QCD Light-cone sum rules: achievements and challenges

Alexander Khodjamirian

Volodya Fest ,
November 27, 2025, Regensburg

Outline

- The early history of LCSRs
- $B \to \pi$ transition form factor: the standard application
- The saga of pion distribution amplitude
- From one pion to two pions ...
- B-meson distribution amplitude enters the field
- Challenges and future goals

- The early history of LCSRs

The very first paper on light-cone sum rules

ЯДЕРНАЯ ФИЗИКА
JOURNAL OF NUCLEAR PHYSICS
т. 44, вып. 6(12), 1986

ПИСЬМА В РЕДАКЦИЮ

РАСПАД $\Sigma^+ \to p\gamma$ В КХД

БАЛИЦКИЙ Я. Я., БРАУН В. М., КОЛЕСНИЧЕНКО А. В.

ЛЕНИНГРАДСКИЙ ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ АН СССР

(Поступило в редакцию 15 мая 1986 г.)

$(qp)/p^2$ разлагать нельзя. Такая ситуация характерна для глубоконеупругого рассеяния или эксклюзивных процессов. Поэтому адекватным формализмом будет операторное разложение на световом конусе, а не традиционное для правил сумм разложение Вильсона.

Действуя в духе правил сумм, рассматриваем коррелятор

$$i^2 \int d^4x d^4y \langle T\{\eta_p(x) j_\mu(y) \bar{\eta}_\Sigma(0)\}\rangle \exp[ip_1(x-y)+ip_2 y] = \\ = i\sigma_{\alpha\beta} p_{1\alpha} p_{2\beta} (p_1+p_2)_\mu [W_R(p_1, p_2) P_R + W_L(p_1, p_2) P_L] + \text{другие структуры}, \quad (1)$$

где $P_{L(R)} = (1 \pm \gamma_5)/2$; η_p, η_Σ – токи с квантовыми числами протона и Σ^+ [3], $j_\mu = e_u \bar{u}\gamma_\mu u + e_d \bar{d}\gamma_\mu d + e_s \bar{s}\gamma_\mu s$. Нетривиально вычисление вкладов в интеграл (1) от области больших $y^2 \gg x^2$ ($\sim p^{-2}$). Для T-произведения барионных токов можно записать разложение вблизи светового конуса. Типичный член разложения имеет вид

$$T\{\eta_p(x)\bar{\eta}_\Sigma(0)\} = -i(72\pi^2 x^4)^{-1} \langle \bar{u}u\rangle \hat{x}\sigma_{\alpha\beta}\hat{x} \Big\{ \int_0^1 d\alpha \langle \bar{u}u\rangle [\bar{d}(\alpha x)\sigma_{\alpha\beta} d(x) P_L + \\ + \bar{s}(0)\sigma_{\alpha\beta} s(\alpha x) P_R] - \bar{u}(0)\sigma_{\alpha\beta} u(x) [\langle \bar{d}d\rangle P_L + \langle \bar{s}s\rangle P_R] \Big\} \quad (2)$$

и соответствует степенной поправке размерности семь на рисунке ∂, e. Слабый гамильтониан для характерных в задаче виртуальностей ~ 1 ГэВ^2 выбран в стандартном виде [4], множители $\sqrt{2} G_F \sin\theta_C \cos\theta_C \cdot c_-$ подразумеваются. В (2) характерно наличие интеграла по дополнительному параметру α, управляющему раздвижкой кварковых операторов. В более сложных диаграммах с глюонами появятся вставки операторов $G_{\mu\nu}$ на прямой линии, соединяющей кварки [5]. При подстановке (2) в интеграл (1) возникает коррелятор, определяющий тензорную волновую функцию (ВФ) фотона

$$\int d^4y\, e^{iqy} \langle T\{j_\mu(y)\bar{\psi}(x)\sigma_{\alpha\beta}\psi(0)\}\rangle = e_\psi(\delta_{\mu\alpha} q_\beta - \alpha \rightleftarrows \beta) \int_0^1 dv\, \varphi(v) e^{iqxv} + \ldots$$

Other early papers on light-cone sum rules

Nuclear Physics B312 (1989) 509–550
North-Holland, Amsterdam

RADIATIVE DECAY $\Sigma^+ \to p\gamma$ IN QUANTUM CHROMODYNAMICS

I.I. BALITSKY, V.M.

Leningrad Nuclear Physics

R

Z. Phys C – Particles and Fields 44, 157–166 (1989)

Zeitschrift für Physik C Particles and Fields

© Springer-Verlag 1989

QCD sum rules in exclusive kinematics and pion wave function

V.M. Braun and I.E. Filyanov

Leningrad Nuclear Physics Institute, Gatchina, SU-188350 Leningrad, USSR

Received 10 April 1989

Nuclear Physics B345 (1990) 137–172
North-Holland

B-MESON EXCLUSIVE DECAYS INTO BARYONS

V.L. CHERNYAK*

Collège de France – Laboratoire de Physique Corpusculaire, 11, Place Marcelin Berthelot, 75 231 PARIS, Cedex 05, France

I.R. ZHITNITSKY

664 003, Irkutsk State University, Irkutsk-03, USSR

Received 9 January 1990

- $B \to \pi$ form factor: the standard application

$B \to \pi$ transition form factors

- hadronic matrix element reduced to two form factors:

  functions of the momentum transfer squared q^2

$$\langle \pi^+(p)|\bar{u}\gamma_\mu b|B(p+q)\rangle = f^+_{B\pi}(q^2)\left[2p_\mu + \left(1 - \frac{m_B^2 - m_\pi^2}{q^2}\right)q_\mu\right] + f^0_{B\pi}(q^2)\frac{m_B^2 - m_\pi^2}{q^2}q_\mu,$$

- observable: differential width

$$\frac{d\Gamma(\bar{B}^0 \to \pi^+ l^- \nu)}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} p_\pi^3 |f^+_{B\pi}(q^2)|^2 + O(m_l^2)$$

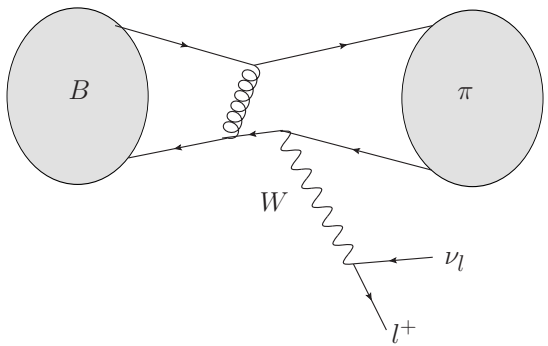
$0 < q^2 < (m_B - m_\pi)^2 \sim 26\ \mathrm{GeV}^2$, p_π -kinematical factor

Form factors in continuum QCD

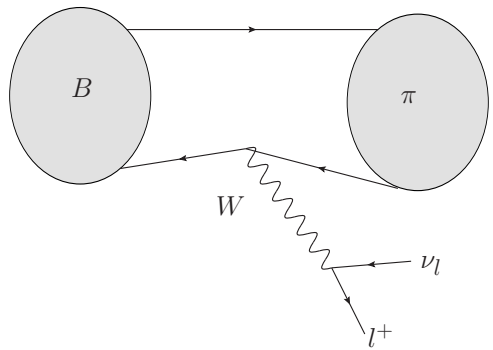
- use of effective theories obtained from QCD in a certain limit:
  heavy-quark limit $\rightarrow$ HQET, large recoil limit $\rightarrow$ SCET
- factorization theorems in $m_b \rightarrow \infty$, (originally for $B \rightarrow \pi\pi$)

  M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda (1999)
- for $B \rightarrow \pi$ form factor at large recoil: ($E_\pi \sim m_B/2$, $q^2 \rightarrow 0$)

  M. Beneke, Th. Feldmann (2001)



HARD, FACTORIZABLE



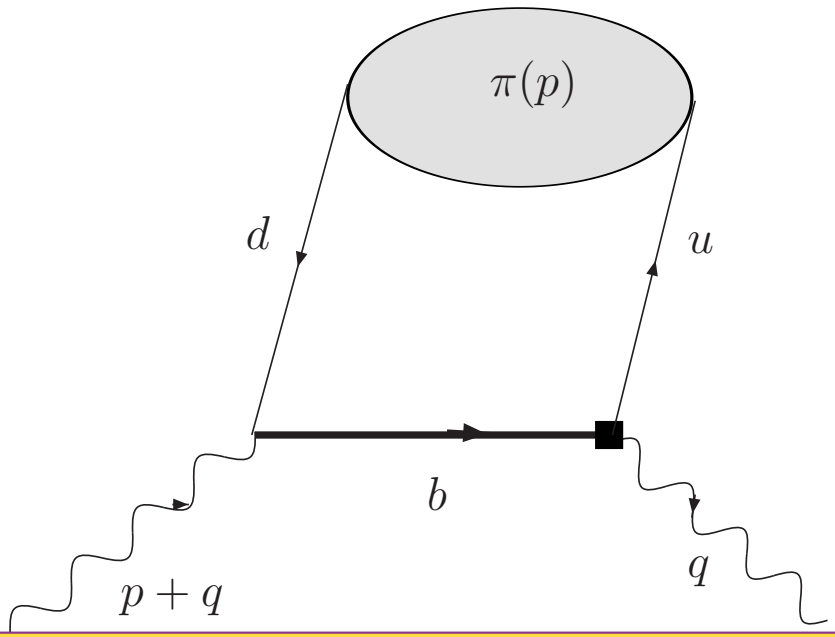
SOFT, NONFACTORIZABLE

$$f_{B\pi}(q^2) \sim \alpha_s(\mu) \int d\omega du \, \phi_B^+(\omega,\mu) T_h(q^2,\omega,u,\mu) \varphi_\pi(u,\mu) + f_{B\pi}^{soft}(q^2)$$

$$\mu = \sqrt{m_b \Lambda}$$

- the main challenge: calculate the soft (overlap) part of the form factor

The correlation function



$q^2, (p+q)^2 \ll m_b^2,$
b-quark highly virtual $\Rightarrow$ $x^2 \sim 0$

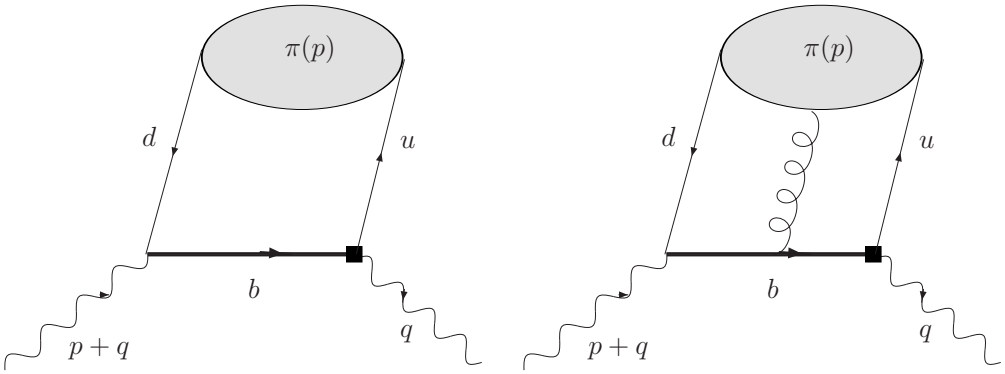
$$F_\lambda(q,p) = i \int d^4x\, e^{iqx} \langle \pi(p) \mid T\{\bar{u}(x)\gamma_\lambda b(x), \bar{b}(0) i\gamma_5 d(0)\} \mid 0\rangle$$

$$\sum_{t=2,3,4,..} \int du\; T^{(t)}(q^2, (p+q)^2, m_b^2, \alpha_s, u, \mu)\, \varphi_\pi^{(t)}(u,\mu)$$

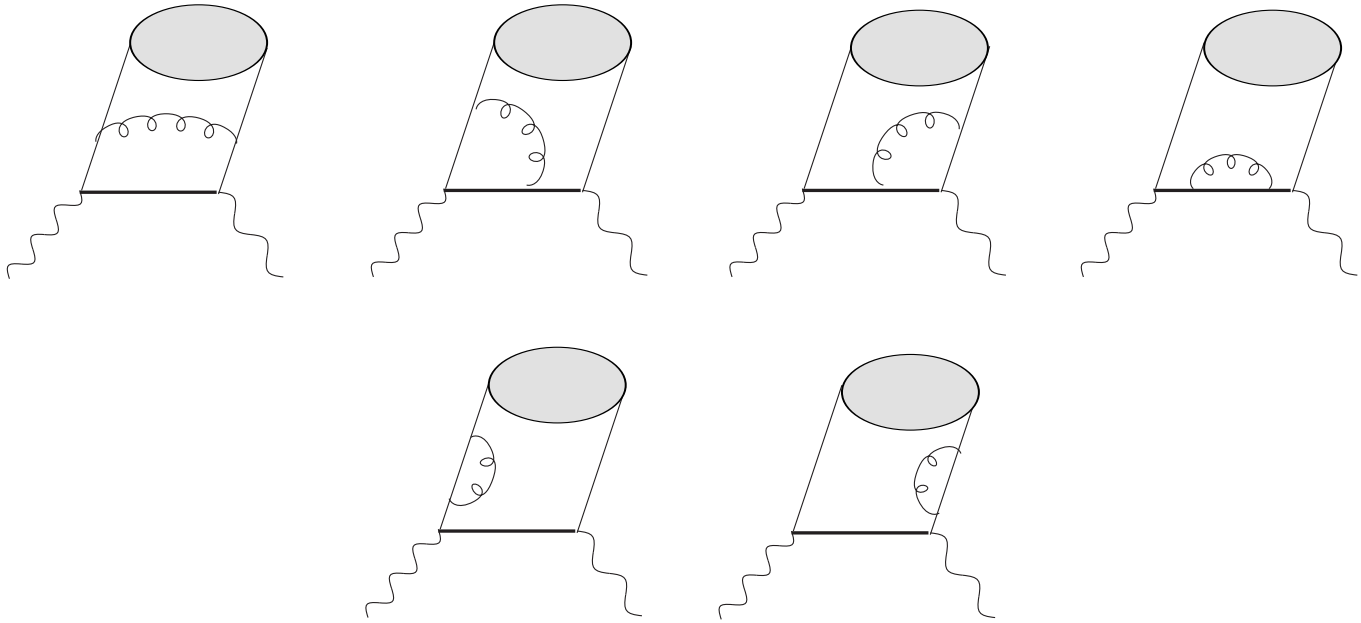
V.Belyaev, A.K., R.Rückl (1993) V.Braun, V.Belyaev, A.K., R.Rückl (1996)
A.K., R.Rückl, S.Weinzierl, O. Yakovlev (1997); E.Bagan, P.Ball, V.Braun (1997)
P. Ball, R. Zwicky (2001) G.Duplancic, A.K., B.Melic, Th.Mannel, N.Offen (2007)

Diagrams

- leading order in α_s (LO) including soft, i.e. low-virtuality gluon



- NLO, $O(\alpha_s)$ contributions

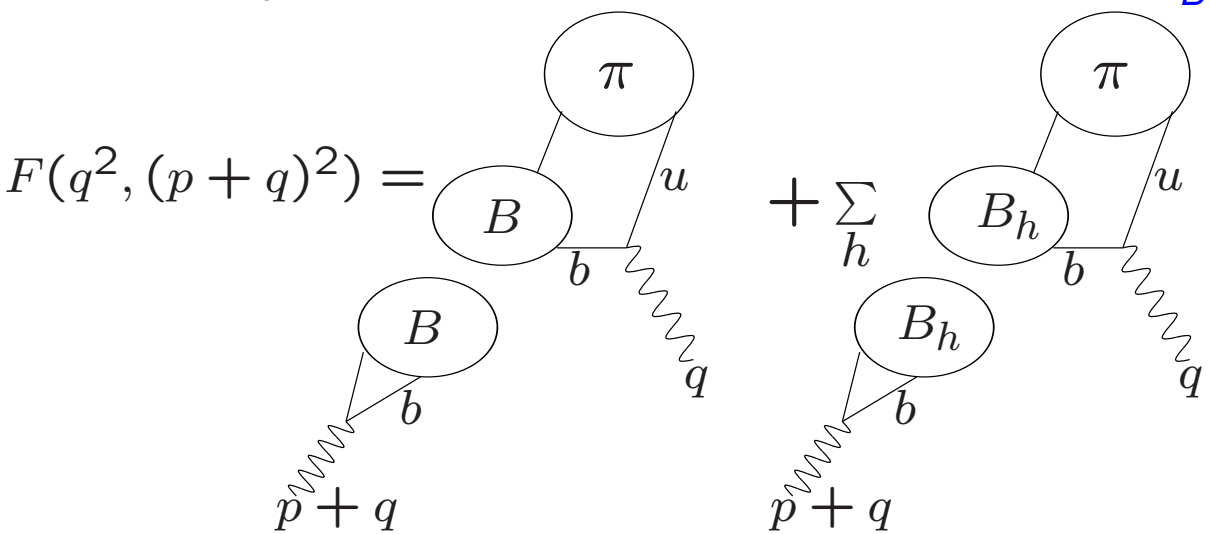


Hadronic dispersion relation

- Analytical continuation of the correlation function in the complex variable $(p+q)^2$ at fixed $q^2 \Longrightarrow$ Cauchy theorem,

$$F((p+q)^2, q^2) = \int_{s_{min}}^{\infty} ds \frac{\mathrm{Im} F(s, q^2)}{s - (p+q)^2 - i\epsilon}$$

- replacing the Im part by the sum over all possible hadronic states with B-meson quantum numbers, located at $s_{min} = m_B^2$ and above



$f_B f_{B\pi}^{+}(q^2)$ $\qquad\qquad$ $\sum_{B_h} \rightarrow duality\ (s_0^B)$

- The saga of the pion distribution amplitude

Basics of the pion DA's

- twist 2 DA: normalized with f_π, expansion in Gegenbauer polynomials

$$\varphi_\pi(u,\mu) = 6u(1-u)\left[1 + \sum_{n=2,4,..} a_n^\pi(\mu) C_n^{3/2}(2u-1)\right],$$

$$a_{2n}^\pi(\mu) \sim [Log(\mu/\Lambda_{QCD})]^{-\gamma_{2n}} \to 0 \quad \text{at} \quad \mu \to \infty$$

[Efremov-Radyushkin-Brodsky-Lepage evolution]

- essential parameters: $a_{2,4}^\pi(\mu_0)$,
  determined from:
  - matching measured pion form factors to LCSRs,
  - two-point QCD sum rules,
  - lattice QCD
- normalization constants and moments of twist 3,4 DA,
  determined mainly from two-point sum rules

[P. Ball, V.Braun, A.Lenz (2006)]

LCSR for pion e.m. form factor

ELSEVIER

Physics Letters B 328 (1994) 457–465

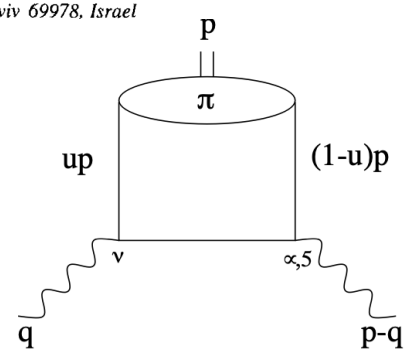
Soft contribution to the pion form factor from light-cone QCD sum rules

Vladimir Braun [1]

Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, D-80805 Munich, Federal Republic of Germany

Igor Halperin

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel



The correlation function:

$$T_{\mu\nu}(p,q) = i\int d^4x\, e^{iqx}\langle 0|T\{(\bar{d}(0)\gamma_\mu\gamma_5 u(0) j_\nu^{em}(x)\}|\pi(p)\rangle\,,$$

the form factor "embedded' in the dispersion relation

$$T_{\mu\nu}(p,q) = 2if_\pi(p-q)_\mu p_\nu F_\pi(Q^2)\frac{1}{m_\pi^2-(p-q)^2} + \int ds\,\frac{\rho_{\mu\nu}(s)}{s-(p-q)^2}\,.$$

LCSR for pion e.m. form factor

[V.Braun, AK, M.Maul, PRD D61 (2000)]

$$F_\pi(Q^2) = F_\pi^{(2)}(Q^2) + F_\pi^{(2,\alpha_s)}(Q^2) + F_\pi^{(4)}(Q^2) + F_\pi^{(6)}(Q^2)\,,$$

$$F_\pi^{(2)}(Q^2) = \int_{u_0^\pi}^1 du\ \varphi_\pi(u,\mu) \exp\left(-\frac{\bar{u}Q^2}{uM^2}\right)\,, \quad u_0^\pi = Q^2/(Q^2+s_0^\pi)$$

$$F_\pi^{(2,\alpha_s)}(Q^2) = \frac{\alpha_s C_F}{\pi}\int_0^1 du \varphi_\pi(u,\mu)\left[\Theta(u-u_0^\pi)\mathcal{F}_{\rm soft}(u,M^2) + \Theta(u_0^\pi-u)\mathcal{F}_{\rm hard}(u,M^2)\right],$$

- "hard" and "soft" contributions to $F_\pi(Q^2)$ identified as different terms of the OPE/LCSR

- rely on quark-hadron duality in the pion channel
(s_0^π taken from QCD SR (SVZ) calculation of f_π)

LCSR for $\gamma\pi$ form factor

- The LCSR relation for $F_{\pi\gamma\gamma^*}(Q^2)$, derived [A.K. (1998)] from a very similar correlation function: $\gamma^*\gamma^* \to \pi^0$

$$\int d^4x e^{-iqx} \langle \pi^0(p)|T\{j_\mu^{em}(x) j_\nu^{em}(0)\}|0\rangle = i\epsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta F^{\gamma^*\pi}(Q^2,(p-q)^2).$$

at Q^2 and $|(p-q)^2|$ sufficiently large,
employ light-cone OPE in terms of the same diagrams $(\gamma_\mu\gamma_5 \to \gamma_\mu)$

$$F^{\pi\gamma^*\gamma^*}(Q^2,(p-q)^2) = \frac{\sqrt{2}f_\pi}{3}\int_0^1 \frac{du\ \varphi_\pi(u)}{\bar{u}Q^2 - u(p-q)^2} + O(\alpha_s) + O(twist4) + \ldots$$

twist-2 [Brodsky-Lepage(1979)]; $O(\alpha_s)$ [Braaten (1983)]; twist-4.

- Next step: dispersion relation for $F^{\pi\gamma^*\gamma^*}$ in $(p-q)^2$, with $F^{\rho\pi\gamma^*}_{LCSR}$ in the ρ- pole term, using duality ansatz:

$$F^{\pi\gamma^*\gamma^*}(Q^2,(p-q)^2) = \frac{\sqrt{2}f_\rho F^{\rho\pi\gamma^*}_{LCSR}(Q^2)}{m_\rho^2 - (p-q)^2} + \frac{1}{\pi}\int_{s_0^\rho}^\infty ds\ \frac{\mathrm{Im}F^{\gamma^*\gamma^*\pi}(Q^2,s)}{s-(p-q)^2}.$$

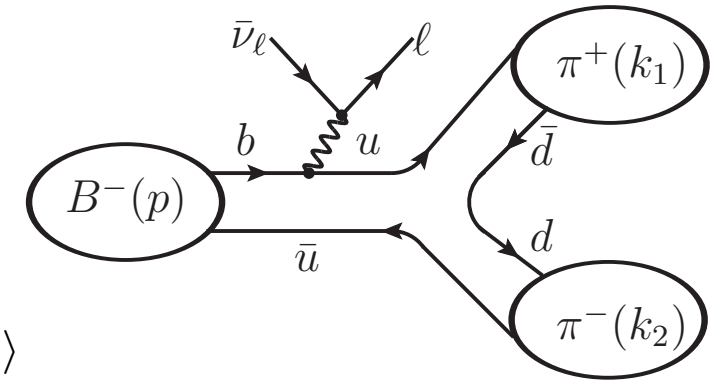
- finally: $(p-q)^2 \to 0$ safely, i.e. $F^{\pi\gamma^*\gamma^*}(Q^2,(p-q)^2) \to F^{\pi\gamma\gamma^*}(Q^2)$

this is not literally LCSR, but a dispersion relation, containing LCSR

- From one pion to two pions

$B \to \pi\pi$ form factors

- semileptonic $B \to \pi\pi\ell\nu_\ell$ decay



- expansion of $B \to \pi\pi$ matrix element:

$$
\begin{aligned}
& i\langle \pi^+(k_1)\pi^0(k_2)|\bar{u}\gamma^\mu(1-\gamma_5)b|\bar{B}^0(p)\rangle \\
& = -F_\perp(q^2,k^2,\zeta)\,\frac{4}{\sqrt{k^2\lambda_B}}\,i\epsilon^{\mu\alpha\beta\gamma}\,q_\alpha\,k_{1\beta}\,k_{2\gamma} \\
& + F_t(q^2,k^2,\zeta)\,\frac{q^\mu}{\sqrt{q^2}} + F_0(q^2,k^2,\zeta)\,\frac{2\sqrt{q^2}}{\sqrt{\lambda_B}}\left(k^\mu - \frac{k\cdot q}{q^2}q^\mu\right) \\
& + F_\parallel(q^2,k^2,\zeta)\,\frac{1}{\sqrt{k^2}}\left(\bar{k}^\mu - \frac{4(q\cdot k)(q\cdot\bar{k})}{\lambda_B}\,k^\mu + \frac{4k^2(q\cdot\bar{k})}{\lambda_B}\,q^\mu\right),
\end{aligned}
$$

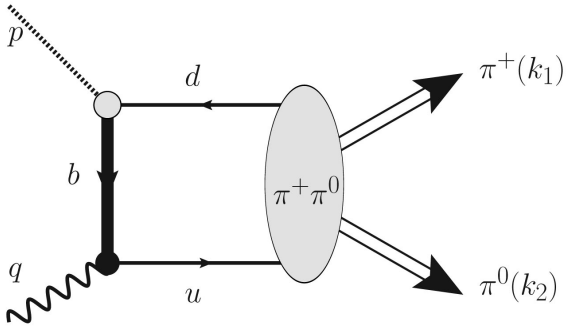
$$k(\bar{k}) = k_1 + (-)k_2$$

- dipion state with $J^P = 0^+, 1^-, 2^+, \ldots$, a rich set of observables
  (decay width distributions, asymmetries etc.)
- final-state pions interact strongly and form resonances,
  $\rho(770)$ with $J^P = 1^-$, $B \to \rho\ell\nu_\ell$ is a model-dependent part of the $B \to 2\pi\ell\nu_\ell$

□ Use of LCSRs with dipion distribution amplitudes

[Ch. Hambrock, AK, (2015)

- consider $\bar{B}^0 \to \pi^+\pi^0\ell^-\nu_\ell$, isospin 1, $L = 1, 3, , ..$
- vacuum $\to$ dipion correlation function



- nonperturbative input: dipion distribution amplitudes (DAs)
- introduced and developed for $\gamma^*\gamma \to 2\pi$ processes

  [M. Diehl, T. Gousset, B. Pire and O. Teryaev, (1998),
  D. Müller, D. Robaschik, B. Geyer, F.-M. Dittes and J. Horejsi, (1994),
  M. V. Polyakov, (1999)]
- only LO, twist-2 approximation for dipion DAs available
- DAs model available only at small $k^2 \sim 4m_\pi^2$
- problems addressed:
  - how important are $L > 1$ partial waves of 2π state in $B \to \pi\pi\ell\nu_\ell$?
  - comparison with $B \to \rho$ FFs calculated from LCSRs with narrow ρ DAs

- *B*-meson distribution amplitude enters the field

B-meson DAs

- definition of two-particle DA in HQET: [A.Grozin, M.Neubert (1997)]

$$\langle 0|\bar{q}_{2\alpha}(x)[x,0]h_{v\beta}(0)|\bar{B}_v\rangle$$
$$= -\frac{if_B m_B}{4}\int_0^\infty d\omega e^{-i\omega v\cdot x}\left[(1+\not{v})\left\{\phi_+^B(\omega)-\frac{\phi_+^B(\omega)-\phi_-^B(\omega)}{2v\cdot x}\not{x}\right\}\gamma_5\right]_{\beta\alpha}$$

$\oplus$ higher twists

- key input parameter: the inverse moment

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty d\omega \frac{\phi_+^B(\omega,\mu)}{\omega}$$

  - possible to extract λ_B from $B\to\gamma\ell\nu_\ell$ using QCDF$\oplus$LCSR

[M.Beneke,V.M. Braun, Y.Ji, Y.-B. Wei (2018)],

  - QCD sum rules in HQET: $\lambda_B(1\text{ GeV}) = 380 \pm 150$ MeV

[V.Braun, D.Ivanov, G.Korchemsky (2004);] [AK,R.Mandal, Th.Mannel (2021)]

- higher twists DAs [V. Braun, Y. Ji, A. Manashov (2017)]
- analogous definition of Λ_B DA in HQET [P. Ball, V. Braun, Y. Ji, E.Gardi (2008)]

LCSRs with B-meson distribution amplitudes (DAs)

[A.K., N. Offen, Th. Mannel (2006)]

"SCET sum rules", [F. De Fazio, Th. Feldmann, T.Hurth (2006)]

- vacuum-to-B-correlation function:

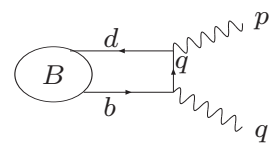
B on-shell state

$$F_{ab}(p,q) = i\int d^4x\, e^{ip\cdot x}\langle 0|\mathrm{T}\{\bar{d}(x)\Gamma_b q(x), \bar{q}(0)\Gamma_a b(0)\}|\bar{B}^0(q+p)\rangle = L_{ab}(p,q)F(p^2,q^2)$$

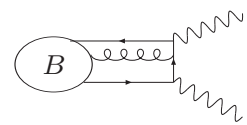
h-interpolating current

transition current

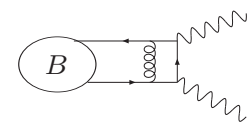
- OPE in terms of B-meson DA's
  defined in HQET,
- valid at $0 < q^2 \ll m_B^2$
- dispersion relation:



(a)



(b)



(c)

$$F(p^2,q^2) = \frac{1}{\pi}\int\limits_{m_h^2}^{\infty}\frac{ds}{s-p^2}\mathrm{Im}F(s,q^2) = F_{OPE}(p^2,q^2)$$

$$\mathrm{Im}F_{ab} = \langle 0|\bar{d}\Gamma_b q|h\rangle\langle h|\bar{q}\Gamma_a b|\bar{B}^0(q+p)\rangle\pi\delta(m_h^2-s) + \ldots = L_{ab}\mathrm{Im}F(s,q^2)$$

$$\boxed{\frac{f_h}{m_h^2-p^2}F^{(B\to h)}(q^2) = \frac{1}{\pi}\int\limits_{m_h^2}^{s_0}\frac{ds}{s-p^2}\mathrm{Im}F_{OPE}(s,q^2)}$$

☐ Accessing $B \to \pi\pi$ form factors

[S.Cheng, AK, J.Virto, (2017)]

- OPE diagrams $\Rightarrow$ invariant amplitudes $\Rightarrow$ dispersion form in k^2:

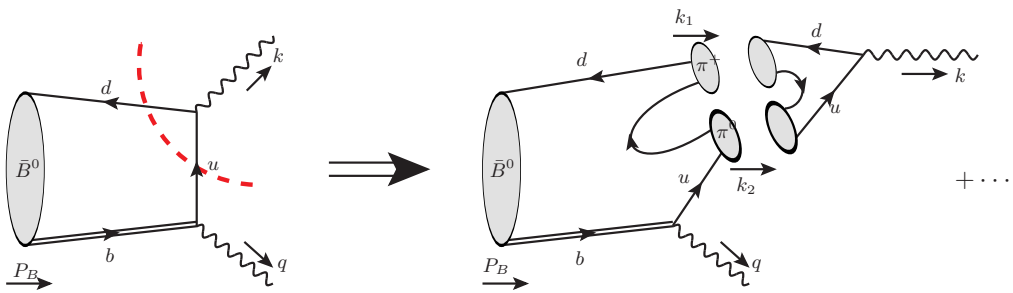
$$F^{\rm OPE}_{(\varepsilon)}(k^2,q^2) = f_B m_B \int_0^\infty d\sigma\, \frac{\phi_+^B(\sigma m_B)}{\bar\sigma(s-k^2)} + \{3-particle\ DAs\}$$

$$s = s(\sigma,q^2) = \sigma m_B^2 - \sigma q^2/\bar\sigma\,, \qquad \bar\sigma \equiv 1-\sigma$$

- hadronic dispersion.relation and unitarity:

$$F_{(\varepsilon)}(k^2,q^2) = \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty} ds\, \frac{{\rm Im}F_{(\varepsilon)}(s,q^2)}{s-k^2}\,.$$

$$2\,{\rm Im}F_{\mu\nu}(k,q) = \int d\tau_{2\pi}\, \underbrace{\langle 0|\bar d\gamma_\mu u\,|\pi^+\pi^0\rangle}_{F_\pi(s)}\, \underbrace{\langle \pi^+\pi^0|\bar u\gamma_\nu(1-\gamma_5)b|\bar B^0(q+k)\rangle}_{B\to 2\pi\ (\ell=1)\ \text{form factors}} + \cdots\,,$$



- Extension to $B \to K\pi$ form factors [S.Descotes-Genon, AK, J.Virto (2019) ...]

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Soft contribution to $B \to \gamma \ell \nu_\ell$ and the B-meson distribution amplitude

V.M. Braun [a], A. Khodjamirian [b,*]

[a] Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany
[b] Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät, Universität Siegen, D-57068 Siegen, Germany

(2013)

- Factorization formula for $B \to \gamma \ell \nu$ M. Beneke, J. Rohrwild, (2011)

$$F_V(E_\gamma) = \frac{e_u f_B m_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \left[\xi(E_\gamma) + \frac{e_b f_B m_B}{2E_\gamma m_b} + \frac{e_u f_B m_B}{(2E_\gamma)^2}\right],$$

- How to estimate the soft form factor $\xi(E_\gamma)$,
  Calculate the same form factor for virtual photon with LC OPE, use dispersion relation:

$$F_{B\to\gamma^*}(q^2, p^2) = \frac{f_\rho F_{B\to\rho}(q^2)}{m_\rho^2 - p^2} + \frac{1}{\pi}\int_{s_0}^{\infty} ds\, \frac{\mathrm{Im}\, F_{B\to\gamma^*}(q^2, s)}{s - p^2},$$

- take the limit $p^2 \to 0$

$$F_{B\to\gamma}(q^2) = \frac{1}{\pi}\int_0^{s_0} \frac{ds}{m_\rho^2}\, \mathrm{Im}\, F^{\mathrm{HQE}}_{B\to\gamma^*}(q^2, s) e^{-(s-m_\rho^2)/M^2} + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{s}\, \mathrm{Im}\, F^{\mathrm{HQE}}_{B\to\gamma^*}(q^2, s).$$

- Challenges and future goals

- two main versions of LCSRs for $B \to h$ form factors:
  with light-hadron DAs and with B DAs.
  both shaped by the results obtained by Volodya !
- LCSRs provide a variety of hadron form factors,
  complements lattice QCD results with independent estimates
- there are hadronic matrix elements where even a 30-40% accuracy would be sufficient, and they are not yet accessible on the lattice
- can we formulate a LCDA formalism for a finite heavy quark mass ?
- to develop further the dimeson DAs (higher twists, 3-particla DAs)
- challlenge common for all sum rule methods: confronting the hadronic spectrum when using dispersion relations, duality approximation, resonance models etc.

I am sure that Volodya will continue his very successful journey near the light cone !