Additional Novel Cosmological Models

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Outline

This is *Work in Progress*

Last year, we presented a vacuum cosmological *exact* solution to the Einstein Field Equations with the surprising property of being independent of the form of the cosmic scale factor: $a(t)$.

Today, we will look at three extensions of this first result.

- Vacuum Plus Cosmological Constant
- Pressureless Dust
- Dust Plus Cosmological Constant

Metric Ansatz

$$g_{\alpha\beta} = \text{diag}\big(-b^2 \,,\, a^2 \,,\, R^2 \,,\, R^2 \sin^2(\theta)\big)$$

with coordinates: $[t \,,\, r \,,\, \theta \,,\, \phi]$

Coefficient functions have dimension of length, coordinates are dimensionless

- $b = b(t \,,\, r)$ NOT $-b^2 = -1$ *à la* FLRW
- $a = a(t)$
- $R = R\big(t \,,\, r\big) = R\big(r\, a(t)\big)$

Trace-Reversed Einstein Field Equations

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R + g_{\alpha\beta}\, \Lambda = \kappa\, T_{\alpha\beta}$$

$$R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, T \right) + g_{\alpha\beta}\, \Lambda$$

- $a^2\, R_{tt} = \ldots = \frac{G\, a^2\, b^2}{2\, a^3} - a^2\, b^2\, \Lambda$
- $\ldots\, R_{tr} = \ldots = 0$
- $b^2\, R_{rr} = \ldots = \frac{G\, a^2\, b^2}{2\, a^3} + a^2\, b^2\, \Lambda$
- $R_{\theta\theta} = \ldots = \frac{G\, R^2}{2\, a^3} + R^2\, \Lambda$

$G = 0, \Lambda = 0$

- $R_{\alpha\beta} = 0$ Ricci Flat vacuum – sourceless
- ... solving ...
- For D a constant of integration (with dimension of length)
- $b(t, r) = D\,\frac{\dot{a}}{a}\,\bar{R} = D\,\frac{\dot{a}}{a}\sqrt{1 + \left(\frac{a\,r}{D}\right)^2}$
- $R(t, r) = \frac{D}{2}\left[\frac{a\,r}{D}\sqrt{1 + \left(\frac{a\,r}{D}\right)^2} + \sinh^{-1}\left(\frac{a\,r}{D}\right)\right] = R(r\,a(t))$
- $a(t) =$ unconstrained!

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- $a(t) =$ unconstrained!

Comments on the Vacuum Solution

- $R^{\alpha}_{\beta\gamma\delta} = 0$ Riemann Flat – Vacuum
- For $u = \frac{a\,r}{D}$, a dimensionless measure of distance

$$b(t\,,\,r) = b(a\,,\,u) = D\,\frac{\dot{a}}{a}\,\sqrt{1+u^2}$$

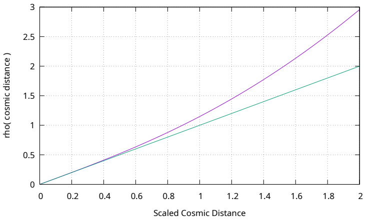
$$R(t\,,\,r) = R(u) = D\,u\,\sqrt{1+u^2}\left[\frac{1}{2} + \frac{\sinh^{-1}(u)}{2\,u\,\sqrt{1+u^2}}\right]$$

- $\lim_{u\to 0}\left[\ldots\right] = 1$ $\lim_{u\to\infty}\left[\ldots\right] = 1/2$

$R \sim D\,u \simeq a\,r$ $R \sim D\,u^2/2 \simeq a^2\,r^2/(2\,D)$

More Comments

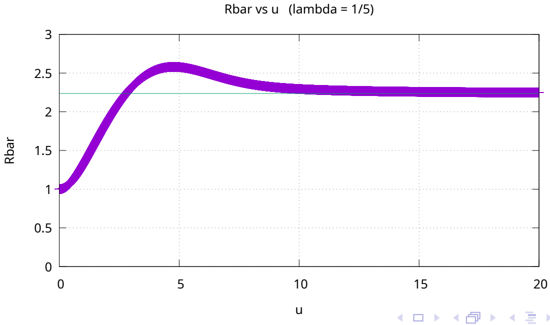
- In contrast to the standard approach to cosmology, there is no Friedmann Equation with which to determine the form of $a(t)$ – this is an exact solution of the Einstein vacuum field equations for any $a(t)$.
- Although it's a vacuum solution, the three-geometry is hyperbolic. There is an excess of spatial volume since $R > a\,r$.



Vacuum $G = 0$, Cosmological Constant $\Lambda \neq 0$

- $a(t) =$ unconstrained
- $b(t, u) = \frac{\dot{a}}{a} \bar{R}$
- $R(u)$

Implicit Solution $\frac{\bar{R}}{D} = \sqrt{1 + u^2 - \lambda \frac{R^2}{D^2}}$ $\lambda = \frac{\Lambda D^2}{3}$



Implications:

Parameter estimates:

$\lambda \sim 1.2 \times 10^{-52}\ \mathrm{m}^2 \qquad D \sim 2\ \mathrm{Gpc} \sim 6 \times 10^{25}\ \mathrm{m}$

$\implies \lambda \sim 1/5 \qquad \frac{1}{\sqrt{\lambda}} \sim \sqrt{5} \simeq 2.24$

For large u, $R(u) \sim \frac{u}{\sqrt{\lambda}} \propto \frac{r\,a}{\sqrt{\lambda}}$

As $u \to 0$, $\bar{R} = 1 \;\Rightarrow\; R \propto u \propto r\,a$

For intermediate values of u there is a transient from $1 \longrightarrow \frac{1}{\sqrt{\lambda}}$

For large u, $b \sim \frac{\dot{a}}{a}\frac{u}{\sqrt{\lambda}} \propto \dot{a}\,\frac{r}{\sqrt{\lambda}}$

As $u \to 0$, $b \sim \frac{\dot{a}}{a}$

For intermediate values of u there is a transient from $1 \longrightarrow \frac{1}{\sqrt{\lambda}}$

Dust $G \neq 0, \Lambda = 0$

Partial solution:

$$b = B(t)\,\bar{R} = \ldots = \frac{\dot{a}}{a}\sqrt{\frac{3\,a^3}{S^2\,D^2\,G + 3\,C\,a^3}}\;S\,D\,\frac{u}{\sqrt{u^2+k^2}}$$

$$R(u) = S\,D\sqrt{u^2 + k^2}$$

Integration constants: S, C, k^2, D

Applying the $\theta\theta$ EFE forces $S = 1$, $C = 0$, $k^2 = 0$. Hence,

- $a(t)$
- $b(t, u) = \sqrt{\frac{3\,a}{G}}\,\dot{a}$
- $R(u) = D\,u = a\,r$

More Comments

$$g_{\alpha\beta} = \mathrm{diag}\left(-\frac{3}{G}\, a\, \dot{a}^2\,,\; a^2\,,\; a^2\, r^2\,,\; a^2\, r^2\, \sin^2(\theta)\right)$$

The spatial part looks flat

Dust $G \neq 0, \Lambda \neq 0$

Same partial solution as for dust alone:

$$b = B(t)\,\bar{R} = \ldots = \frac{\dot{a}}{a}\sqrt{\frac{3\,a^3}{S^2\,D^2\,G + 3\,C\,a^3}}\,S\,D\,\frac{u}{\sqrt{u^2+k^2}}$$

$$R(u) = S\,D\,\sqrt{u^2 + k^2}$$

Integration constants: S, C, k^2, D

Applying the $\theta\theta$ EFE forces $S = 1$, $C = \frac{\Lambda\,D^2}{3} = \lambda$, $k^2 = 0$. Hence,

- $a(t)$
- $b(t,u) = \frac{\dot{a}}{a}\sqrt{\frac{3\,a^3}{D^2\,G + \Lambda\,D^2\,a^3}}\,D = \dot{a}\sqrt{\frac{3\,a}{G + \Lambda\,a^3}}$
- $R(u) = D\,u = a\,r$

More Thoughts

$$g_{\alpha\beta} = \mathrm{diag}\left(-\frac{3\,a}{G+\Lambda\,a^3}\,\dot{a}^2\,,\,a^2\,,\,a^2\,r^2\,,\,a^2\,r^2\,\sin^2(\theta)\right)$$

The spatial part looks flat

Possible Next Steps

- Confirm these solutions
- Check for generalisations
- Constrain $a(t)$ by appeal to data

Extend the analysis to include radiation as source