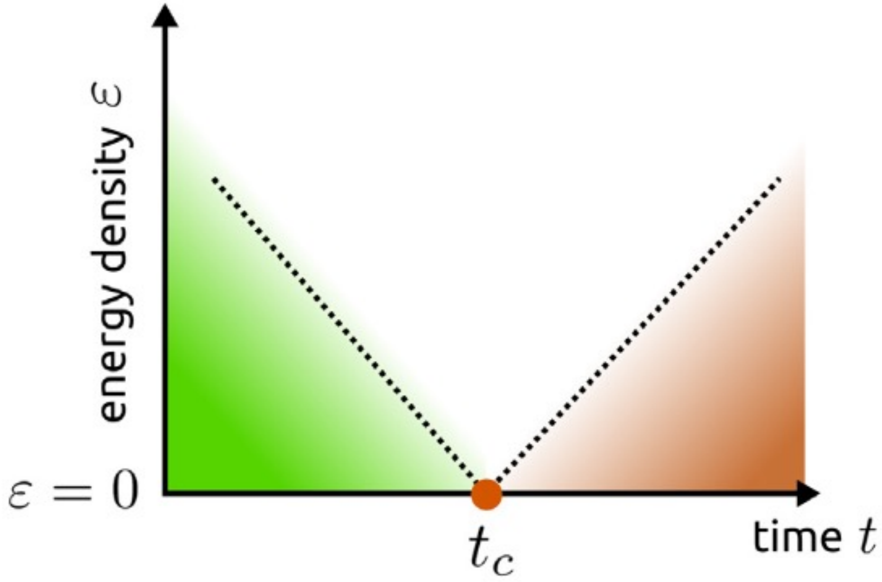
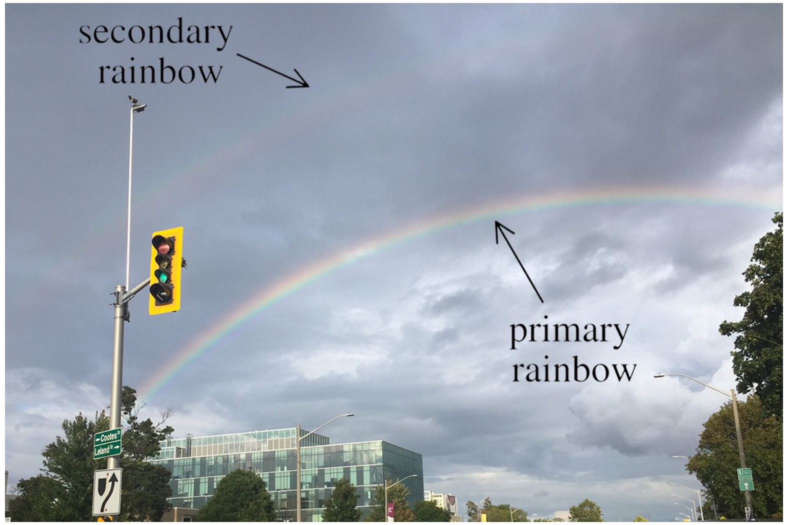
Quantum dark bands and dynamical phase transitions





Valentin Link, Walter Strunz, DOD New J. Phys. 26, 103021 (2024)

Duncan O'Dell, McMaster University

CAP congress, Ottawa 2026

Acknowledgments

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→ Heidelberg

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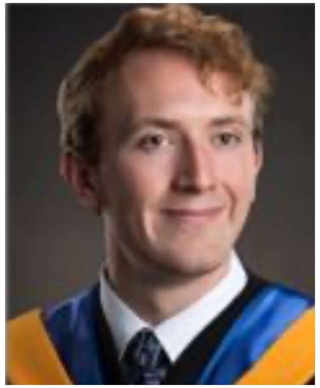


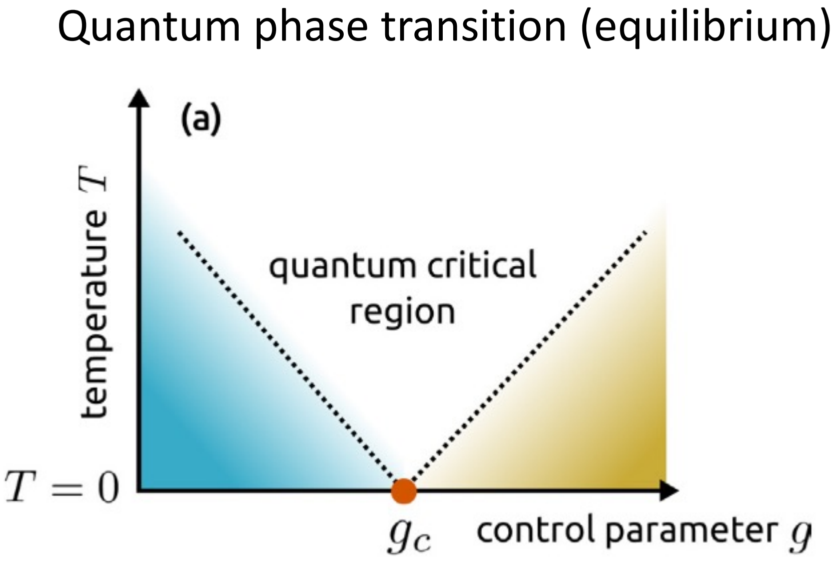
Nevan Keating

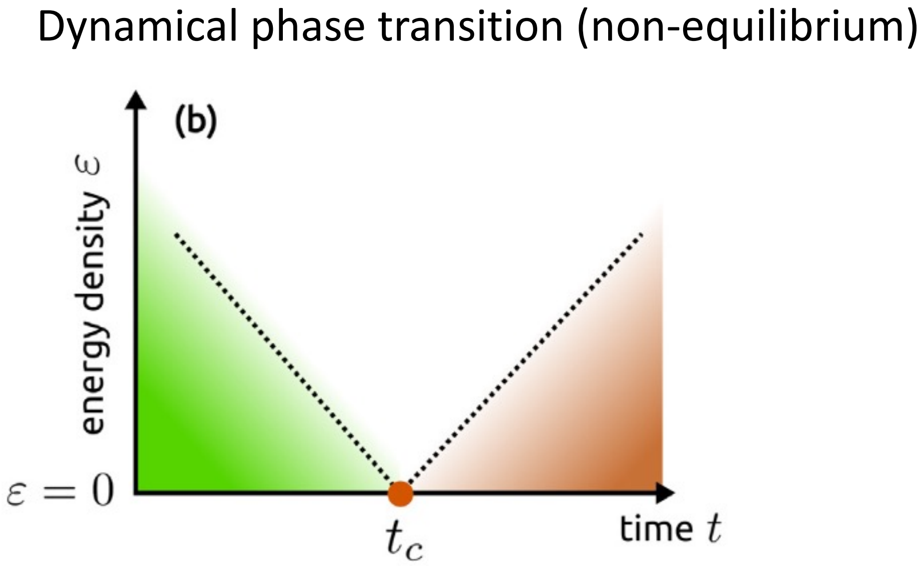
Hung Nguyen

Jay Mehta

Valentin Link and Walter Strunz
(TU Dresden)

Equilibrium vs Dynamical quantum phase transitions





First paper: M. Heyl, A. Polkovnikov, and S. Kehrein, Phys. Rev. Lett. 110, 135704 (2013).

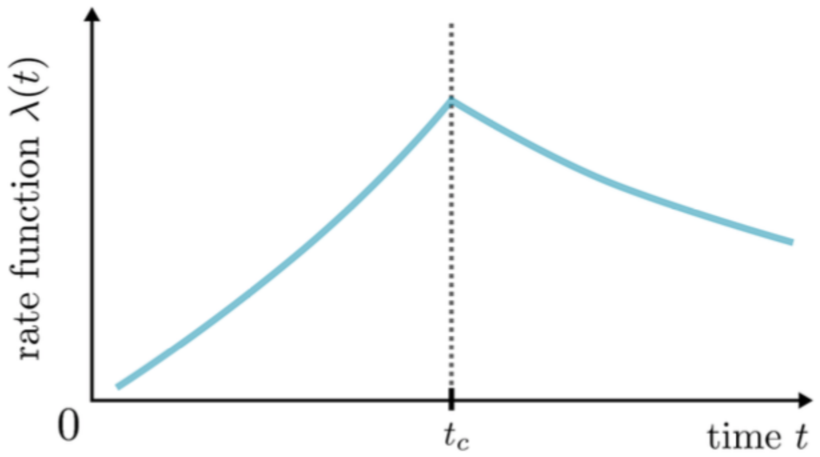
Loschmidt echo

$$L(t) = |\langle \psi_0 | \psi_t \rangle|^2$$

$$r(t) \equiv -\frac{1}{N} \log L(t)$$ **rate function**

or $$L(t) = e^{-N\,r(t)}$$

dynamical phase transition:



Compare:

$$\langle \psi_0 | \psi_t \rangle = \langle \psi_0 | e^{-iHt/\hbar} | \psi_0 \rangle$$

With: $Z = \mathrm{tr}\, e^{-\beta H}$

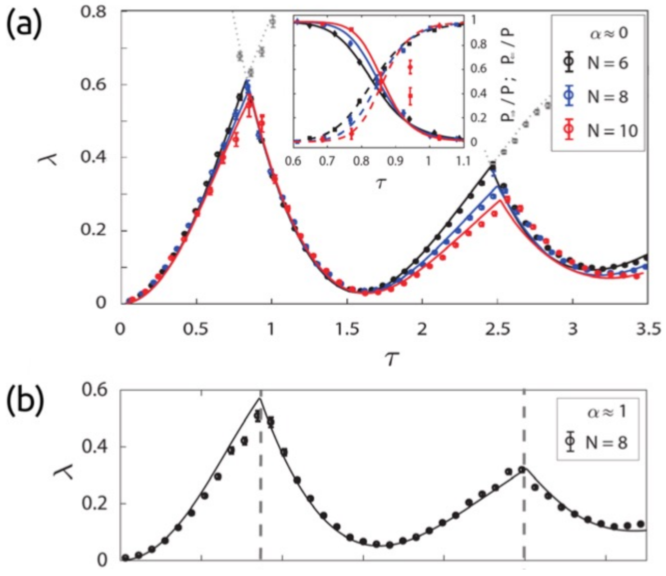
(partition function)

$$Z = e^{-\beta F} = e^{-\beta N f}$$

F= free energy
f=F/N= free energy density

At an equilibrium phase transition, thermodynamic potentials such as the free energy F become nonanalytic as a function of the respective control parameter.

Experiment showing dynamical phase transition following a sudden quench



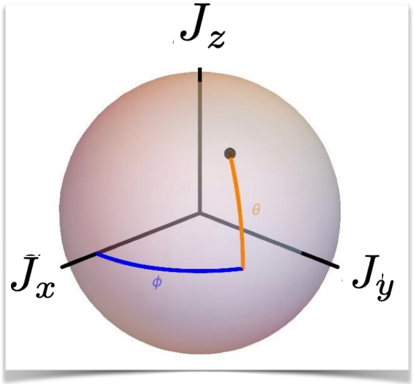
(a) Measured data for the Loschmidt echo rate function $\lambda(t)$ at $\alpha \approx 0$ and different system sizes as a function of dimensionless time $\tau = ht$ displaying clearly nonanalytic behavior. The colored data points show $\lambda(t)$, obtained by taking its dominant contribution $\lambda(t) = \min\eta{=}\uparrow,\downarrow \lambda\eta(t)$, whereas the grey data points refer to the respective subleading ones. (b) The experimental result for $\lambda(t)$ at a larger interaction exponent $\alpha \approx 1$.

Jurcevic P, Shen H, Hauke P, Maier C, Brydges T, Hempel C, Lanyon B P, Heyl M, Blatt R and Roos C F
Phys. Rev. Lett. 119 080501 (2017)

MODEL USED IN THIS TALK:

$$H = \frac{G}{2N}\sum_{i,j=1}^{N}\frac{\sigma_z^i\sigma_z^j}{4} + \Omega\sum_{i=1}^{N}\frac{\sigma_x^i}{2}$$

Fully connected transverse field Ising model



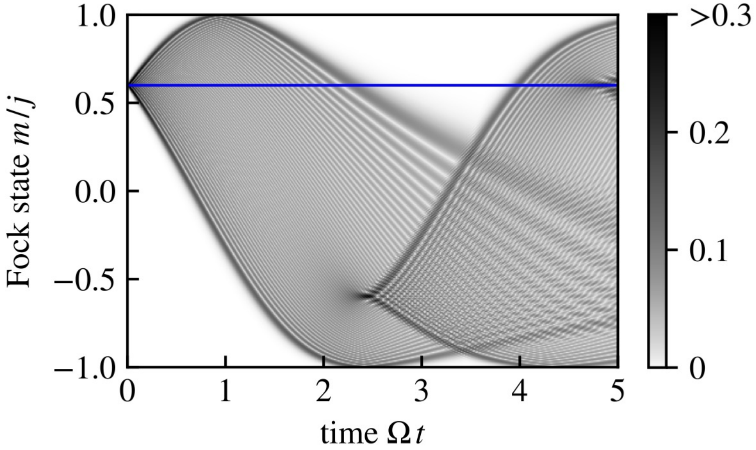
Map onto a giant spin of length **j=N/2**

$$\vec{J} = \frac{1}{2}\sum_{i=1}^{N}\vec{\sigma}^i \quad \Rightarrow \quad H = \frac{G}{2j}J_z^2 + \Omega J_x$$

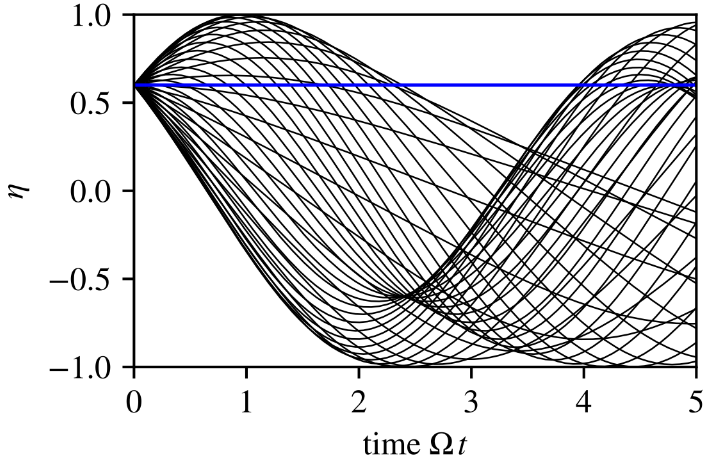
Work in Fock basis: $J_z|j,m\rangle = m|j,m\rangle, \quad m = -j, -j+1, ..., j.$ (magnetization along z)

Dynamics starting from a single Fock state (sudden quench)

Probability density in Fock space



Classical trajectories



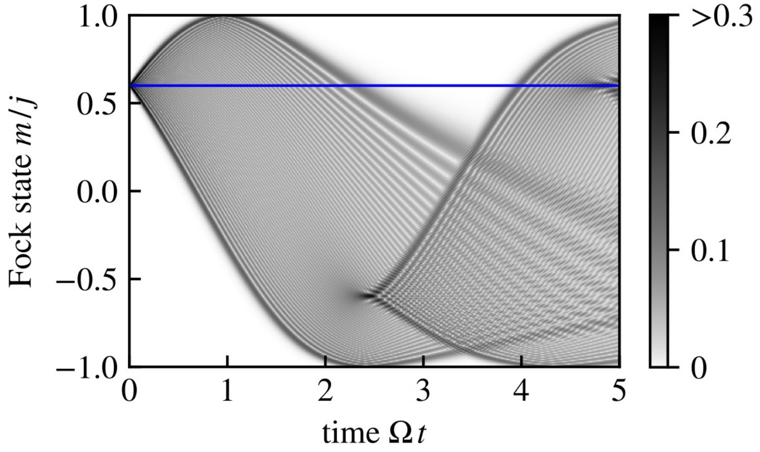
$$H = \frac{G}{2}\eta^2 + \Omega\sqrt{1-\eta^2}\cos\phi$$

$$\dot{\phi} = \frac{\partial H}{\partial \eta} \qquad \dot{\eta} = -\frac{\partial H}{\partial \phi}$$

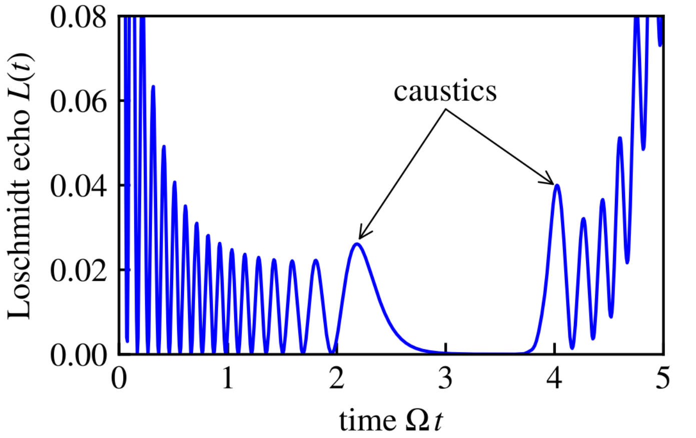
$$\eta = m/j$$

Dynamical Phase Transition: caustics in Fock space

Probability density in Fock space

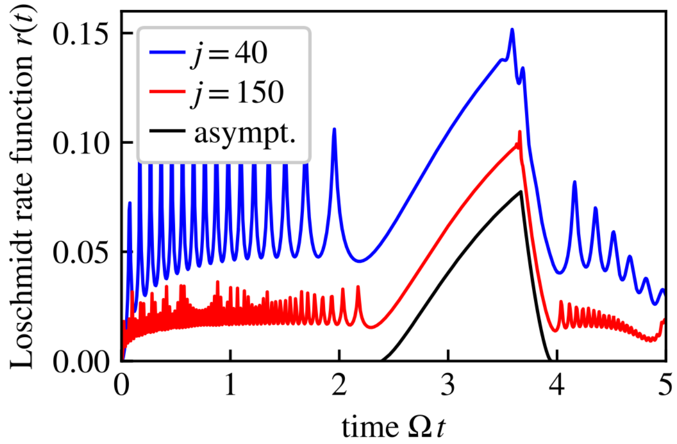


Loschmidt echo

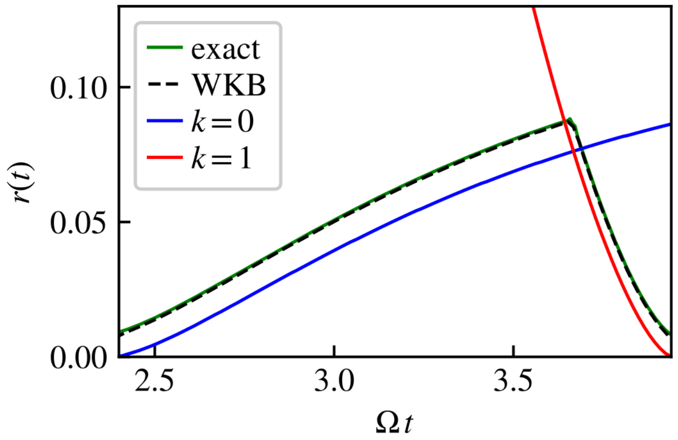


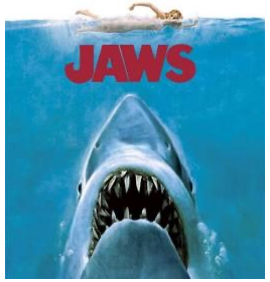
DPT at $t_c=3.7/\Omega$

Rate function

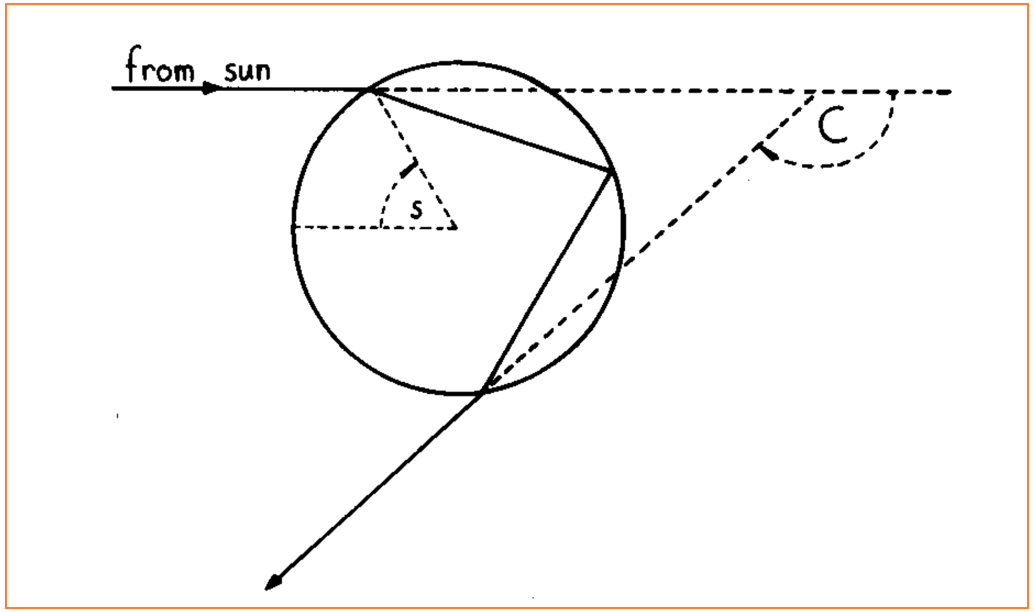


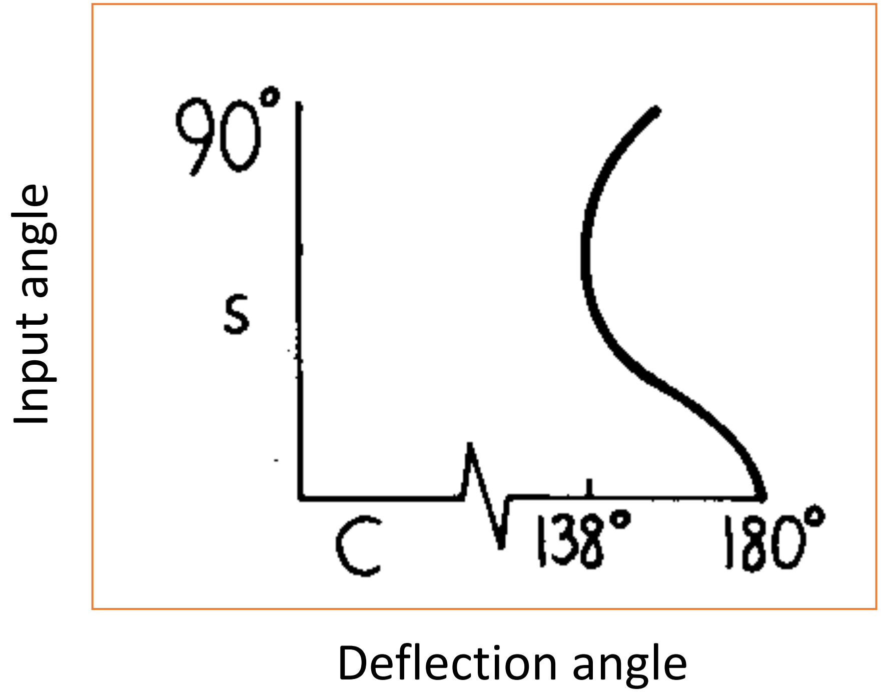
Analytical approximation (WKB)





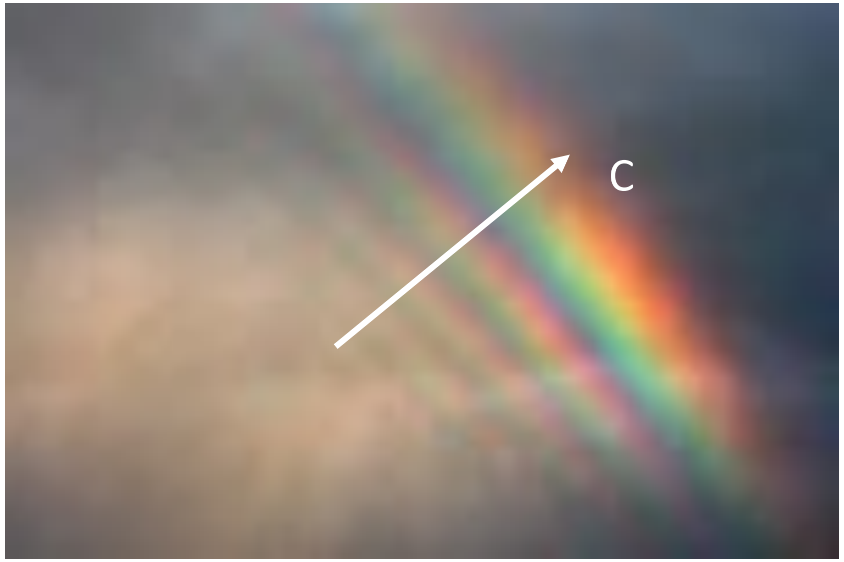
Rainbows: ray theory





Rainbow is a **structurally stable caustic** called a **fold catastrophe** described by catastrophe theory
Michael Berry, *Singularities in waves and rays,* Physics of Defects, vol XXXV ,
ed R Balian (North-Holland Publishing) (1981)

Rainbows: wave theory

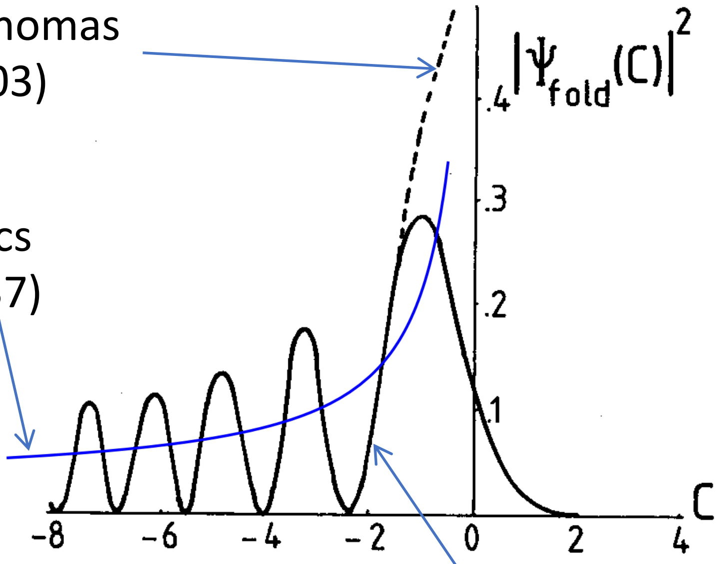


Supernumerary arcs = Airy fringes made by white light

Airy fringes for **one** colour (e.g. yellow)

“WKB” (Thomas Young 1803)

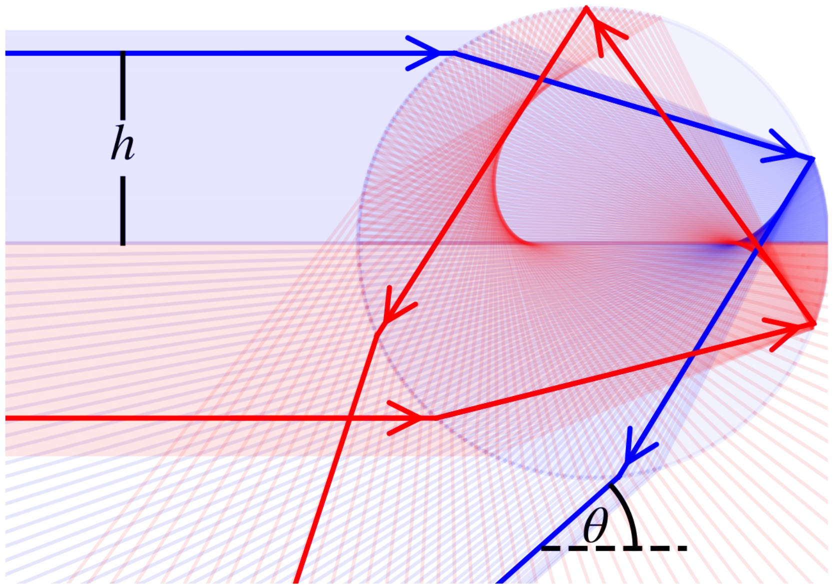
Geometrical optics (R. Descartes 1637)

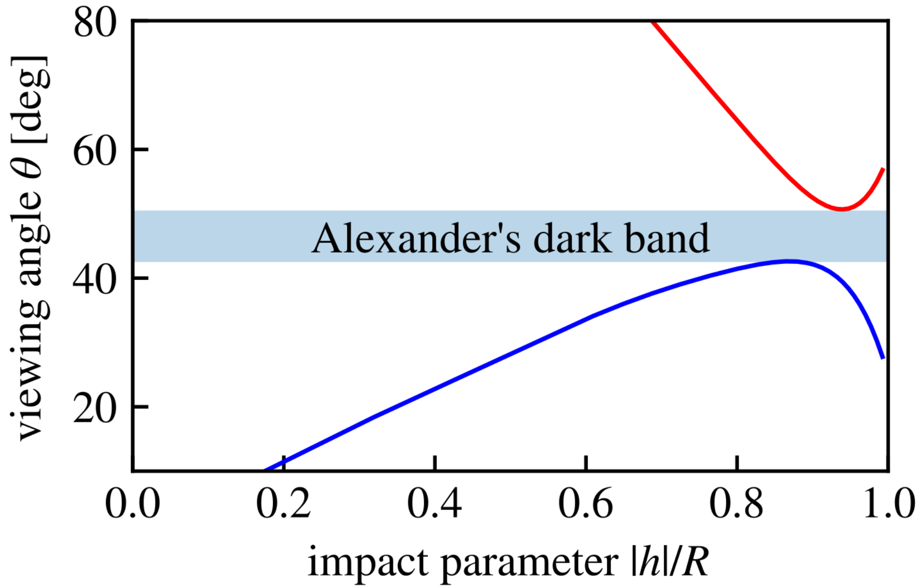


Airy function (GB Airy 1838)

G.B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Trans. Camb. Phil. Soc. **6**, 379 (1838)

Double rainbows: ray theory

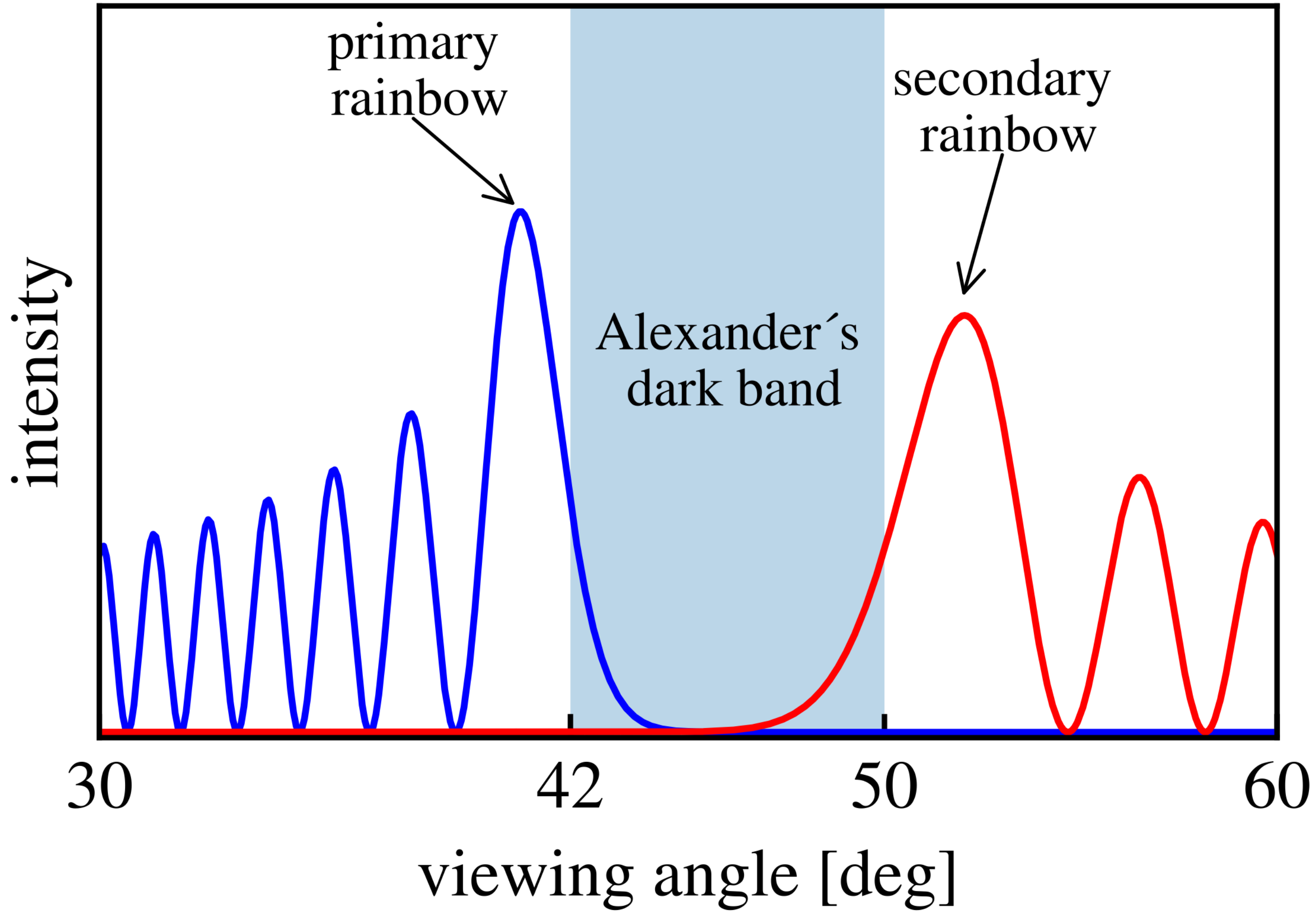




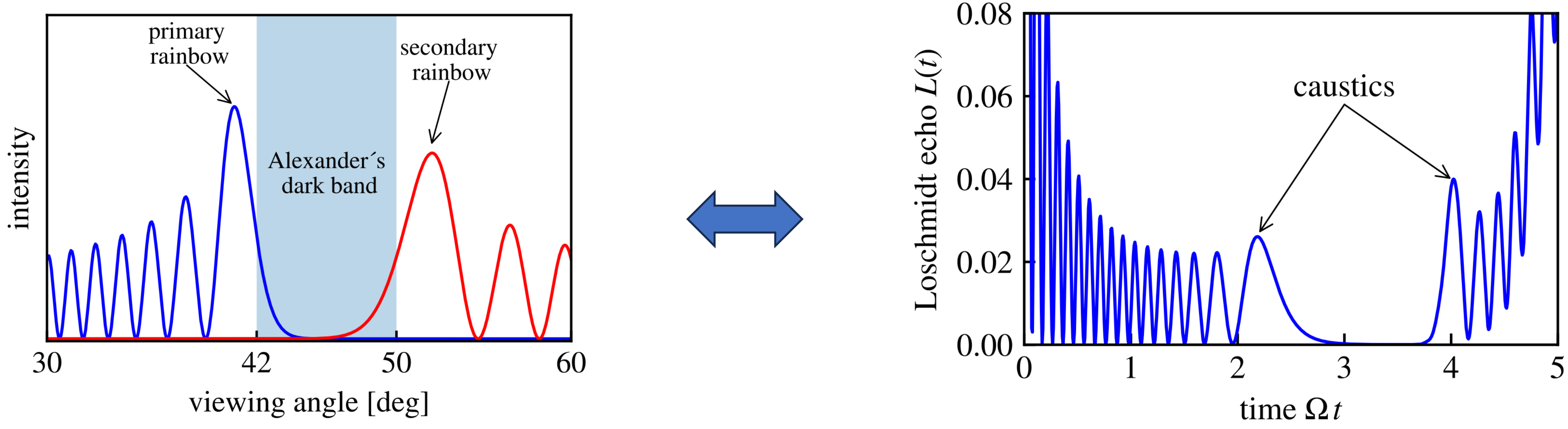
θ_1=42°
θ_2=50°

Alexander of Aphrodisias, *Commentary on Book IV of Aristotle's* Meteorology (circa AD 200)

Double rainbow wave theory

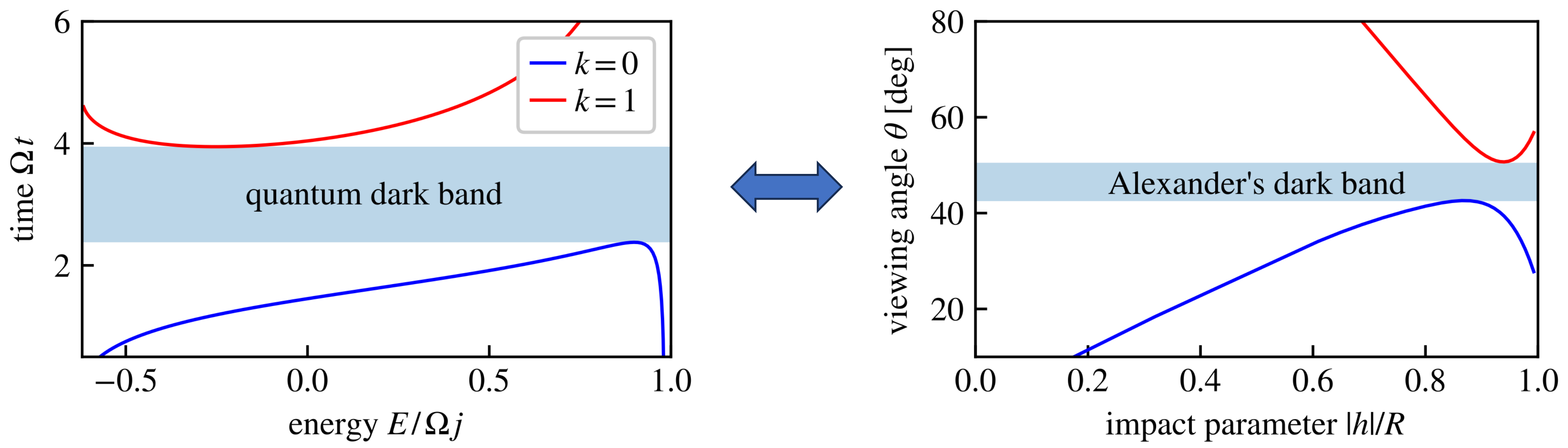


Quantum dark bands



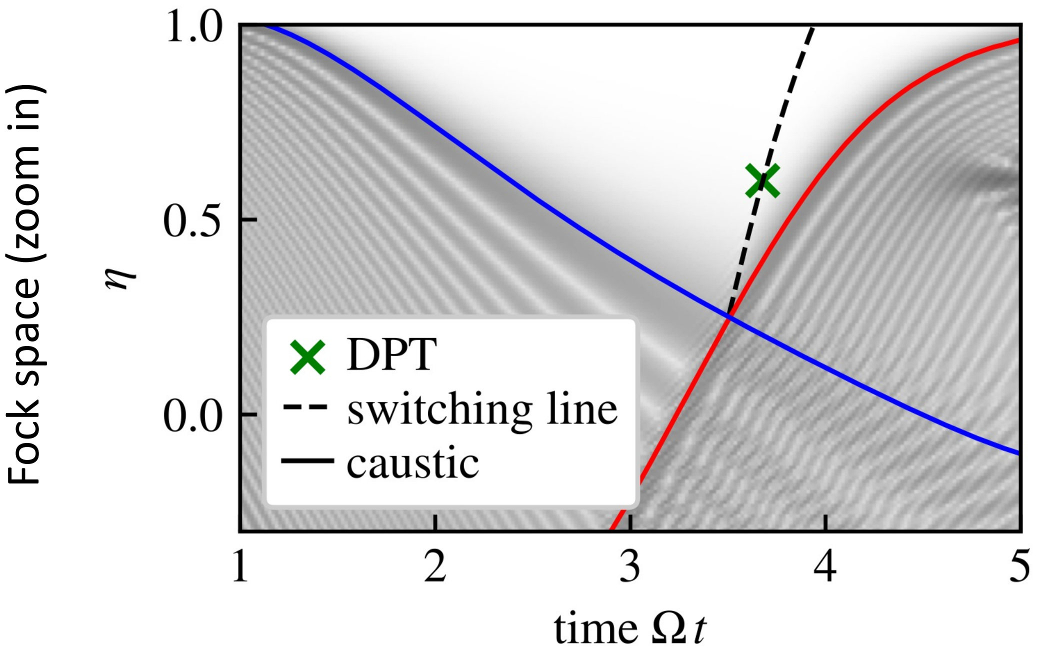
Dynamical Phase Transitions, Caustics, and Quantum Dark Bands
V. Link, W. T. Strunz, DOD
New J. Phys. 26, 103021 (2024)

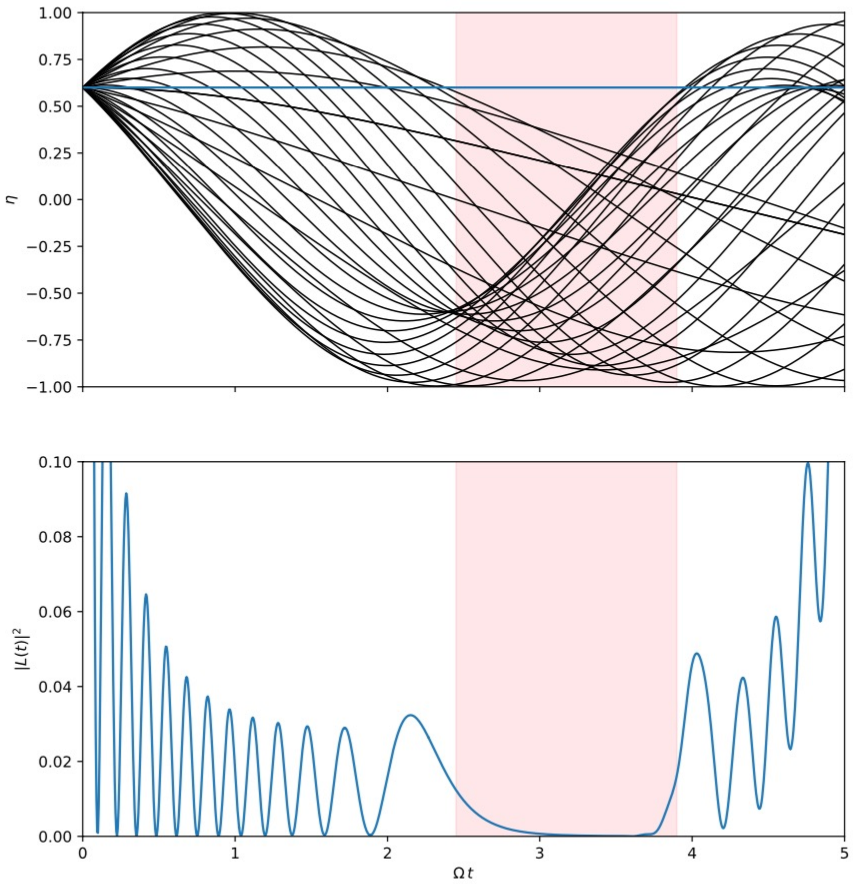
Quantum dark bands



Quantum dark bands (and Alexander’s dark band) are structurally stable and hence generic (according to catastrophe theory)

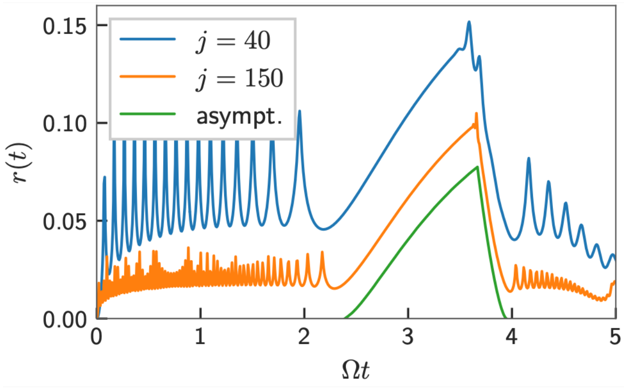
Switching lines

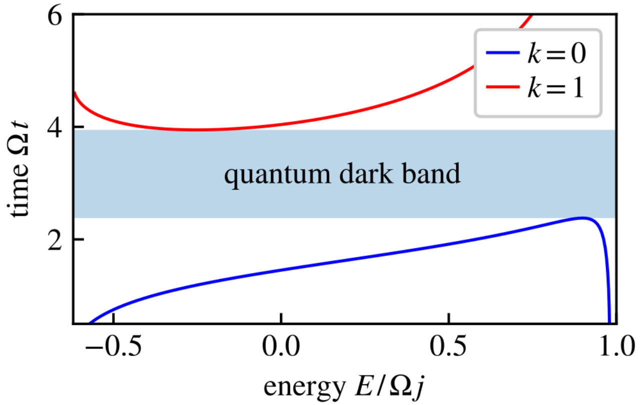




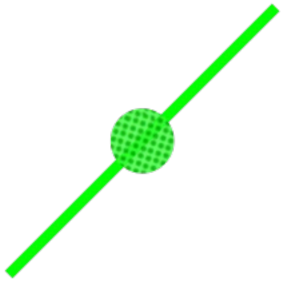
Conclusions

- **Quantum dark bands** are enclosed by **caustics in Fock space**
- Dynamical phase transition occurs when the Loschmidt echo crosses the switching line between tails of back-to-back Airy functions in Fock space
- Universality due to singularity (Airy functions)
- Close analogy between **Alexander's Dark Band** in double rainbows (back-to-back caustics)

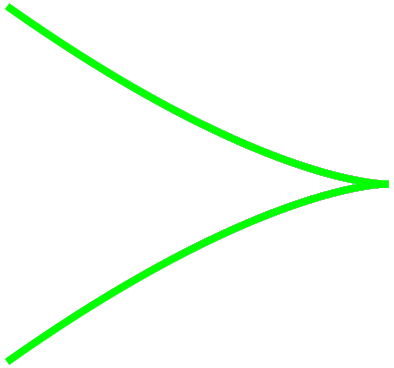




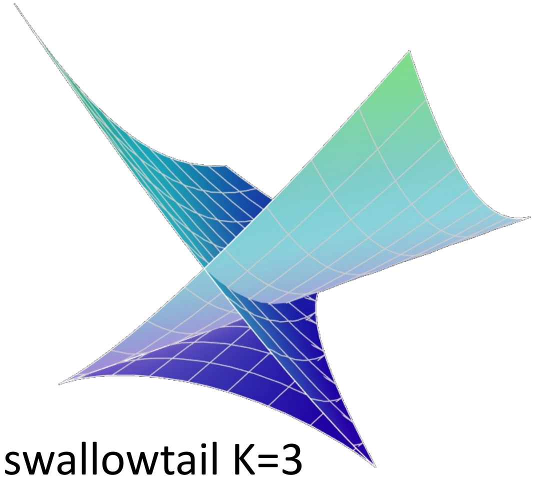
Catastrophe theory: structurally stable singularities



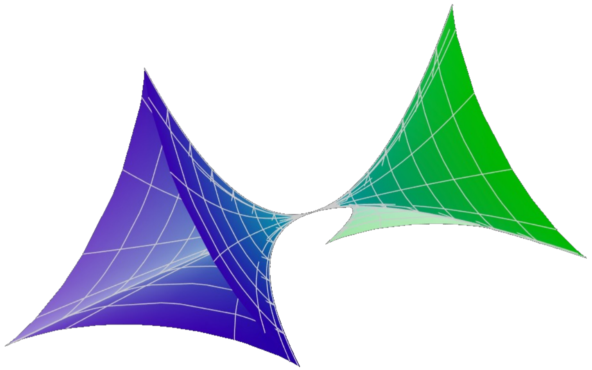
fold K=1



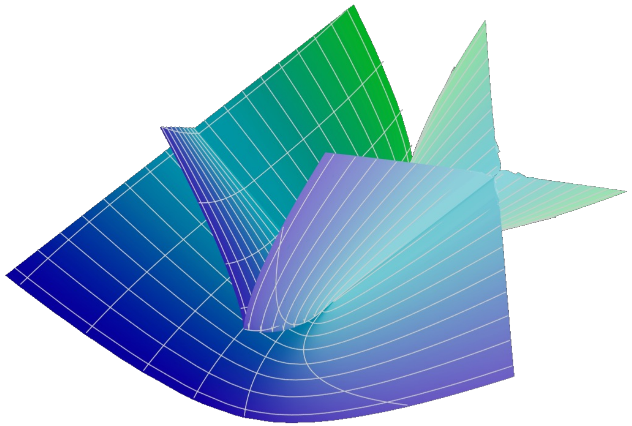
cusp K=2



swallowtail K=3

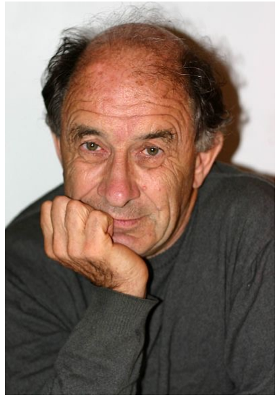


elliptic umbilic K=3



hyperbolic umbilic K=3

René Thom

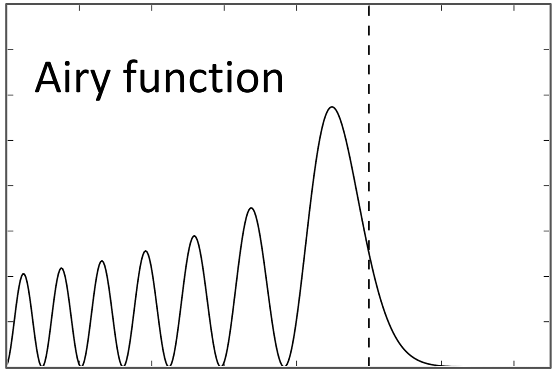


Vladimir Arnold

Wave catastrophes

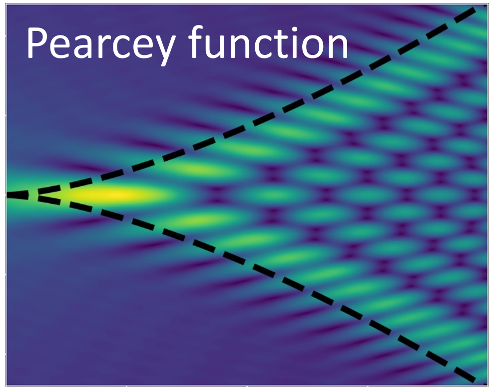
$$\Psi_{\text{fold}}(C) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s^3/3+Cs)}\ ds$$
$$= \sqrt{2\pi}\,\text{Ai}[C]$$

G.B. Airy, Trans. Camb. Phil. Soc. **6**, 379 (1838)



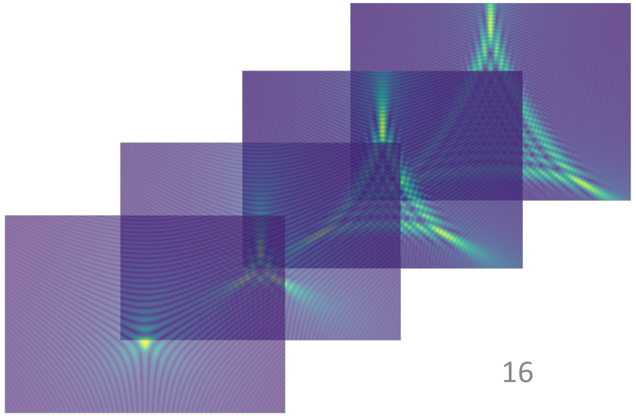
$$\Psi_{\text{cusp}}(C_1, C_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s^4/4+C_2 s^2/2+C_1 s)} ds$$

T. Pearcey, Phil. Mag. **37**, 311 (1946)

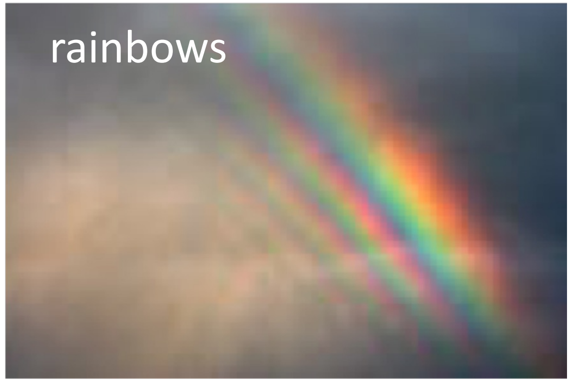


$$\text{U}_E(C_1, C_2, C_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{e}^{\text{i}(s_1^3-3s_1 s_2^2-C_3(s_1^2+s_2^2)-C_2 s_2-C_1 s_1)} \text{d}s_1 \text{d}s_2$$

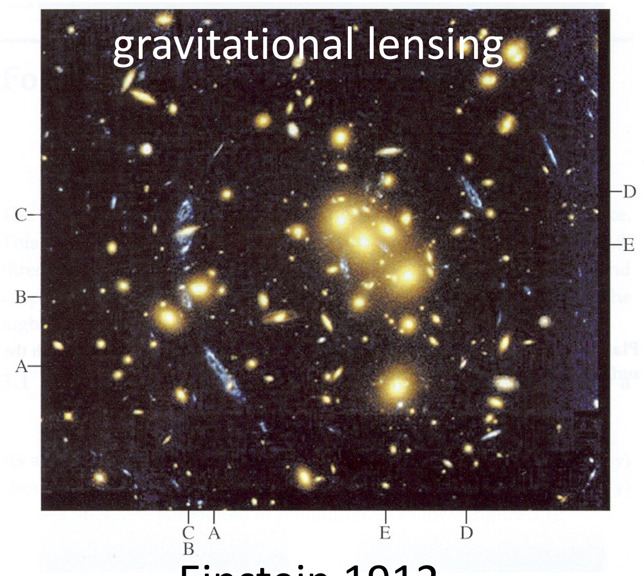
Berry, Nye & Wright, Phil. Trans. R. Soc. A. **291**, 453 (1979)



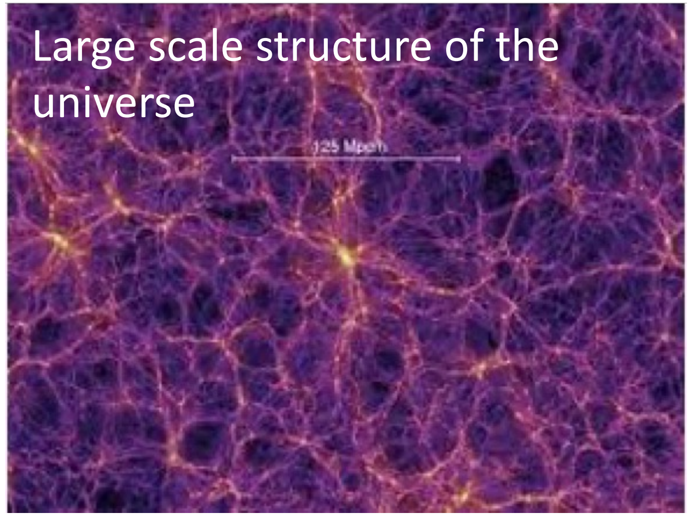
Natural focusing: caustics



Descartes 1637



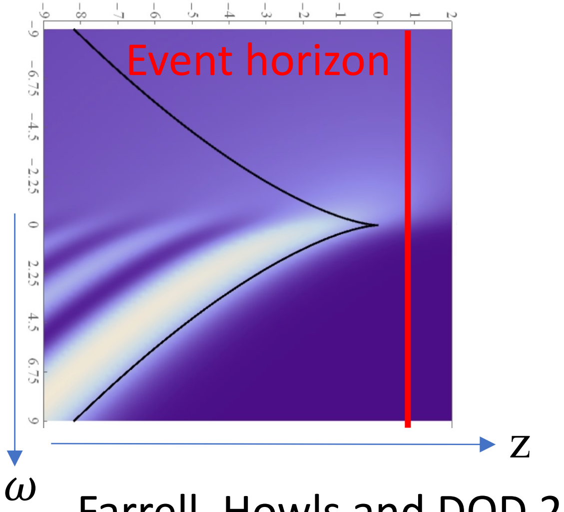
Einstein 1912



Zeldovich 1981

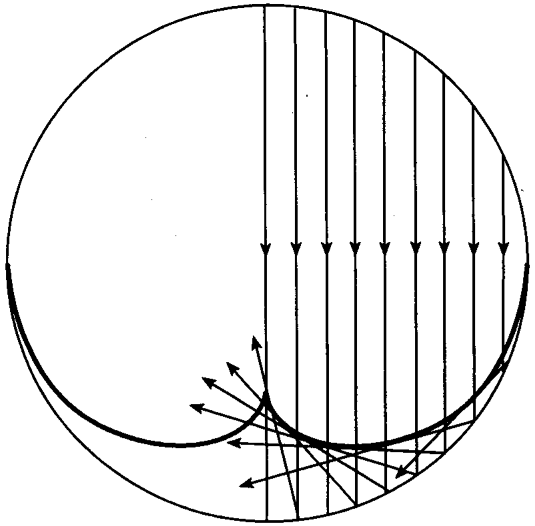
Pelinovsky 2003

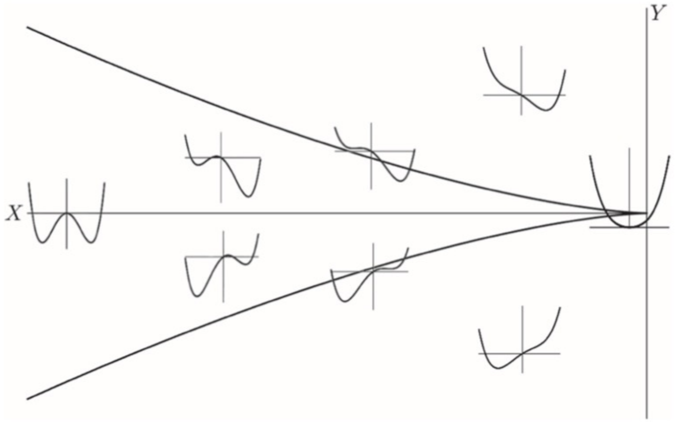
Berry 2005



Farrell, Howls and DOD 2023

Caustics and catastrophe theory





- Singular
- Structurally stable
- Generic (do not need fine tuning)

Generating functions for catastrophes up to K=4

NAME	CODIMENSION K	GENERATING FUNCTION
fold	1	$s^3 + C s$
cusp	2	$s^4/4 + C_2 s^2/2 + C_1 s$
swallowtail	3	$s^5/5 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
elliptic umbilic	3	$s_1^3 - 3 s_1 s_2^2 - C_3 (s_1^2 + s_2^2) - C_2 s_2 + C_1 s_1$
hyperbolic umbilic	3	$s_1^3 + s_2^3 - C_3 s_1 s_2 - C_2 s_2 + C_1 s_1$
butterfly	4	$s^6/6 + C_4 s^4/4 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
parabolic umbilic	4	$s_1^4 + s_1 s_2^2 + C_4 s_2^2 + C_3 s_1^2 + C_2 s_2 + C_1 s_1$

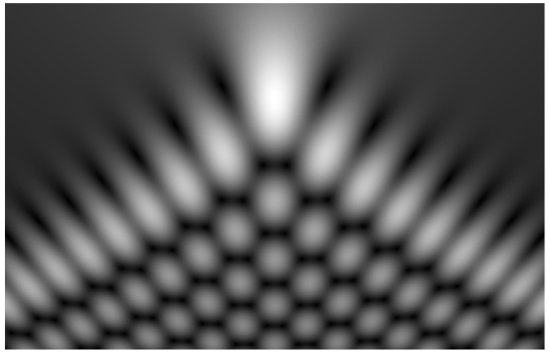
R. Thom Structural Stability and Morphogenesis (Benjamin, 1975)

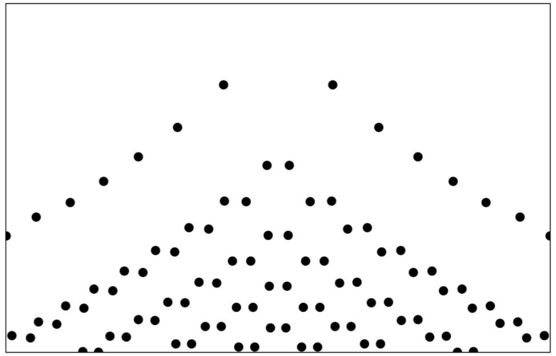
V.I. Arnold, Russ. Math. Survs. 30 (5) (1975)

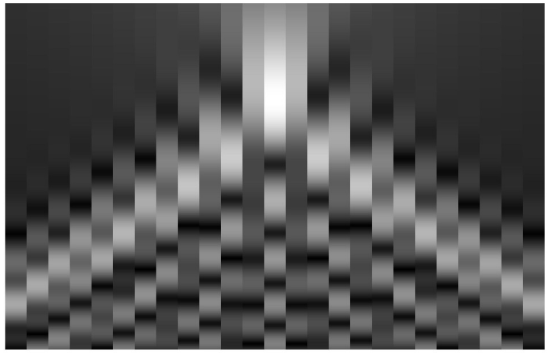
Self-similarity: scaling exponents

catastrophe	Arnold index β	Berry indices σ_{j}	Berry index γ
fold	1/6	2/3	2/3
cusp	1/4	3/4 , 1/2	5/4
swallowtail	3/10	4/5 , 3/5, 2/5	9/5
elliptic umbilic	1/3	2/3 , 2/3 , 1/3	5/3
hyperbolic umbilic	1/3	2/3 , 2/3, 1/3	5/3
butterfly	1/3	5/6 , 2/3 , 1/2 , 1/3	7/3
parabolic umbilic	3/8	5/8 , 3/4 , 1/2 , 1/4	17/8

Quantum caustics







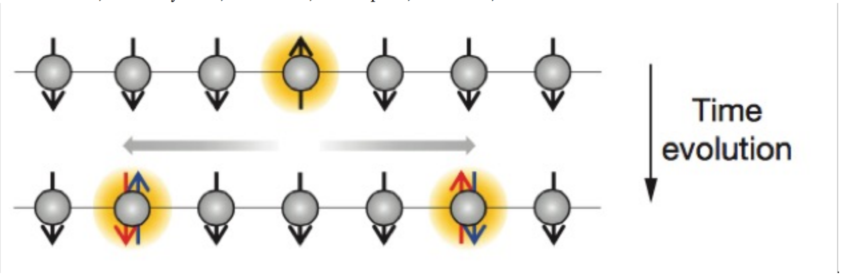
“Light” cones in quantum **quench** experiments

LETTER

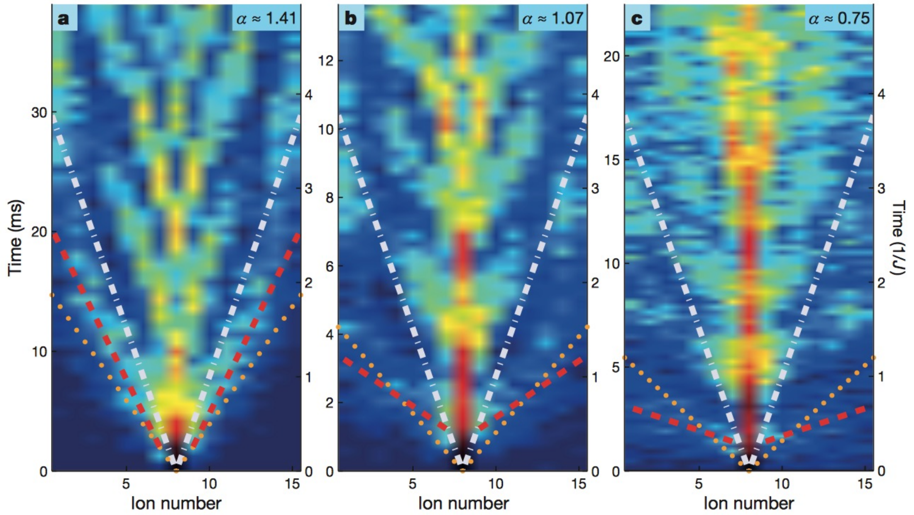
doi:10.1038/nature13461

Quasiparticle engineering and entanglement propagation in a quantum many-body system

P. Jurcevic[1,2]*, B. P. Lanyon[1,2]*, P. Hauke[1,3], C. Hempel[1,2], P. Zoller[1,3], R. Blatt[1,2] & C. F. Roos[1,2]



Measured magnetization: $\langle \sigma_i^z(t) \rangle$



LETTER

doi:10.1038/nature10748

Light-cone-like spreading of correlations in a quantum many-body system

Marc Cheneau[1], Peter Barmettler[2], Dario Poletti[2], Manuel Endres[1], Peter Schauß[1], Takeshi Fukuhara[1], Christian Gross[1], Immanuel Bloch[1,3], Corinna Kollath[2,4] & Stefan Kuhr[1,5]

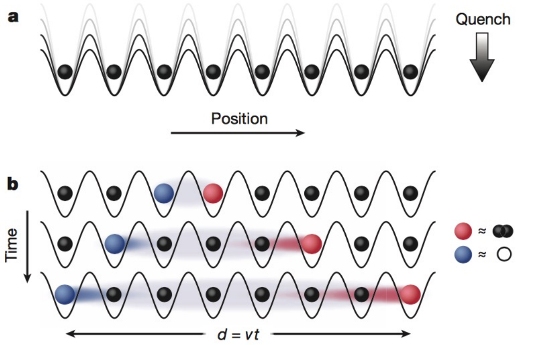


Figure 1 | Spreading of correlations in a quenched atomic Mott insulator.

$$C_d(t) = \langle \hat{s}_j(t)\hat{s}_{j+d}(t)\rangle - \langle \hat{s}_j(t)\rangle\langle \hat{s}_{j+d}(t)\rangle$$

Bose-Hubbard system

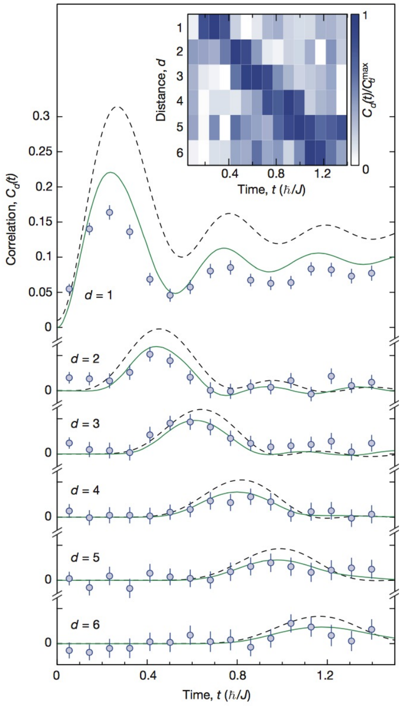


Figure 2 | Time evolution of the two-point parity correlations. After the

Light cones are caustics!

Wyatt Kirkby

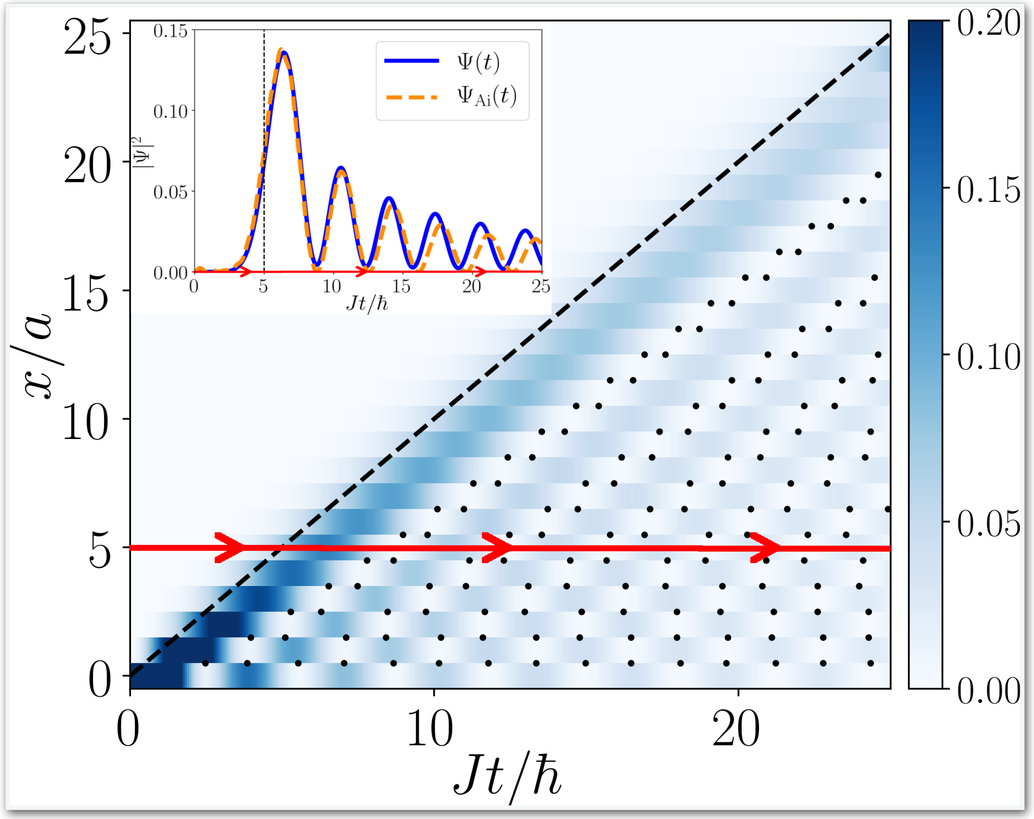
$$H = -J\sum_{\langle ij\rangle}\sigma_i^x\sigma_j^x - h\sum_i \sigma_i^z$$

Transverse field Ising model

$$\Psi_{\mathrm{Ai}}(C^j;J) \approx \frac{1}{2\pi t^{1/3}}\left(\frac{2Jg^{\frac{2-j}{3}}}{v_{\mathrm{I}}\hbar}\right)^{\frac{1}{2}}\int_{\alpha_j}^{\beta_j}\mathrm{d}s_j\, \mathrm{e}^{\frac{\mathrm{i}J}{\hbar}\Phi_1(s_j;C^j)} \overset{t\to\infty}{\propto} \left(\frac{J}{\hbar}\right)^{\frac{1}{6}}\mathrm{Ai}\left[\left(\frac{J}{\hbar}\right)^{\frac{2}{3}}C^j\right]$$

$$C_j(x,t) = 2(x/v_{\mathrm{I}} - t)(g^{2-j}/\sqrt{t})^{2/3} \qquad j = \{1,2\} \text{ when } \{g>1, g<1\}$$

$$v_{\mathrm{I}} \equiv \begin{cases} \frac{2Jag}{\hbar} & 0<|g|<1 \\ \frac{2Ja}{\hbar} & 1<|g| \end{cases} \qquad g = h/J$$



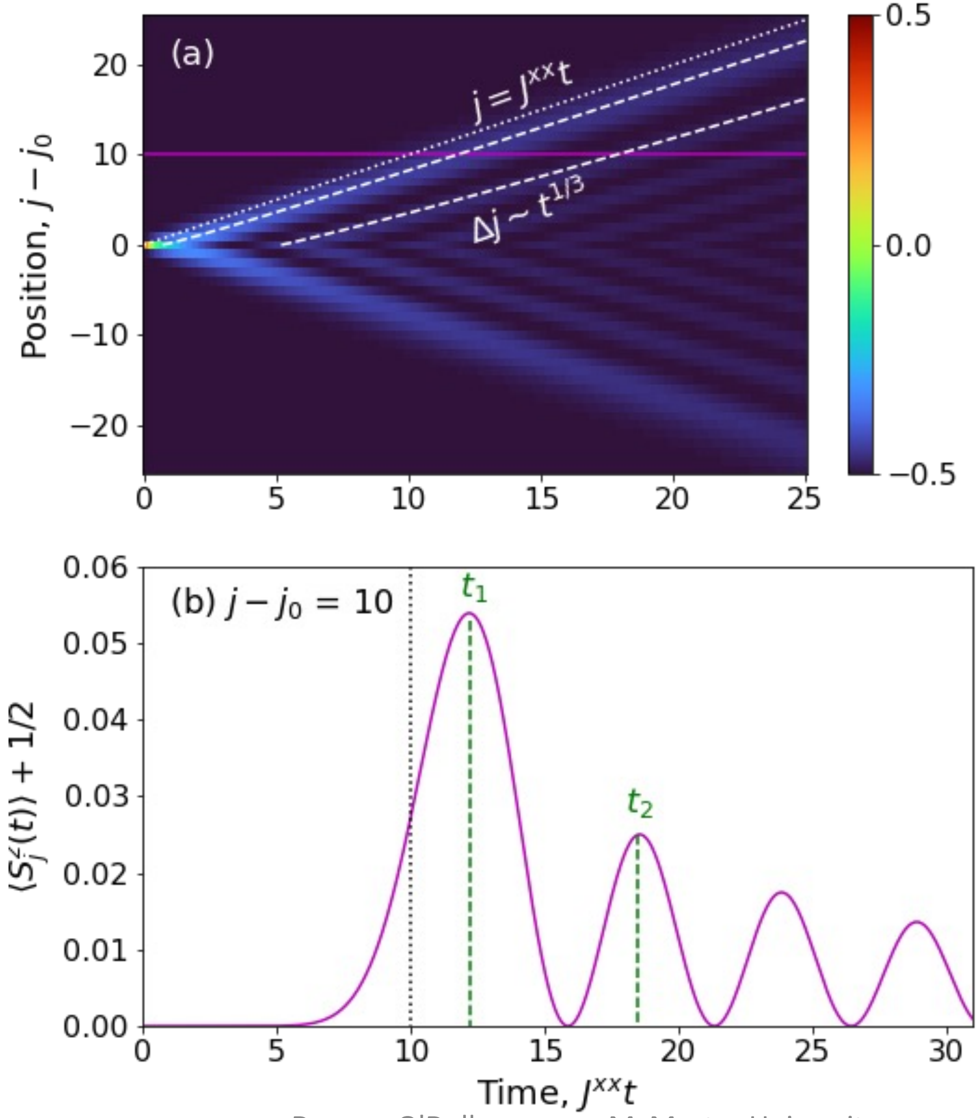
g = 0.8

W. Kirkby, J. Mumford, and DOD, Phys. Rev. Research 1, 033135 (2019)

Predict subdiffusive scaling of the wavefront (TFIM)

Local magnetization as a function of position and time on a spin chain

Slice at fixed position on chain



If wavefront spread diffusively we would expect $t^{1/2}$

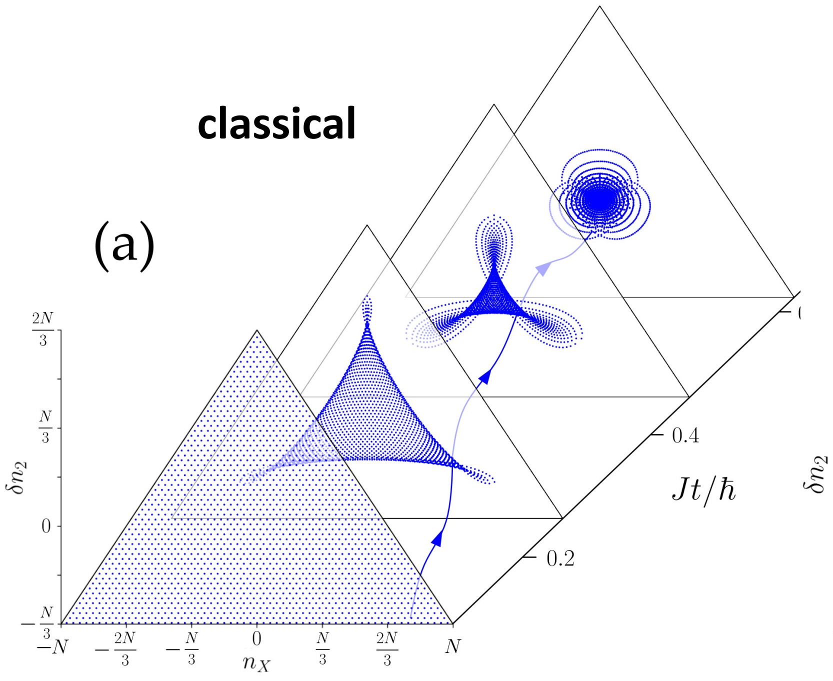
Monalisa Singh Roy, Jesse Mumford, DOD, Emanuele G. Dalla Torre
arXiv:2410.06803

Caustics in Fock space: Triple-well Bose-Hubbard model

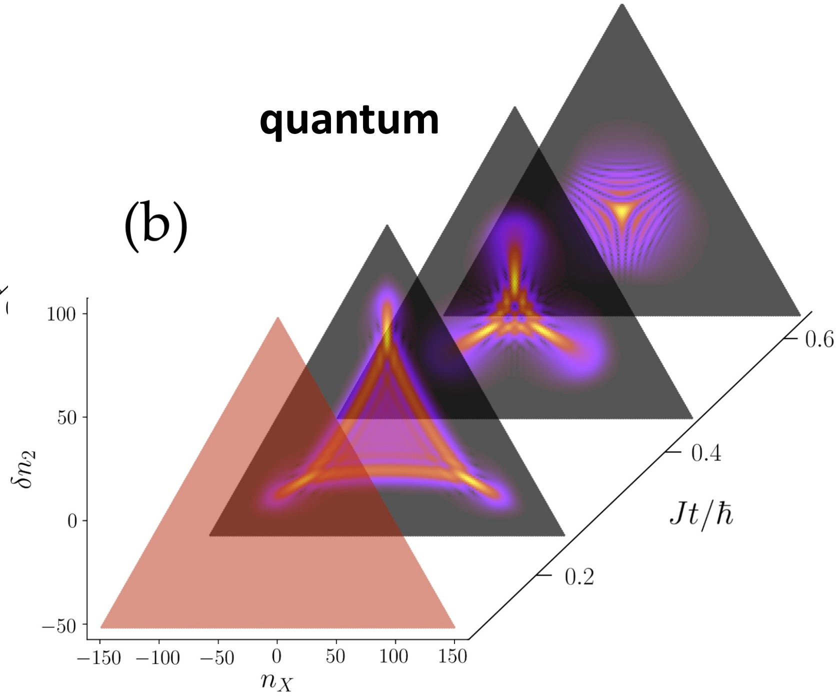
W. Kirkby, Y. Yee, K. Shi and DOD, Phys. Rev. Res. 4, 013105 (2022).

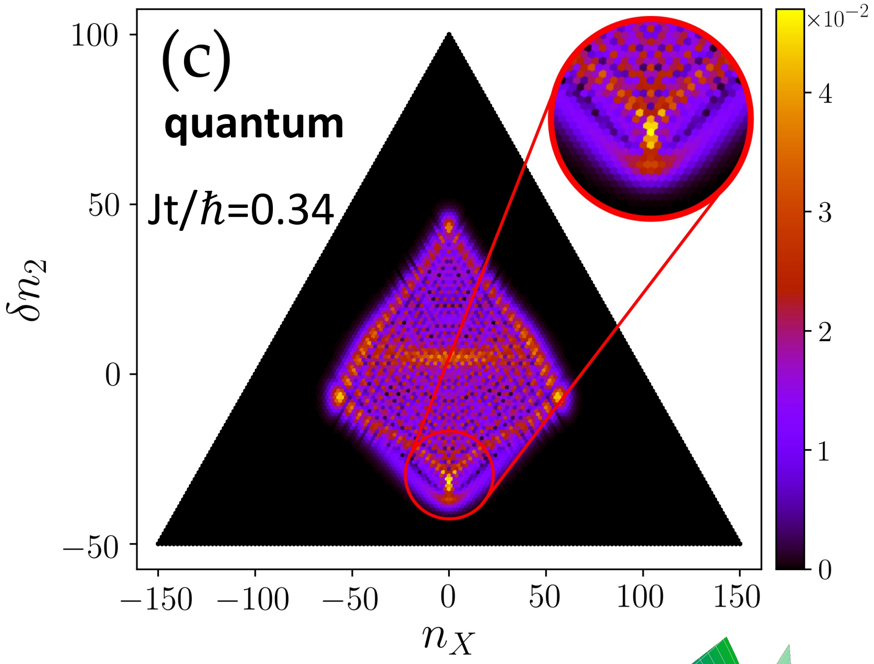
$$\hat{H} = -K_L(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) - K_R(\hat{a}_2^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_2) - K_X(\hat{a}_3^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_3) + \frac{U}{2}\sum_{i=1}^{3} \hat{n}_i(\hat{n}_i - 1) + \sum_{i=1}^{3} \epsilon_i \hat{n}_i$$

classical



quantum

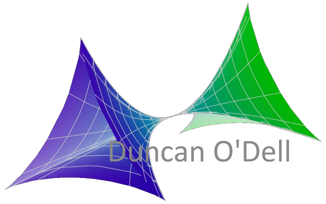




$n_X = n_1 - n_3$ $\delta n_2 = n_2 - N/3$

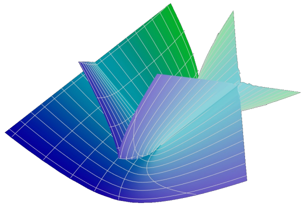
Initial state = coherent state

N=150, K_L=K_R=K_X=U/100



elliptic umbilic

Initial state=Fock state



N=150, K_L=K_R=4U, K_X=0

hyperbolic umbilic

Light cones on spin chains

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar} b^\dagger_{x=0}|0\rangle_b = \frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{-\mathrm{i}\epsilon_k t/\hbar}|k\rangle$$

Continuum approximation: $$\Psi_{\mathrm{CA}}(x,t) = \frac{\sqrt{a}}{2\pi} \int_{-\pi/a}^{\pi/a} \mathrm{d}k \; \mathrm{e}^{\mathrm{i}\Phi(k;x,t)}$$

$$\Phi(k;x,t) = kx - \epsilon_k t/\hbar$$

$$L = k\dot{x} - \epsilon_k/\hbar$$

$$v_g = (1/\hbar)|\mathrm{d}\epsilon_k/\mathrm{d}k|$$

Lieb-Robinson bound: $$v_{\mathrm{LR}} = \max_k |\mathrm{d}\epsilon_k/\mathrm{d}k|$$

$\Rightarrow$

$$\partial\Phi/\partial k = 0 \quad \Rightarrow \frac{x}{t} = v_g$$

$$\partial^2\Phi/\partial k^2 = 0$$

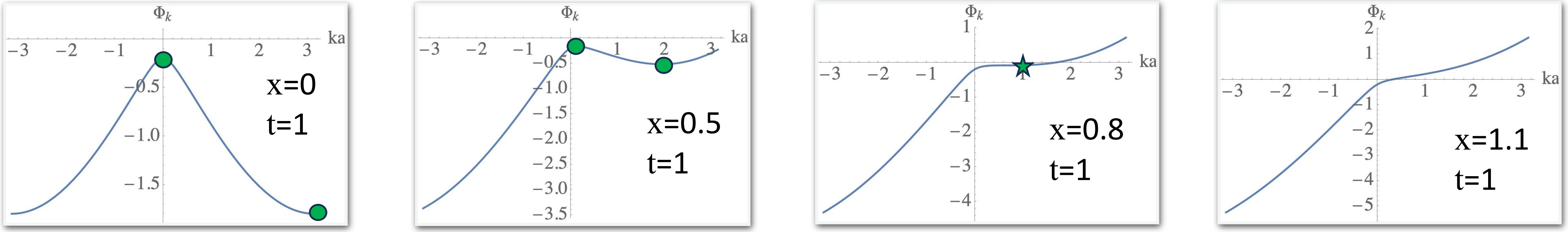
Light cone conditions match conditions for a caustic!

Transverse field Ising model

$$H = -J\sum_{\langle ij\rangle}\sigma_i^x\sigma_j^x - h\sum_i \sigma_i^z$$

Dispersion relation: $\epsilon_k = 2J\sqrt{(\cos(ka)-g)^2+\sin^2(ka)}$ $\qquad g = h/J$

Generating function: $\Phi(k,x,t) = kx - 2Jt\sqrt{(\cos(ka)-g)^2+\sin^2(ka)}/\hbar$



Two coalescing stationary points: fold catastrophe

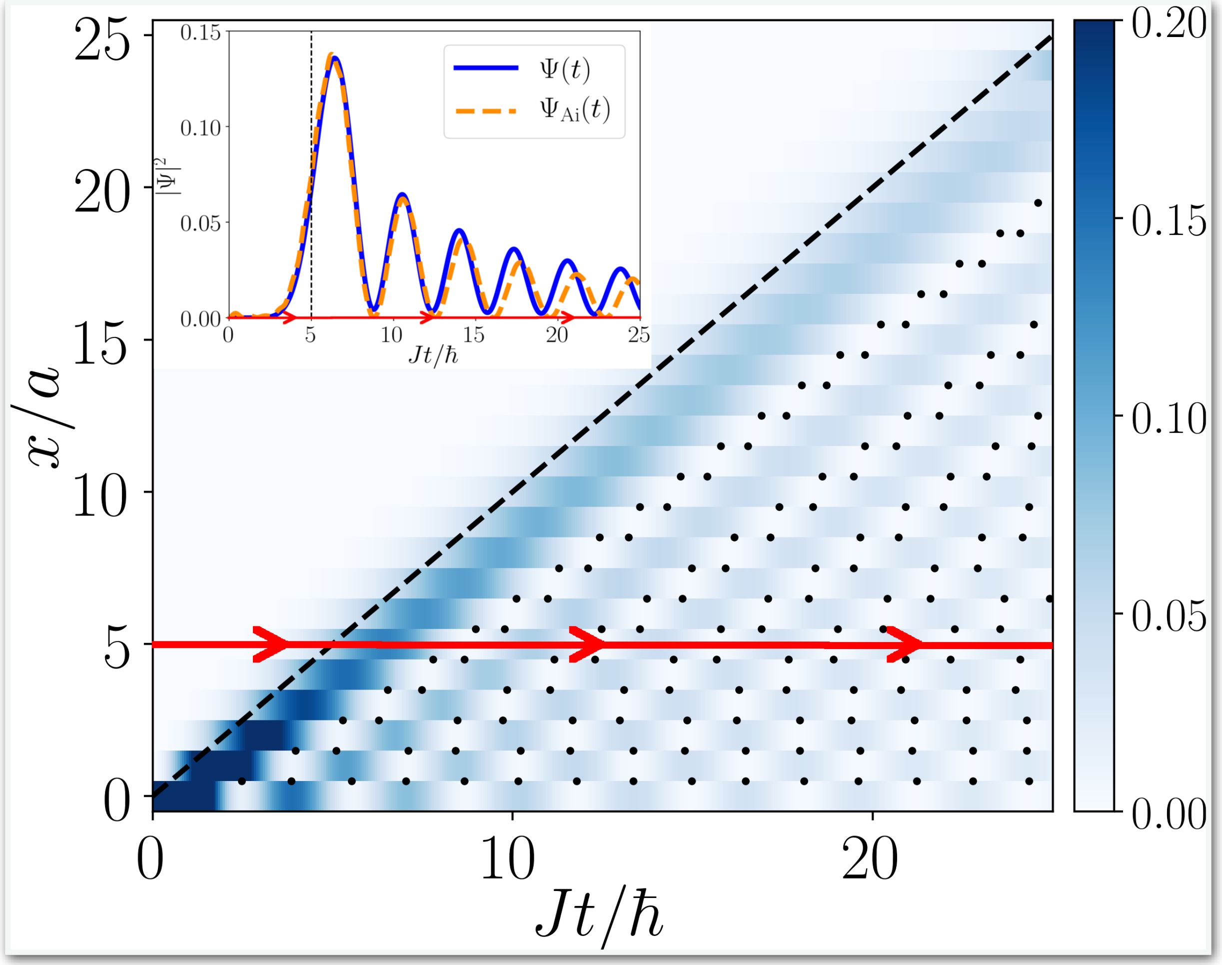
W. Kirkby, J. Mumford, and DOD, Phys. Rev. Research 1, 033135 (2019)

TFIM light cone

$$C_j(x,t) = 2(x/v_\mathrm{I} - t)(g^{2-j}/\sqrt{t})^{2/3}$$

$$j = \{1,2\} \text{ when } \{g>1, g<1\}$$

$$v_\mathrm{LR} \equiv \begin{cases} \frac{2Jag}{\hbar} & 0<|g|<1 \\ \frac{2Ja}{\hbar} & 1<|g| \end{cases}$$



$$\Psi_\mathrm{Ai}(C^j;J) \approx \frac{1}{2\pi t^{1/3}}\left(\frac{2Jg^{\frac{2-j}{3}}}{v_\mathrm{I}\hbar}\right)^{\frac{1}{2}} \int_{\alpha_j}^{\beta_j} \mathrm{d}s_j\, \mathrm{e}^{\frac{\mathrm{i}J}{\hbar}\Phi_1(s_j;C^j)} \overset{t\to\infty}{\propto} \left(\frac{J}{\hbar}\right)^{\frac{1}{6}} \mathrm{Ai}\left[\left(\frac{J}{\hbar}\right)^{\frac{2}{3}} C^j\right]$$

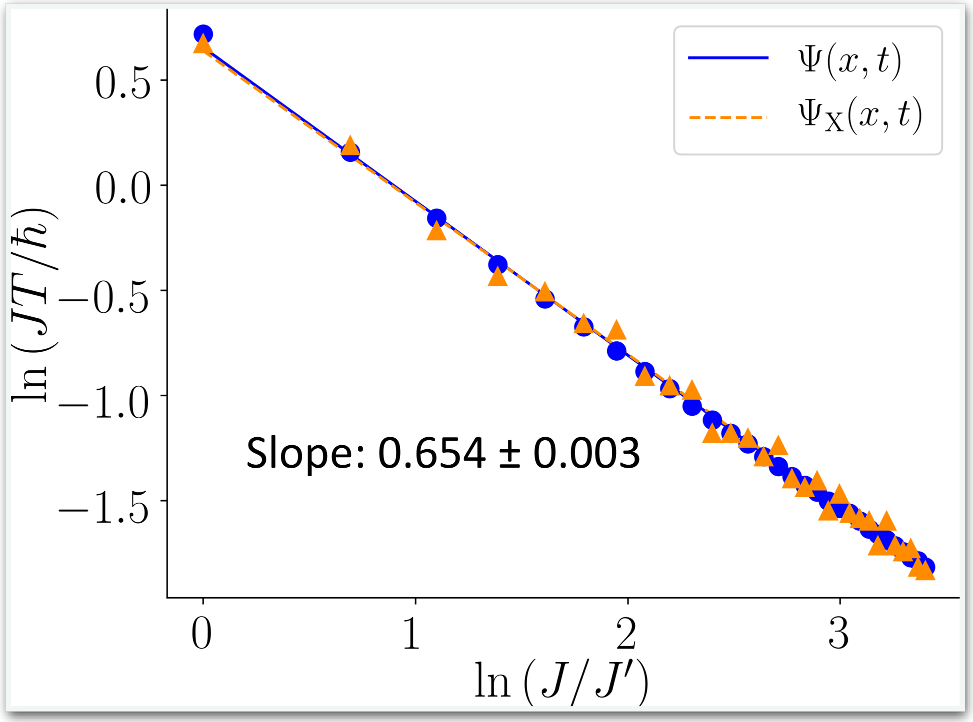
Self-similar scaling

Arnold index

Berry index

$$\Psi_{\mathrm{Ai}}(C^j; J) \overset{t\to\infty}{\propto} \left(\frac{J}{\hbar}\right)^{\frac{1}{6}} \mathrm{Ai}\left[\left(\frac{J}{\hbar}\right)^{\frac{2}{3}} C^j\right]$$

numerical results:



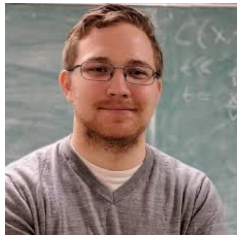
$\Psi(x,t)$ Single Bogoliubov quasiparticle

$\Psi_X(x,t)$ Initial state = spin flip = superposition of Bogoliubov quasiparticles

T = oscillation period inside the light cone

Breaking integrability

Heisenberg spin-1/2 chain


Jonathon Riddell


Wyatt Kirkby


Erik Sørensen

$$\hat{H}_\lambda = \hat{H}_f + \lambda \hat{H}_I$$

$$\hat{H}_f = J_1 \sum_{j=1}^{L-1} \left(\hat{S}_j^+ S_{j+1}^- + \text{h.c} \right)$$

$$\hat{H}_I = \Delta \sum_{j=1}^{L-1} \hat{S}_j^Z \hat{S}_{j+1}^Z \; + J_2 \sum_{j=1}^{L-2} \left(\hat{S}_j^+ S_{j+2}^- + \text{h.c} \right) + \gamma \sum_{j=1}^{L-2} \hat{S}_j^Z \hat{S}_{j+2}^Z$$

Out of time order correlator (OTOC): $C(x,t) = \langle [\hat{A}(t), \hat{B}]^\dagger [\hat{A}(t), \hat{B}] \rangle$

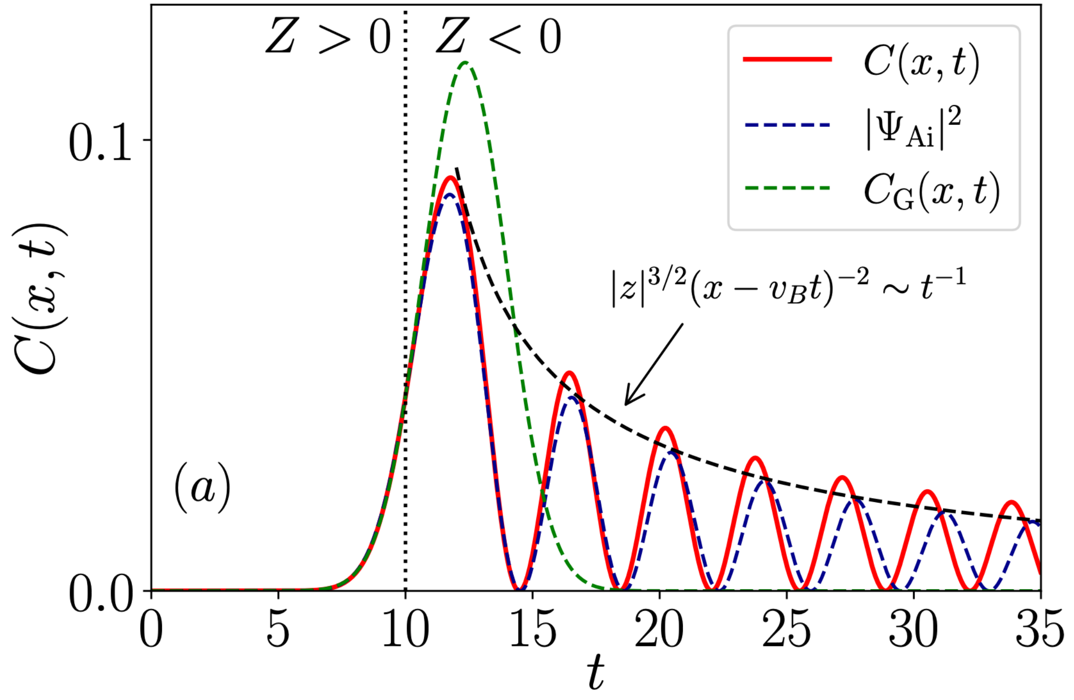
Choose: $\hat{A}(t) = \sigma_1^Z$ $\hat{B} = \sigma_m^Z$ x = distance between sites 1 and m

$\langle \ldots \rangle$ = average over thermal ensemble restricted to eigenstates with zero magnetization $m_z = \sum_{j=1}^{L} \langle \hat{S}_j^Z \rangle = 0$

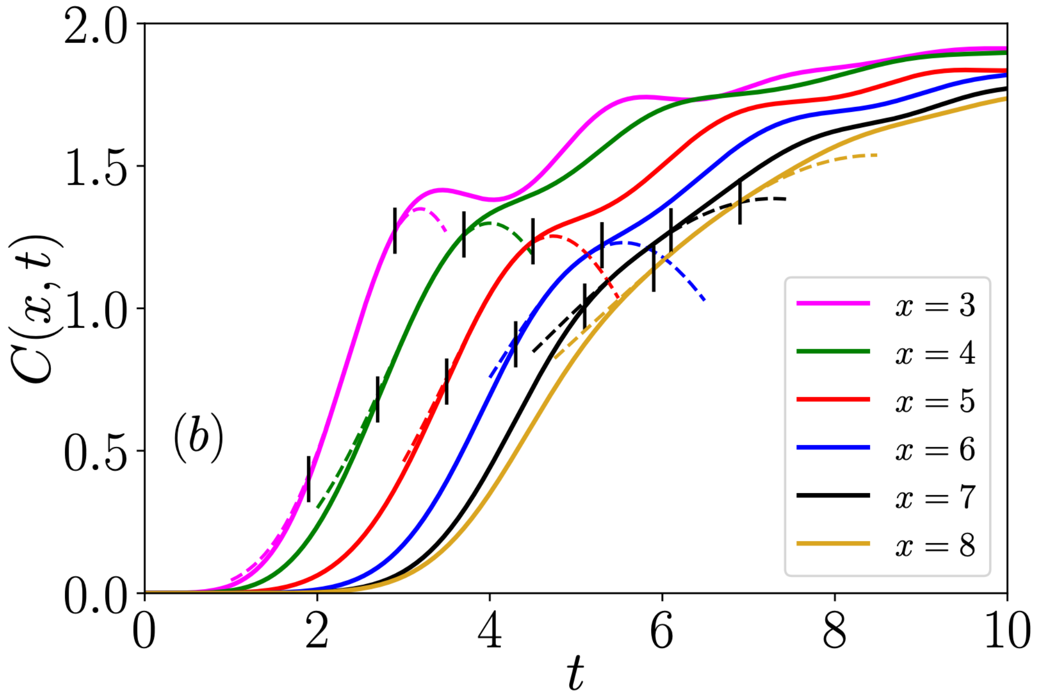
and β=1

OTOC dynamics

Free case $x = 10$



ETH case*



$$\Psi_{\mathrm{Ai}}(x,t) \sim \sqrt{a}\left(\frac{-2}{\partial_k^3 \epsilon(k_c) t}\right)^{1/3} \mathrm{e}^{\mathrm{i}\Phi(k_c,x,t)} \,\mathrm{Ai}(Z)$$

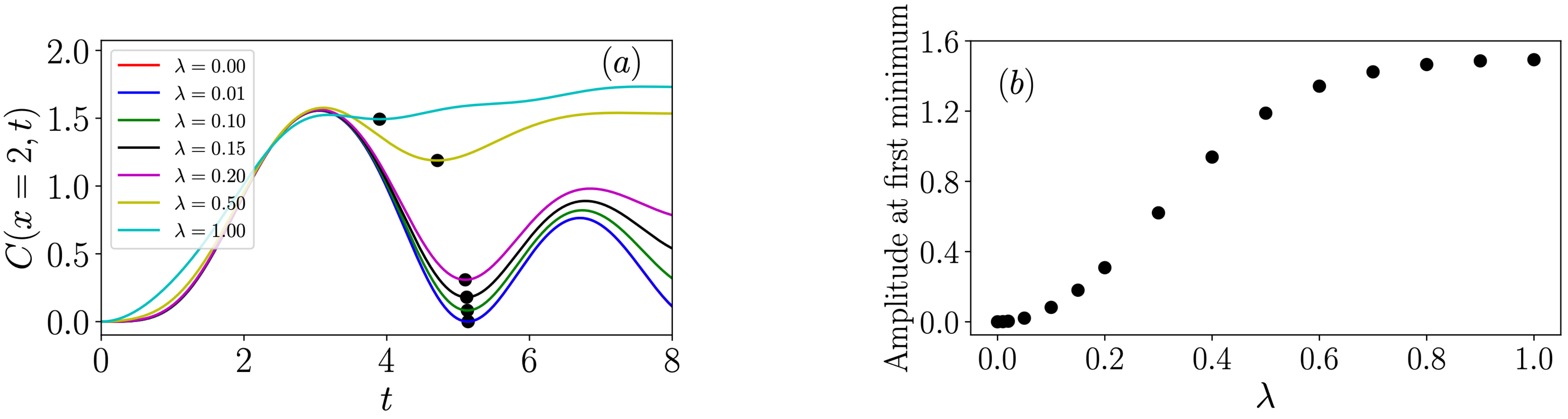
$$Z = (x - v_B t)\left|t\partial_k^3 \epsilon(k_c)/2\right|^{-1/3} \qquad \epsilon(k) = 2J_1 \cos ka$$

*LeBlond *et al* Phys. Rev E 100, 062134 (2019)

Along wavefront $x = v_B t$ amplitude decays as $x^{-2/3}$ and width grows as $t^{1/3}$

Riddell, Kirkby, DOD and Sorensen, PRB 108, L121108 (2023)

Evidence for a quantum Kolmogorov-Arnold-Moser (KAM) theorem



$$\hat{H}_\lambda = \hat{H}_f + \lambda \hat{H}_I \qquad J_1 = -0.5 \qquad \Delta = \lambda \qquad J_2 = -0.2\lambda \qquad \gamma = 0.5\lambda$$

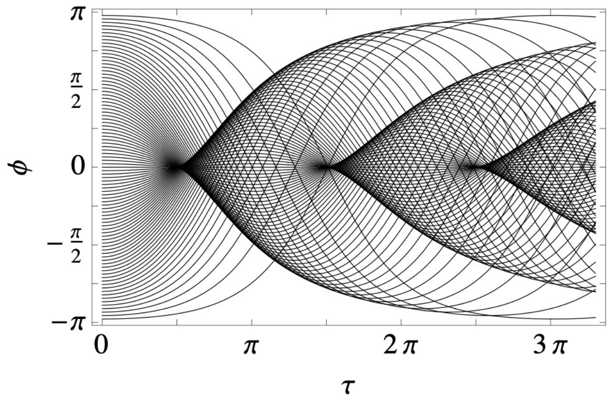
$$\hat{H}_f = J_1 \sum_{j=1}^{L-1} \left(\hat{S}_j^+ S_{j+1}^- + \mathrm{h.c} \right) \qquad \hat{H}_I = \Delta \sum_{j=1}^{L-1} \hat{S}_j^Z \hat{S}_{j+1}^Z \; + J_2 \sum_{j=1}^{L-2} \left(\hat{S}_j^+ S_{j+2}^- + \mathrm{h.c} \right) + \gamma \sum_{j=1}^{L-2} \hat{S}_j^Z \hat{S}_{j+2}^Z$$

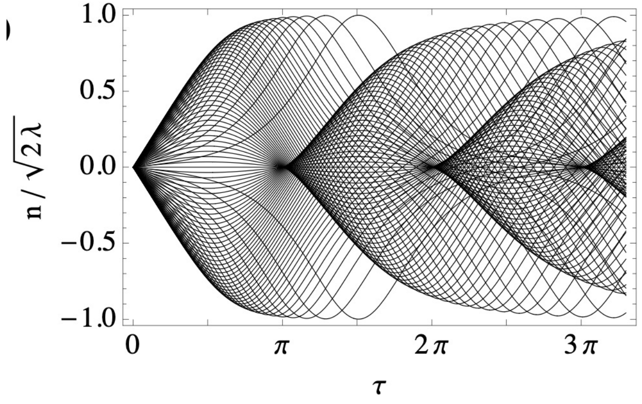
Riddell, Kirkby, DOD and Sorensen, PRB 108, L121108 (2023)

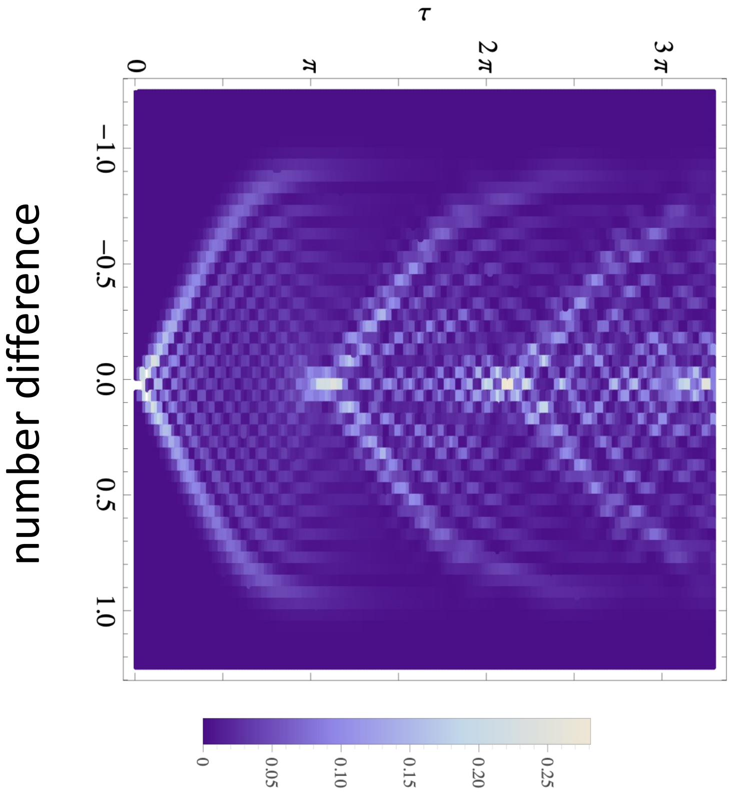
Unkicked case: Josephson junction

Number difference Phase difference

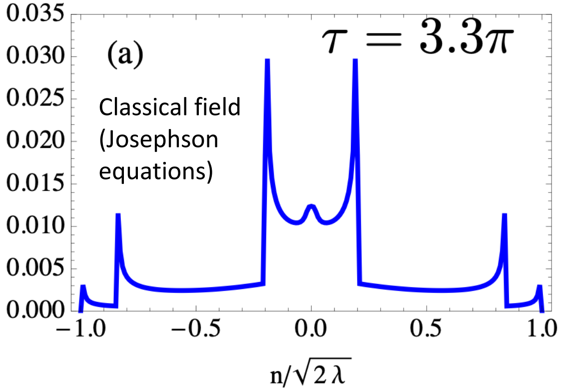
$$H = \frac{E_c}{2} n^2 - E_J \cos\phi \qquad n \equiv \frac{1}{2}(n_l - n_r) \quad , \quad \phi \equiv \phi_l - \phi_r \qquad \lambda \equiv 2E_J/E_c$$

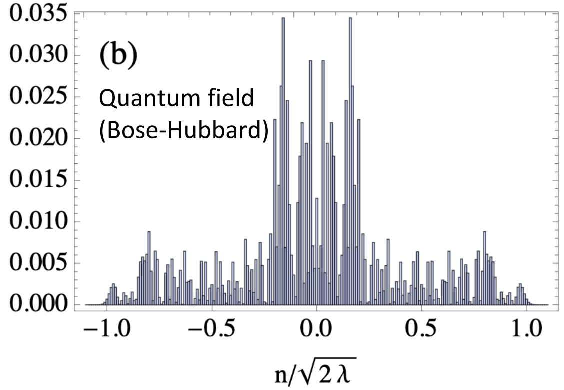






$$\lambda \equiv \frac{2E_J}{E_c} = 200$$

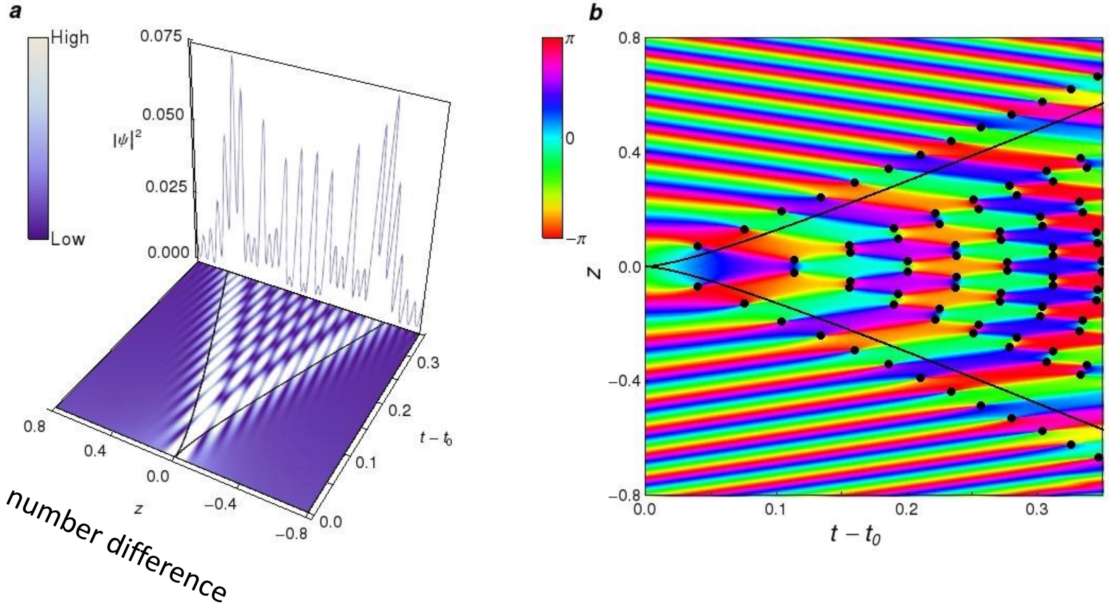




$$\lambda \equiv \frac{2E_J}{E_c} = 5000$$

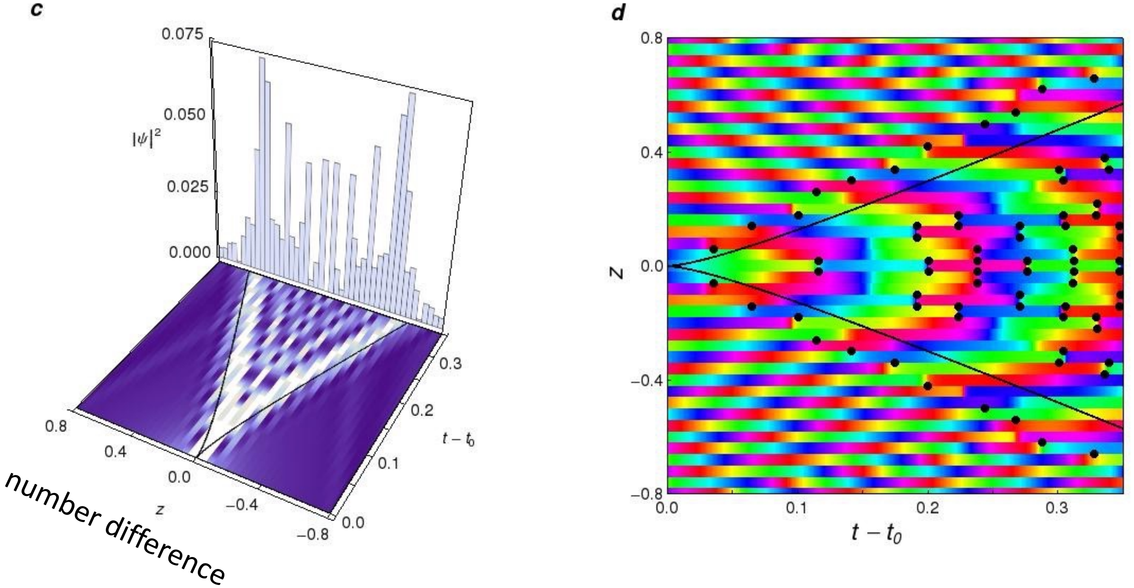
DOD, PRL **109**, 150406 (2012)

Morphology of a quantum catastrophe

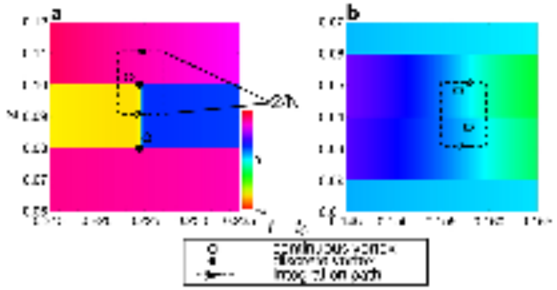


Standard wave catastrophe (Pearcy function)

$$z \equiv \frac{n_l - n_r}{N}$$



Quantum catastrophe (discrete)



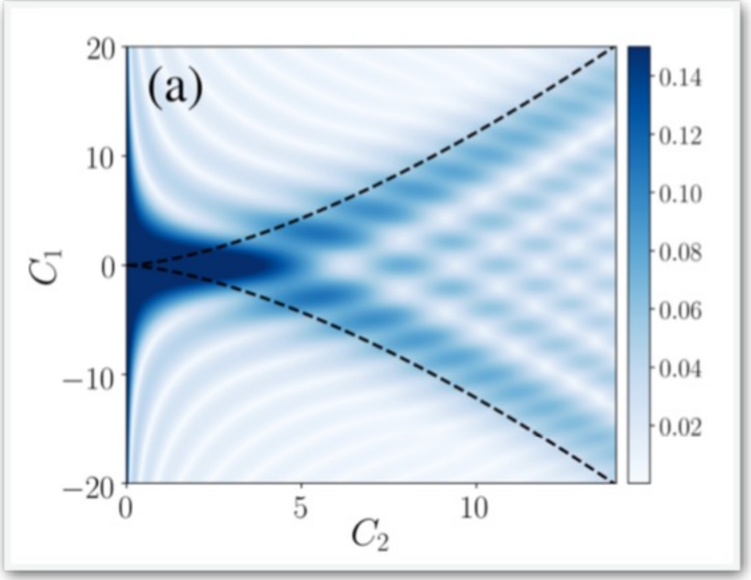
$$\oint \nabla\theta \cdot d\mathbf{l} = \pm 2\pi$$

Coreless vortices

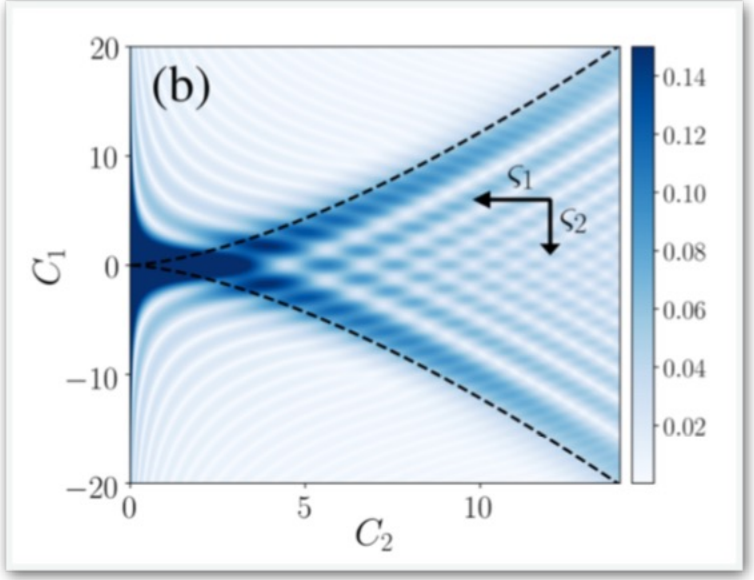
Mumford, Turner, Sprung, DOD, PRL **122**, 170402 (2019)

Scaling of the inner cone

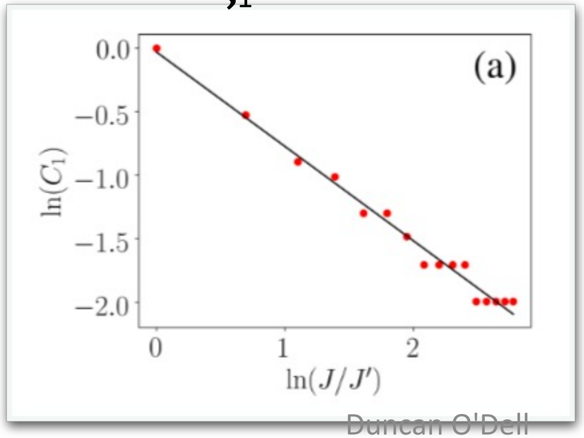
$$\Psi_{\rm Pe}(C_1,C_2;J) \approx \frac{1}{2\pi}\left(\frac{J(\gamma^2+g-1)}{\hbar v_{\rm I}(g-1)C_2}\right)^{\frac{1}{2}}\int_{-S}^{S} \mathrm{d}s\, \mathrm{e}^{-\frac{\mathrm{i}J}{\hbar}\Phi_2(s;C_1,C_2)} \overset{t\to\infty}{\propto} \left(\frac{J}{\hbar}\right)^{\frac{1}{4}}\mathrm{Pe}\left[\left(\frac{J}{\hbar}\right)^{\frac{3}{4}}C_1, \left(\frac{J}{\hbar}\right)^{\frac{1}{2}}C_2\right]$$



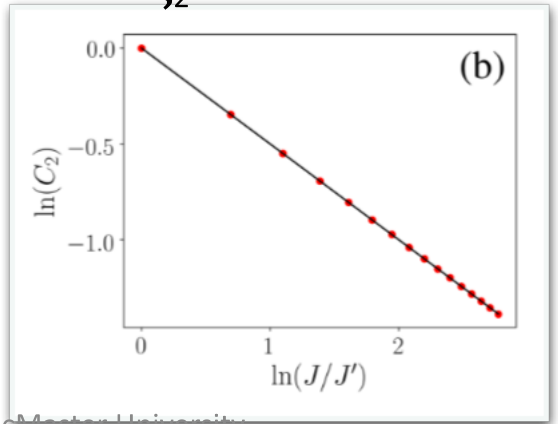
Increasing J



$\zeta_1=0.743\pm0.002$

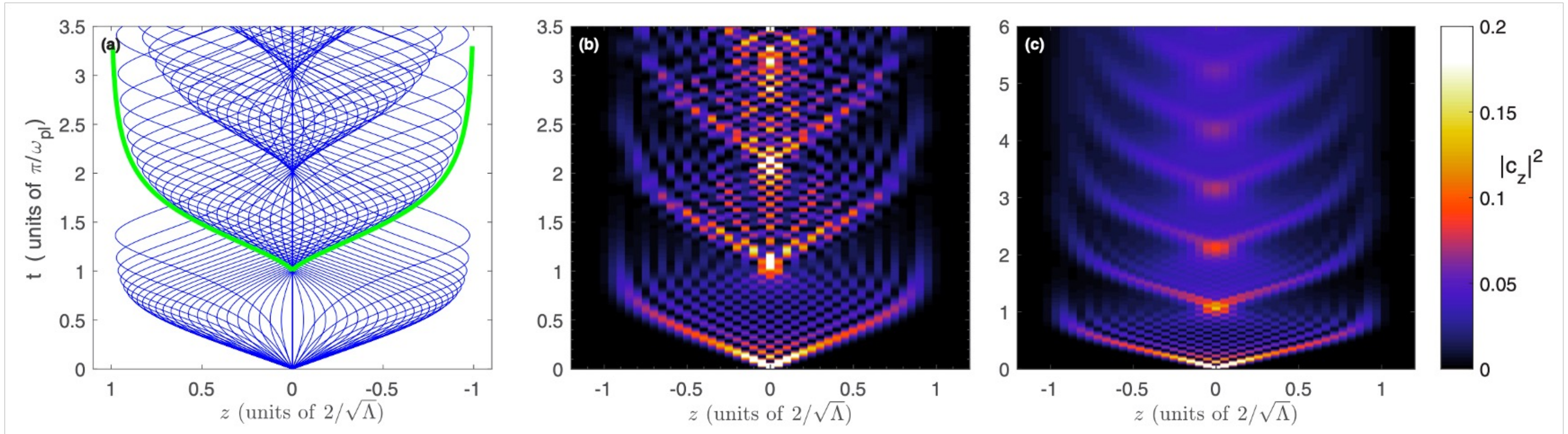


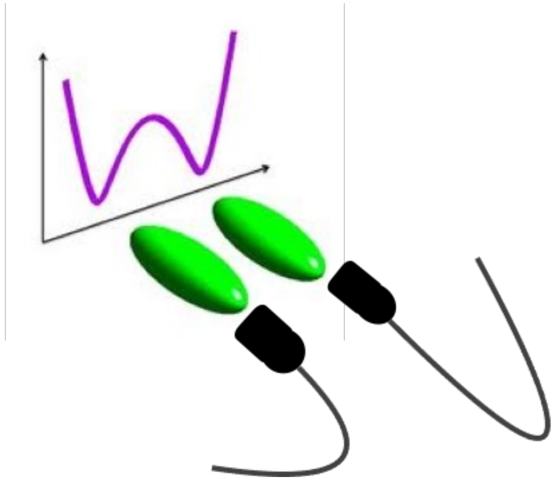
$\zeta_2=0.500\pm0.001$



Stability against decoherence

classical | Quantum (no decoherence) | decoherence





$$\frac{\partial \hat{\rho}}{\partial \tau} = i[\hat{S}_x, \hat{\rho}] - i\frac{\Lambda}{N}[\hat{S}_z^2, \hat{\rho}] - D\frac{\Lambda}{N}[\hat{S}_z, [\hat{S}_z, \hat{\rho}]]$$

A. Z. Goldberg, A. Al-Qasimi, J. Mumford, and DOD, Phys. Rev. A 100, 063628 (2019)