

MASS COMPARISON THEOREMS FOR TORIC GRAVITATIONAL INSTANTONS

Hari Kunduri

McMaster University
Mathematics and Statistics & Physics and Astronomy

CAP 2026
University of Ottawa 2026

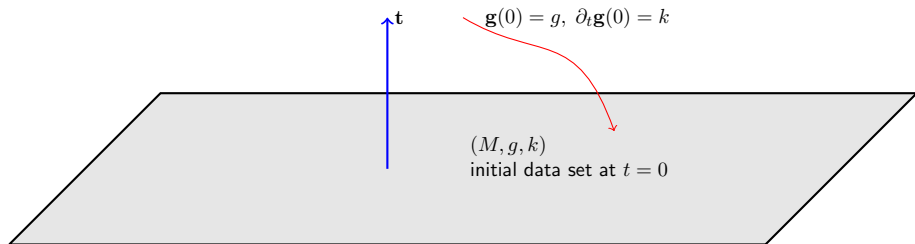
Outline

- 1 Positive mass theorem for asymptotically Euclidean manifolds
- 2 Gravitational instantons
- 3 Comparison theorem for toric ALE and ALF manifolds
(Alaee, Khuri, HK) [arxiv.2605.12352](https://arxiv.org/abs/2605.12352)

Initial Data

Einstein eqns \Rightarrow constraint eqns on data + evolution eqns

Spacetime (\mathbf{M}, \mathbf{g})
evolves from initial data



Initial data consists of a Riemannian manifold (M, g) and rank 2 tensor k

k measures how M curves within an ambient spacetime \mathbf{M} :

Constraint equations

Einstein onstraint equations

An **initial data set** (M, g, k) consists of a Riemannian manifold (M, g) and symmetric 2-tensor k satisfying

$$R_g - |k|_g^2 + \text{Tr}_g k = 16\pi\mu \quad \text{Hamiltonian Constraint}$$

$$\text{div}_g k - \nabla(\text{Tr}_g k) = 8\pi J \quad \text{Momentum Constraint}$$

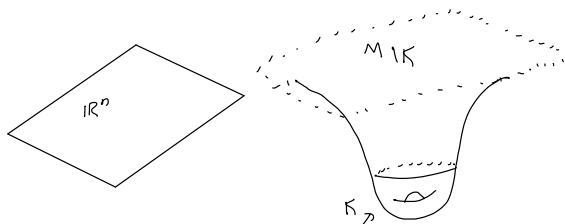
Time symmetric data: $k = 0$. Then constraints reduce to

$$R_g = 16\pi\mu \geq 0.$$

Asymptotically Euclidean manifolds

Isolated gravitating system: described by a geodesically complete (M, g) that 'approaches' (\mathbb{R}^n, δ) in an asymptotic region.

- 1 Topology: $M \setminus K \cong \mathbb{R}^n \setminus \text{Ball}$ for some compact set $K \subset M$.
- 2 Geometry: the metric g approaches flat Euclidean metric δ with some precise decay rate, i.e. $g_{ij} - \delta_{ij} = O_2(r^{-p})$ for $p > (n - 2)/2$.

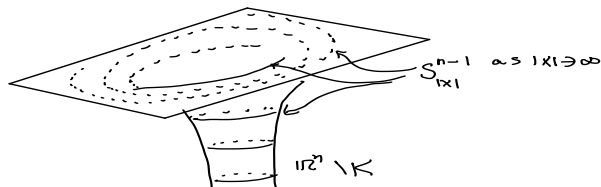


The ADM mass

An isolated gravitating system (e.g. black hole) evolves from AE initial data. For an AE Riemannian manifold, define

$$m := c_n \int_{S_\infty^{n-1}} (\operatorname{div} g - \nabla \operatorname{Tr} g) \cdot \nu \, dS$$

The integral is taken in the limit $r \rightarrow \infty$ of asymptotic S^{n-1} 'boundary sphere at infinity' with unit normal ν that encloses M .



m is finite and a **geometric invariant** (BARTNIK 86) Intuitively this captures the decay rate of g at $O(r^{-(n-2)})$.

The Positive mass theorem

Theorem (Positive Mass Theorem - SCHOEN & YAU 81 , WITTEN 81)

The mass of a complete, AE Riemannian manifold (M, g) with non-negative scalar curvature $R_g \geq 0$ satisfies

$$m \geq 0$$

with equality iff (M, g) is isometric to (\mathbb{R}^n, δ) .

Holds for $n < 8$ or if M is spin.

Note that a space with $R_g = \mu = 0$ can still have $m > 0$ (e.g. Schwarzschild initial data).

Gravitational instantons

Definition

A **gravitational instanton** is a smooth, geodesically complete (non-compact) connected **Ricci flat** Riemannian 4-manifold.

- characterized by their asymptotic geometry and topology. The only AE gravitational instanton is (\mathbb{R}^4, δ) .
- analogues of (anti-) self dual Yang-Mills instantons in gauge theory which minimize energy. We can impose self-dual curvature:

$$\star_g \text{Riem} = \pm \text{Riem}$$

Such (M, g) are *hyperKähler*. The Einstein-Hilbert action is not positive definite, so we drop this condition.

Many explicit examples exist by 'Euclideanizing' black hole metrics.

Schwarzschild instanton

Consider the Schwarzschild exterior metric with $t \rightarrow i\tau$:

$$g_S = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2}$$

Extends to a global metric on $\mathbb{R}^2 \times \mathbb{S}^2$ provided we identify:

$$\tau \sim \tau + \beta, \quad \beta = 8\pi m$$

ensures $T = \partial_\tau$ smoothly degenerates at $r = r_+$.

Asymptotically flat: as $r \rightarrow \infty$,

$$g \rightarrow d\tau^2 + g_{\mathbb{R}^3}$$

the canonical flat metric on $\mathbb{S}^1 \times \mathbb{R}^3$. Note $|T|$ is bounded as $r \rightarrow \infty$.

Taub-NUT instanton

Hyperkähler Taub-NUT metric;

$$\mathbf{g}_{TN} = H^{-1}L^2 \left(d\psi + k \cos^2 \left(\frac{\theta}{2} \right) d\phi \right)^2 + H(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)),$$
$$H = 1 + \frac{kL}{2r}, \quad r > 0, \theta \in (0, \pi)$$

Surfaces of constant r are \mathbb{S}^1 bundles over \mathbb{S}^2 provided $k \in \mathbb{Z}$ and we identify the (ψ, ϕ) -plane as:

$$(\psi, \phi) \sim (\psi + 2\pi, \phi) \sim (\psi, \phi + 2\pi)$$

Regularity as $r \rightarrow 0$ fixes $k = 1 \Rightarrow \mathbf{g}_{TN}$ is a smooth metric on \mathbb{R}^4 .

Eguch-Hanson instanton

Complete hyperkähler 1-parameter family of metrics on $T^*\mathbb{S}^2$

$$\mathbf{g}_{EH} = \frac{dr^2}{U} + \frac{r^2}{4} [U(d\psi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2],$$
$$U = 1 - \frac{a^4}{r^4}, \quad a \geq 0$$

with $r \in (a, \infty)$ and (θ, ϕ) standard coords on \mathbb{S}^2 .

- If $a = 0$ then \mathbf{g}_{EH} is Euclidean metric on \mathbb{R}^4 when $\psi \sim \psi + 4\pi$.
- if $a > 0$, ∂_ψ degenerates at $r = a$. Regularity fixes $\psi \sim \psi + 2\pi$.
Surfaces of constant r have topology $L(2, 1)$.
Asymptotically $\mathbf{g}_{EH} \rightarrow$ flat metric on $\mathbb{R}^4 \setminus \mathbb{Z}_2$.

ALF and ALE manifolds

We consider Riemannian manifolds with prescribed asymptotic behaviour.

- 1 (M, g) is **asymptotically locally Euclidean** (ALE) if it has an asymptotic end with the topology $\mathbb{R} \times L(p, q)$ and $g \rightarrow g_E$ (Euclidean metric). AE corresponds to $p = 1$. The volume grown $\sim r^4$.
- 2 (M, g) is **asymptotically locally flat** (ALF) if it has an asymptotic end with the topology $\mathbb{R} \times L$ where L is an \mathbb{S}^1 -bundle over \mathbb{S}^2 , or more generally $L(p, q)$ but with cubic volume growth $\sim r^3$.
- 3 if (M, g) is ALF and $L = \mathbb{S}^1 \times \mathbb{S}^2$ (trivial bundle) then we say it is **asymptotically flat** (AF).

ALF and AF spaces have a **bounded** dimension.

Equilibrium geometries

- Positive mass thm for complete ALE/ALF space (M, g) with $R_g \geq 0$?
Not possible - explicit examples with $m < 0$.
- Expect gravitational instantons to be natural **ground states** for a fixed asymptotic class (role played by Euclidean \mathbb{R}^4 for AE spaces)
- **Problem:** how to define a 'comparison' model geometry for given (M, g) ? **Toric symmetry** provides required structure.

Mass comparison theorem (Alaee, Khuri, HK 2026)

Theorem (Mass comparison)

Let (M, g) be a simply connected toric ALE/ALF manifold with $R_g \geq 0$ and (M, g_0) be the corresponding equilibrium gravitational instanton sharing the same ALE/ALF structure and the same toric data \mathcal{D} . Then

$$\text{mass}_b(M, g) - \text{mass}_b(M, g_0) \geq 0.$$

with equality iff (M, g) is isometric to (M, g_0) .

- Result generalizes to include (M, g) with conical singularities.
- b is a suitable reference background used to define the mass.

Toric geometries

(M, g) has a torus symmetry if it admits a $\mathbb{T}^2 = U(1) \times U(1)$ isometry group (possibly with fixed points) The metric is

$$g = \hat{g}_{ab} dx^a dx^b + G_{ij} (d\phi^i + A_a^i dx^a) (d\phi^j + A_b^j dx^b)$$

where $\phi^i \sim \phi^i + 2\pi$. The torus metric is the matrix

$$G_{ij} = g(\eta_i, \eta_j), \quad \eta_i := \frac{\partial}{\partial \phi^i}$$

$\det G = 0 \iff$ integer linear combination of η_i vanish.

- The $\{x^a\}$ are coordinates on the **orbit space** $\mathcal{B} := M/\mathbb{T}^2$ which is a 2D manifold with boundary and corners.
- The topology of M is characterized by **toric data** \mathcal{D} encoding how the \mathbb{T}^2 acts on M .

Toric gravitational instantons

A gravitational instanton is also Ricci flat, $R_{ab} = 0$. Define the unit determinant, positive definite matrix

$$\Phi_{ij} := (\det G)^{-1/2} G_{ij}.$$

The Ricci flat condition reduces to

$$\nabla \cdot (\Phi^{-1} \nabla \Phi) = 0$$

where ∇ is the divergence with respect to the auxiliary flat \mathbb{R}^3 . Φ defines a **harmonic map** from $\mathbb{R}^3 \rightarrow \mathbb{H}^2$, i.e. gravitational instantons are critical points of the (divergent) harmonic energy

$$E[\Phi] := \int_{\mathbb{R}^3} \text{Tr}[(\Phi^{-1} \nabla \Phi)^2] dx$$

Rough idea of proof

Consider a general (M, g) . Compute R_g and integrate over \mathbb{R}^3 :

$$\text{Boundary terms} = \mathcal{I}[\Phi] + \int_{\mathbb{R}^3} [R_g + \text{positive terms}] dx$$

Here $\mathcal{I}[\Phi]$ is the *renormalized energy* (a finite reduction of $E[\Phi]$).

- Careful analysis shows the boundary terms contain the mass m .
- Use convexity of harmonic energy (Schoen-Zhou) to prove RHS is globally minimized by the gravitational instanton with sane \mathcal{D} as the original (M, g) .
- Existence and uniqueness of the ground state has been established previously (Lucietti, HK 2021 & 2026)

Open questions

- While a PMT is not possible for ALE/ALF manifolds, can prove mass comparison-type results in the toric setting using an appropriate ‘equilibrium ground state’ geometry.
- Applications to general relativity: given a gravitational instanton (M, g) , one can produce a vacuum spacetime: $(\mathbb{R} \times M, \mathbf{g}_5)$:

$$\mathbf{g}_5 = -dt^2 + \underbrace{g_{ij}dx^i dx^j}_{\text{grav. instanton}}$$

Minkowski spacetime occurs for $(M, g) = (\mathbb{R}^4, \delta_4)$

- Open problem: prove a positive mass theorem for spacetimes asymptotic to these vacua.