theta^{(0)}_{24}\right) + \mathrm{Cl}_2\left(2\theta^{(1)}_{24} - 2\theta^{(0)}_{24}\right) - \mathrm{Cl}_2\left(4\theta^{(1)}_{24}\right) \\ & + \mathrm{Cl}_2\left(2\theta^{(1)}_{34} + 2\theta^{(0)}_{34}\right) + \mathrm{Cl}_2\left(2\theta^{(1)}_{34} - 2\theta^{(0)}_{34}\right) - \mathrm{Cl}_2\left(4\theta^{(1)}_{34}\right) \Big]. \end{aligned}$$

Four denominator integral: More masses

- shorthand notation: $J^{(m),d=6}_{1111}(v_i, v_j, v_k, v_l) \equiv J^{(m)}(i,j,k,l)$
- auxiliary massless vectors:

$$v_{(ij)} \equiv (1-\lambda_{(ij)})\, v_i + \lambda_{(ij)}\, v_j\,, \text{ where } \lambda_{(ij)} = \frac{v_{ij} - v_{ii} - \sqrt{v_{ij}^2 - v_{ii}\, v_{jj}}}{2v_{ij} - v_{ii}\, v_{jj}}$$

- multi-mass integrals – very compact:

$$J^{(2)}(1,2,3,4) = J^{(0)}((12),(21),3,4) - J^{(1)}(1,(12),3,4) - J^{(1)}(2,(21),3,4)\,,$$
$$J^{(3)}(1,2,3,4) = J^{(1)}(1,(23),(32),4) - J^{(2)}(1,2,(23),4) - J^{(2)}(1,3,(32),4)\,,$$
$$J^{(4)}(1,2,3,4) = J^{(2)}(1,2,(34),(43)) - J^{(3)}(1,2,3,(34)) - J^{(3)}(1,2,4,(43))\,.$$

Reminder: Clausen function

Definition (Clausen 1832)

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} = \mathrm{Im}\,\mathrm{Li}_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$

- To taste, $\mathrm{Cl}_2(2\arctan z)$ can equally be written as dilogarithms or in GPL form:

$$\begin{aligned}\mathrm{Cl}_2(2\arctan z) &= \frac{1}{2\mathrm{i}}\left[\mathrm{Li}_2\left(\frac{\mathrm{i}-z}{\mathrm{i}+z}\right) - \mathrm{Li}_2\left(\frac{\mathrm{i}+z}{\mathrm{i}-z}\right)\right] \\ &= \mathrm{i}\ln 2\left[\mathrm{G}(\mathrm{i};z) - \mathrm{G}(-\mathrm{i};z)\right] + \mathrm{i}\left[\mathrm{G}(\mathrm{i},0;z) - \mathrm{G}(-\mathrm{i},0;z)\right] \\ &\quad + \frac{1}{2\mathrm{i}}\left[\mathrm{G}(\mathrm{i},\mathrm{i};z) + \mathrm{G}(\mathrm{i},-\mathrm{i};z) - \mathrm{G}(-\mathrm{i},\mathrm{i};z) - \mathrm{G}(-\mathrm{i},-\mathrm{i};z)\right].\end{aligned}$$

- We like Cl_2 because it is a single-variable, real-valued function $\rightarrow$ ideal for fast numeric evaluation

3. IBP without Laporta

IBP without Laporta

- Index lowering identity (general n):

$$I_{j_1\dots j_n} = \frac{1}{X_n(j_k-1)}\Bigg\{\Bigg[(j+1-d)X_{4,\bar{k}} + (j_k-1)X_4^{(k,k)} + \sum_{i\neq k} j_i X_n^{(k,i)}\Bigg]\hat{\mathbf{j}}_{\mathbf{k}}^{-}$$
$$+\sum_{i\neq k}(j_k-1)X_n^{(k,i)}\hat{\mathbf{j}}_{\mathbf{i}}^{-} + \sum_{i\neq k}(d-j-1)X_n^{(k,i)}\hat{\mathbf{j}}_{\mathbf{k}}^{-}\hat{\mathbf{j}}_{\mathbf{i}}^{-} + \sum_{\substack{i\neq k, l\neq k\\ i\neq l}} j_l X_n^{(k,i)}\hat{\mathbf{j}}_{\mathbf{k}}^{-}\hat{\mathbf{j}}_{\mathbf{i}}^{-}\hat{\mathbf{j}}_{\mathbf{l}}^{+}$$
$$+(d-j-1)X^{(1,1)}\left(\hat{\mathbf{j}}_{\mathbf{k}}^{-}\right)^2 + \sum_{i\neq k} j_i X_n^{(k,k)}\left(\hat{\mathbf{j}}_{\mathbf{k}}^{-}\right)^2\hat{\mathbf{j}}_{\mathbf{i}}^{+}\Bigg\} I_{j_1\dots j_n}$$

- Similar identity for index raising.
- $X_n^{(k,i)}$ are cofactors of the GD (k-th column, i-th row deleted)

Some Laporta at least

Hyperspherical Integration and the Triple-Cross Vertex Graphs (*).

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(ricevuto il 28 Febbraio 1994; approvato il 24 Marzo 1994)

- work from the 90s on hyperspherical integrals for 3-loop corrections to $g-2$
- very similar angular integrals with 1, 2, and 3 denominators, but in d spatial dimensions (after Wick rotation) compared to $d-1$ here
- relation $d \to d \pm 1$? (regular dimensional shift is $d \to d \pm 2$)

4. Scale reduction via “branch integrals”

Scale reduction of master integrals

- Angular integrals depend on $n(n-1)/2+m$ scales
- Can we express it as integrals with fewer scales?
- **Idea:** Dimensional shift formula:

$$I_{\vec{1}_n}(d) = \sum_{i=1}^{n} x_{n,i} \underbrace{I_{\vec{1}_{n,i}}(d)}_{n-1 \text{ denom.}} + \frac{n+1-d}{d-3}\, y_n\, I_{\vec{1}_n}(d+2)\,,$$

where we use the compact notation $\vec{1}_n = \overbrace{1,\ldots,1}^{n}$, $\vec{1}_{n,i} = 1\ldots\underset{\substack{\uparrow\\ i\text{-th}}}{0}\ldots 1$, $x_{n,i} = X_{n,i}/X_n$, and $y_n = Y_n/X_n$.

- Iterate this equation by plugging it in for $I_{\vec{1}_n}(d+2)$.

From dimensional shift to dimensional recurrence

- Dimensional shift formula:

$$I_{\vec{1}_n}(d) = \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d) + \frac{n+1-d}{d-3} y_n I_{\vec{1}_n}(d+2)$$

From dimensional shift to dimensional recurrence

- Iterate once:

$$I_{\vec{1}_n}(d) = \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d) + \frac{n+1-d}{d-3} y_n \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d+2) + \frac{(n+1-d)(n-1-d)}{(d-3)(d-1)} y_n^2 I_{\vec{1}_n}(d+4)$$

From dimensional shift to dimensional recurrence

- Iterate twice:

$$I_{\vec{1}_n}(d) = \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d) + \frac{n+1-d}{d-3} y_n \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d+2) + \frac{(n+1-d)(n-1-d)}{(d-3)(d-1)} y_n^2 I_{\vec{1}_n}(d+4)$$

From dimensional shift to dimensional recurrence

- Iterate twice:

$$I_{\vec{1}_n}(d) = \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d) + \frac{n+1-d}{d-3} y_n \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d+2)$$
$$+ \frac{(n+1-d)(n-1-d)}{(d-3)(d-1)} y_n^2 \sum_{i=1}^{n} x_{n,i} I_{\vec{1}_{n,i}}(d+4)$$
$$+ \frac{(n+1-d)(n-1-d)(n-3-d)}{(d-3)(d-1)(d+1)} y_n^3 I_{\vec{1}_n}(d+6)$$

From dimensional shift to dimensional recurrence

- Keep on iterating:

$$
\begin{aligned}
I_{\vec{1}_n}(d) &= \sum_{i=1}^{n} x_{n,i}\, I_{\vec{1}_{n,i}}(d) + \frac{n+1-d}{d-3}\, y_n \sum_{i=1}^{n} x_{n,i}\, I_{\vec{1}_{n,i}}(d+2) \\
&\quad + \frac{(n+1-d)(n-1-d)}{(d-3)(d-1)}\, y_n^2 \sum_{i=1}^{n} x_{n,i}\, I_{\vec{1}_{n,i}}(d+4) + \dots \\
&\quad + \frac{(n+1-d)(n-1-d)\dots(n+1-2k-d)}{(d-3)(d-1)\dots(d+2k-3)}\, y_n^k \sum_{i=1}^{n} x_{n,i}\, I_{\vec{1}_{n,i}}(d+2k) + \dots \\
&= \underbrace{\boxed{\sum_{i=1}^{n} x_{n,i} \sum_{k=0}^{\infty} \frac{\left(\frac{d-n-1}{2}\right)_k}{\left(\frac{d-3}{2}\right)_k} (-y_n)^k\, I_{\vec{1}_{n,i}}(d+2k)}}_{\text{series over } n-1 \text{ denom. int.}} .
\end{aligned}
$$

where $(a)_n = a\,(a+1)\,\dots\,(a+n-1)$ are Pochhammer symbols.

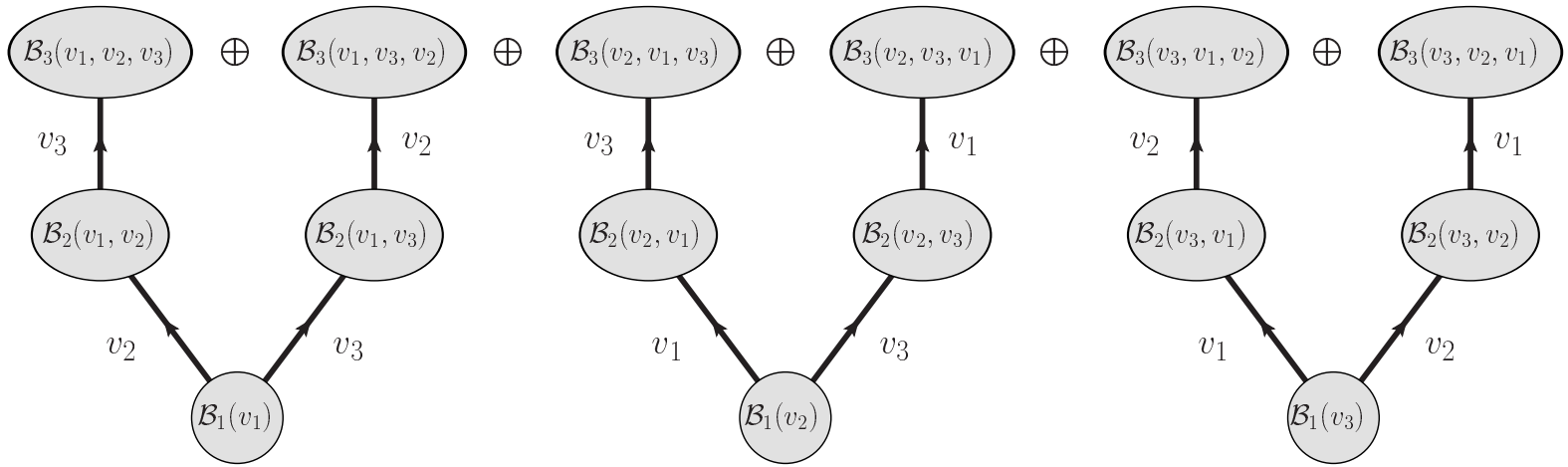
From dimensional recurrence to splitting into branches

- Have expressed n denom. int. as a sum of n series over $n-1$ denom. int.
- **Second round of iterations:** Plug in the same series representation for the $n-1$ denominator integral; results in $n(n-1)$ double series over $n-2$ denominator integrals
- Keep iterating until you get to one denominator integrals
- Original integral represented as sum of $n!$ $n-1$-fold series over one-denominator integrals we call **Branch integrals** $\mathcal{B}_n$

Branch decomposition

$$I_{\vec{1}_n}^{d=4-2\varepsilon}(v_1,\ldots,v_n) = \sum_{\sigma\in S_n}\left(\prod_{i=2}^{n} x_{i,i}\left(v_{\sigma(1)},\ldots,v_{\sigma(i)}\right)\right)\mathcal{B}_n(v_{\sigma(1)},\ldots,v_{\sigma(n)};\varepsilon)$$

Branch integrals

$I_{111} =$ 

- Branch integrals are recursively given by

$$\mathcal{B}_{n+1}(v_1, \ldots, v_{n+1}; \varepsilon) = \sum_{k_{n+1}=0}^{\infty} c(\varepsilon, n+1, k_{n+1}) (-y_{n+1}(v_1, \ldots, v_{n+1}))^{k_{n+1}} \times \mathcal{B}_n(v_1, \ldots, v_n; \varepsilon - k_{n+1}).$$

with summation coefficients $c(\varepsilon, n, k) = \frac{\left(\frac{3-n}{2} - \varepsilon\right)_k}{\left(\frac{1}{2} - \varepsilon\right)_k}$.

- $\mathcal{B}_n$ depends only on the $n-1$ variables y_i $(i = 2, \ldots n)$ and the "root vectors" mass v_{11}

Example: Massless two denominator integral

- **Massless root integral:** $\mathcal{B}_1^{(0)}(v_1;\varepsilon) = -\frac{\pi}{\varepsilon}$
- Plug into recursive formula: $(-y_2 = 1 - 2/v_{12})$

$$\mathcal{B}_2^{(0)}(v_1,v_2;\varepsilon) = \sum_{k_2=0}^{\infty}(-y_2)^{k_2}\mathcal{B}_1^{(0)}(v_1;\varepsilon-k_2) = \underbrace{\pi\sum_{k_2=0}^{\infty}\frac{(-y_2)^{k_2}}{k_2-\varepsilon}}_{\text{good for further iteration}}$$

- ε-expansion (all orders):

$$\mathcal{B}_2^{(0)}(v_1,v_2;\varepsilon-k_2) \underset{\substack{\uparrow\\ \text{geom. series}}}{=} \pi\left[-\frac{1}{\varepsilon}+\sum_{n=0}^{\infty}\varepsilon^n\,\mathrm{Li}_2\left(1-\frac{2}{v_{12}}\right)\right]$$

$$\underset{\substack{\uparrow\\ \text{resum } \log v_{12}}}{=} \pi\left(\frac{v_{12}}{2}\right)^{-\varepsilon}\left[-\frac{1}{\varepsilon}+\sum_{N=1}^{\infty}\varepsilon^N\sum_{m=1}^{N}(-1)^m\mathrm{S}_{N-m+1,m}\left(1-\frac{v_{12}}{2}\right)\right]$$

All-order result for three denominators (beautiful)

$$
\begin{aligned}
I^{(0)}_{111}(\varepsilon) &= 2\pi \sum_{i=1}^{3} \frac{x_{3,i}}{w_i} \left(\frac{w_i}{2}\right)^{-\varepsilon} \left[-\frac{1}{\varepsilon} + \sum_{N=1}^{\infty} \varepsilon^N \mathcal{S}_{N+1}\left(1-\frac{w_i}{2}\right)\right] \\
&+ 4\pi \frac{\sqrt{Y_3}}{X_3} \frac{4^\varepsilon \Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \sum_{i=1}^{3} \Bigg\{ \sum_{N=1}^{\infty} \varepsilon^N \sum_{\{\pm\}_N} \Big[\mathcal{G}_{N+1}(\delta_i,\beta) - \mathcal{G}_{N+1}(\delta_i,1)\Big] \\
&\quad + (1+y_3)^\varepsilon \arctan\frac{1}{\delta_i} \left[-1 + \sum_{N=2}^{\infty} \varepsilon^N \mathcal{S}_N\left(\frac{y_3}{y_3+1}\right)\right] + \left(\frac{w_i}{2}\right)^{-\varepsilon} \left[\arctan\frac{1}{\delta_i}\right. \\
&\quad \left. + \varepsilon\, \mathcal{G}_2(\delta_i,1) + \sum_{N=2}^{\infty} \varepsilon^N \left(\mathcal{G}_{N+1}(\delta_i,1) - \sum_{k=0}^{N-2} \mathcal{G}_{k+1}(\delta_i,1) \mathcal{S}_{N-k}\left(1-\frac{w_i}{2}\right)\right)\right]\Bigg\},
\end{aligned}
$$

with abbreviations $w_1 = v_{23}$, $w_2 = v_{13}$, $w_3 = v_{12}$ as well as

$$
\delta_i = \frac{-w_i + w_j + w_k}{\sqrt{Y_3}} \quad (i,j,k \text{ pairwise different}), \quad \beta = \sqrt{\frac{1+y_3}{y_3}},
$$

$$
\mathcal{S}_N(z) \equiv \sum_{m=1}^{N-1} (-1)^m \mathrm{S}_{N-m,m}(z) \text{ and } \mathcal{G}_{N+1}(\delta_i,\beta) \equiv (-1)^N \sum_{\{\pm\}_N} \mathrm{Im}\, \mathrm{G}(\mathrm{i}\delta_i, \pm\beta, \underbrace{\pm 1, \ldots, \pm 1}_{N-1}; 1).
$$

5. Conclusion

Conclusion

Structurally:

- Lee-Pomeransky like Euler representation for angular integrals
- IBP relations turned to explicit recursion relations for arbitrary number of denominators
- scale reduction of master integrals $n(n+1)/2 \to n$

Explicit:

- $\mathcal{O}(\varepsilon)$ of four denominator integral
  (confirmed by independent MB calculation of [Ahmed, Hasan, and Rapakoulias '25])
- all-order ε-expansion for three-denominator integral

Thank you for your attention!

6. Backup

Simplified Euler representation for angular integrals

- Start from Somogyi's Mellin-Barnes representation
- Directly convert to parametric Euler integral
- Result is (up to factors of 2 and π) of the form of a Lee-Pomeransky representation (dimension: $d = 2 - 2\varepsilon$, loop number: $l = 1$):

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\varepsilon) = \frac{2^{2-j-2\varepsilon}\pi^{1-\varepsilon}\,\Gamma(1-\varepsilon)}{\Gamma(2-j-2\varepsilon)\prod_{k=1}^{n}\Gamma(j_k)} \times \int_0^\infty \frac{\prod_{k=1}^{n} \mathrm{d}t_k\, t_k^{j_k-1}}{\underbrace{\left(1+\sum_{k=1}^{n} t_k + \sum_{k\le l=1}^{n} \tilde{v}_{kl}\, t_k\, t_l\right)}_{\text{"Lee-Pomeransky" polynomial}}}^{1-\varepsilon}$$

with $\tilde{v}_{kl} = v_k \cdot v_l/2$ for $k \neq l$ and $\tilde{v}_{kk} = v_k^2/4$.

Salvatori's result: genuine four denominator part

$$
\begin{aligned}
C_2 =& \mathrm{G}\left(0, Z_1^+; 1\right) - \mathrm{G}\left(0, Z_1^-; 1\right) \\
& - \mathrm{G}\left(\frac{v_{13}}{v_{13}-v_{14}}, Z_1^+; 1\right) + \mathrm{G}\left(\frac{v_{13}}{v_{13}-v_{14}}, Z_1^-; 1\right) \\
& - \mathrm{G}\left(\frac{v_{23}}{v_{23}-v_{24}}, Z_1^+; 1\right) + \mathrm{G}\left(\frac{v_{23}}{v_{23}-v_{24}}, Z_1^-; 1\right) \\
& + \mathrm{G}\left(\frac{2v_{34}}{2v_{34}-v_{44}}, Z_1^+; 1\right) - \mathrm{G}\left(\frac{2v_{34}}{2v_{34}-v_{44}}, Z_1^-; 1\right) \\
& - \mathrm{G}\left(0; \frac{v_{14}}{v_{12}}\right) \left[\mathrm{G}\left(0, Z_2^+\right) - \mathrm{G}\left(0; Z_2^-\right)\right] \\
& + \mathrm{G}\left(0; \frac{v_{44}}{2v_{24}}\right) \left[\mathrm{G}\left(0; Z_2^+\right) - \mathrm{G}\left(0; Z_2^-\right)\right]
\end{aligned}
$$

with the abbreviations: $Z_1^{\pm} = \dfrac{2v_{13}v_{23}}{v_{12}v_{34} + 2v_{13}v_{23} - v_{13}v_{24} - v_{14}v_{23} \pm \sqrt{X_4}}$

$$
Z_2^{\pm} = \frac{v_{13}v_{24} + v_{14}v_{23} - v_{12}v_{34} \pm \sqrt{X_4}}{2v_{13}v_{23}}.
$$

IBP without Laporta

Algorithm for reduction to master integrals

1. Set n to the number of non-zero indices.
2. `IF` there is at least one negative index, use index raising identity for n denominators to raise the least negative index. This will eventually reduce all negative indices to zero and lower n.
3. `ELSE IF` there is an index larger than 1, use index lowering identity for n denominators to reduce the largest index. This will eventually reduce all positive indices to 1.
4. `ELSE` return the integral as a master integral.