Fractional Skyrmions ?

by

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Lemma 1.1. *Let X be connected simple topological space. Let $j \geq 1$ be an integer. Endow the j-sphere S^j with a basepoint s. Suppose $f, g : S^j \to X$ are two maps, and γ is a path from $f(s)$ to $g(s)$. Then f is homotopic to g if and only if $\psi_\gamma([f]) = [g] \in \pi_j(X, g(s))$.*

1.2. **A representation.** Suppose $n \geq 2$. The adjoint representation $\mathrm{SU}(n) \to \mathrm{SL}(n^2-1; \mathbf{C})$ is given by letting $A \in \mathrm{SU}(n)$ act on the complex n^2-1-dimensional vector space of $n \times n$ trace-0 matrices by conjugation:

$$A \cdot M = A^\dagger M A.$$

The kernel of this representation is the centre $Z(\mathrm{SU}(n)) \cong \mathbf{Z}/(n)$, so that we obtain a faithful representation of $\mathrm{PU}(n)$.

It is a general fact that since $\mathrm{SU}(n)$ is compact, the image of the representation lies in a maximal compact subgroup of $\mathrm{SL}(n^2-1; \mathbf{C})$. Such subgroups are the subgroups that preserve complex inner products. In this case, we can identify one such complex inner product on the space of $n \times n$ trace-0 matrices:

$$\langle M, N \rangle = \mathrm{Tr}(MN^\dagger). \tag{1}$$

(To see that this is positive-definite, observe that if $M \neq 0$, then $MM^\dagger$ is a hermitian matrix having at least one positive entry on the diagonal).

We observe that the action of A also preserves hermitian matrices, i.e., those for which $M^\dagger = M$. The space of trace-0 hermitian $n \times n$-matrices is a real vector space of dimension n^2-1, and the restriction of (3) to this vector space is a positive-definite inner product. Therefore, our representation $\mathrm{PU}(n) \to \mathrm{SU}(n^2-1)$ factors as

$$\mathrm{PU}(n) \xrightarrow{\phi_n} \mathrm{SO}(n^2-1) \xhookrightarrow{\text{incl.}} \mathrm{SU}(n^2-1). \tag{2}$$

We will write ϕ for ϕ_n when n is clear from the context.

Later we will want to know that ϕ_n is the restriction to maximal compact subgroups of a homomorphism of complex algebraic groups. This is addressed in Section 4.

Topological Charge and Skyrmions

- A Skyrmion corresponds to a field configuration defined on space, $\mathbb{R}^3$, that is of finite energy and stable and consequently goes to a constant at infinity.
- The field configuration has a non-trivial, topological "quantum" number, which is ultimately responsible for its stability.
- The field takes values in an interesting "target space", T, the Skyrmion corresponds to of maps: $\mathbf{S}^3 \to T$
- This defines the homotopy group $\Pi_3(T)$

Usual Skyrmions

- The usual Skyrmions correspond to maps:

$$\mathbf{S}^3 \to (SU_L(2) \times SU_R(2))/SU_V(2) = SU(2)$$

- A possible Lagrangian that defines dynamics, energy, etc. is given by:

$$\mathcal{L} = -f_\pi Tr\left(U^\dagger \partial_\mu U U^\dagger \partial^\mu U\right) + \lambda Tr\left([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U][U^\dagger \partial^\mu U, U^\dagger \partial^\nu U]\right)$$

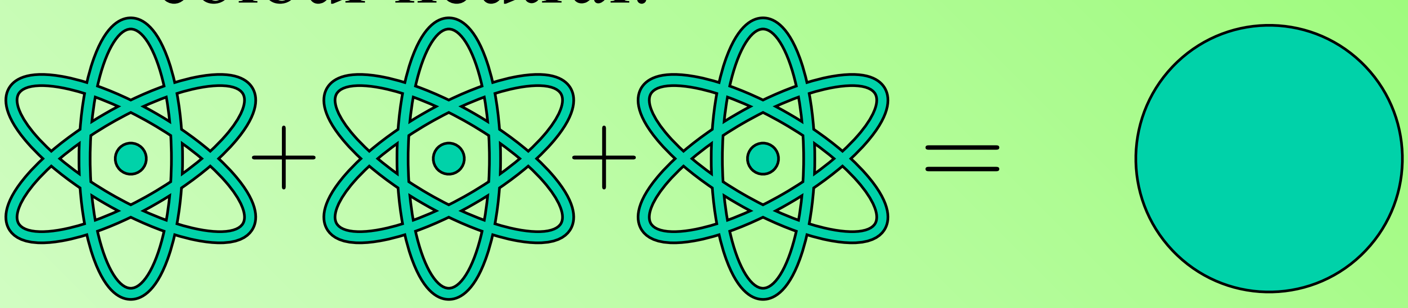
$$\Pi_3(SU(2)) = \mathbb{Z}$$

Fractional Skyrmions

- What if $\Pi_3(T)$ was more interesting.
- We considered the possibility that

$$\Pi_3(T) = \mathbb{Z}_3$$

- In such a case, combining three such Skyrmions together would give a trivial configuration. If the quantum number was 1/3, then it would be something analogous to colour confinement, three quarks can be colour neutral.



Some mathematics

For monopoles we have the exact sequence:

$$0 = \Pi_2(G) \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) = 0$$

which means that:

$$\Pi_2(G/H) = \Pi_1(H)$$

We need a similar identification but for one higher dimension

$$\Pi_3(G) \to \Pi_3(G/H) \to \Pi_2(H) \to \Pi_2(G)$$

- Usually,

$$\Pi_2(G) = \Pi_2(H) = 0$$

- But the exact sequence extended to the left gives:

$$\Pi_3(H) \to \Pi_3(G) \to \Pi_3(G/H) \to \Pi_2(H) \to \Pi_2(G)$$

- and $\quad \Pi_3(H) = \Pi_3(G) = \mathbb{Z}$
- so we get

$$\mathbb{Z} \to \mathbb{Z} \to \Pi_3(G/H) \to 0 \to 0$$

- The arrows are group homomorphisms. With a little thought, you should be able to convince yourself that the only group homomorphisms from the integers to the integers simply correspond to multiplication by a fixed integer, called the Dynkin index.

- But then we have the following, the first homomorphism corresponds to multiplication by a fixed integer j
- all integers of the form nj are mapped to the identity by the next map as the image of the previous map is the kernel of the next map
- then where do $nj+1, \quad nj+2,$ etc get mapped.
- it becomes clear that the homotopy group

$$\Pi_3(G/H) = \mathbb{Z}_j$$

- There is no other possibility and therefore $\mathbb{Z}_j$ Skyrmions exist!

Constructing Fractional Skyrmions

- We consider the following possibility for $\mathbb{Z}_N$
- Skyrmions. We take

$$H = SU(N)_{adj.} = SU(N)/\mathbb{Z}_N \equiv PU(N)$$

- and $G = SO(N^2 - 1)$
- The point is that $SU(N)$ has $N^2 - 1$ generators so the adjoint representation fits nicely into the orthogonal group.
- But note $\Pi_1(PU(N)) = \mathbb{Z}$ which we will use later.

Indirect construction of $\mathbb{Z}_3$ Skyrmions

- It is easy to construct an $SO(8)$ Skyrmion. This corresponds to a non trivial smooth map $\mathbb{R}^3 \to g(r,\theta,\phi) \in SO(8)$
- Then take the vector representation $\mathcal{U}_V(g(r,\theta,\phi))$ and compute
$$\mathcal{I} = \frac{1}{24\pi^2}\int d^3x \epsilon^{ijk} Tr\left(\mathcal{U}^\dagger \partial_i \mathcal{U}\mathcal{U}^\dagger \partial_j \mathcal{U}\mathcal{U}^\dagger \partial_k \mathcal{U}\right)$$
- One finds $\mathcal{I} = 2N, \quad N \in \mathbb{Z}$
- Changing $g(r,\theta,\phi) \to g \cdot h(r,\theta,\phi)$
- One gets $\mathcal{I} \to 2N + 6M, \;\; M \in \mathbb{Z}$

Hints from magnetic monopoles

- It is known how to make Skyrmions from magnetic monopoles (sort of !).
- It is also known how to make $\mathbb{Z}_3$ monopoles.
- Monopoles exist because of a slightly different topological analysis. They form in the spontaneous breaking of a group: $G \to H$
- Their topological stability is due to the homotopy group: $\Pi_2(G/H)$
- The 'Hooft-Polyakov monopole corresponds to

$$G = SU(2) \qquad H = U(1)$$
$$G/H = SU(2)/U(1) = \mathbf{S}^2$$

- From the “short exact sequence”:

$$0 = \Pi_2(G) \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) = 0$$

$$0 = \Pi_2(SU(2)) \to \Pi_2(SU(2)/U(1)) \to \Pi_1(U(1)) \to \Pi_1((SU(2)) = 0$$

- it follows that:

$$\Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$$

- Then for monopoles we ask what group has:

$$\Pi_1(H) = \mathbb{Z}_3$$

- Indeed:

$$\Pi_1(SU(3)_{\mathrm{adj.}}) = \mathbb{Z}_3$$

- $SU(3)_{\text{adj.}}$ is 8 dimensional, it sits inside $SO(8)$
- Can we spontaneously break SO(8) to SU(3)?

$$SO(8) \to SU(3)_{\text{adj.}}$$

- Then we would look for non-trivial homotopy group $\Pi_2(SO(8)/SU_{\text{adj.}}(3)) = \mathbb{Z}_3$ in the case of monopoles and $\Pi_3(SO(8)/SU_{\text{adj.}}(3)) = \mathbb{Z}_3$ Skyrmions.
- Do they exist? The answer is of course yes! The construction uses some special properties of representations of $SO(8)$
- There are three, inequivalent, irreducible 8 dimensional representations of $SO(8)$ call them

$$U_V \quad U_L \quad U_R$$

Skyrmions from monopoles

- the ‘Hooft-Polyakov monopole consists of a hedgehog scalar field at spatial infinity: $\vec{\phi} = v\hat{r}$
- This field is obtained by the action of a rotation on the scalar field which points in the 3 direction at the north pole $\vec{\phi} = vR(\theta,\varphi)\hat{z}$
- This can be written as $\vec{\phi}\cdot\vec{\tau} = vU^\dagger(\theta,\varphi)(\hat{z}\cdot\vec{\tau})U(\theta,\varphi)$
- where $U(\theta,\varphi) = e^{i\theta\hat{\varphi}\cdot\vec{\tau}/2}$
- this matrix is singular at the south pole, however, it is completely inside the unbroken subgroup there.
- We will exploit this artefact in our construction.

- This data can be used to construct a Skyrmion by:

$$\mathcal{U}(r,\theta,\varphi) = U^{\dagger}(\theta,\varphi)e^{iv(r)(\hat{z}\cdot\vec{\tau})}U(\theta,\varphi) = e^{iv(r)(\hat{r}\cdot\vec{\tau})}$$

- with the condition

$$v(0) = 0 \quad v(\infty) = \pi$$

- Note that at the south pole, $U(\theta,\varphi)$ is singular, and defines a non-trivial element of $\Pi_1(H) = \Pi_1(U(1)) = \mathbb{Z}$.

Skyrmion winding number

- There is a topological invariant associated to this configuration called the winding number $\mathcal{I}$

$$\mathcal{I} = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} Tr \left(\mathcal{U}^\dagger \partial_i \mathcal{U} \mathcal{U}^\dagger \partial_j \mathcal{U} \mathcal{U}^\dagger \partial_k \mathcal{U} \right)$$

- This integral is always an integer and is invariant under arbitrary deformations of $\mathcal{U}$ that are continuous and keep the boundary conditions at infinity.
- The integral is equal to 1 for the previous configuration.

$\mathbb{Z}_3$ Skyrmions

- We will perform the analogous manipulations to convert $\mathbb{Z}_3$ monopoles to $\mathbb{Z}_3$ Skyrmions!
- The monopoles are constructed through an $SO(8)$ group element defined on the sphere at infinity:

$$g(\theta,\varphi) = U_V(\theta,\varphi)$$

- The three 8 dimensional representations are special that they can be taken to be identical on an $SU_{\text{adj.}}(3)$ subgroup: $U_V(h) = U_L(h) = U_R(h) \qquad h \in SU_{\text{adj.}}(3)$
- Then the product $\mathcal{U}(g) = U_L(g)U_R^\dagger(g)$ is actually an element of the coset space $SO(8)/SU_{\text{adj.}}(3)$

$$\mathcal{U}(gh) = U_L(g)U_L(h)U_R^\dagger(h)U_R^\dagger(g) = U_L(g)U_R^\dagger(g)$$

- The analogous data for a $\mathbb{Z}_3$ monopole is

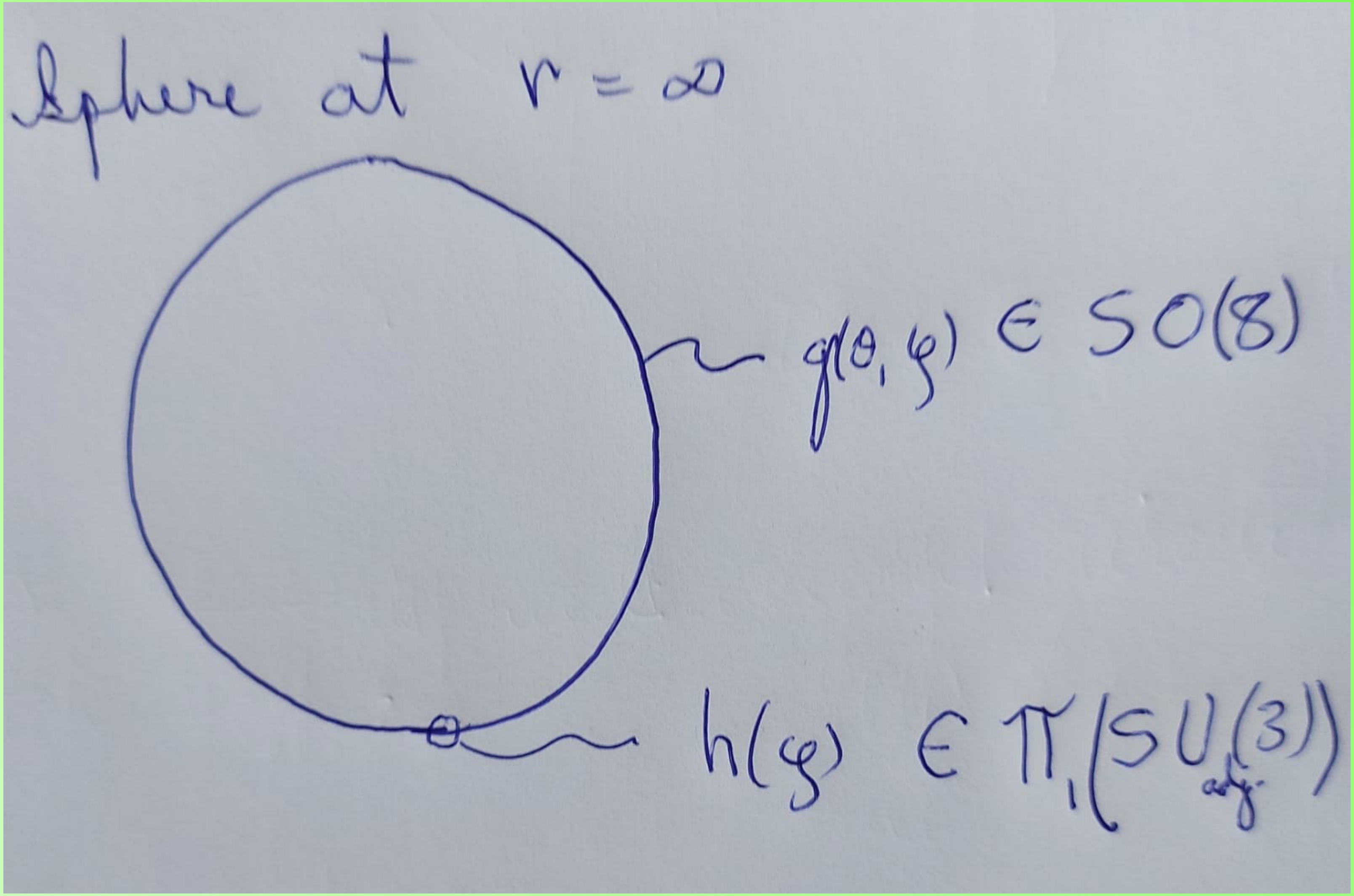
$$\mathcal{U}(g(\theta,\varphi)) = U_L(g(\theta,\varphi))U_R^\dagger(g(\theta,\varphi))$$

- where $g(\theta,\varphi)$ is defined on the sphere at spatial infinity such that $g(\theta,\varphi) \to h(\varphi)$ and $h(\varphi)$ corresponds to a loop around the z - axis at the south pole that is non-trivial in $\Pi_1(SU_{\mathrm{adj.}}(3)) = \mathbb{Z}_3$
- As all representations are identical on $SU_{\mathrm{adj.}}(3)$ we have

$$\mathcal{U}(h(\varphi)) = U_L(h(\varphi))U_R^\dagger(h(\varphi)) = \mathbb{I}$$

- Now we have to extend this data into the interior to construct the corresponding Skyrmion

monopole data



Skyrmions from the monopole

- To make the Skyrmion we do the analogous construction to the one done for the 'Hooft-Polyakov monopole and make an $SO(8)$ Skyrmion

$$\mathcal{U}(r,\theta,\varphi) = U_L(\theta,\varphi)V(r)U_R^\dagger(\theta,\varphi)$$

- However with $V(r)$ designed so that

$$\mathcal{U}(0,\theta,\varphi) = const. \quad \mathcal{U}(\infty,\theta,\varphi) = const.$$

- This can be explicitly constructed and we obtain, depending on the non-trivial loop at the south pole, a topologically stable Skyrmion one to one with the group $\mathbb{Z}$

$\mathbb{Z}_3$ Skyrmions

- to construct the Skyrmions, we simply think of elements of $SO(8)$:

$$\mathcal{U}(r,\theta,\varphi) = U_L(\theta,\varphi)V(r)U_R^\dagger(\theta,\varphi) = \tilde{\mathcal{U}}_V(g(r,\theta,\varphi))$$

- Then to obtain the corresponding element of the coset space $SO(8)/SU_{\text{adj.}}(3)$ we construct the corresponding elements of the other two 8 dimensional representations $\tilde{\mathcal{U}}_L(g(r,\theta,\varphi))$, $\tilde{\mathcal{U}}_R(g(r,\theta,\varphi))$ and construct

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_L(g(r,\theta,\varphi))\tilde{\mathcal{U}}_R^\dagger(g(r,\theta,\varphi))$$

$\mathbb{Z}_3$ -Skyrmion specifics

- We use an $SU(2)$ subgroup sitting inside $SO(8)$ to construct explicit configurations

$$T^i \ni [T^i, T^j] = i\epsilon^{ijk} T^k$$

$$T^1 = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&i&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad T^2 = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&i&0&0&0\\ 0&0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&i&0&0&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\mathcal{U}(r,\theta,\varphi) = U_L(\theta,\varphi)V(r)U_R^\dagger(\theta,\varphi) = \tilde{\mathcal{U}}_V(g(r,\theta,\varphi))$$

- T^2 is an element of $SU(3)$ while T^1 and T^3 are not.
- $V(r) = e^{i\zeta(r)T^2}$ goes from the $-\mathbb{I}$ to $\mathbb{I}$ as $r : 0 \to \infty$
- this gives the constant values

$$\mathcal{U}(r=0) = -e^{i(4\pi/3)T^2}$$

$$\mathcal{U}(r=\infty) = e^{i(4\pi/3)T^2}$$

$\mathbb{Z}_3$ -Skyrmion specifics

- The fields correspond to $\mathcal{U}_V = e^{i\varphi T^2} e^{i(\pi-\theta)T^3} e^{i\varphi T^2}$ with corresponding $\mathcal{U}_L$ and $\mathcal{U}_R$
- With $V(r) = e^{i\zeta(r)T^2}$ and $\zeta(0) = \frac{-2\pi}{3}, \quad \zeta(\infty) = \frac{4\pi}{3}$
- Then we construct the $SO(8)$ Skyrmion as

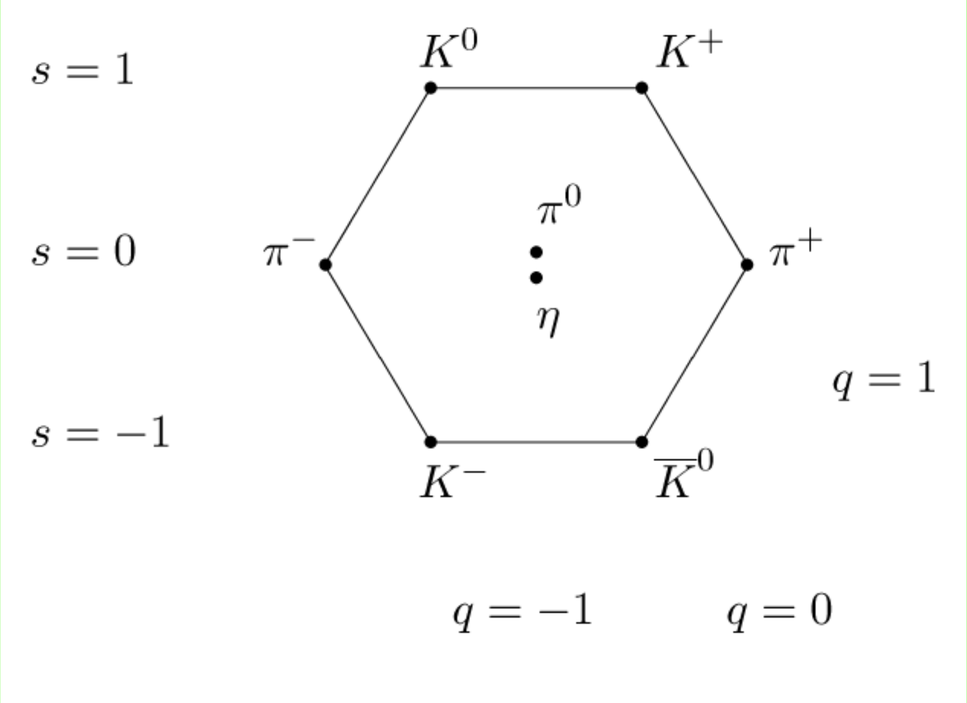
$$\mathcal{U}(r,\theta,\varphi) = U_L(\theta,\varphi)V(r)U_R^\dagger(\theta,\varphi) = \tilde{\mathcal{U}}_V(g(r,\theta,\varphi))$$

- The $\mathbb{Z}_3$-Skyrmion ensues via “the machine”

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_L(g(r,\theta,\varphi))\tilde{\mathcal{U}}_R^\dagger(g(r,\theta,\varphi))$$

A possible application

- the hadronic mesons come in an octet



- there are 8 scalar fields with an obvious $SO(8)$ transformation group, broken to $SU_{\text{adj.}}(3)$

Conclusions

- We have shown the existence of topological $\mathbb{Z}_3$ configurations.
- We must address the stability and excitations of such such objects within a possible dynamics given by a Skyrme type Lagrangian.
- Applications to phenomenology must also be addressed. Highly speculative ideas correspond to thinking about these Skyrmions as quarks. One would have to motivate the baryon number is somehow