Perturbative QCD on the light-cone

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Partonic operators on the light-cone

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EVOLUTION EQUATIONS FOR QCD STRING OPERATORS

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It is shown that all of the usual programs for the operator expansion can be performed in terms of string operators on the light cone; namely, the separation of contributions from large and small distances, the study of higher twist corrections and the renormalization group analysis. Evolution equations for the leading-twist QCD string operators such as $\bar{\psi}(x)P\exp(ig\int_C A_\mu \mathrm{d}x_\mu)\psi(0)$ are studied in the coordinate space, which has an advantage of preserving explicitly the Lorentz and (in one-loop) conformal invariance of the theory. The solution is found, relating the two string operators in different normalization points. Its short-distance expansion reproduces conventional results for the summation of leading logs in deep inelastic scattering and evolution of hadron wave functions. The light-cone expansion of string operators provides an effective covariant technique for a separation of higher twist effects. As an illustration we calculate the twist-three and twist-four corrections for the deep inelastic scattering and confirm in this way the recent results on the renormalization of twist-three operators.

Operator matrix elements

- QCD applications to hard processes use nonlocal operators of partons at light-like separation

$$\mathcal{O}_\mu(x; z_1, z_2) = \bar\psi(x + z_1)\gamma_\mu[z_1, z_2]\psi(x + z_2)$$

  - quark and anti-quark fields joined by Wilson line along '$+$'-direction

$$[z_1, z_2] = \text{Pexp}\left[ig\int_{z_2}^{z_1} dt A_+(t)\right]$$

- Short-distance expansion yields light-ray operators $\mathcal{O}_\mu(x; z_1, z_2)$ with light-like direction n

$$[\mathcal{O}](x; z_1, z_2) \quad \equiv \quad \sum_{m,k} \frac{z_1^m z_2^k}{m!k!}\left[\bar\psi(x)(\overleftarrow{D}\cdot n)^m \not{n}(n\cdot\overrightarrow{D})^k\psi(x)\right]$$

  - local operators with multiplicative renormalization $[\mathcal{O}] = Z\mathcal{O}$

Light-ray operators

- Light-ray operators satisfy renormalization group equation Balitsky, Braun '87

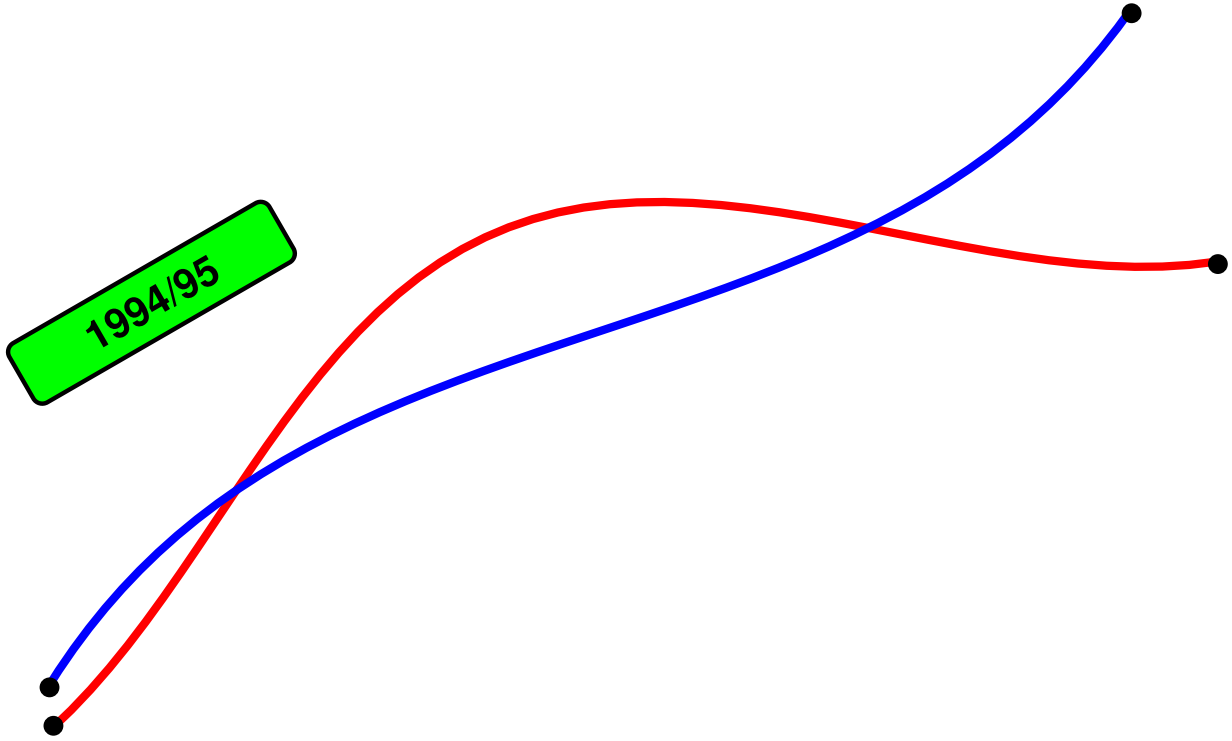
$$\Big(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a)\Big)[\mathcal{O}](x; z_1, z_2) = 0$$

- Integral operator $\mathbb{H}(a)$ with evolution kernel $h(\alpha, \beta)$
- Eigenvalues $\gamma(N) = \int d\alpha d\beta\, h(\alpha, \beta)(1-\alpha-\beta)^N$ are anomalous dimensions of leading-twist local operators with $N = m + k$ derivatives

Applications

- Formalism applicable to all QCD evolution equations
  - PDFs Altarelli, Parisi; $\gamma(N)$
  - meson light-cone distribution amplitudes Efremov, Radyushkin, Brodsky, Lepage
  - general evolution equation for GPDs Belitsky, Müller; $h(\alpha, \beta)$

Worldlines



QCD instantons

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PHYSICS LETTERS B

Instanton-induced contributions to structure functions of deep inelastic scattering

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We identify and calculate the instanton-induced contributions to deep inelastic scattering which correspond to nonperturbative exponential corrections to the coefficient functions in front of parton distributions of the leading twist.

QCD instantons in DIS

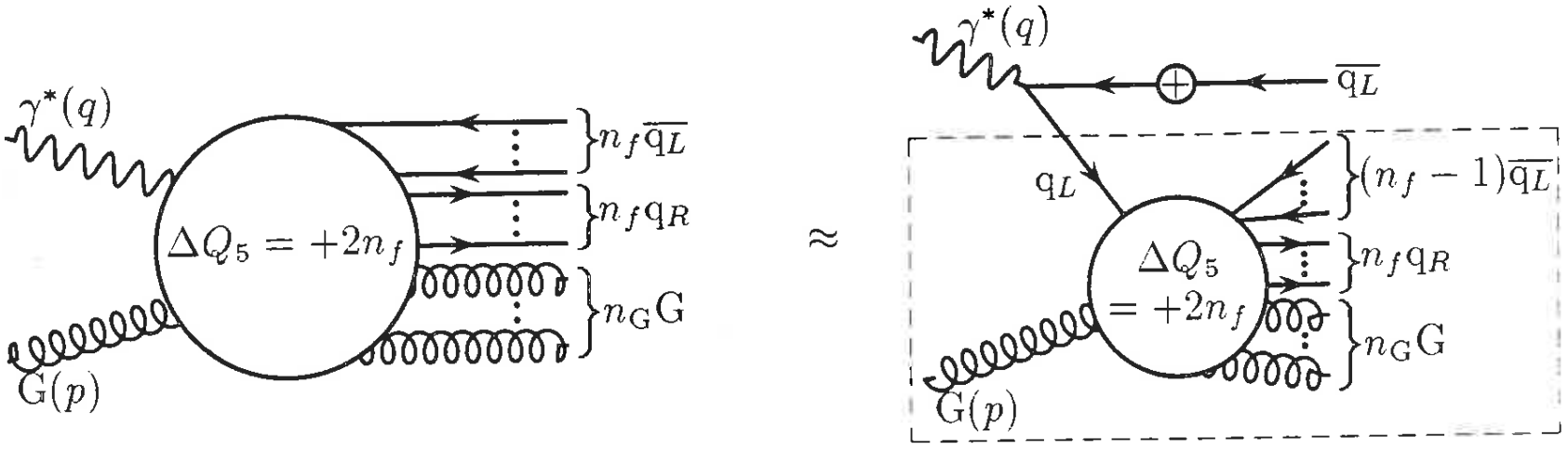
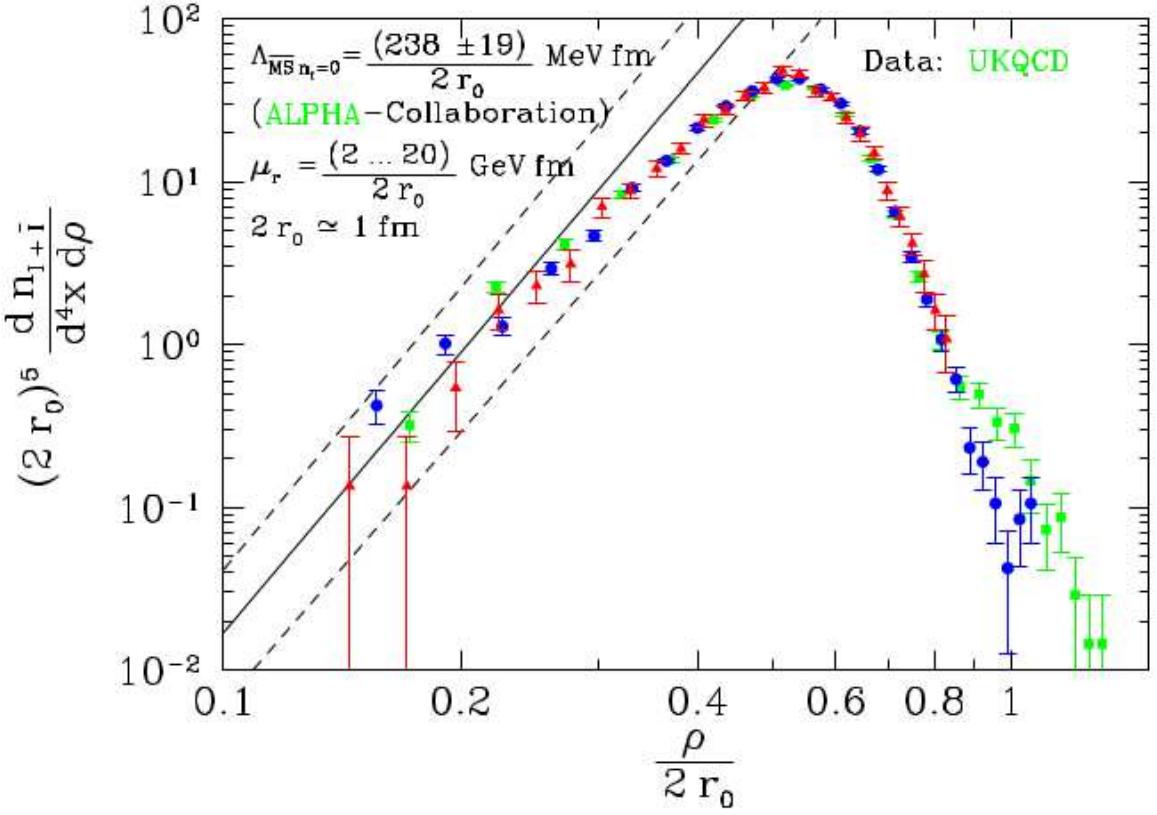


Figure 2.4: Exclusive deep-inelastic scattering of a virtual photon off a gluon with *chirality violation*.

- Chirality violating processes (vertex with n_f right-handed quarks
- Spherical emission of many gluons ("fireball" signature)
- Structure function $F_2(x,Q^2) \simeq \exp\left(-\frac{4\pi}{\alpha_s(\rho)} S_{\mathrm{I\bar{I}}}\right)$ with scale of strong coupling set by instaton size $\rho \sim 1/Q$

Density of QCD instantons

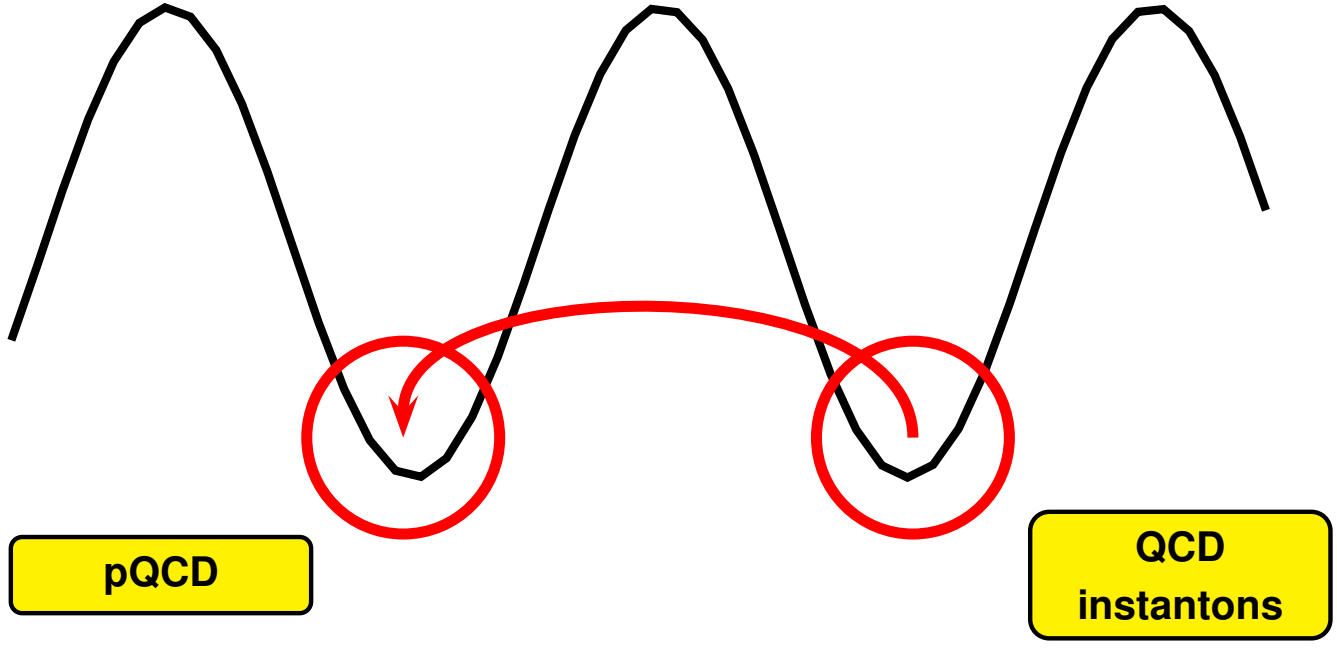


- QCD instanton density $dn_{i+\bar{I}}/d\rho$
  - drives the DIS cross section
  - can be measured on the lattice as function of instanton size
  - agrees for small ρ nicely with instanton perturbation theory

Ringwald, Schrmepp '99

QCD vacuum

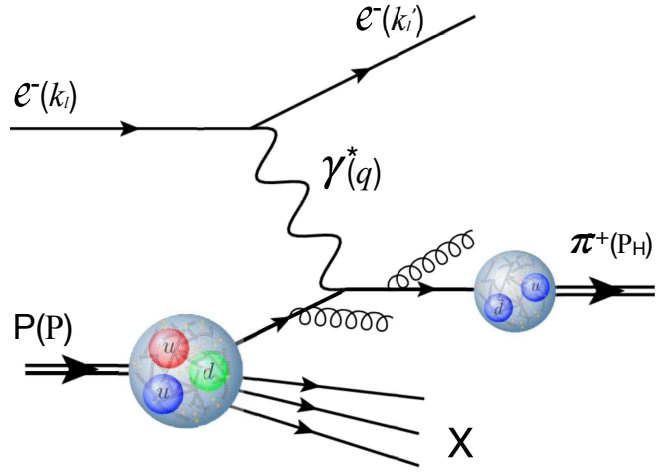
- Tunneling transition do happen



Deep-inelastic scattering

- DIS structure functions
  - unpolarized F_2, F_L, F_3
  - spin dependent g_1

- Semi-inclusive DIS
  - production of identified hadrons in DIS
  - multiple hadron species: π, K, D, p, n, Λ, . .
  - probe of hadron structure
    in broad kinematic range



- QCD factorization at scale μ^2

$$\sigma_{\gamma H\to H'} = \sum_{ij} f_{i/H}(\mu^2) \otimes \hat{\sigma}_{\gamma i\to j}\left(\alpha_s(\mu^2), Q^2, \mu^2\right) \otimes D_{H'/j}(\mu^2)$$

  - parton distribution function (PDF) $f_{i/H}(x,\mu^2)$
  - parton-to-hadron fragmentation function (FF) $D_{H'/j}(z,\mu^2)$

Parton evolution

- Evolution equations for parton distributions
  - non-singlet valence PDFs $q_{\rm ns}^{\rm v} = \sum_f (q_f - \bar{q}_f)$
  - flavor asymmetries $q_{{\rm ns},ff'}^{\pm} = (q_f \pm \bar{q}_f) - (q_{f'} \pm \bar{q}_{f'})$

$$\frac{d}{d\ln\mu^2} q_{\rm ns}^{\pm,{\rm v}} \quad = \quad P_{\rm ns}^{\pm,{\rm v}} \otimes q_{\rm ns}^{\pm,{\rm v}}$$

  - quark-flavor singlet PDFs $q_{\rm s} = \sum_f (q_f + \bar{q}_f)$ and gluon PDF g
  - 2x2 matrix equation

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad = \quad \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Splitting functions P up to N^{3}LO (work in progress)

$$P_{\rm ij} \quad = \quad \underbrace{\alpha_s\, P_{\rm ij}^{(0)} \,+\, \alpha_s^2\, P_{\rm ij}^{(1)} \,+\, \alpha_s^3\, P_{\rm ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4\, P_{\rm ij}^{(3)} + \ldots$$

- Anomalous dimensions (Mellin transform)

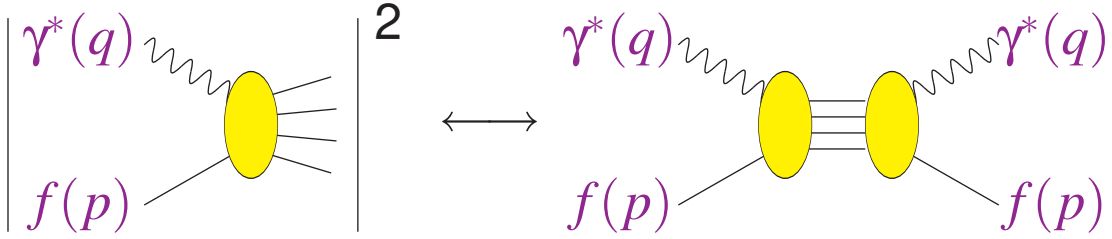
$$\gamma_{\rm ij}(N) = -\int_0^1 dx\, x^N\, P_{\rm ij}(x) = \alpha_s\, \gamma_{\rm ij}^{(0)} + \alpha_s^2\, \gamma_{\rm ij}^{(1)} + \alpha_s^3\, \gamma_{\rm ij}^{(2)} + \alpha_s^4\, \gamma_{\rm ij}^{(3)} + \ldots$$

Operator product expansion (I)

- Direct computation of physical observable
  - structures functions in deep-inelastic scattering (DIS)

Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - Bjorken variable $x = Q^2/(2p\cdot q)$ and momentum transfer $Q^2 = -q^2$



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu} = \mathrm{i}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\, T\left(j^\dagger_\mu(z) j_\nu(0)\right)|\mathrm{P}\rangle$

  $$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3(x,Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Wilson '72; Christ, Hasslacher, Mueller '72

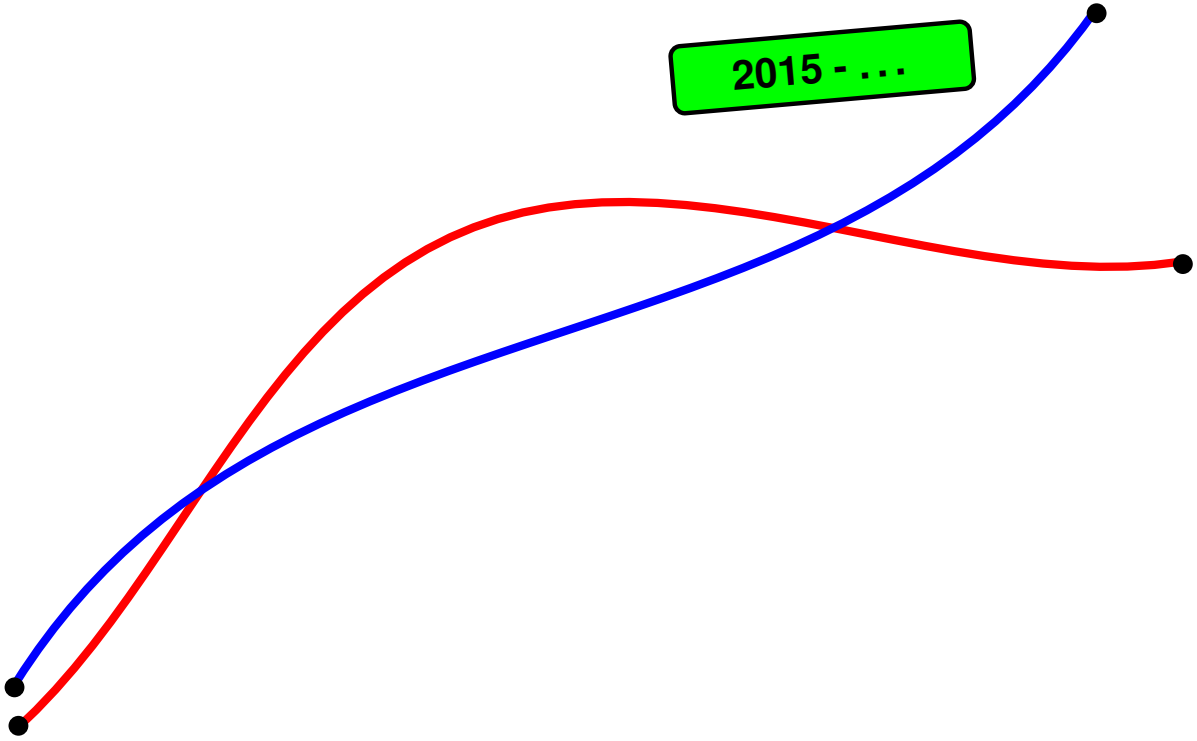
Operator product expansion (II)

- OPE for parton states gives coefficient functions in Mellin space $C_{a,i}^N$

$$T_{\mu\nu,\mathrm{k}} = \sum_{N,\mathrm{j}} \left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C_{L,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) + d_{\mu\nu}\, C_{2,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p\cdot q} C_{3,\mathrm{j}}^N \left(\frac{Q^2}{\mu^2}, \alpha_s\right) \right] A_{\mathrm{jk}}^N\left(\mu^2\right) + \text{ higher twists}$$

- Operator matrix elements $A_{\mathrm{ij}}^N = \langle j | O_{\mathrm{i}}^N | j \rangle$ in parton state
- Anomalous dimensions $\gamma_{\mathrm{ij}}(N)$ from collinear singularities of Compton amplitude $T_{\mu\nu}$ after mass factorization
  - established computational approach through four loops
    one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99;
    three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
- Versatile calculation method
  - photon-DIS $\longrightarrow \gamma_{\mathrm{qq}}, \gamma_{\mathrm{qg}}$
  - Higgs (scalar)-DIS $\longrightarrow \gamma_{\mathrm{gq}}, \gamma_{\mathrm{gg}}$
  - graviton-DIS $\longrightarrow \Delta\gamma_{\mathrm{ij}}$ (polarized quantities) S.M., Vermaseren, Vogt '14

Worldlines



DFG research unit FOR2926

- DFG research unit FOR2926
  Next Generation Perturbative QCD for Hadron Structure: Preparing for the Electron-Ion Collider

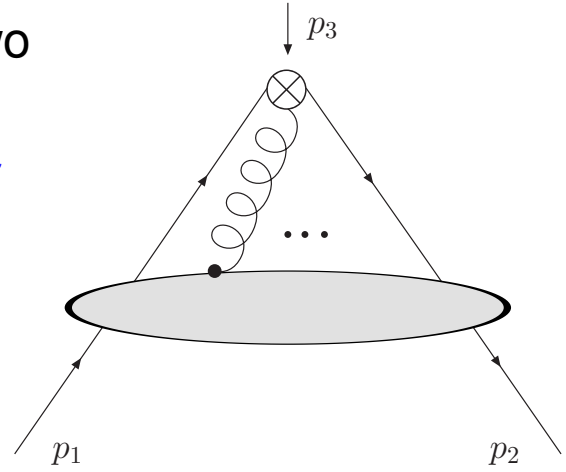
  - joint collaboration Hamburg, Regensburg, Tübingen and DESY
  - advancing theoretical studies in strong interactions
  - second phase ending in 2026

Operator matrix elements

- Scalar singlet operators of spin-N and twist two from contraction with light-like vector Δ_μ
  - quarks ψ, field strenghth $F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}$
  - N covariant derivatives $D = \Delta_\mu D^\mu$

$$\mathcal{O}_{\rm q}^{(N)} = \frac{1}{2}\mathrm{Tr}\left[\overline{\psi}\not{\Delta} D^{N-1}\psi\right]$$

$$\mathcal{O}_{\rm g}^{(N)} = \frac{1}{2}\mathrm{Tr}\left[F_\nu D^{N-2} F^\nu\right]$$



- Direct computation of OMEs $A_{\rm ij}^N = \langle j|O_{\rm i}^N|j\rangle$ in parton state
  - anomalous dimensions $\gamma_{\rm ij}(N)$ from renormalization of operators
- Physical operators mix under renormalization with alien operators

  Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76

Computer algebra

Workflow

- Zero-momentum transfer through operator gives 2-point functions
- Feynman diagrams generation with **Qgraf** Nogueira '91
- Four-loop IBP reduction with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and **TForm** Tentyukov, Vermaseren '07

Quark pure-singlet splitting function $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

Moments of pure-singlet splitting function

- Moments $N = 2, \ldots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

$$\begin{aligned}
\gamma_{\rm ps}^{(3)}(N=2) &= -691.5937093\,n_f + 84.77398149\,n_f^2 + 4.466956849\,n_f^3\,, \\
\gamma_{\rm ps}^{(3)}(N=4) &= -109.3302335\,n_f + 8.776885259\,n_f^2 + 0.306077137\,n_f^3\,, \\
&\ldots \\
\gamma_{\rm ps}^{(3)}(N=20) &= -0.442681568\,n_f + 0.805745333\,n_f^2 - 0.020918264\,n_f^3\,.
\end{aligned}$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd} d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-n_f parts agree with all-N results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- ζ_4 terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-π^2 theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Checked by n_f^2 terms at all-N Gehrmann, von Manteuffel, Sotnikov, Yang '23

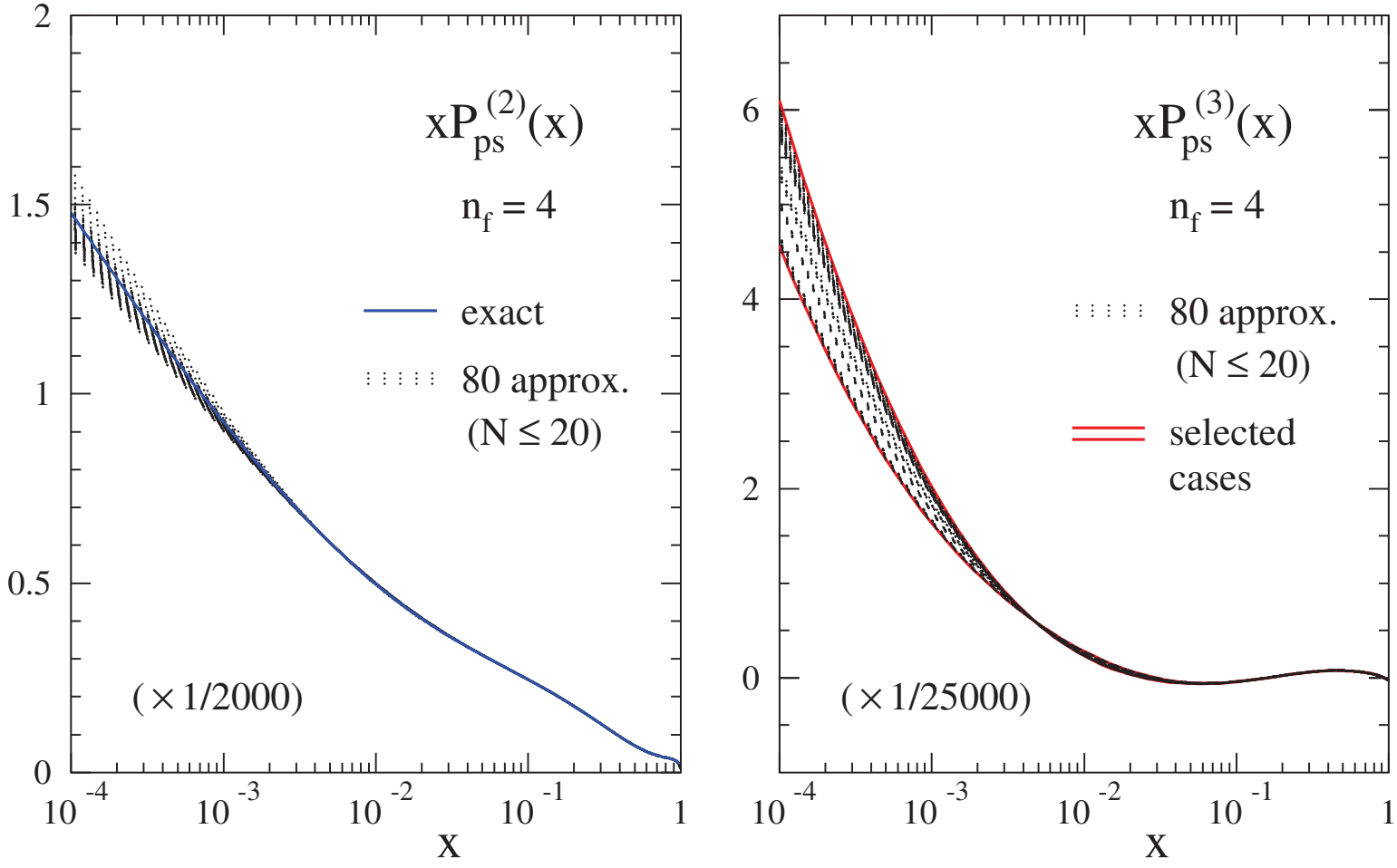
Outlook

- Higher moments $N = 22, \ldots$ to be published

Approximations in x-space

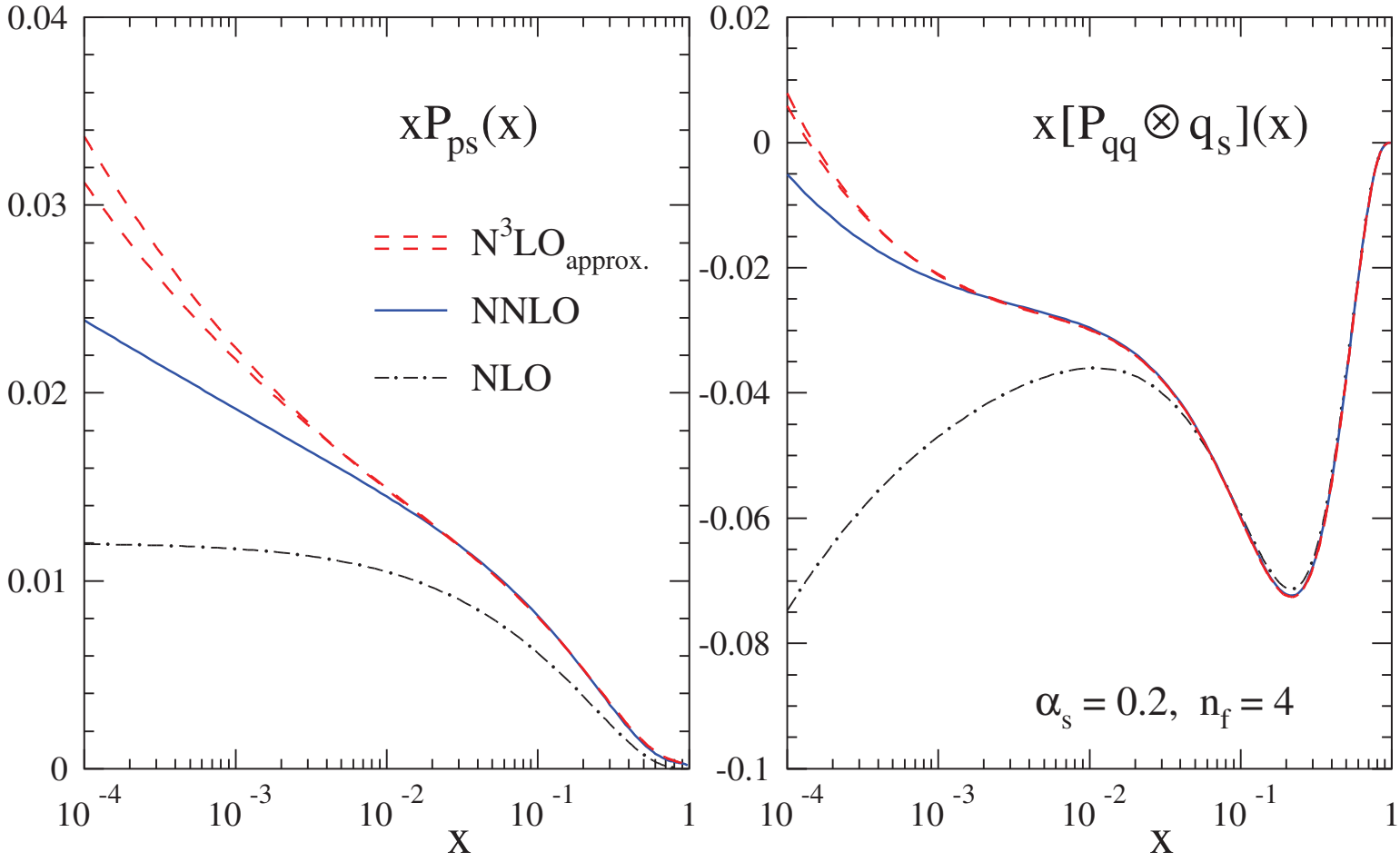
- Large- and small-x information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ Davies, Kom, S.M., Vogt '22
  - leading large-x terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-x terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-x terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)
- Approximations for phenomenology with fixed $n_f = 3, 4, 5$
  - easy-to-use
  - no correlations between different n_f dependenct terms accounted for

Pure-singlet splitting function



- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: four-loops ($n = 3$) with remaining uncertainty

Pure-singlet splitting function



- Left: results for $P_{\rm ps}(x)$ up to N^{3}LO; $\alpha_s = 0.2$ fixed, $n_f = 4$
- Right: contribution to evolution kernel $d\ln q_{\rm s}/d\ln\mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$xq_{\rm s}(x,\mu_0^2) = 0.6\,x^{-0.3}(1-x)^{3.5}\left(1+5.0\,x^{0.8}\right)$$

Scale stability of evolution (I)

- PDF evolution
  - splitting functions enter PDF evolution via convolution

$$\frac{d}{d\ln\mu^2}\, f_{\rm i}(x) \;=\; \sum_{\rm j}\int_x^1 \frac{dz}{z}\, P_{\rm ij}(z)\, f_{\rm j}\left(\frac{x}{z}\right)$$

- Interplay between $P(z\sim x\to 0)$ and $f(\frac{x}{z}\to 1)$
  - $P(\mathbf{z\sim x\to 0})$ has largest uncertainty
  - $f(\frac{\mathbf{x}}{\mathbf{z}}\to \mathbf{1})$ is suppressed
- Model singlet PDFs

$$\begin{aligned} x{\rm q}_{\rm s}(x,\mu_0^2) &= 0.6\, x^{-0.3}(\mathbf{1-x})^{\mathbf{3.5}}\left(1+5.0\, x^{0.8}\right)\\ x{\rm g}(x,\mu_0^2) &= 1.6\, x^{-0.3}(\mathbf{1-x})^{\mathbf{4.5}}\left(1-0.6\, x^{0.3}\right)\end{aligned}$$

- Residual small-x uncertainty in four-loop splitting functions at $x\sim\mathcal{O}(10^{-4})$ affects PDFs only at $x\sim\mathcal{O}(10^{-5})$
  - edge of LHC parton kinematics (low scales, forward region)
  - $x\sim 10^{-5}$ corresponds to $y\sim 4$ and $Q\sim 10$ GeV

Analytic reconstruction (I)

- Sufficiently many Mellin moments allow for reconstruction of analytic all-N expressions through solution of Diophantine equations

  Lenstra, Lenstra, Lovász '82

- Harmonic sums define basis in space of functions for $\gamma_{\rm ij}(N)$

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - at weight w there are $2 \cdot 3^{w-1}$ harmonic sums
  - l-loop $\gamma_{\rm ij}^{(l-1)}(N)$ contains harmonic sums up to weight $2l-1$
    $\longrightarrow$ numbers grow quickly: $2, 18, 162, 1458$ sums for $l = 1, 2, 3, 4$

- Some applications in QCD
  - three-loop non-singlet transversity $\gamma_{\rm tr}^{(2)}$ Velizhanin'12
  - three-loop polarized $\Delta\gamma_{\rm ij}^{(2)}$ S.M., Vermaseren, Vogt '14
  - four-loop non-singlet $\gamma_{\rm ns}^{(3)\pm}$ (large-n_c) S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - four-loop non-singlet DIS $C_{\rm ns}^{(4)}$ (large-n_f)

    Basdew-Sharma, Pelloni, Herzog, Vogt '22
  - ...

Analytic reconstruction (II)

Conformal symmetry and integrability

- Gribov-Lipatov reciprocity relation (RR)
  - diagonal splitting functions $P_{\rm ii}^{(0)}(x)$ invariant under mapping $x \to \frac{1}{x}$
- RR realized for universal $\gamma_{\rm u}(N)$ in $N = 4$ SYM theory
  - uniform transcendentality sums with $w = 2l - 1$ only at l-loops
- RR in N-space for QCD implies $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(\alpha_s)\right)$
- RR constraints for $\gamma_{\rm u}$ reduce number to 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - $2^{w+1} - 1$ objects with denominators $1/(N+1)$ added (255 at $w = 7$)

Example

- Large-n_c limit of $\gamma_{\rm ns}^{(3)\pm}$ only needs harmonic sums with positive index
  - weight w RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ (87 at $w = 7$)
- Additional 46 constraints from large-x/small-x ($N \to \infty$/$N \to 0$) limit
- Solution becomes feasible with 18 Mellin moments for $\gamma_{\rm ns}^{(3)\pm}$

Analytic reconstruction (III)

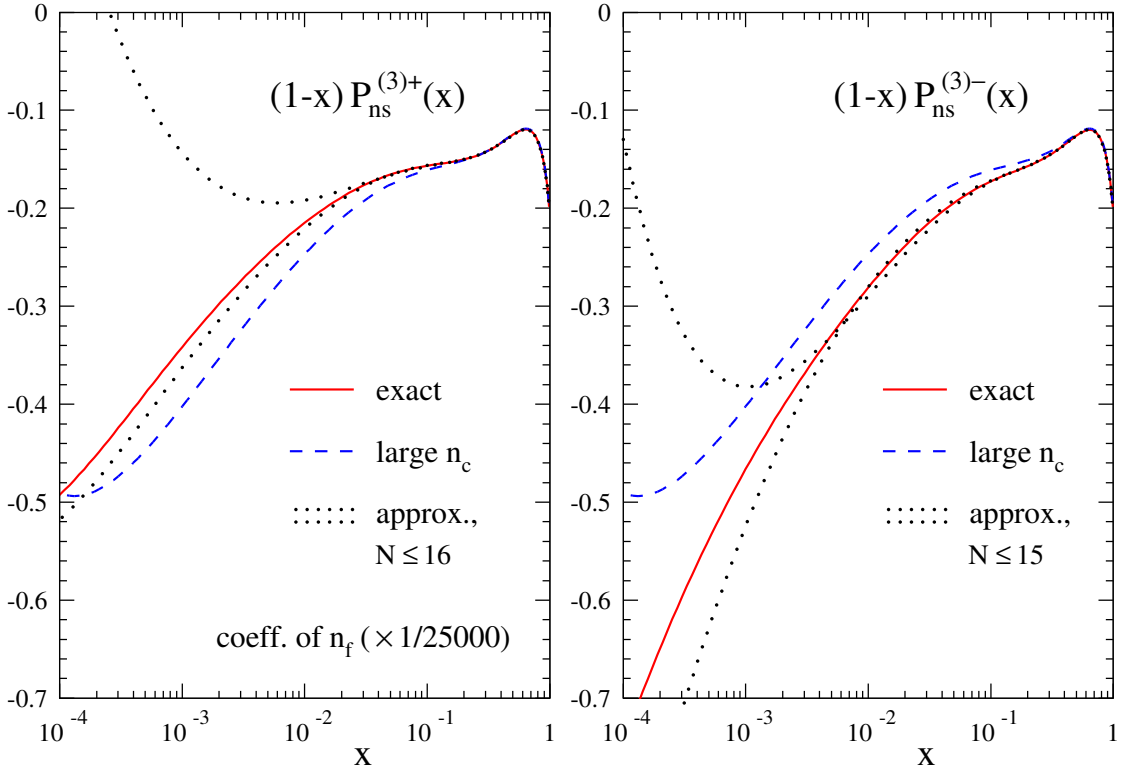
- Mellin moments suffice to determine all-N result for parts of $\gamma_{\rm ps}^{(3)}(N)$
  - harmonic sums and Riemann ζ_n terms up to total weight $w = 7$
- Terms proportional to ζ_5 are particularly simple
  - N-dependent terms respect RR
  - RR implies invariance under mapping $N \to -N-1$
- Combinations of denominators $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\begin{aligned}
\gamma_{\rm ps}^{(3)}(N)\Big|_{\zeta_5} \;=\;& 160\,n_f\,C_F^3\left(9\,\eta + 6\,\eta^2 - 4\,\nu\right) + 80/3\,n_f\,C_A\,C_F^2\left(-9\,\eta - 6\,\eta^2 + 4\,\nu\right) \\
& +40/9\,n_f\,C_A^2\,C_F\left(-1 - 214\,\eta - 144\,\eta^2 + 104\,\nu\right) \\
& +320/3\,n_f\,\frac{d_R^{abcd}d_R^{abcd}}{n_c}\left(-1 + 56\,\eta + 36\,\eta^2 - 16\,\nu\right)
\end{aligned}$$

- Inverse Mellin transformation generates additional terms with ζ_n
  - ζ_n in N-space $\neq$ ζ_n in x-space

Four-loop non-singlet splitting functions

- Four-loop $P_{\rm ns}^{(3)\pm}(x)$ (n_f^1 terms)
- Comparison to large-n_c limit and uncertainty bands from approximations



Analytic results

- Contributions to non-singlet splitting functions
  - n_f-terms (n_f^3 Gracey '94; n_f^2 Davies, Vogt, Ruijl, Ueda, Vermaseren '16)
  - leading n_c terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23
  - all n_f terms Kniehl, S.M., Velizhanin, Vogt '25

    analytic reconstruction based on moments $N \leq 24$ for $n_f d_{FF}^{(4)}$ and $N \leq 32$ for $n_f C_F^2 C_A$

Splitting functions at large x

The large x-limit: $x \to 1$

- Structure of diagonal splitting functions $P_{\rm ii}$ (for $\rm i = q, g$) at large x

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_n^{\rm i}}{(1-x)_+} \,+\, B_n^{\rm i}\,\delta(1-x) \,+\, \ldots$$

- Cusp anomalous dimension $A_n^{\rm i}$
  - known from $1/\epsilon^2$-poles of QCD form factor
- Four loop results in QCD

Large-n_c (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17); n_f (Grozin '18; Henn, Peraro, Stahlhofen, Wasser '19); n_f^2 (Davies, Ruijl, Ueda, Vermaseren, Vogt '16; Lee, Smirnov, Smirnov, Steinhauser '17); n_f^3 (Gracey '94; Beneke, Braun, '95); quartic colour (Lee, Smirnov, Smirnov, Steinhauser '19; Henn, Korchemsky, Mistlberger '19)

- Virtual anomalous dimension $B_n^{\rm i}$
  - parts related to $1/\epsilon$-poles of QCD form factor

Quark virtual anomalous dimension

- Four loop result (up to one unknown $b^{\rm q}_{4,FA}$) Kniehl, S.M., Velizhanin, Vogt '25

$$
\begin{aligned}
B^{\rm q}_4 \;=\;
&C_F^4\left(\frac{4873}{24}-450\zeta_2-\frac{684}{5}\zeta_2^2-\frac{16888}{35}\zeta_2^3+2004\zeta_3-120\zeta_3\zeta_2+\frac{128}{5}\zeta_3\zeta_2^2-1152\zeta_3^2-2520\zeta_5-384\zeta_5\zeta_2+5880\zeta_7\right)\\
&+C_F C_A^3\left(-\frac{371201}{648}-\frac{1}{24}b^{\rm q}_{4,FA}+\frac{4582}{3}\zeta_2-\frac{22388}{135}\zeta_2^2+\frac{48368}{315}\zeta_2^3-\frac{153670}{81}\zeta_3+\frac{472}{3}\zeta_3\zeta_2+\frac{16}{5}\zeta_3\zeta_2^2+528\zeta_3^2\right.\\
&\left.+\frac{11372}{9}\zeta_5+504\zeta_5\zeta_2-2870\zeta_7\right)+C_F^2C_A^2\left(\frac{29639}{36}-\frac{46771}{27}\zeta_2-\frac{24340}{27}\zeta_2^2-\frac{21988}{35}\zeta_2^3+\frac{129662}{27}\zeta_3+\frac{2096}{9}\zeta_3\zeta_2\right.\\
&\left.-\frac{64}{5}\zeta_3\zeta_2^2-\frac{7102}{3}\zeta_3^2+\frac{5354}{9}\zeta_5-2104\zeta_5\zeta_2+8610\zeta_7\right)+C_F^3C_A\left(-\frac{2085}{4}+1167\zeta_2+\frac{4334}{5}\zeta_2^2+\frac{317188}{315}\zeta_2^3\right.\\
&\left.-3260\zeta_3-\frac{1988}{3}\zeta_3\zeta_2+\frac{256}{5}\zeta_3\zeta_2^2+3220\zeta_3^2-976\zeta_5+2064\zeta_5\zeta_2-10920\zeta_7\right)+\frac{d_F^{abcd}d_A^{abcd}}{n_F}\left(b^{\rm q}_{4,FA}\right)\\
&+n_fC_F^3\left(32+162\zeta_2-\frac{408}{5}\zeta_2^2-\frac{51472}{315}\zeta_2^3-308\zeta_3-\frac{256}{3}\zeta_3\zeta_2+224\zeta_3^2+912\zeta_5\right)\\
&+n_fC_F^2C_A\left(-\frac{7751}{54}-\frac{3892}{27}\zeta_2+\frac{55708}{135}\zeta_2^2+\frac{2808}{35}\zeta_2^3-\frac{15400}{27}\zeta_3+\frac{2672}{9}\zeta_3\zeta_2-\frac{1232}{3}\zeta_3^2-\frac{7432}{9}\zeta_5\right)\\
&+n_fC_FC_A^2\left(\frac{20027}{108}-\frac{41092}{81}\zeta_2+\frac{2468}{45}\zeta_2^2-\frac{4472}{135}\zeta_2^3+\frac{9554}{27}\zeta_3-\frac{580}{3}\zeta_3\zeta_2+\frac{416}{3}\zeta_3^2+\frac{1130}{9}\zeta_5\right)\\
&+n_f\frac{d_F^{abcd}d_F^{abcd}}{n_F}\left(-192+\frac{1888}{3}\zeta_2-\frac{704}{15}\zeta_2^2+\frac{2048}{45}\zeta_2^3-\frac{992}{3}\zeta_3+64\zeta_3\zeta_2+256\zeta_3^2-1120\zeta_5\right)\\
&+n_f^2C_F^2\left(-\frac{188}{27}+\frac{1244}{27}\zeta_2-\frac{4208}{135}\zeta_2^2+\frac{56}{27}\zeta_3-\frac{160}{9}\zeta_3\zeta_2+\frac{368}{9}\zeta_5\right)+n_f^2C_FC_A\left(-\frac{193}{54}+\frac{3170}{81}\zeta_2-\frac{32}{9}\zeta_2^2\right.\\
&\left.-\frac{320}{9}\zeta_3+\frac{80}{3}\zeta_3\zeta_2-\frac{88}{9}\zeta_5\right)+n_f^3C_F\left(-\frac{131}{81}+\frac{32}{81}\zeta_2-\frac{64}{135}\zeta_2^2+\frac{304}{81}\zeta_3\right)
\end{aligned}
$$

The end . . .

. . . is open

It has been a lot fun . . .

Thank you, Volodya, for collaboration and inspiration.

The best wishes and dedication for many more projects to come . . .