Generalized Uncertainty Principle in Non-Relativistic Quantum Field Theory

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2026 CAP Congress, June 22, 2026

Overview

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Introduction

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Source: Wikipedia

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  - Quantum Mechanics and Quantum Field Theory
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- Minimal length

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- Generalized Uncertainty Principle (GUP) (Ali, Das & Vagenas, 2009)

$$[x_i, p_j] = i\hbar\left(\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \beta\left(p^2\delta_{ij} + 3p_i p_j\right)\right)$$

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- For any length scales $\alpha_0\,\ell_P$ and $\sqrt{\beta_0}\,\ell_P$ between electroweak ($\ell_{EW} \approx 10^{-18}\,\mathrm{m}$) and Planck ($\ell_P \approx 10^{-35}\,\mathrm{m}$) scales

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- Apply GUP to quantum mechanical and gravitational systems
- Goals
  - Derivation of GUP from path integrals
  - Is there a version of GUP for non-relativistic QFT?

Introduction

- Path integrals in quantum mechanics
  - Configuration space

$$\langle x', t' | x, t \rangle = \int_{x(t)=x}^{x(t')=x'} \mathcal{D}x \, \exp\left(\frac{i}{\hbar} S[x]\right)$$

$$\mathcal{D}x = \lim_{N\to\infty} \frac{1}{\sqrt{\frac{i2\pi\hbar\Delta t}{m}}} \prod_{j=1}^{N-1} \frac{\mathrm{d}x_j}{\sqrt{\frac{i2\pi\hbar\Delta t}{m}}}, \text{ with } \Delta t = \frac{t'-t}{N}$$

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    $$\langle x', t' | x, t \rangle = \int_{x(t)=x}^{x(t')=x'} \mathcal{D}x \, \mathcal{D}p \, \exp\left(\frac{i}{\hbar} S[p, x] \right)$$
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  - Feynman-Matthews-Salam (FMS) formula
    $$\langle x', t' | T[F[\hat{x}]] | x, t\rangle = \int_{x(t)=x}^{x(t')=x'} \mathcal{D}x \, F[x] \exp\left(\frac{i}{\hbar} S[x]\right) = \langle F[x]\rangle_S$$

Introduction

- Path integrals in non-relativistic QFT

$$\langle \psi', t_f | \psi', t_i \rangle = \int \mathcal{D}\psi^* \mathcal{D}\psi \exp\left(\frac{i}{\hbar} S[\psi^*, \psi]\right) \exp\left(\int \mathrm{d}^3\mathbf{r}\, \psi_f'^*(\mathbf{r}) \psi(\mathbf{r}, t_f)\right)$$

$$S[\psi^*, \psi] = \int_{t_i}^{t_f} \mathrm{d}t \left(i \int \mathrm{d}^3\mathbf{r}\, \psi^*(\mathbf{r}, t) \frac{\partial}{\partial t} \psi(\mathbf{r}, t) - H + \mu N + \mathbf{v} \cdot \mathbf{P} \right)$$

  - $H = \int \mathrm{d}^3\mathbf{r}\, \psi^*(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} + V_1(\mathbf{r}) \right) \psi(\mathbf{r}) + H_{int}$
  - $N = \int \mathrm{d}^3\mathbf{r}\, \psi^*(\mathbf{r}) \psi(\mathbf{r})$
  - $\mathbf{P} = -i\hbar \int \mathrm{d}^3\mathbf{r}\, \psi^*(\mathbf{r}) \nabla \psi(\mathbf{r})$

Overview

Example in standard quantum mechanics

- GUP modification of the Hamiltonian

$$H_{GUP}(x,p) = \frac{p^2}{2m} - \frac{\alpha}{m}p^3 + \frac{5\beta}{2m}p^4 + V(x)$$

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  - $\dot{q} = \dot{x} = \frac{\partial H_{GUP}}{\partial p} = \frac{p}{m} - \frac{3\alpha}{m}p^2 + \frac{10\beta}{m}p^3$
  - $p = m\dot{x} + 3\alpha m^2\dot{x}^2 + (18\alpha^2 + 10\beta)m^3\dot{x}^3$

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$$L_{GUP}(x,\dot{x}) = \frac{m\dot{x}^2}{2} + \alpha m^2\dot{x}^3 + \left(\frac{9}{2}\alpha^2 - \frac{5}{2}\beta\right)m^3\dot{x}^4 - V(x)$$

Example in standard quantum mechanics

- The Schwinger-Dyson equation

$$\left\langle \frac{\delta F}{\delta x_k} \right\rangle_S = -\frac{i}{\hbar} \left\langle F \frac{\delta S}{\delta x_k} \right\rangle_S$$

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- Arrive at the GUP commutator in 1D, using FMS formula

$$\boxed{[x, p] = i\hbar(1 - 2\alpha p + 4\beta p^2)}$$

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GUP in non-relativistic QFT

- GUP modification

  $\mathbf{P} = \mathbf{P}_0 \left(1 - \alpha P_0 + 2\beta P_0^2\right)$, where $P_0 = \sqrt{P_{0i} P_{0i}}$

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- The GUP modifications in NR-QFTs appear only in the case where there is some intrinsic motion in the system, i.e., the term $\mathbf{v} \cdot \mathbf{P}$

- We define $\mathbf{X} = \int \mathrm{d}^3\mathbf{r}\, \mathbf{r}\, \psi^*(\mathbf{r})\psi(\mathbf{r})$ and set $\mathbf{X} = \mathbf{X}_0$

- As operators (FMS formula), $\hat{\mathbf{X}}_0$ and $\hat{\mathbf{P}}_0$ satisfy $[\hat{X}_{0i}, \hat{P}_{0j}] = i\hbar\delta_{ij}\hat{N}$

GUP in non-relativistic QFT

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- GUP modified commutator algebra

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{R})] = \delta^{(3)}(\mathbf{r} - \mathbf{R}) - \alpha i\hbar \hat{\psi}^\dagger(\mathbf{r}) \left(\nabla \hat{\psi}(\mathbf{R}) \cdot \hat{\mathbf{P}}_0 \right) \hat{P}_0^{-1} + 4\beta i\hbar \hat{\psi}^\dagger(\mathbf{r}) \left(\nabla \hat{\psi}(\mathbf{R}) \cdot \hat{\mathbf{P}}_0 \right)$$

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{R})] = 0 \quad \text{and} \quad [\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{R})] = 0$$

Overview

Phenomenology in statistical mechanics

- Partition function

$$Z = \int \mathcal{D}\psi'^* \mathcal{D}\psi' \langle \psi', \beta | \psi', 0 \rangle_E, \quad (E \to \text{Euclidean: } t \to -i\hbar\beta)$$

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$$g(\epsilon) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Z(\beta) e^{\beta\epsilon} \mathrm{d}\beta = \frac{V(2m)^{3/2}\varepsilon^{1/2}}{4\pi^2\hbar^3}(1 + 16\alpha\sqrt{m}\,\varepsilon^{1/2} - 25\beta m\varepsilon)$$

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- Observables in statistical mechanics

$$\langle Y \rangle = \int_0^\infty Y(\varepsilon)g(\varepsilon)f_{BE(FD)}(\varepsilon)\mathrm{d}\varepsilon$$

Phenomenology in statistical mechanics

Bose-Einstein condensate

- Number density of Bose-Einstein condensate

$$n = \frac{1}{V}\int_0^\infty g(\epsilon) f_{BE}(\varepsilon)\,\mathrm{d}\epsilon$$

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$$n = \frac{1}{V}\int_0^\infty g(\epsilon) f_{BE}(\varepsilon)\,\mathrm{d}\epsilon$$

- Critical temperature

$$T_c = \frac{2\pi\hbar^2}{k_B m \zeta(\frac{3}{2})^{2/3}} n^{2/3} - \alpha \frac{32\sqrt{8}\pi^3\hbar^3}{9k_B m\zeta(\frac{3}{2})^2} n + \beta \frac{100\pi^2\hbar^4\zeta(\frac{5}{2})}{k_B m\zeta(\frac{3}{2})^{7/3}} n^{4/3}$$

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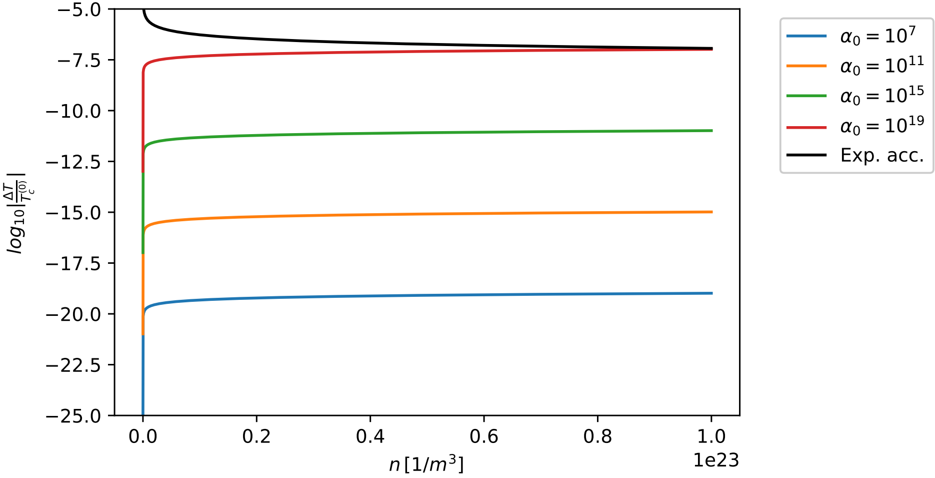
- Fraction of bosons in the ground state

$$f_0 = 1 - \left(\frac{T}{T_c}\right)^{3/2} + \alpha \frac{16\pi^{3/2}}{3\zeta(\frac{3}{2})}\sqrt{m k_B}\left[\frac{T^{3/2}}{T_c} - \frac{T^2}{T_c^{3/2}}\right] - \beta \frac{75}{2}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} m k_B \left[\frac{T^{3/2}}{T_c^{1/2}} - \frac{T^{5/2}}{T_c^{3/2}}\right]$$

Phenomenology in statistical mechanics

Bose-Einstein condensate



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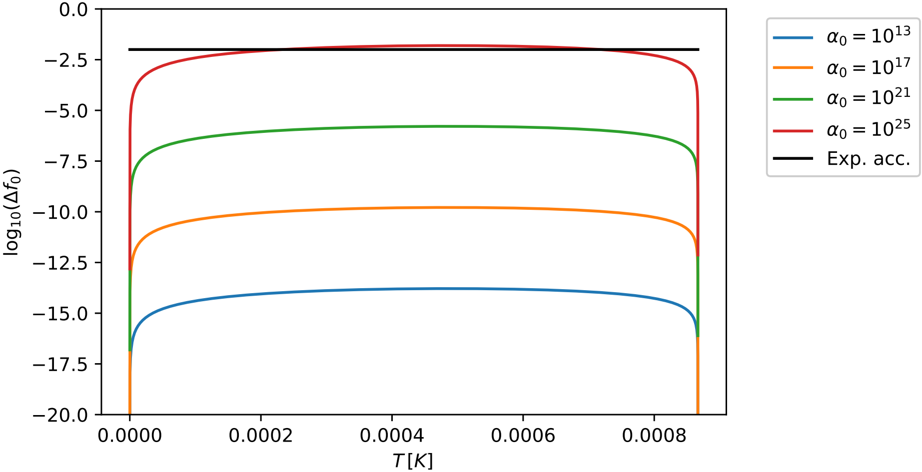

Overview

Conclusion

- Derivation of modified uncertainty relations from the path integral formalism
- Novel field operator algebra modifications
- Such modifications can be widely used in condensed matter physics and statistical mechanics to search for quantum gravity signatures
- Consideration of relativistc BECs in Das & Fridman, 2021
- Constraints $\alpha_0 < 10^{19}$ and $\beta_0 < 10^{46}$ weaker than the electro-weak scale
- With increasing experimental precision, BECs are a viable way to test quantum gravity signatures

Thank you for your attention!