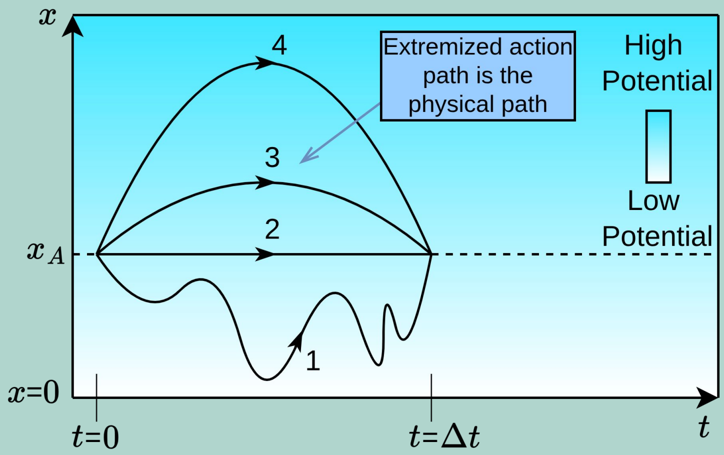
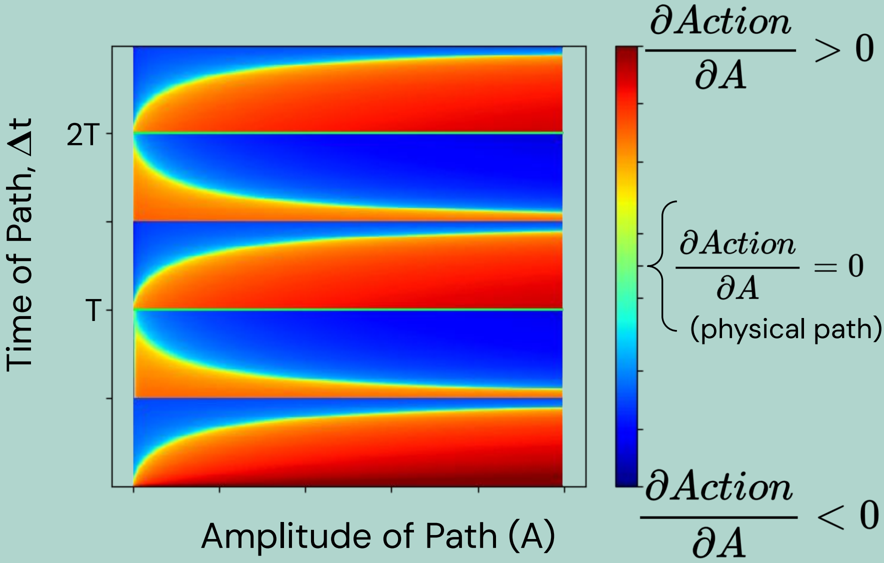
Visualizing the Principle of Least Action

Presented by Ciaran McDonald-Jensen

Work supervised by Dr. Daniel Stolarski

Carleton University Department of Physics



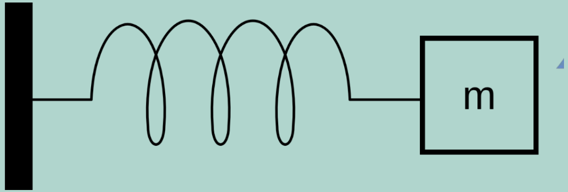


Content overview

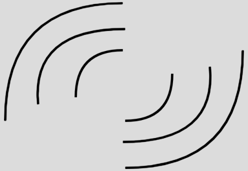
1. What is the Principle of Least Action and why are we looking at it?

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy \Big) dt$$

2. Case Study – 1D Spring



3. Vibration path variation



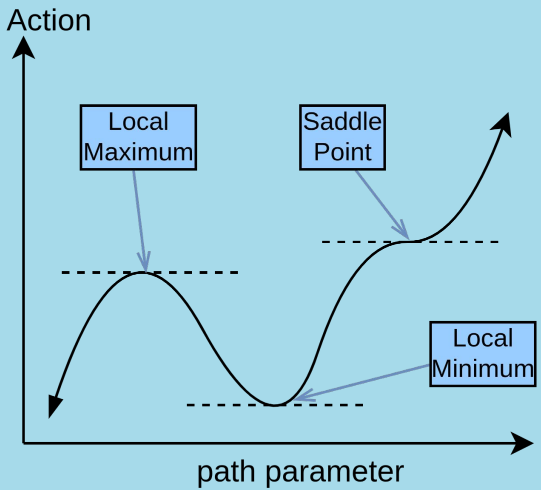
Summary

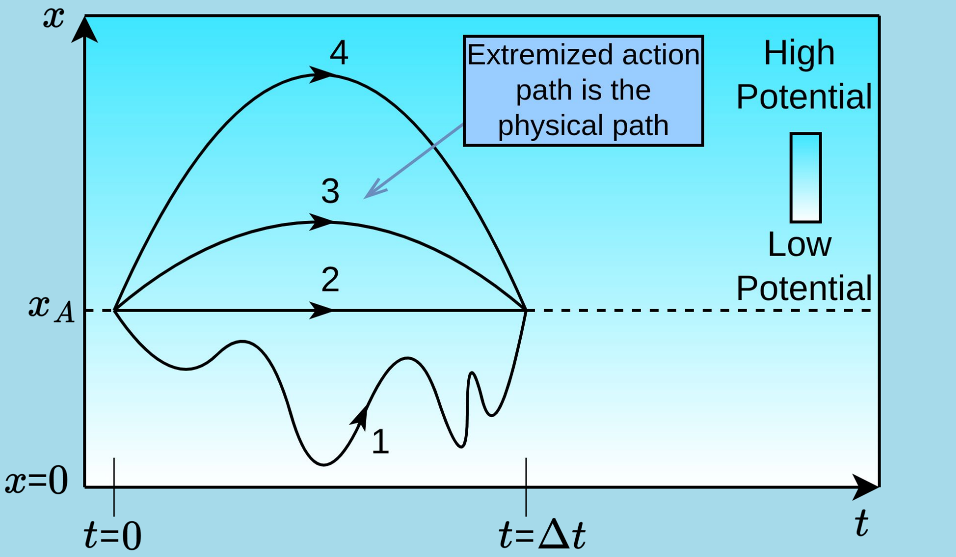
What is the Principle of Least Action?

- Axiom that is everywhere in physics
- True Path = Path of extremized action

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$

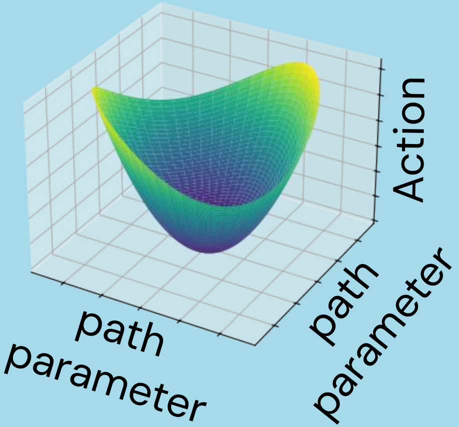
$$\frac{\partial(Action)}{\partial(path\ variation)} = 0$$



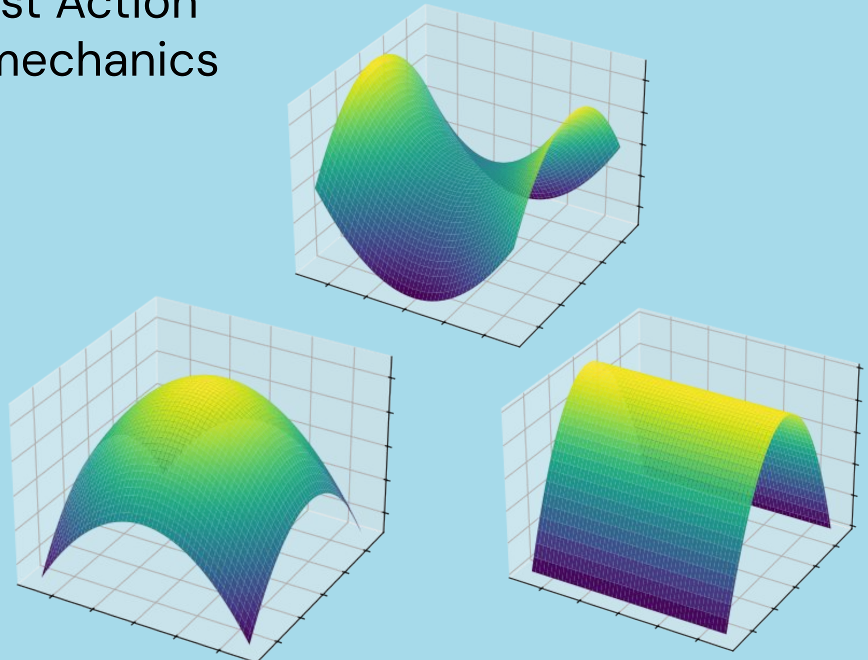


Goals of Presentation

- Build intuition on Axiom
- See subtleties of Principle of Least Action
  -> No Newtonian or Lagrangian mechanics



VS.



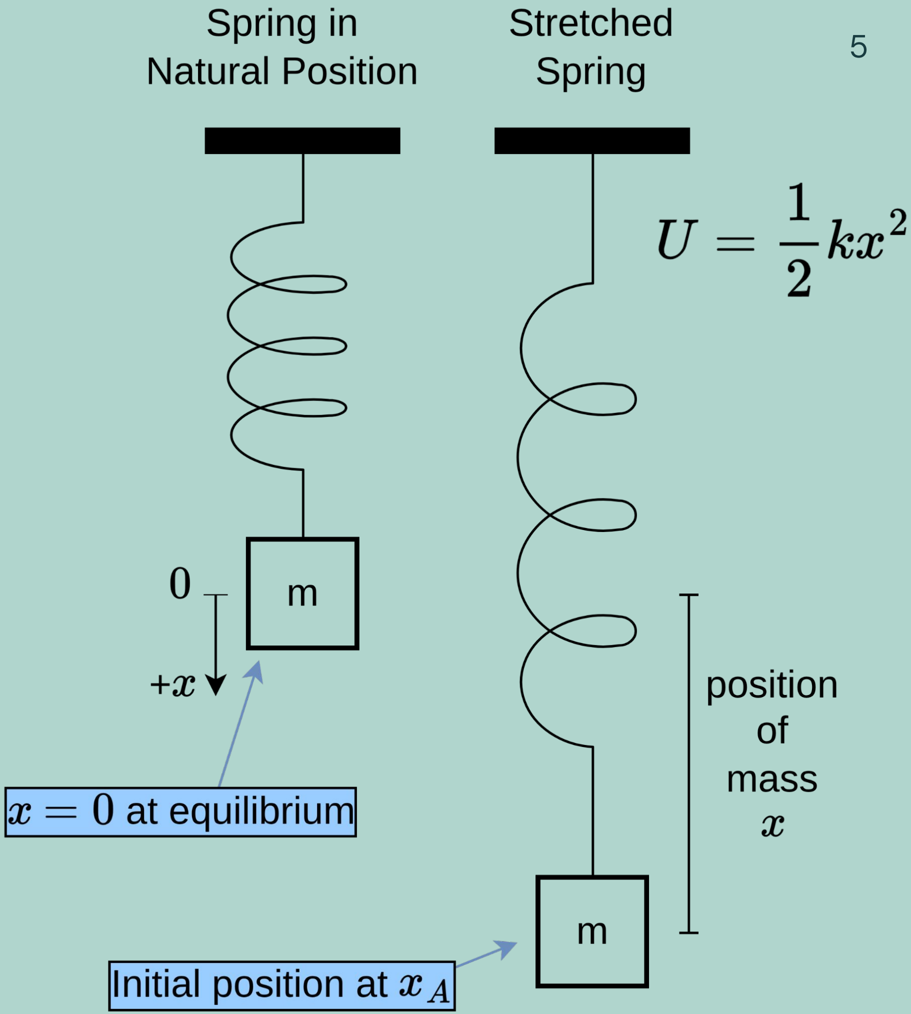
1D Spring Analysis Introduction

Look at sinusoidal paths

$$x(t) = Asin(\omega t + \phi)$$

Boundary Conditions:

$$x(0) = x(\Delta t) = x_A$$



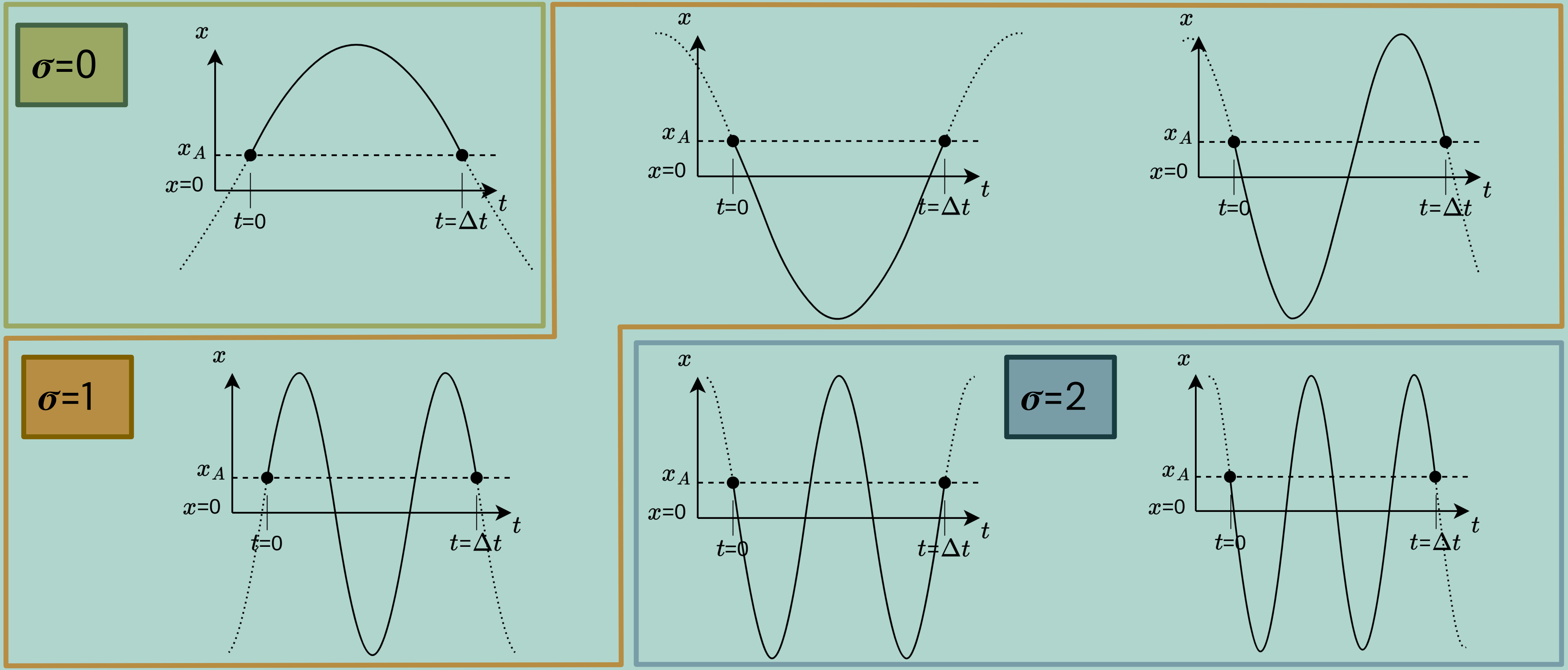
Path Parameterization

Path: $x(t) = Asin(\omega t + \phi)$

Case 1
Bit more than a full period

Case 2
Bit less than a full period

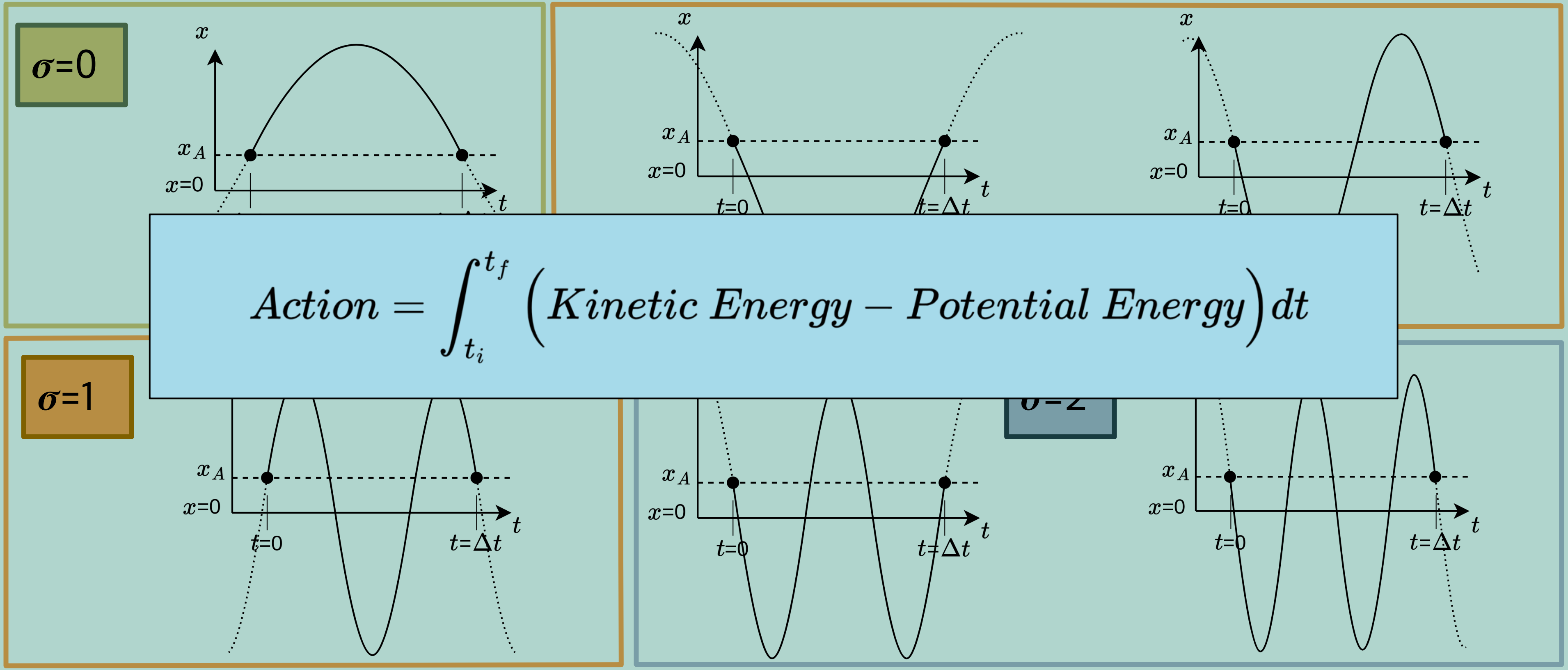
Case 3
Exactly a full period



Path Parameterization

Path: $x(t) = A sin(\omega t + \phi)$

Case 1	Case 2	Case 3
Bit more than a full period	Bit less than a full period	Exactly a full period



$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big)\, dt$$

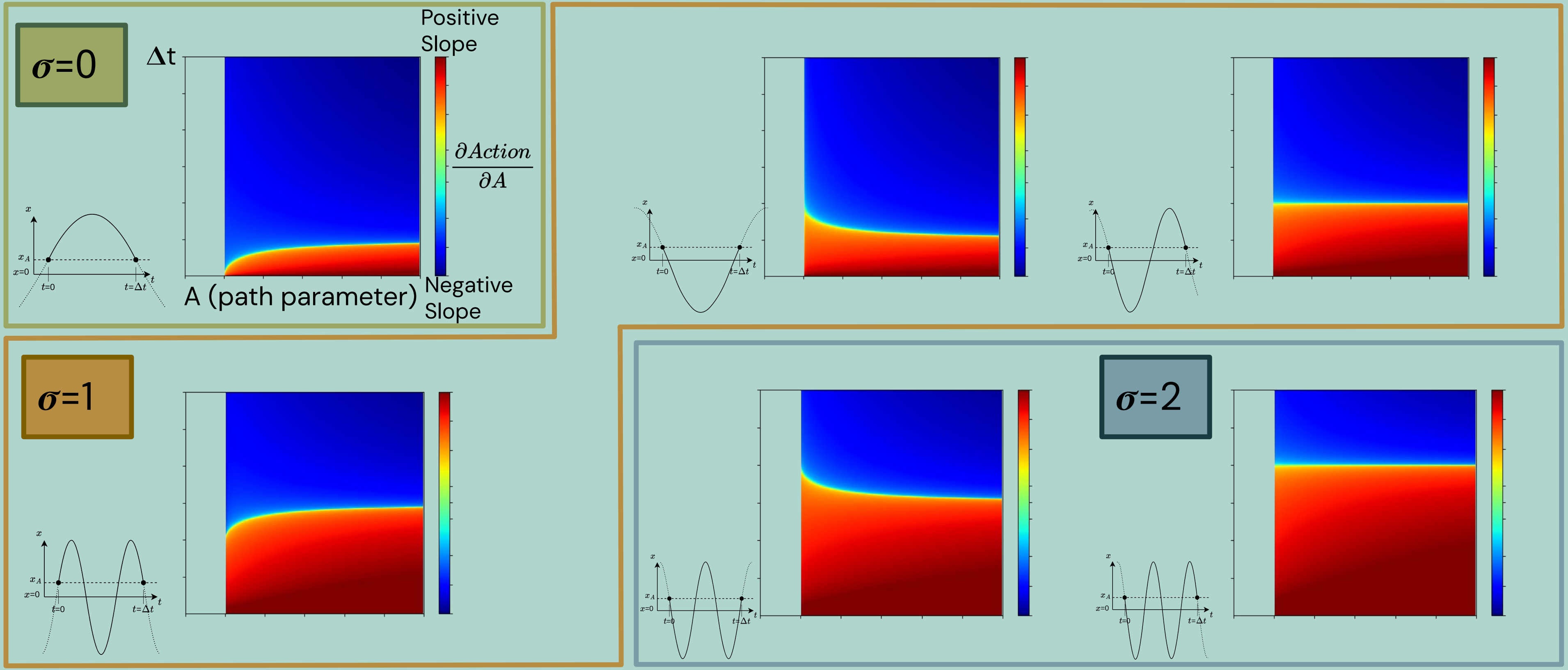
Action of the Paths

Path: $x(t) = \boxed{A} sin(\omega t + \phi)$

Case 1
Bit more than a full period

Case 2
Bit less than a full period

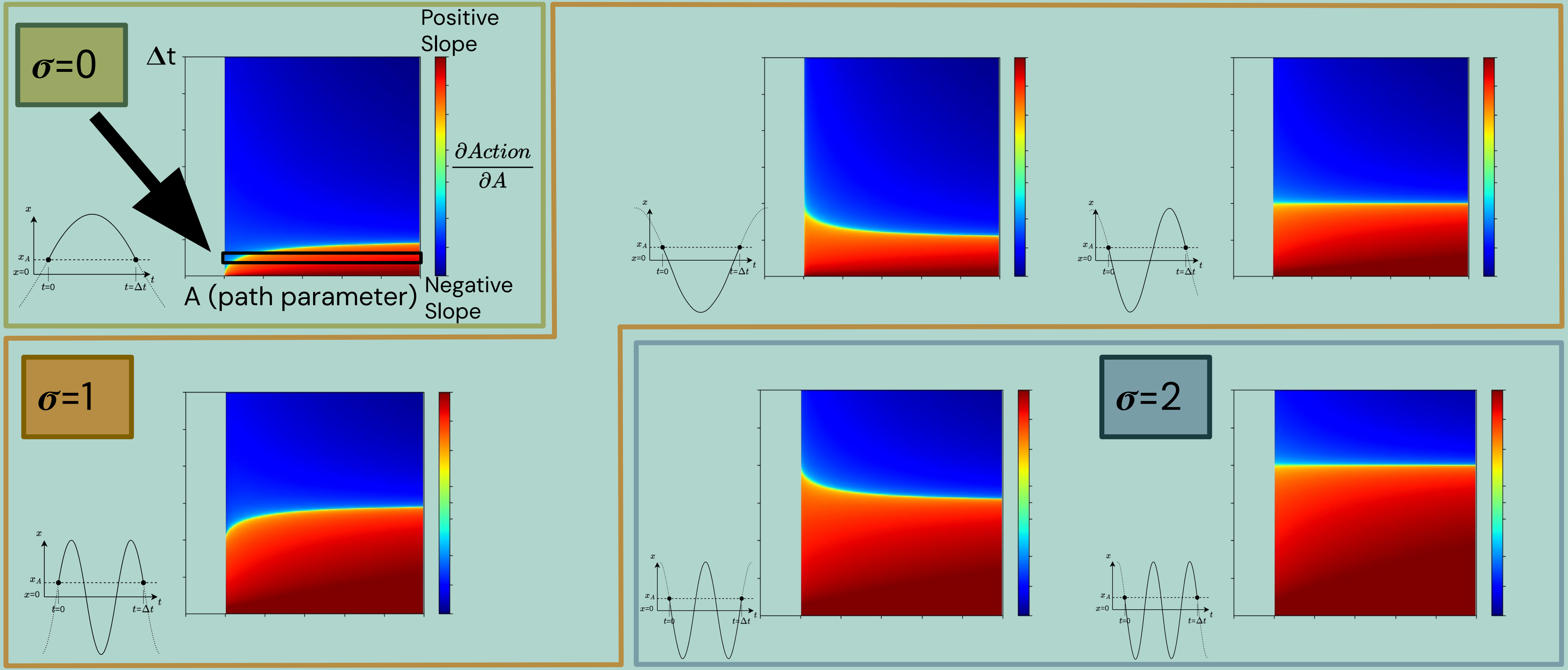
Case 3
Exactly a full period



Action of the Paths

Path: $x(t) = Asin(\omega t + \phi)$

Case 1	Case 2	Case 3
Bit more than a full period	Bit less than a full period	Exactly a full period



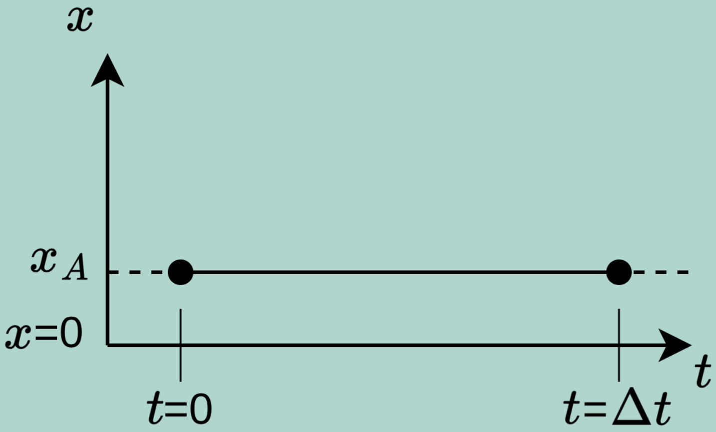
Minimizing Action

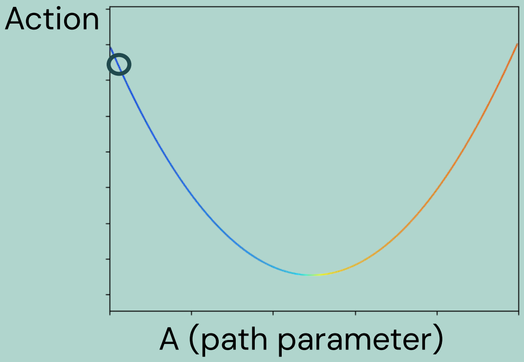
👎 👍

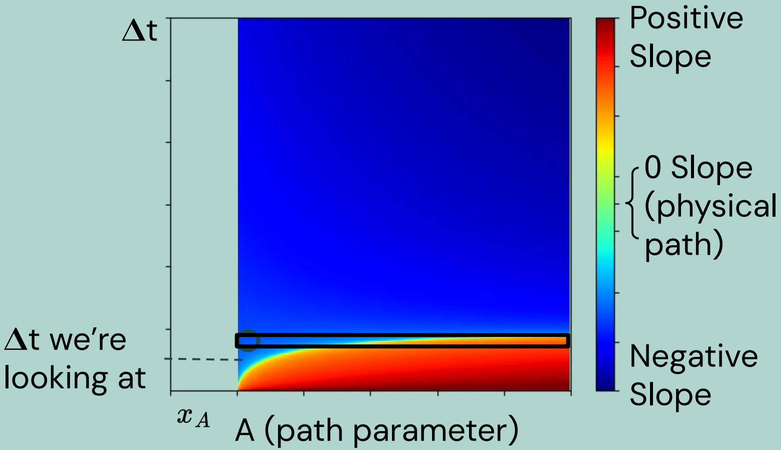
$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$

Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$

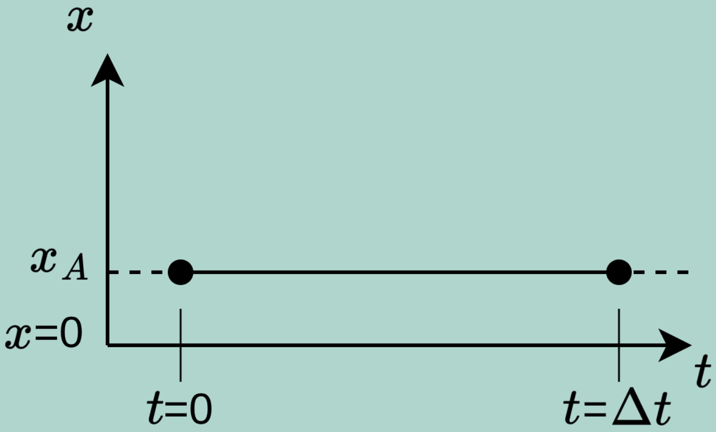


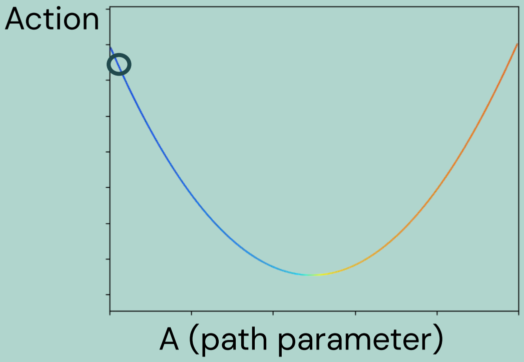


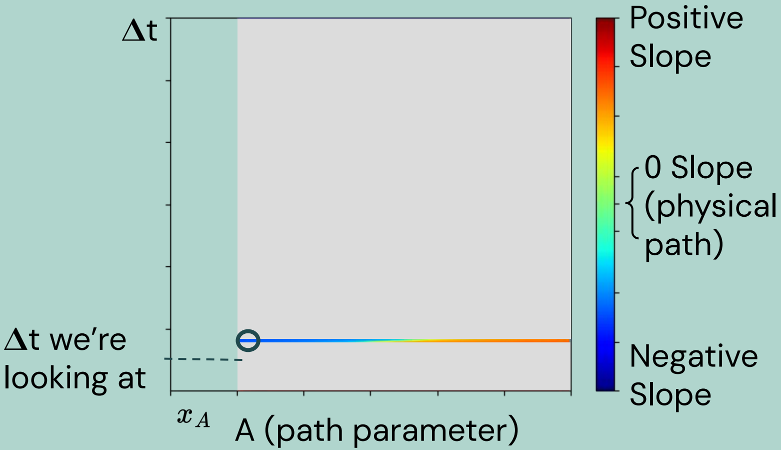


Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$

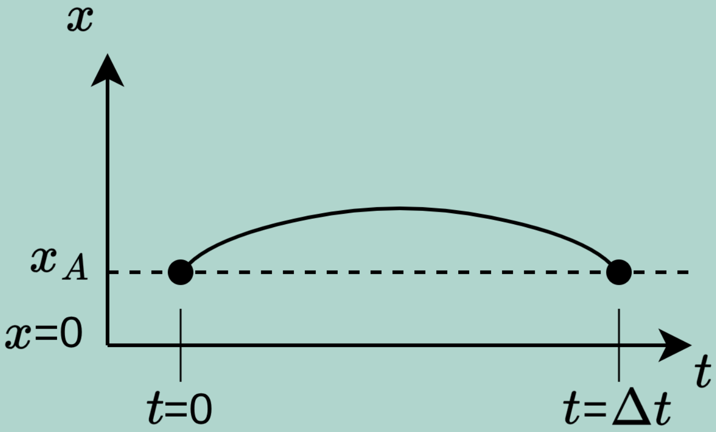


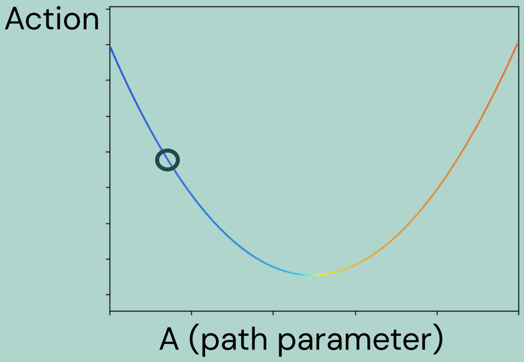


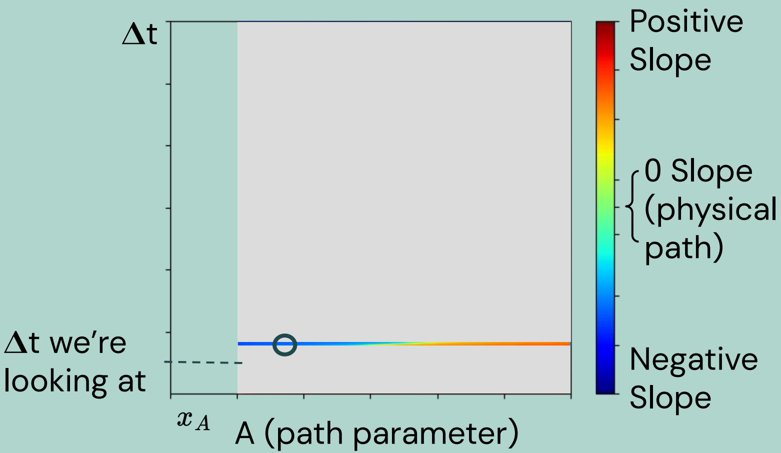


Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$

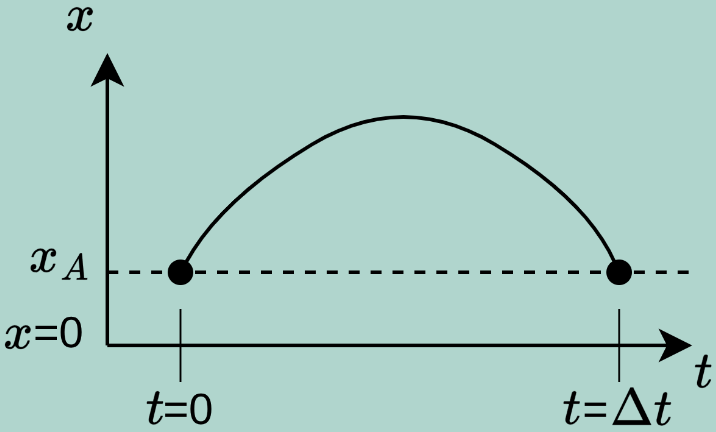


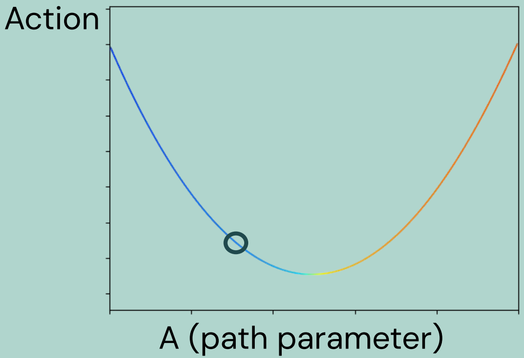


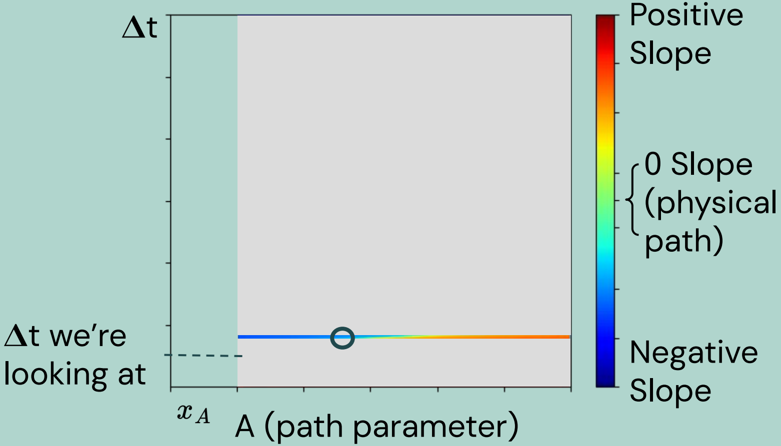


Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$

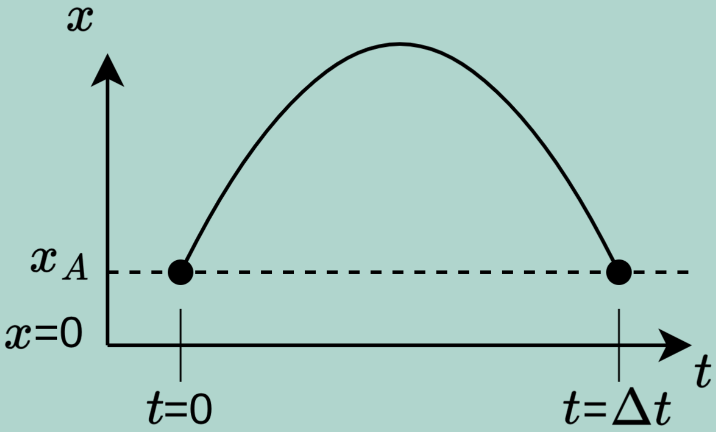


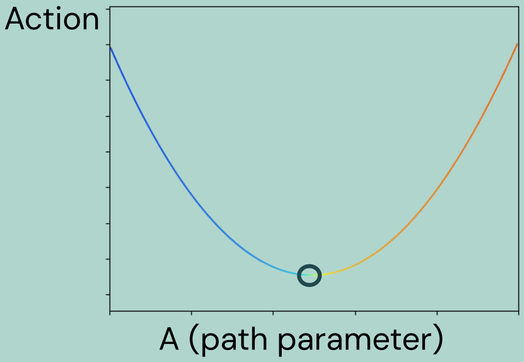


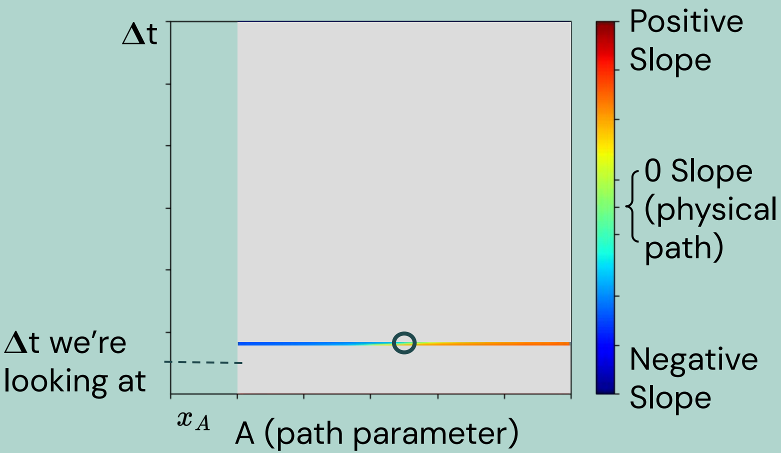


Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big)dt$$

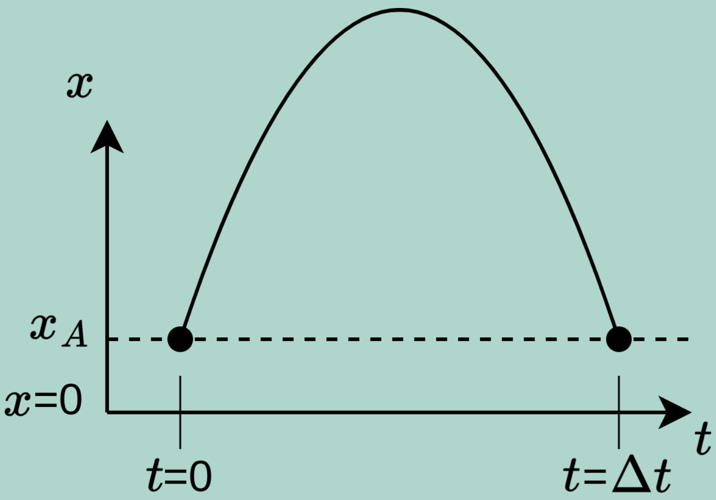


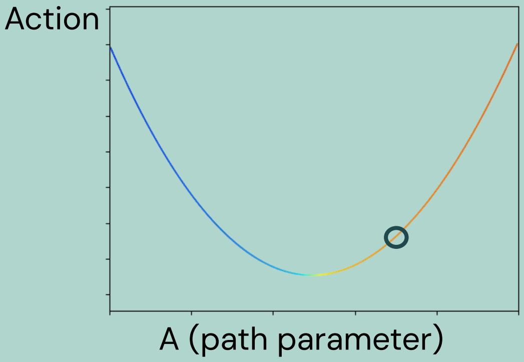


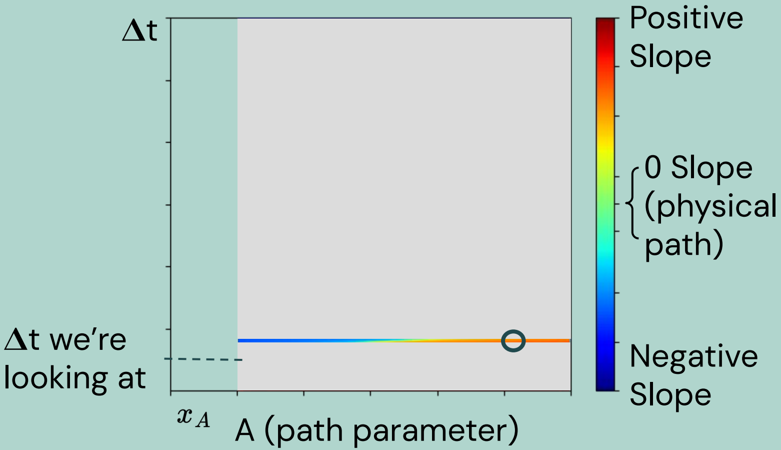


Action Calculation – Case 1

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$



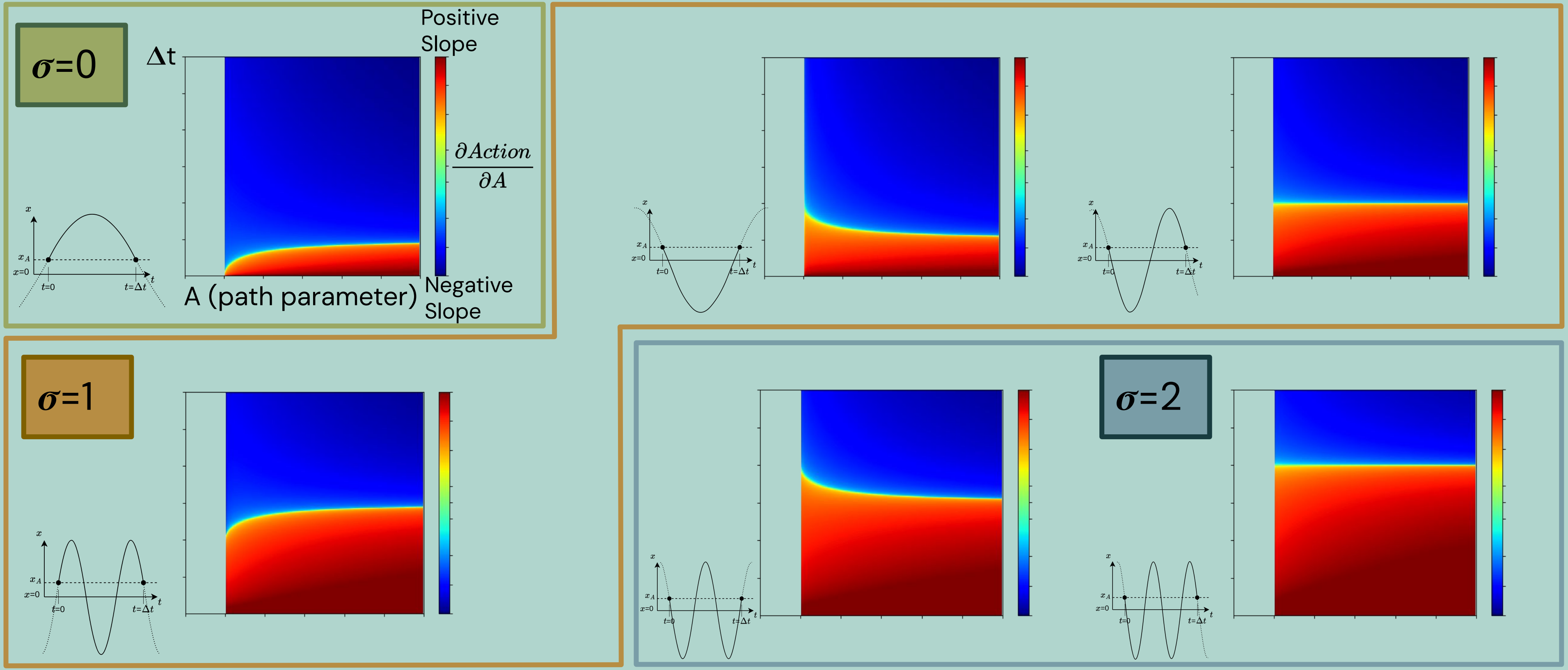




Action of the Paths

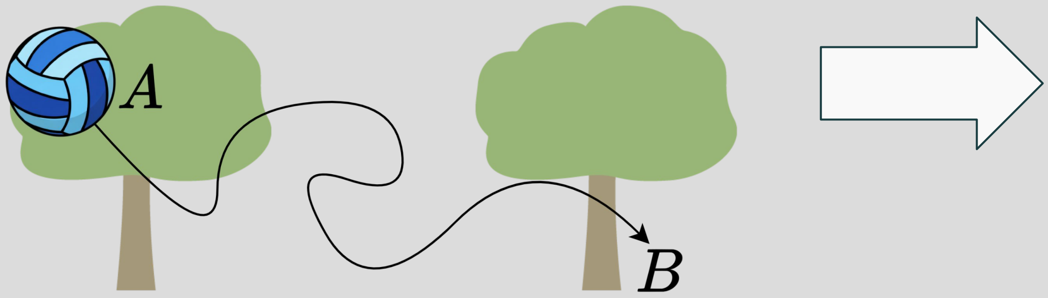
Path: $x(t) = Asin(\omega t + \phi)$

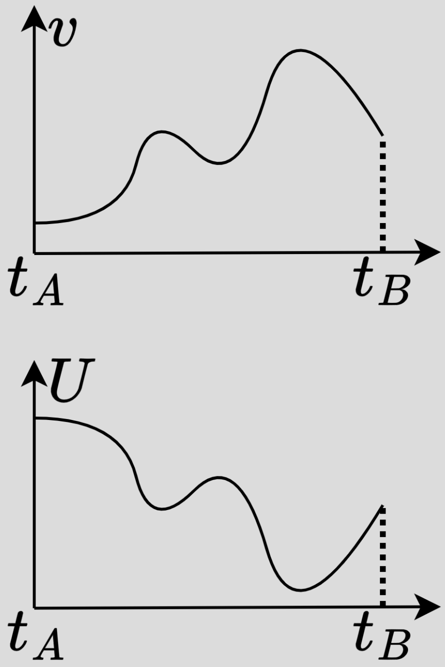
Case 1	Case 2	Case 3
Bit more than a full period	Bit less than a full period	Exactly a full period



Vibration Path Change – Introduction

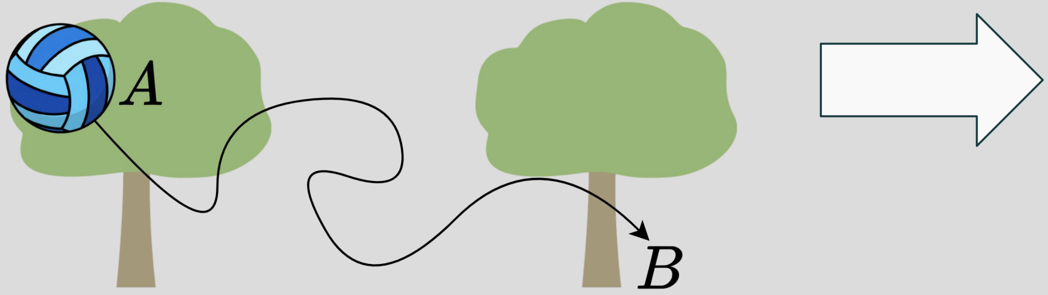
- Time path variation for any arbitrary path in any number of dimensions
  - Doesn't change physical path through space, only rate of traversal

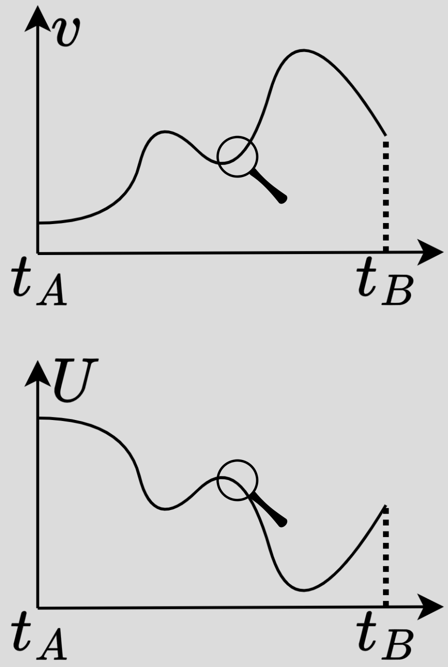




Vibration Path Change – Introduction

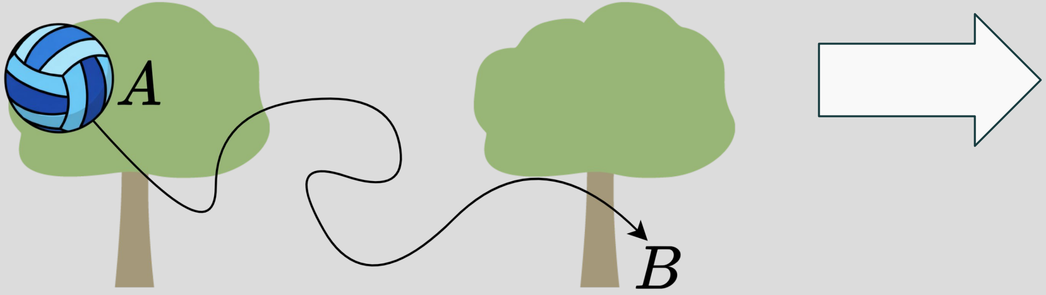
- Time path variation for any arbitrary path in any number of dimensions
  - Doesn't change physical path through space, only rate of traversal

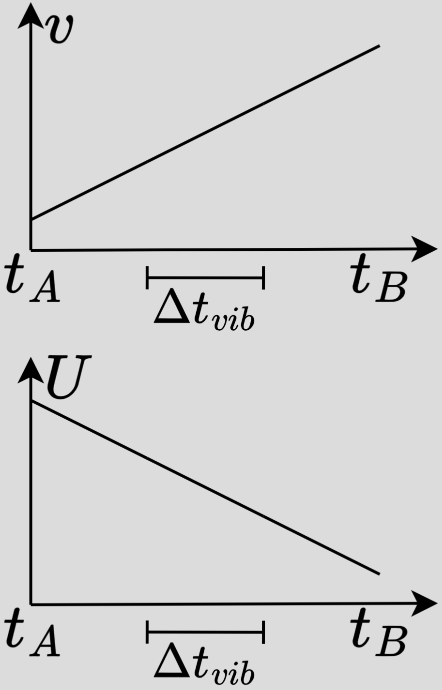




Vibration Path Change – The Vibration

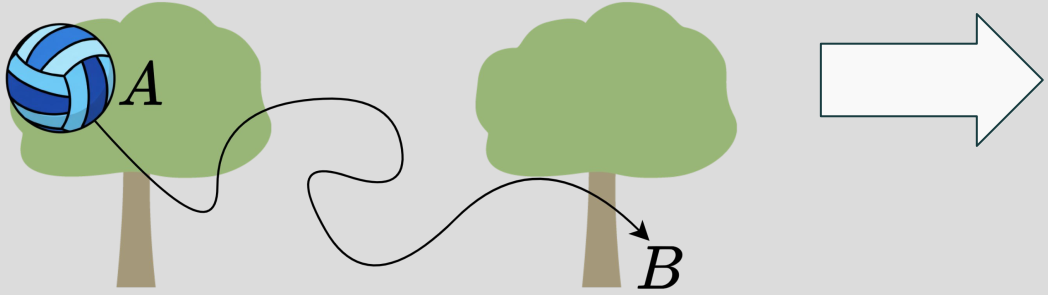
- Speed up for Δt then slow down for Δt

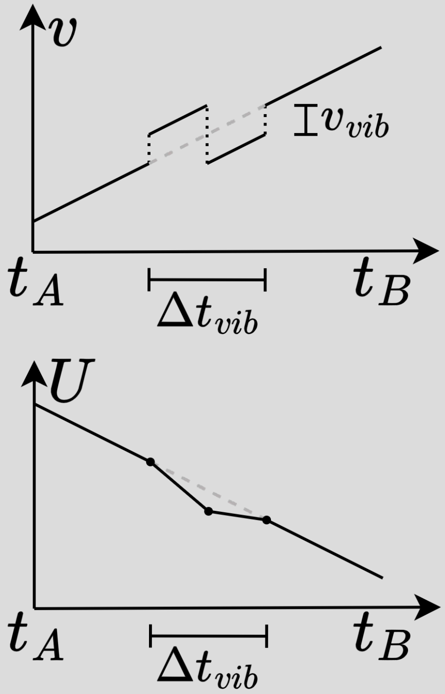




Vibration Path Change – The Vibration

- Speed up for Δt then slow down for Δt



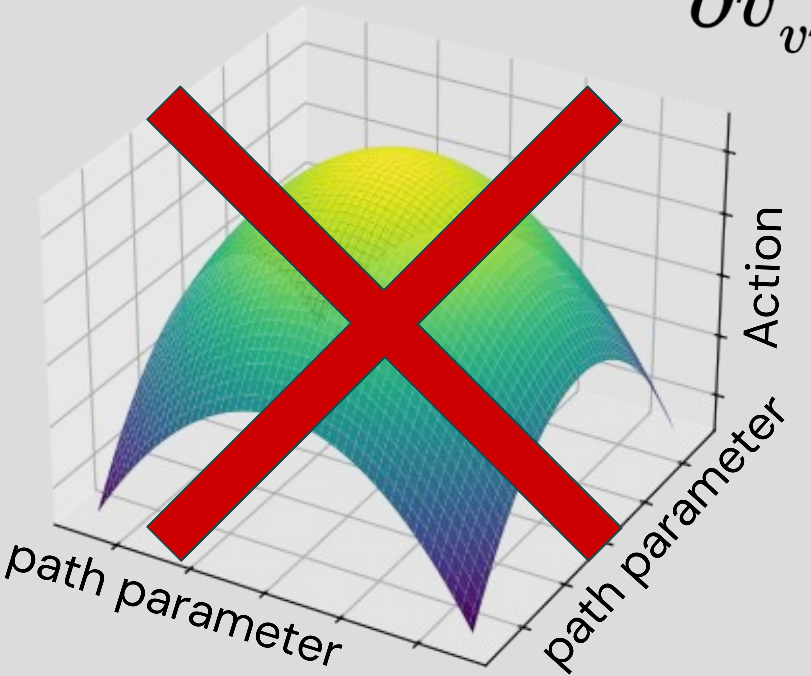


Conclusion 1

$$\frac{\partial^2 Action}{\partial v_{vib}^2} \geq 0$$

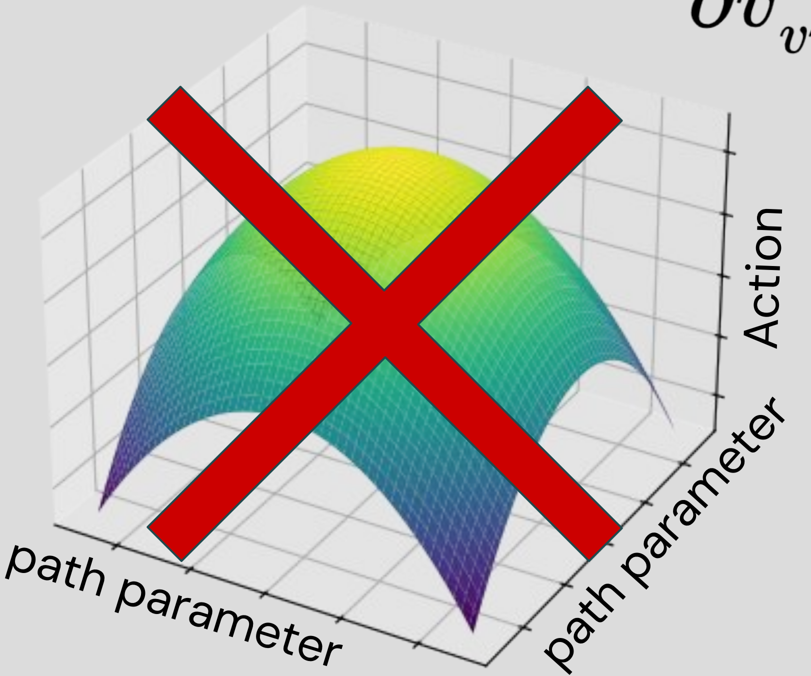
Conclusion 1

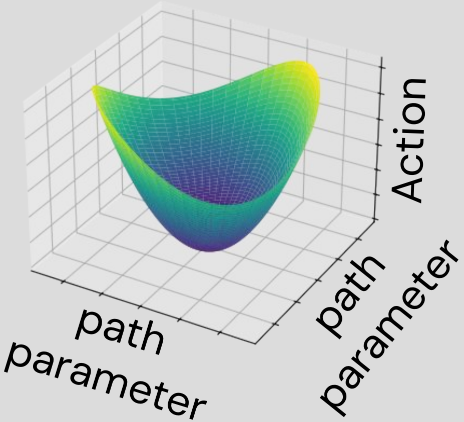
$$\frac{\partial^2 Action}{\partial v_{vib}^2} \geq 0$$

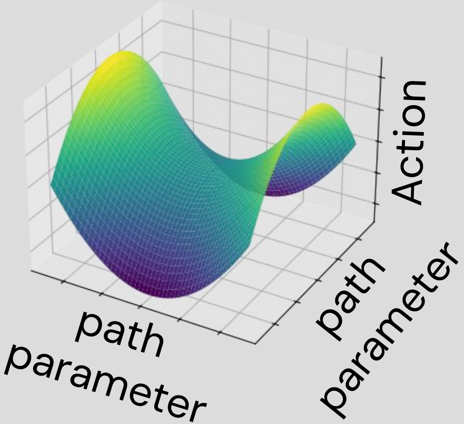


Conclusion 1

$$\frac{\partial^2 Action}{\partial v_{vib}^2} \geq 0$$







Conclusion 2

$$\frac{\partial Action}{\partial v_{vib}} = 0$$

only for paths where

$$\frac{1}{2}mv^2 + U = constant$$

Conclusion 2

$$\frac{\partial Action}{\partial v_{vib}} = 0$$

only for paths where

$$\frac{1}{2}mv^2 + U = constant = E$$

What you learned (hopefully)

Specific:

- Another way to think about energy conservation
- There are **no** situations where there is a true maxima action path
- Easy concrete examples of saddle points or flat action

General:

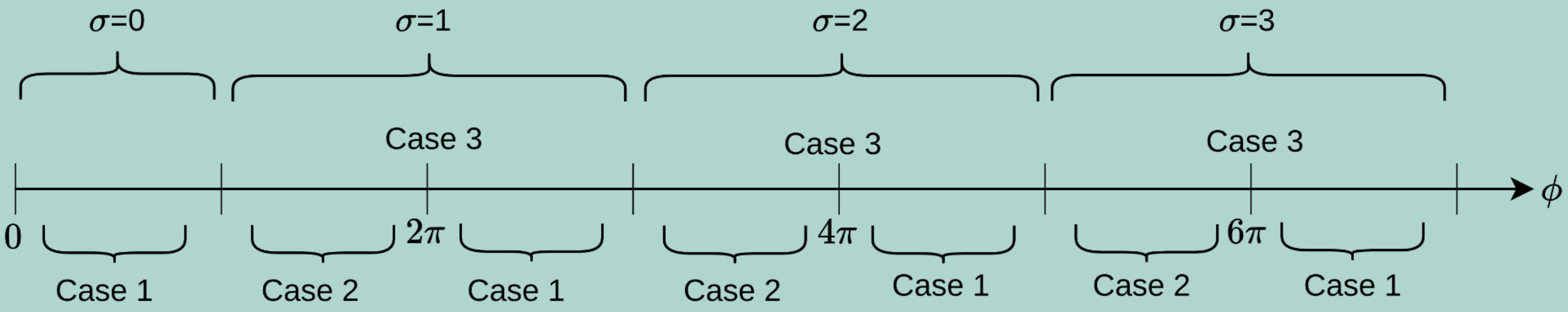
- **Intuition**
- **There are subtleties**

Presented by Ciaran McDonald-Jensen
ciaranmcdonaldjensen@cmail.carleton.ca

Work supervised by Dr. Daniel Stolarski

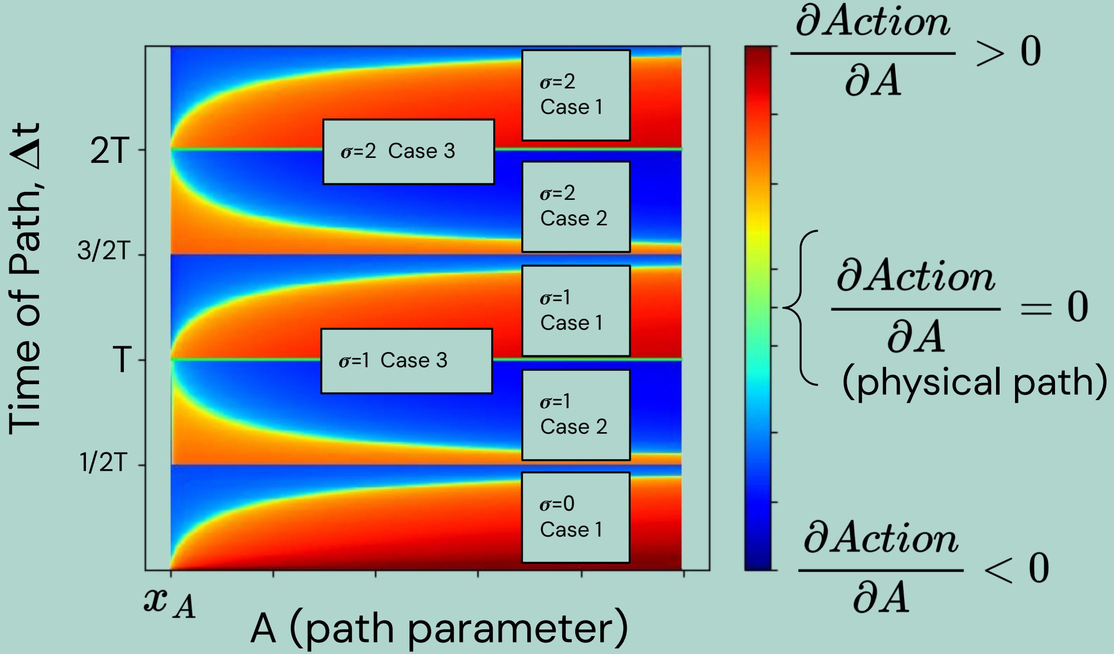
Carleton University Department of Physics

EXTRA SLIDES

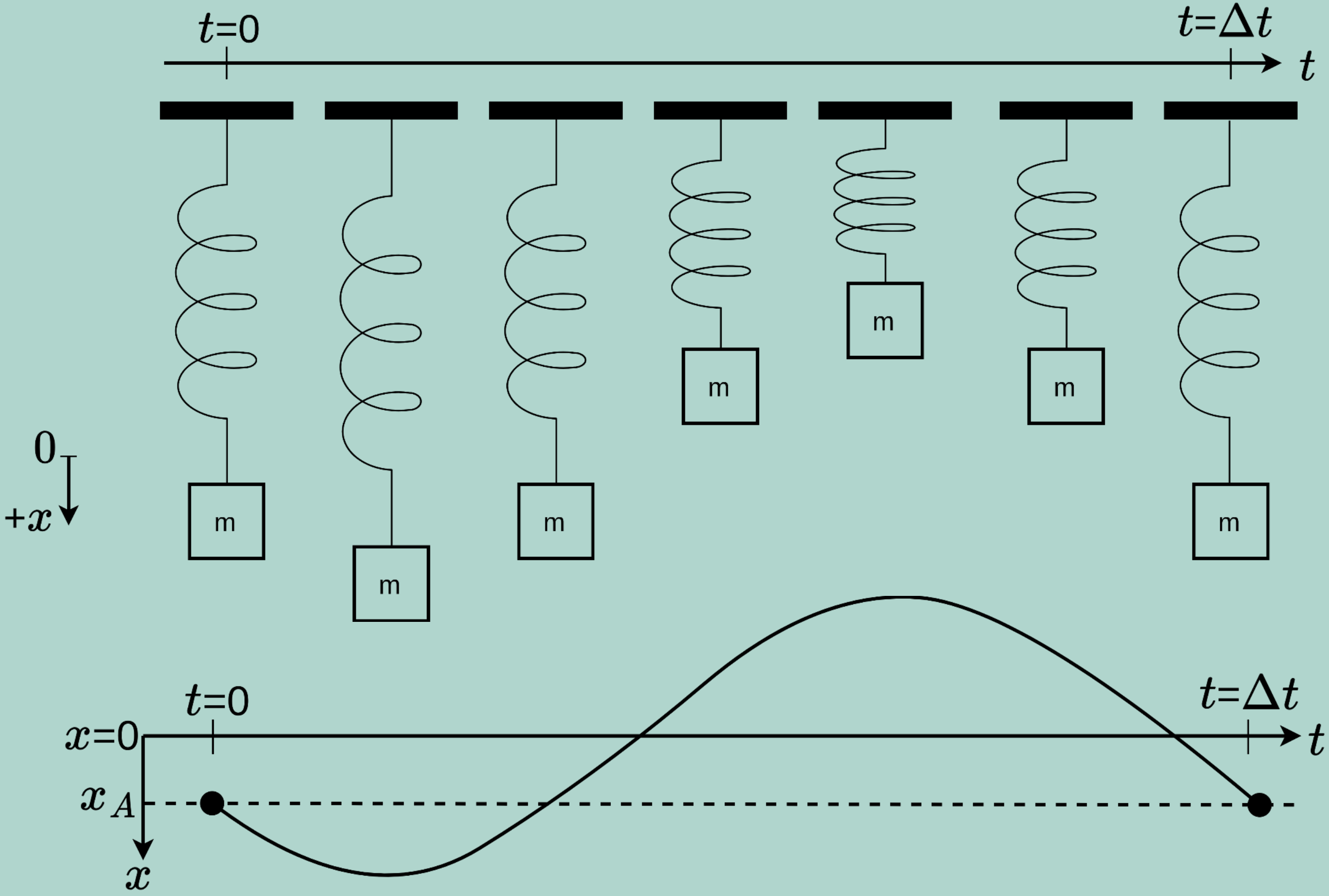

σ=0
σ=1
σ=2
σ=3
Case 3
Case 3
Case 3
ϕ
0
2π
4π
6π
Case 1
Case 2
Case 1
Case 2
Case 1
Case 2
Case 1

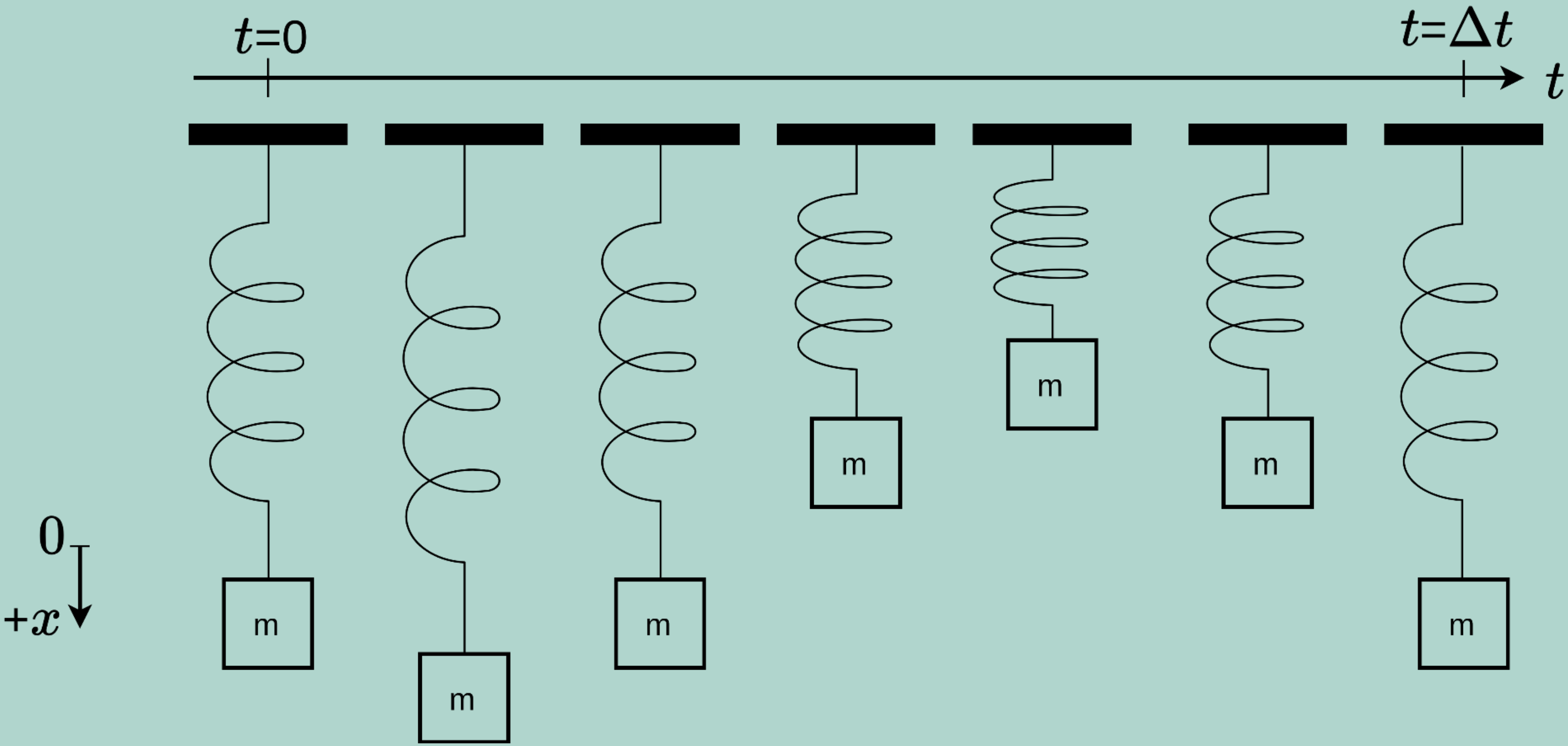
1D Spring Analysis – Action Calculation

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$

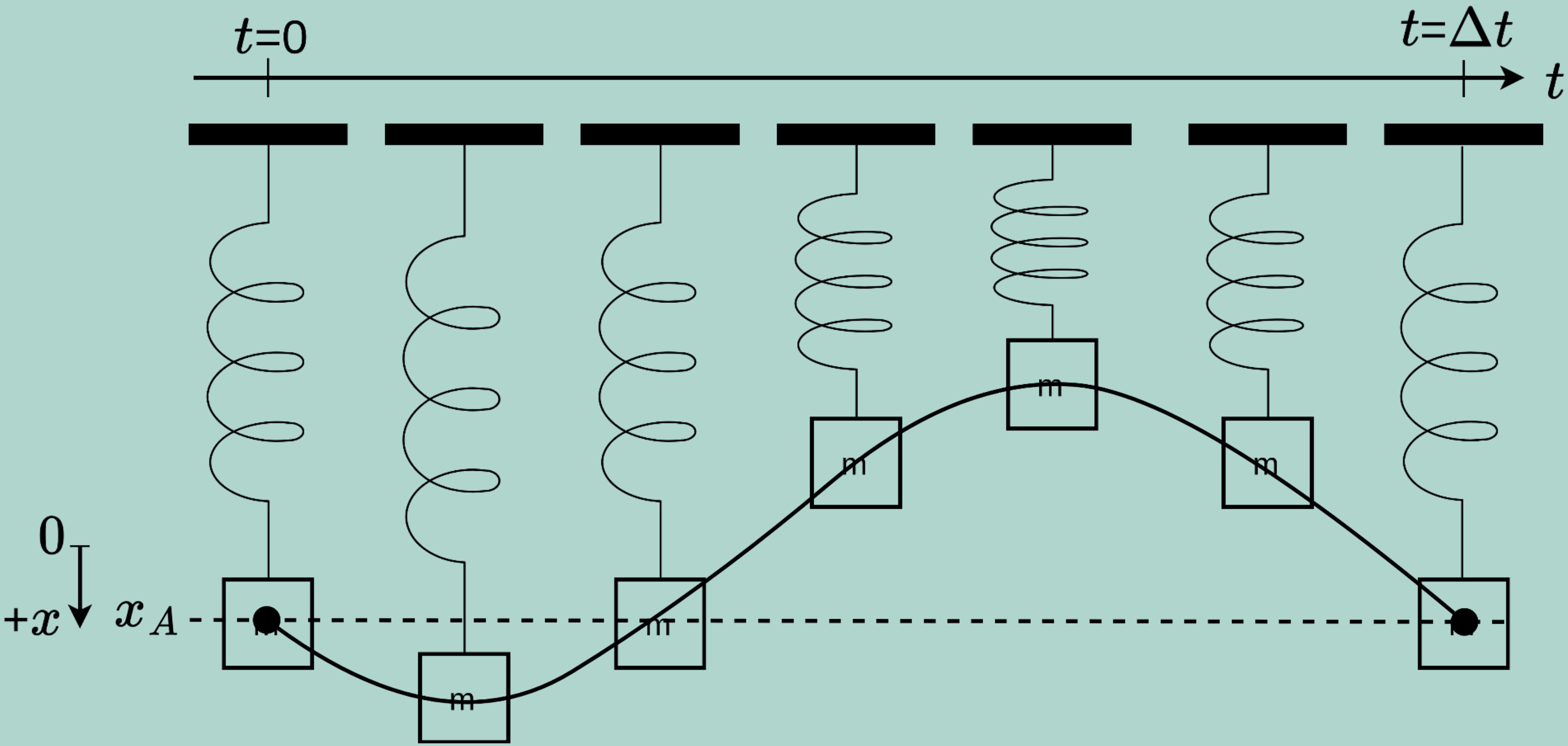


Visualizing a Path

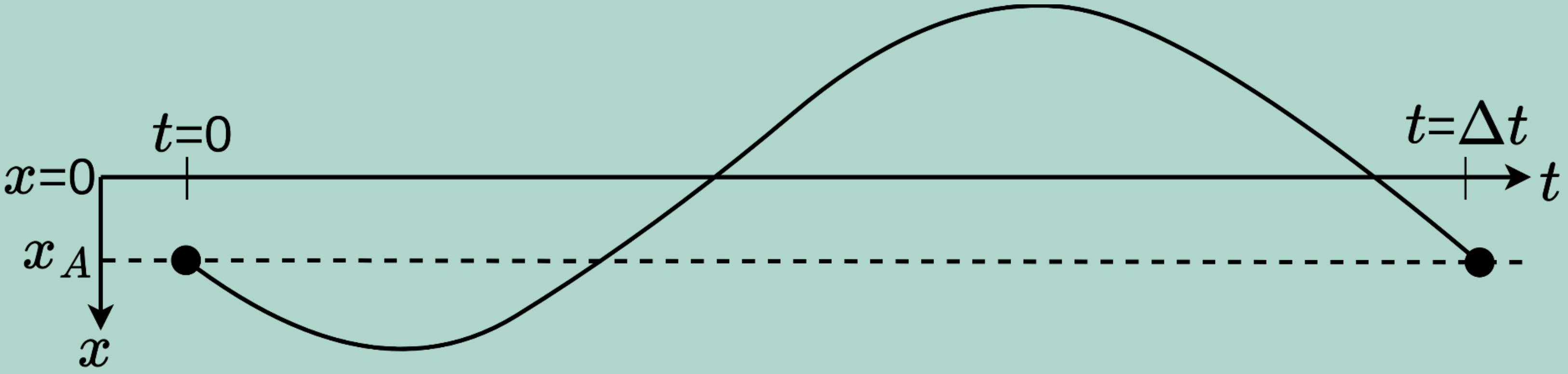




t=0
t=Δt
t
m
m
m
m
m
m
m
0
$+x$



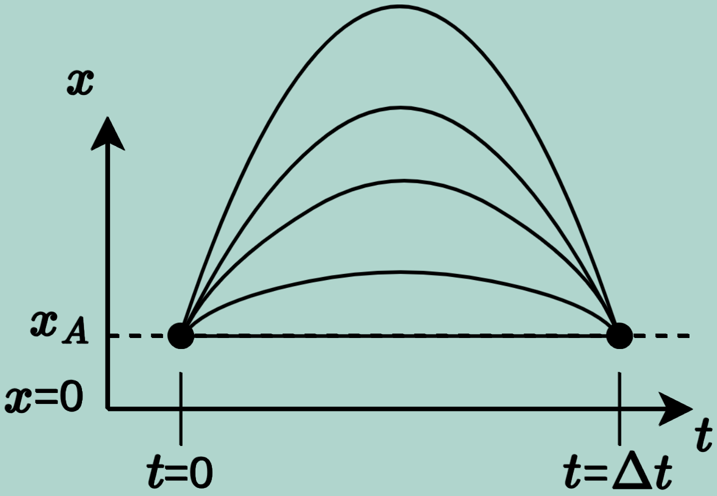
t=0
t=Δt
t
m
m
m
m
m
m
m
0
+x
x A

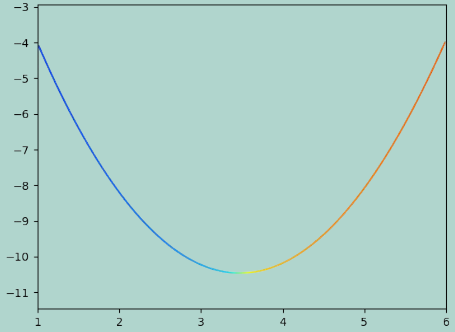


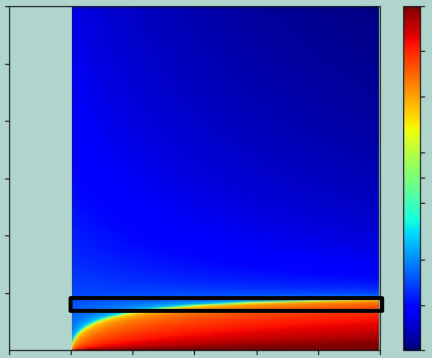
t=0
t=Δt
x=0
t
x_A
x

Action Calculation – Case 1 – Medium Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$

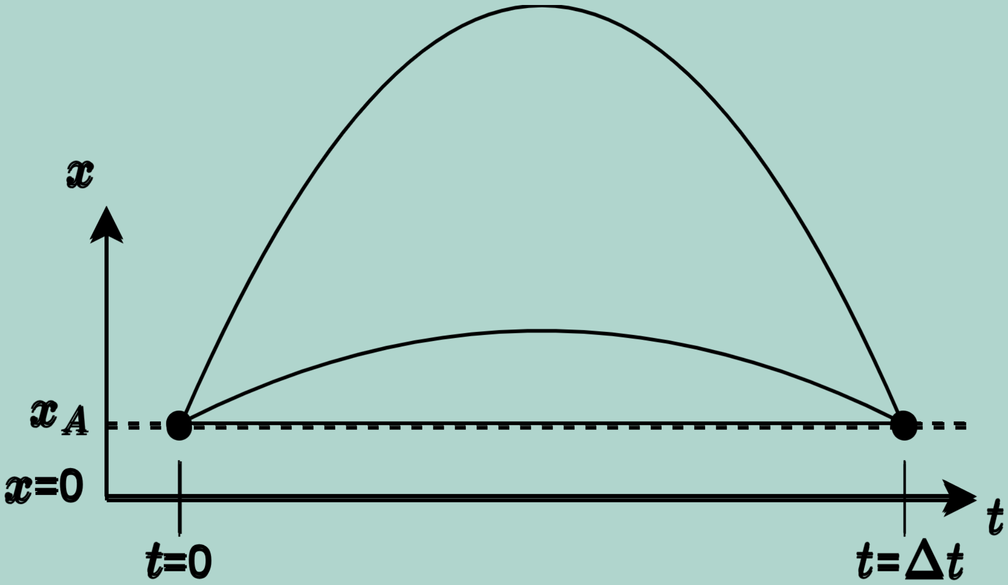


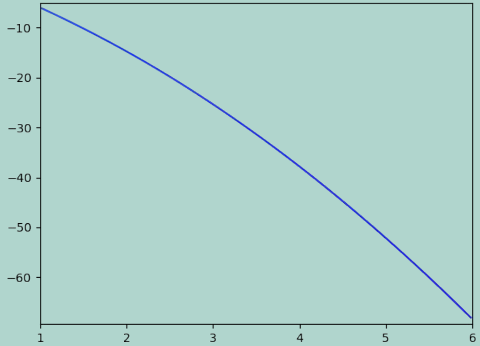


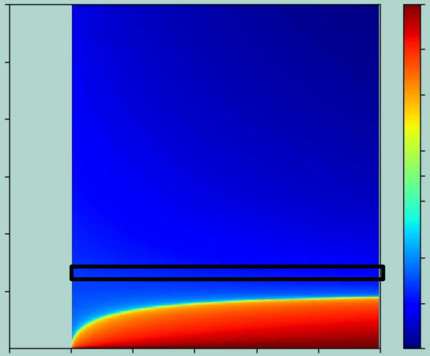


Action Calculation – Case 1 – Long Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$

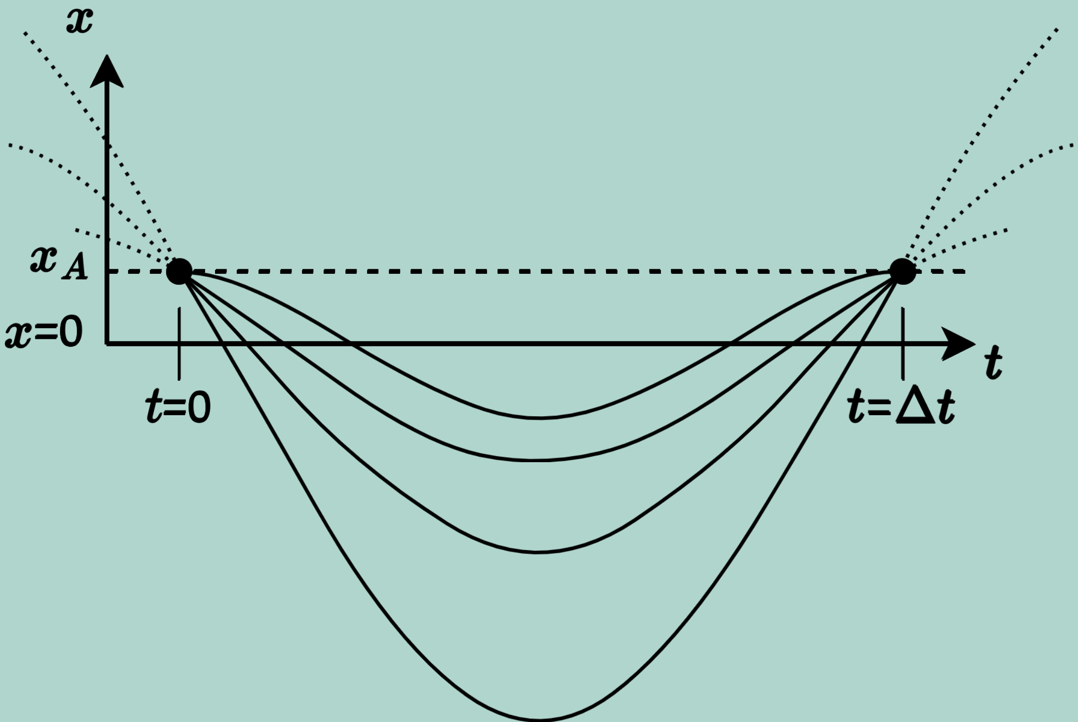


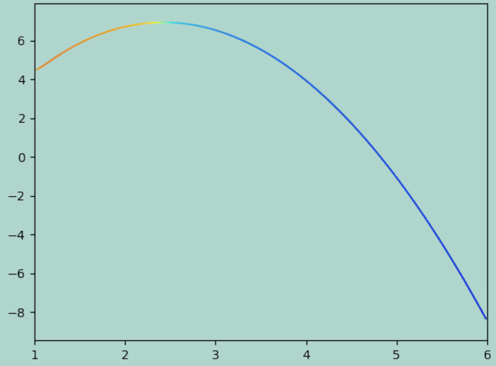


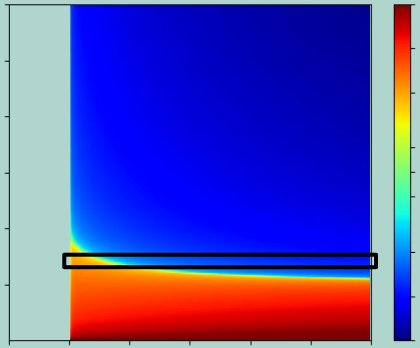


Action Calculation – Case 2

$$Action = \int_{t_i}^{t_f} \left(Kinetic\ Energy - Potential\ Energy\right) dt$$

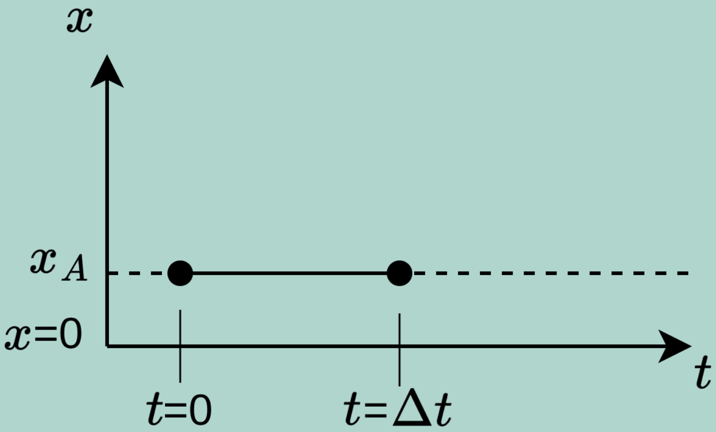


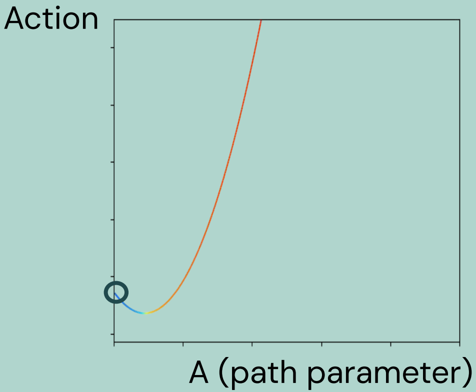


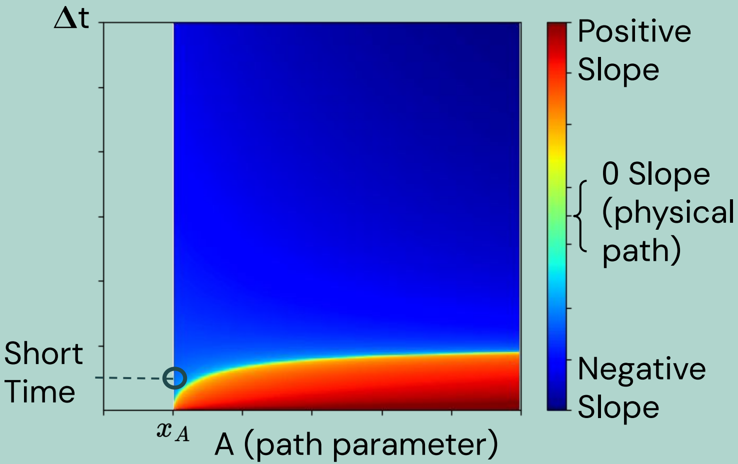


Action Calculation – Case 1 – Short Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big)dt$$

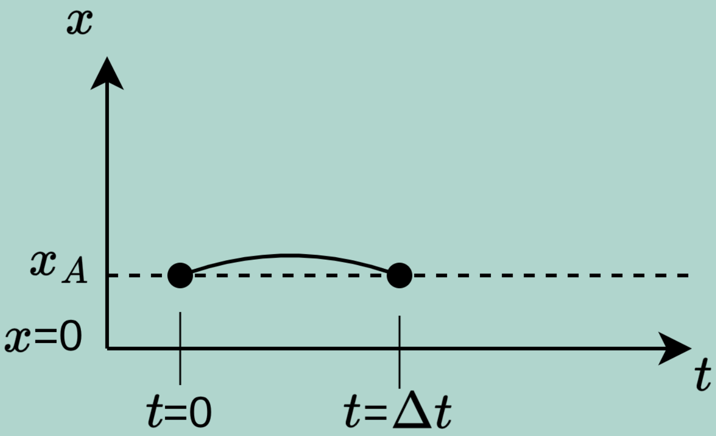


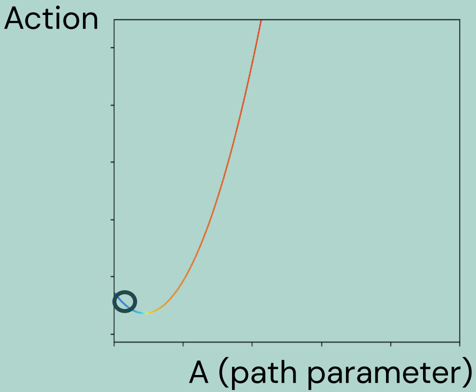


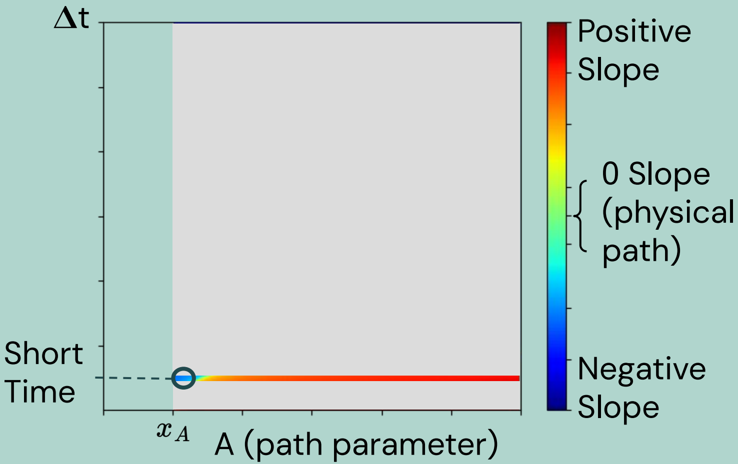


Action Calculation – Case 1 – Short Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big)dt$$

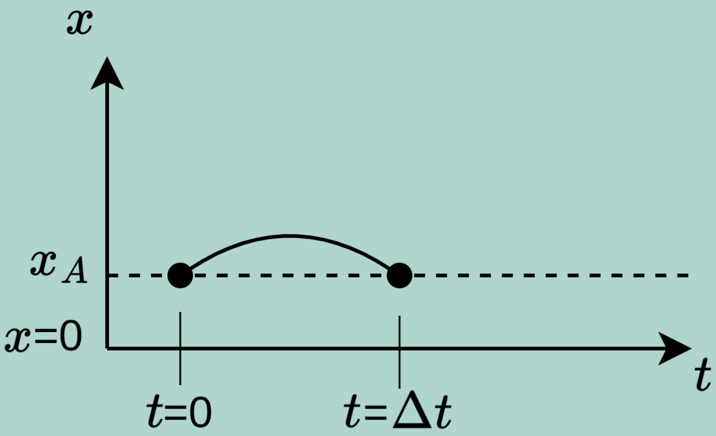


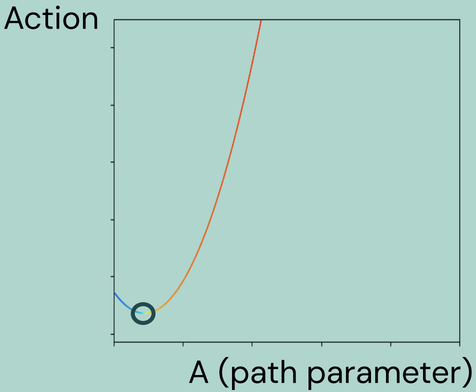


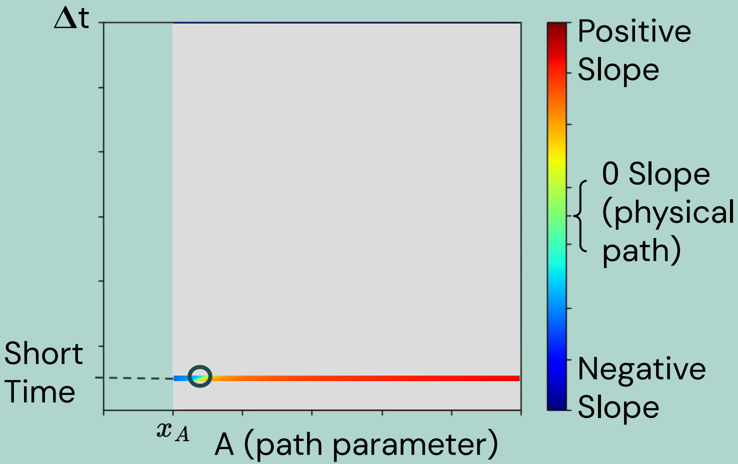


Action Calculation – Case 1 – Short Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy \Big) dt$$

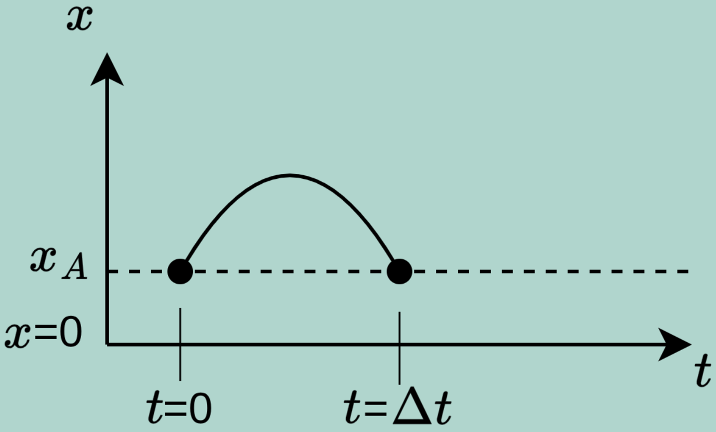


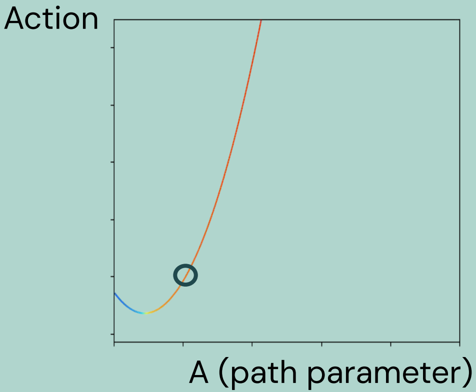


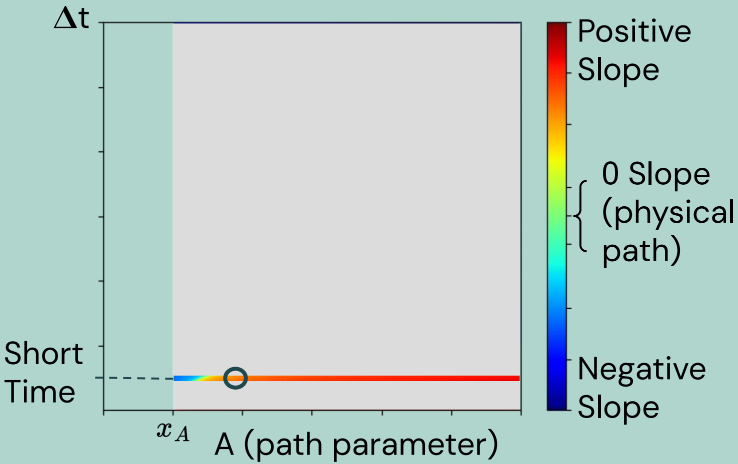


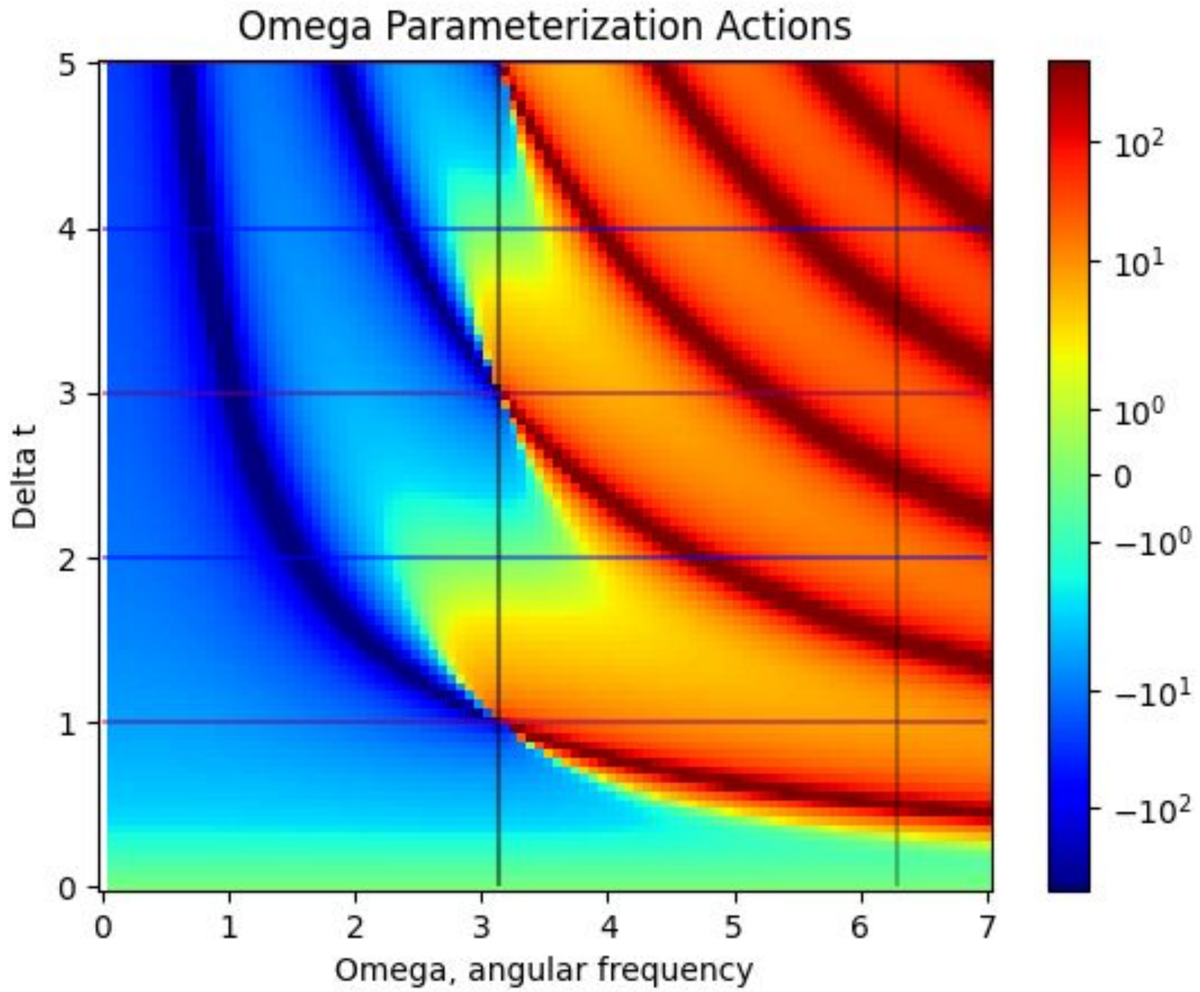
Action Calculation – Case 1 – Short Time

$$Action = \int_{t_i}^{t_f} \Big(Kinetic\ Energy - Potential\ Energy\Big) dt$$








Omega Parameterization Actions
Delta t
Omega, angular frequency
0
1
2
3
4
5
6
7
10^2
10^1
10^0
0
-10^0
-10^1
-10^2