Gluons and glueballs from functional equations

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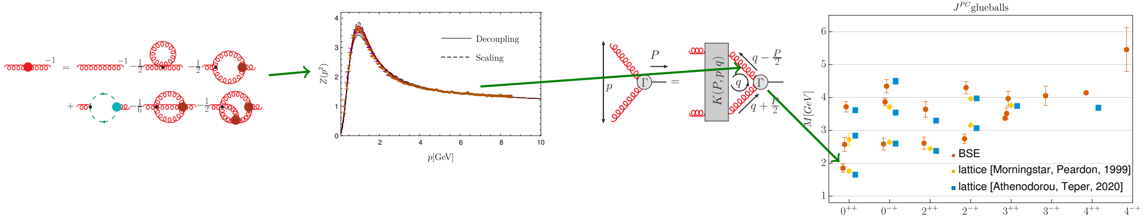

15th International Workshop on the Physics of Excited Nucleons
Sept. 9, 2026
Seville, Spain

$\rightarrow$ MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85, 2503.03821 (2025)

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Glueball calculations on the lattice

Nonperturbative first-principles method

- discretized space-time $\rightarrow$ continuum limit
- computationally expensive
- quark masses

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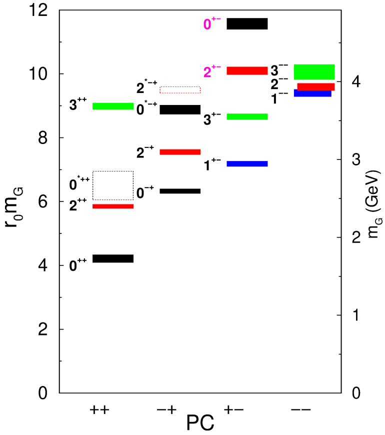
Quantitative results for pure gauge theory (no quarks):

- [Morningstar, Peardon, Phys. Rev. D60 (1999)]
- [Athenodorou, Teper, JHEP11 (2020)]

improved statistics, more states

$\boxed{\rightarrow \text{Benchmark}}$ for other methods



Functional methods: Overview

Systems of equations relying on effective actions:

- Functional renormalization group (FRG): effective average action Γ_k
- Dyson-Schwinger equations (DSEs): eqs. of motion of 1PI effective action
- eqs. of motion of nPI effective action, $n > 1$

Similar eqs.

- bound state equations (BSEs) (Bethe-Salpeter, Faddeev, Yakubovsky)

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spectra

- Infinitely large systems: (in)finitely many eqs. of (in)finite size
- Nonperturbative
- Truncations
- Applied for hadron physics, QCD phase diagram, condensed matter, gravity, ...

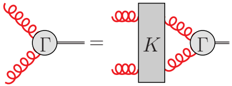
$\longrightarrow$ [Eichmann, "Hadron physics with functional methods", 2503.10397, link; MQH, "A beginner's guide to functional methods in particle physics", 2510.18960, link]

Glueball bound state equations

Details of derivation

n-particle bound state: Pole in a 2*n*-point function ***G*** (or the scattering matrix ***T***).

Generic form:



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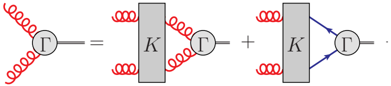

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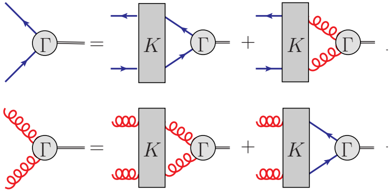

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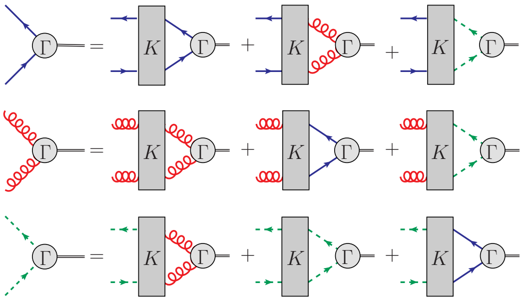

Gluons, quarks (u, d, s, . . .) and ghosts all couple.

(Landau gauge)

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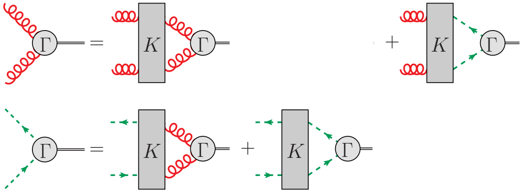
Focus on pure glueballs.



Gluons, quarks (u, d, s, . . .) and ghosts all couple.

(Landau gauge)

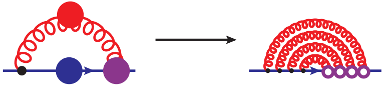
- Conceptually simpler: fewer terms, only glueballs (ghosts *do not* lead to additional states)
- Benchmark: Lattice results

Bottom-up approach: Rainbow-ladder

"Severe" truncations, but phenomenologically successful.

Need the gluon propagator ($\boldsymbol{Z(k^2)}$) and the quark-gluon vertex ($\boldsymbol{h_i(k;p,q)}$).

- $\boldsymbol{\Gamma^{a,\nu}(k;p,q)} \propto \gamma^\nu h_1(k;p,q)$
- $\frac{g^2}{4\pi} Z(k^2) h_1(k;p,q) \propto \boldsymbol{\alpha(k^2)}$



Iteration $\rightarrow$ only 'rainbow-like' diagrams

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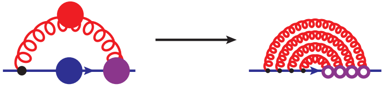
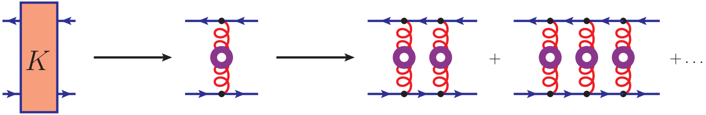


Iteration $\rightarrow$ only 'rainbow-like' diagrams

Scattering kernel: “all interactions which are two-particle irreducible with respect to two horizontal lines”

Simplest realization:



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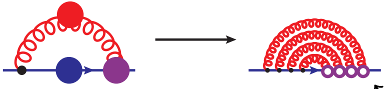

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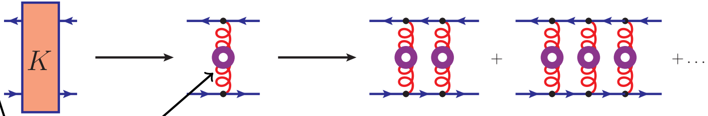
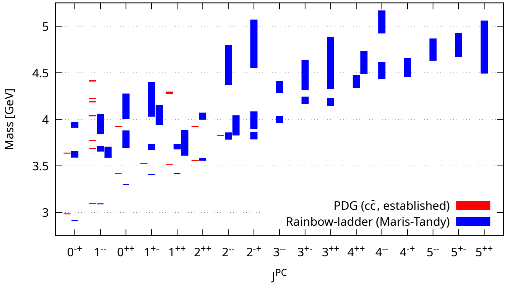


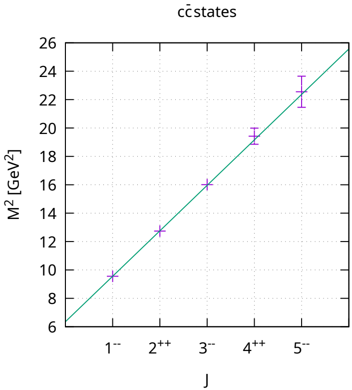
Iteration $\to$ 'ladder-like' diagrams

'model interaction'

Rainbow-ladder: Charmonium

[Hagel, Fischer, MQH, Eur.Phys.J.A 62 (2026)]





- Good description for some states
- Regge trajectory

Model based BSE calculations for glueballs ($J = 0$)

- [Meyers, Swanson, Phys.Rev.D87 (2013)]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
- [Souza et al., Eur.Phys.J.A56 (2020)]
- [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

$\rightarrow$ Calculations possible, but no quantitative prediction yet.

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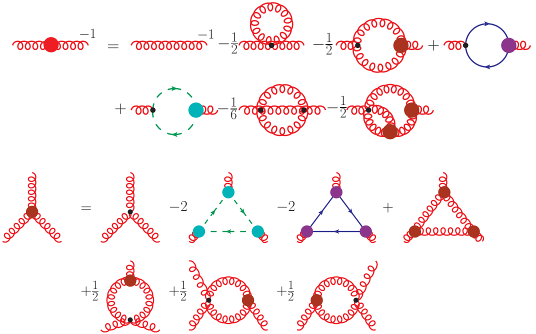
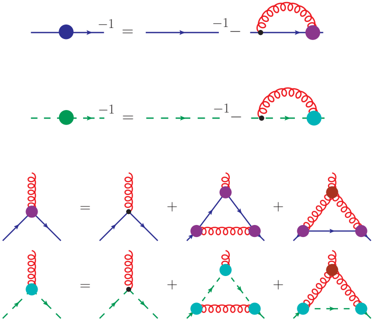
Better models? Alternative: Self-consistently calculated input.

Equations of motion from 3PI 3-loop

Stationarity conditions: $\frac{\delta\Gamma}{\delta D}=0$, $\frac{\delta\Gamma}{\delta\Gamma^{(3)}}=0 \to$ Equations of motion (already truncated):





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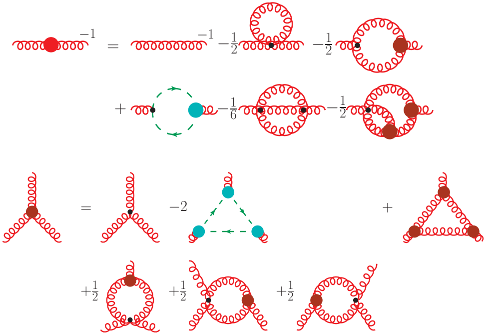

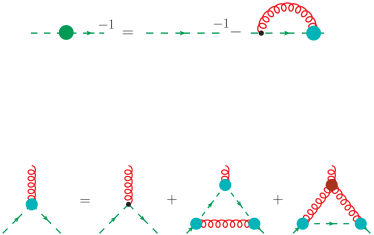

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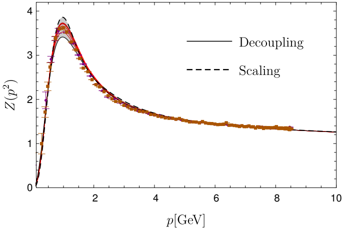
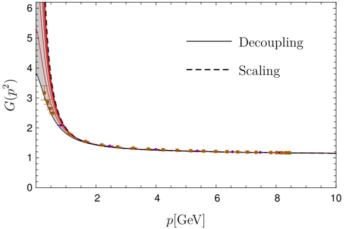
Input for bound state equations: Pure gauge theory

Gluon dressing function:
$\rightarrow$ Talk by Papavassiliou

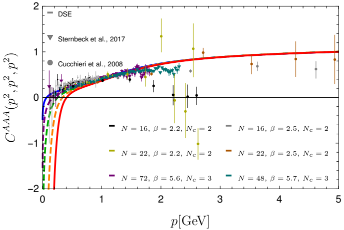


Ghost dressing function:



Continuum: [MQH, Phys. Rev. D 101 (2020)]

Three-gluon vertex:



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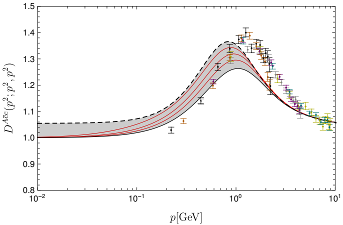


Lattice:
[Sternbeck, hep-lat/0609016; Sternbeck et al., PoS LATTICE2016 349 (2017); Cucchieri, Maas, Mendez, Phys. Rev. D77 (2008); Maas, SciPost Phys. 8 (2020)]

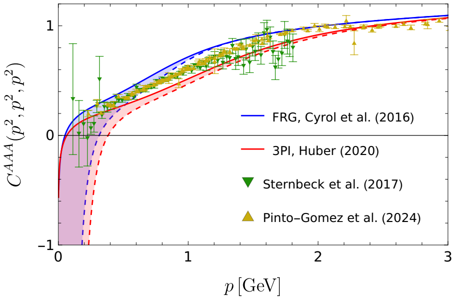
Correlation functions

Efforts of recent years resulted in agreement for many correlation functions (propagators, three- and four-point functions) using different methods:

- Lattice calculations
- Dyson-Schwinger equations
- *n*PI equations
- Functional renormalization group equations

In particular, lattice and continuum methods have complementary challenges.

Example: Three-gluon vertex



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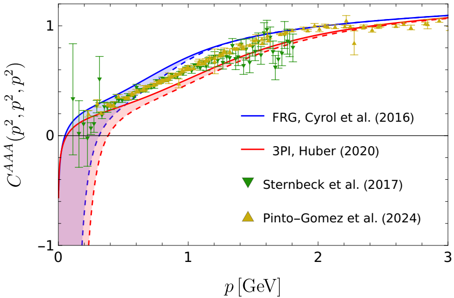


Functional methods: Truncations $\rightarrow$ Reliability of results?

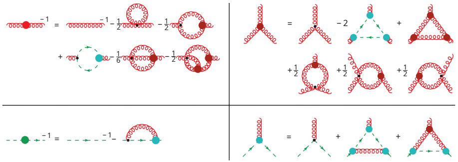
Testing truncations

Pragmatic approach

Enlarging the truncation until changes small enough. $\rightarrow$ Apparent convergence

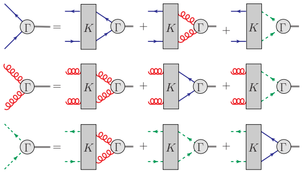
<u>Two sets to test:</u>

Correlation functions



$\rightarrow$ List of tests in [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C85 (2026)]

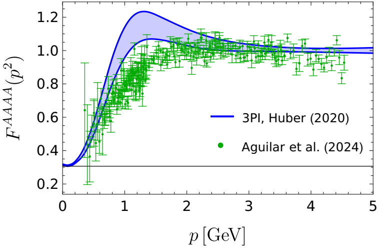
Bound state equations



Some checks for correlation functions

Extensions beyond the truncation used for the glueball calculation:

- More tensors: Three-gluon vertex (4 transverse dressings): Tree-level dressing dominant, others subleading [Eichmann, Williams, Alkofer, Vujinovic, Phys.Rev.D89 (2014); Pinto-Gómez et al., 2208.01020] ✓
- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓
- Four-gluon vertex [MQH, Phys. Rev. D 101 (2020); Aguilar et al., Phys.Lett.B 858 (2024)] (talk by Pinto Gomez): ✓



- Two-ghost-two-gluon and four-ghost vertices [MQH, Eur. Phys.J.C77 (2017)] ✓

Approximations in kernels

Systematic derivation from 3PI loop expansion

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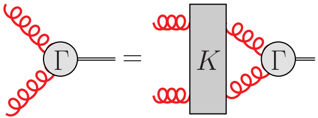
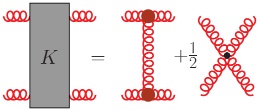
- No ghosts (exact for P-odd states)





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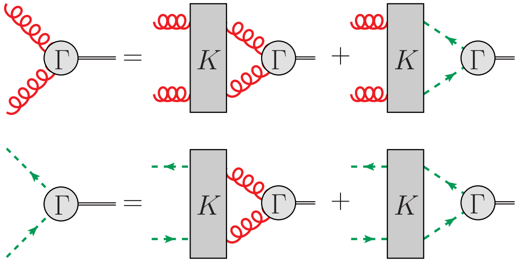

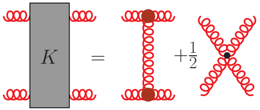

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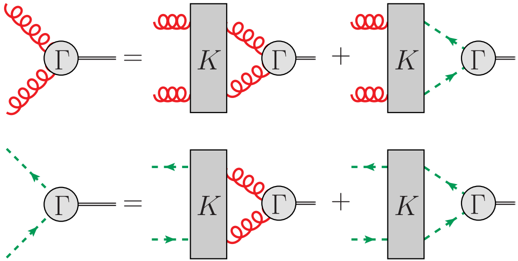
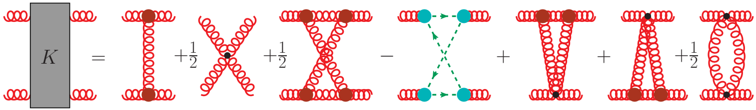
- Gluonic two-loop diagrams (2025)





Extrapolation of $\lambda(P^2)$

$$\boldsymbol{\lambda(P)}\Gamma(P) = \mathcal{K} \cdot \Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:

1. Solve for $\lambda(P)$.
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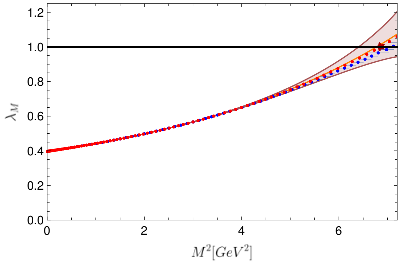
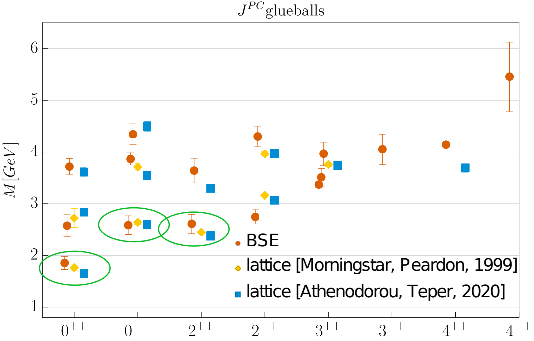


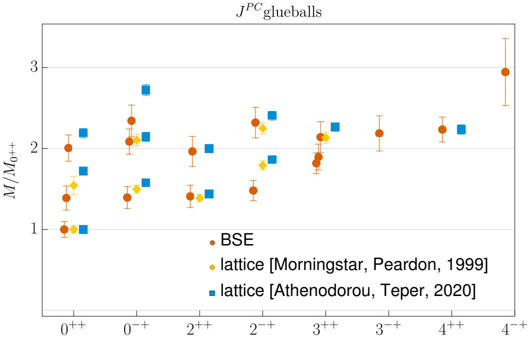
Test extrapolation for solvable system: [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]

Extrapolation method

- Do purely Euclidean calculation.
- λ for time-like P^2 using Schlessinger's continued fraction method (performs here better than default Padé approximants) [Schlessinger, Phys.Rev.167 (1968)]
- Average over extrapolations using subsets of points for error estimate

Glueball results





[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 81 (2021)]

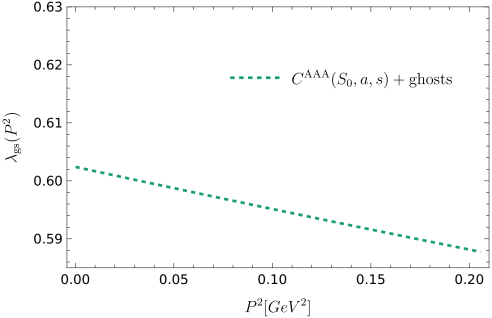
- Agreement with lattice results
- Consensus: **0^{++}** is lightest state
- Next: 0^{-+}, 2^{++}, ${0^{++}}^*$

- Cf. experimental states $f_0(1370)$, $f_0(1500)$, $f_0(1710)$ as scalar and $X(2370)$ as pseudoscalar glueball candidates

Effect of approximations (for 0^{++})

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85 (2025)]

Ground state:



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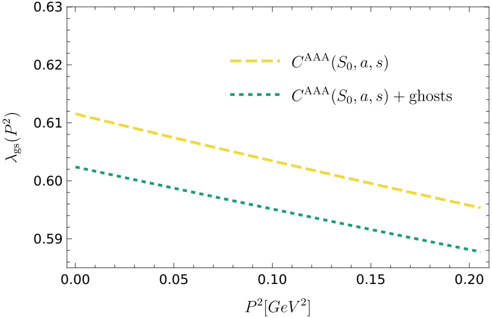

- Ghost diagrams essential: $\Delta m \approx 200 - 350\,\mathrm{MeV}$

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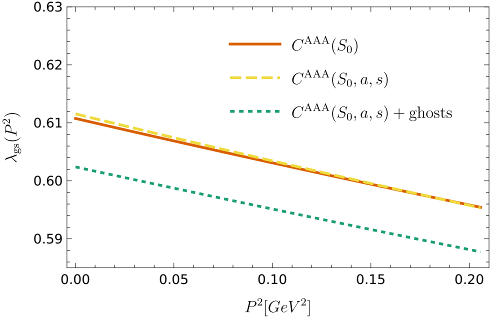

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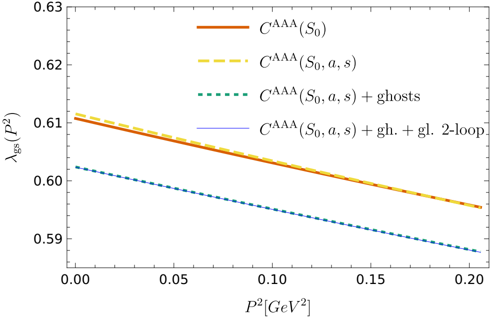

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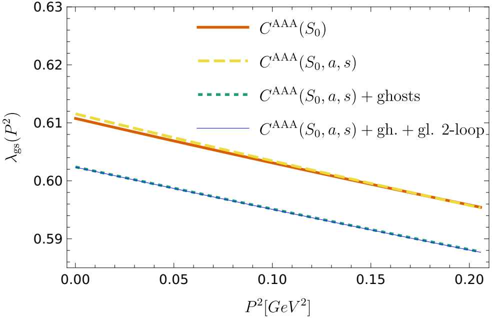
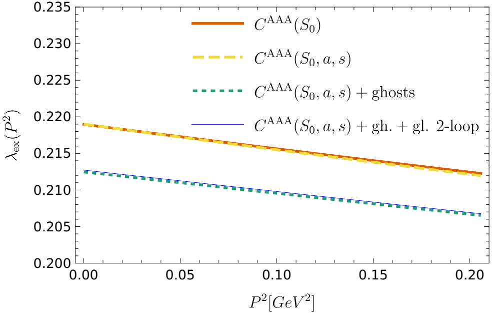


First excited state:



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Summary and conclusions

- Functional methods are versatile tools:
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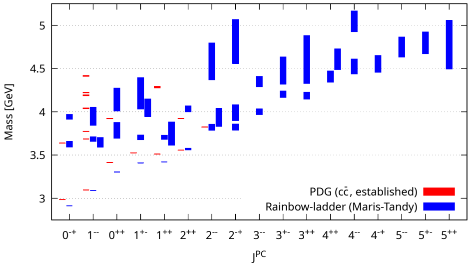

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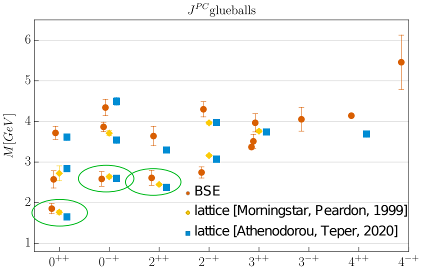

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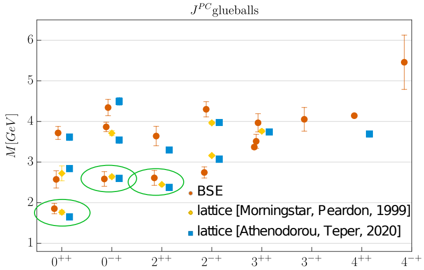

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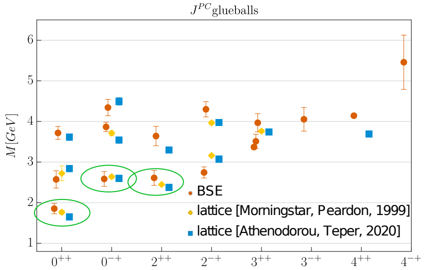

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- Agreement with lattice spectrum
- Two-loop diagrams negligible ($0^{\pm +}$, 2^{++})

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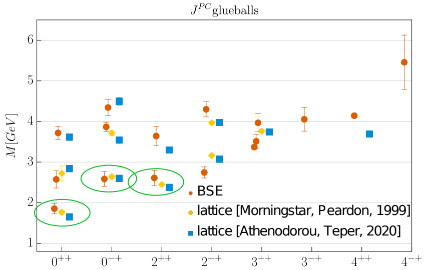

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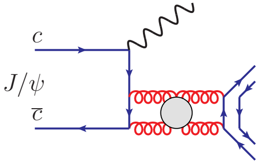
Thank you for your attention.

Glueballs

Experiment

Production in glue-rich environments, e.g.,

- $p\bar{p}$ annihilation (PANDA)
- pomeron exchange in pp (central exclusive production, CMS, ALICE)
- radiative J/ψ decays (BESIII)



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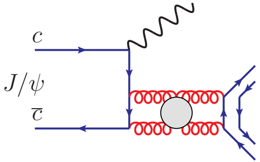

Scalar glueball candidate: Coupled-channel analyses of exp. data (BESIII)

- +add. data, largest overlap with $f_0(1770)$ [Sarantsev, Denisenko, Thoma, Klempt, Phys. Lett. B 816 (2021)]
- largest overlap with $f_0(1710)$ [JPAC Coll., Rodas et al., Eur.Phys.J.C 82 (2022)]

Pseudoscalar glueball candidate:

- $X(2370)$ [Ablikim et al. (BESIII), PRL132 (2024)]

Derivation of bound state equations I

For more details see [Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

n-particle bound state

Pole in a $2n$-point function.

For simplicity here $n = 2$.

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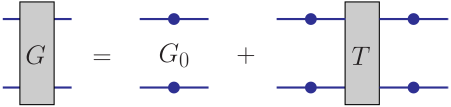
$$G = G_0 + G_0\, T\, G_0$$



disconn. connected part

$\rightarrow$ scattering matrix T (amputated, conn. part of G)

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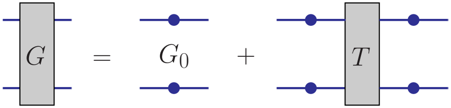

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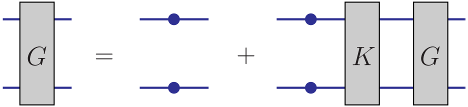
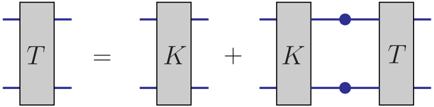
Dyson equations: nonperturbative resummations! Compare:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = 1 + x\, f(x) = 1 + x + x^2\, f(x)$$

$$G = G_0 + G_0\, K\, G$$



$$T = K + K\, G_0\, T$$



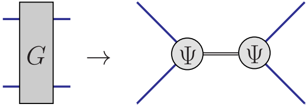
Scattering kernel K: 2-particle irreducible with respect to horizontal quarks lines (rest created by iteration)

Derivation of bound state equations II

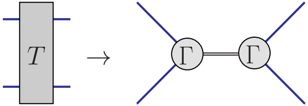
Back

On-shell: at pole position $P^2 = -M^2$

$$G \to \frac{\Psi\,\overline{\Psi}}{P^2 + M^2}$$



$$T \to \frac{\Gamma\,\overline{\Gamma}}{P^2 + M^2}$$



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Pole position: mass M

Residues: $\Gamma\,\overline{\Gamma}$, $\Psi\,\overline{\Psi}$

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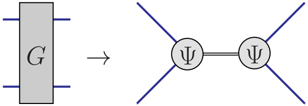

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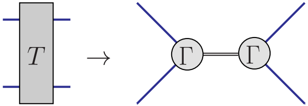

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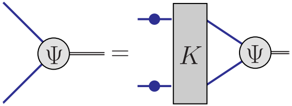
Plug into Dyson equations: homogeneous Bethe-Salpeter equations

$$\Psi = G_0\, K\, \Psi$$



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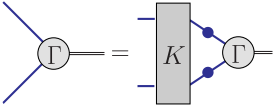

Hadrons as bound states

Hadron masses from correlation functions of gauge invariant operators.

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Examples:
$J^{PC} = 0^{-+}$ meson $\rightarrow O(x) = \overline{\psi}(x)\gamma_5\psi(x)$ $\rightarrow \overline{\psi}^2\psi^2$
$J^{PC} = 0^{++}$ glueball $\rightarrow O(x) = F_{\mu\nu}(x)F^{\mu\nu}(x)$ $\rightarrow A^4, A^5, A^6, A^7, A^8$

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Lattice: Mass from exponential Euclidean time decay

$$\lim_{t\to\infty}\langle O(x)O(0)\rangle \sim e^{-t\,M}$$

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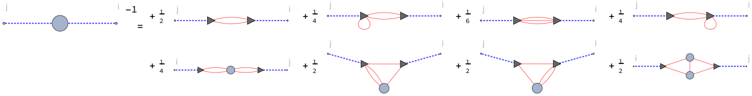
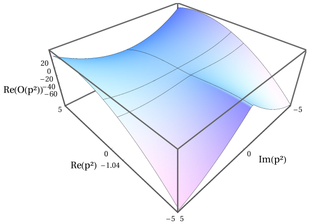
Functional approach:

Glueball:



+ 3-loop diagrams [MQH, Cyrol, Pawlowski, Comput.Phys.Commun. 248 (2020)]

Not feasible. (For now?)



Leading order: [Windisch, MQH, Alkofer, Phys.Rev.D87 (2013)]

BSE calculations for glueballs

- Model based ($J = 0$):
  - [Meyers, Swanson, Phys.Rev.D87 (2013)]
  - [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
  - [Souza et al., Eur.Phys.J.A56 (2020)]
  - [Kaptari, Kämpfer, Few Body Syst.61 (2020)]

  $\rightarrow$ Calculations possible, but no quantitative prediction yet.

- Self-consistently calculated input:
  - [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]: $0^{\pm+}$
  - [MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]: $0^{\pm+}$,$1^{\pm+}$,$2^{\pm+}$,$3^{\pm+}$,$4^{\pm+}$
  - [MQH, Fischer, Sanchis-Alepuz, Eur. Phys. J. C 85 (2025)]: more diagrams
  - [Pawlowski, Schneider, Turnwald, Urban, Wink, Phys.Rev.D 108 (2023)]: $0^{\pm+}$, spectral reconstruction

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Convergence: Pragmatic approach

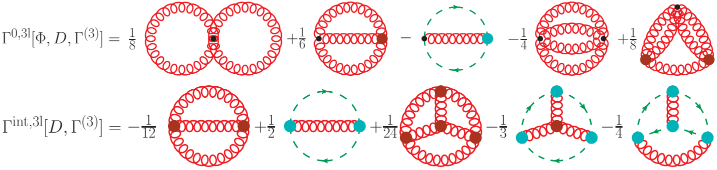
Enlarging the truncation until changes small enough. $\rightarrow$ Apparent convergence

3PI effective action

Back

- 3PI: propagators and three-point functions as dynamic quantities
- Natural expansion scheme in loops.

In general: *m*-loop for *n*PI, $m \geq n$, for a consistent description.



[Berges, Phys. Rev. D 70 (2004); Carrington, Gao, Phys. Rev. D 83 (2011)]

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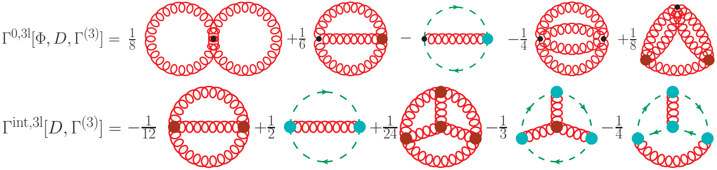
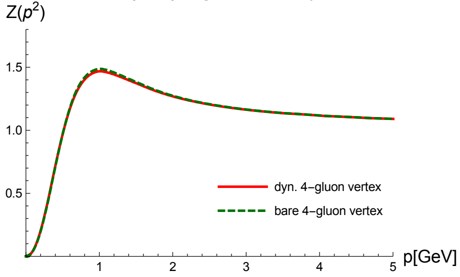


[Berges, Phys. Rev. D 70 (2004); Carrington, Gao, Phys. Rev. D 83 (2011)]

- Equations of motion from stationarity conditions $\rightarrow$ correlation functions ✓
- Kernels from stability conditions $\rightarrow$ BSEs ✓

More checks: Four-point functions II

- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓



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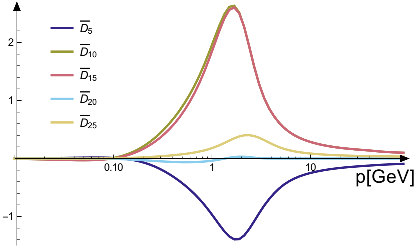
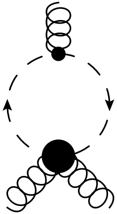
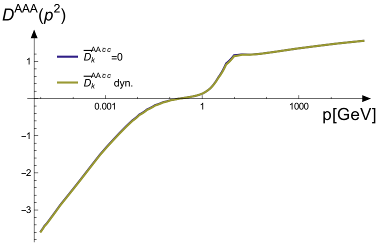
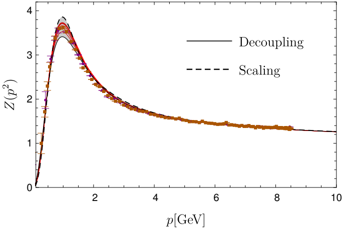
- Two-ghost-two-gluon vertex with 25 dressings [MQH, Eur. Phys.J.C77 (2017)]: ✓
  (FRG: [Corell, SciPost Phys. 5 (2018)])

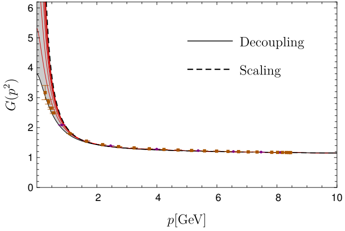


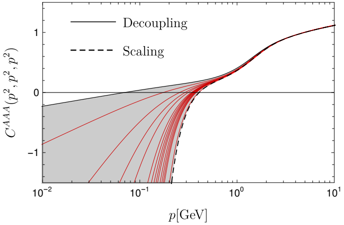




Family of solutions







- Family of solutions [von Smekal, Alkofer, Hauck, PRL79 (1997); Aguilar, Binosi, Papavassiliou, Phys.Rev.D 78 (2008); Boucaud et al., JHEP06 (2008); Fischer, Maas, Pawlowski, Ann.Phys. 324 (2008); Alkofer, MQH, Schwenzer, Phys. Rev. D 81 (2010)]
- Nonperturbative completions of Landau gauge [Maas, Phys. Lett. B 689 (2010)]?

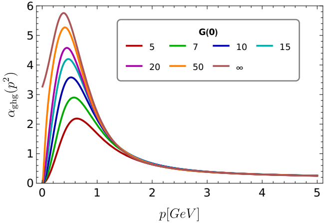
Spectrum for family of solutions

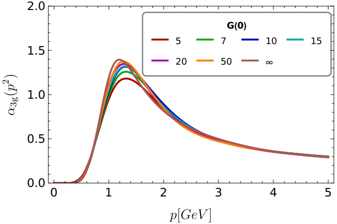
Couplings:

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85]

$$\alpha_{3g}(p^2) = \alpha(\mu^2)\left[D^{3g}(p^2)\right]^2\left[Z(p^2)\right]^3$$

$$\alpha_{ghg}(p^2) = \alpha(\mu^2)\left[D^{ghg}(p^2)\right]^2 Z(p^2)\left[G(p^2)\right]^2$$





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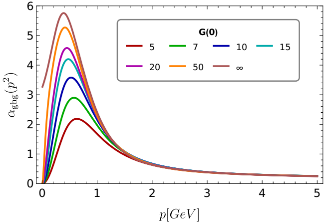
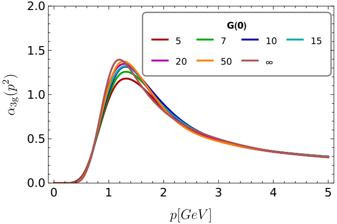
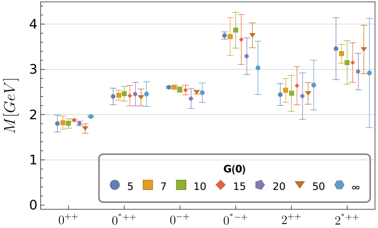






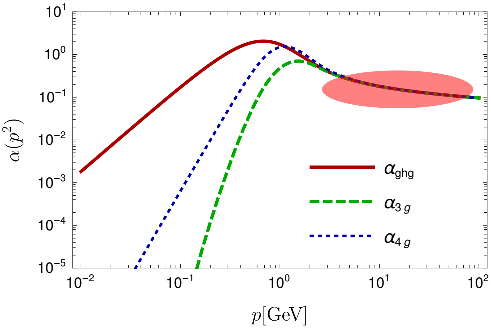
- Invariant spectrum → Nonperturbative gauge hypothesis supported.
- Invariance lost when deforming the input. → Consistency relevant.

Gauge invariance

[MQH, Phys. Rev. D 101 (2020)]

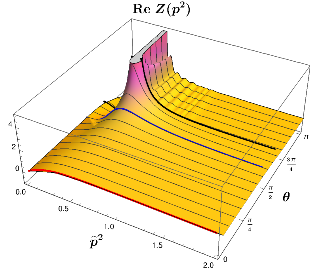
Couplings can be extracted from each vertex.

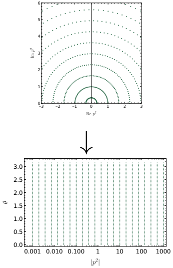
- Slavnov-Taylor identities (gauge invariance): Agreement perturbatively (UV) necessary.
  [Cyrol et al., Phys.Rev.D 94 (2016)]
- Difficult to realize: Small deviations $\rightarrow$ Couplings cross and do not agree.
- Here: Vertex couplings agree down to GeV regime (IR can be different).



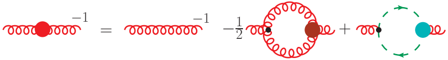
Correlation functions in the complex plane

Gluon [Fischer, MQH, Phys.Rev.D 102 (2020)]





Simpler truncation:



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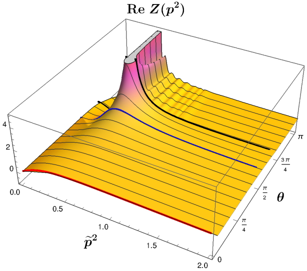

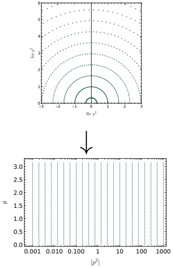

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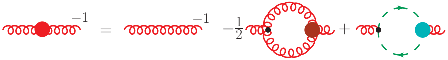
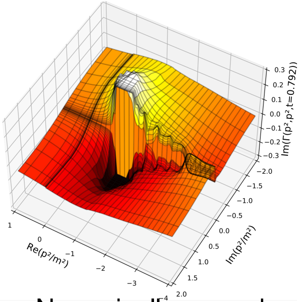


Vertex (scalar toy theory)

[MQH, Kern, Alkofer, Phys.Rev.D107 (2023)]

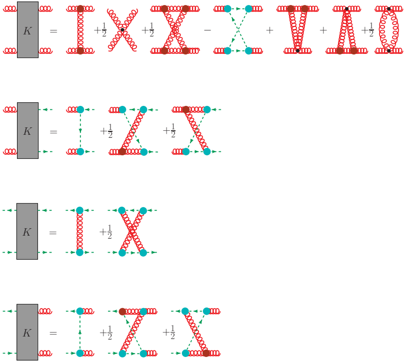


- Numerically more demanding due to calculations close to cuts.

Kernels from the 3PI 3-loop effective action

3PI effective action

Systematic derivation from 3PI effective action [Berges, PRD70 (2004); Carrington, Guo, PRD83 (2011)].



[Fukuda, Prog. Theor. Phys 78 (1987); McKay, Munczek, Phys. Rev. D 40 (1989); Sanchis-Alepuz, Williams, J. Phys: Conf. Ser. 631 (2015); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 80 (2020)]

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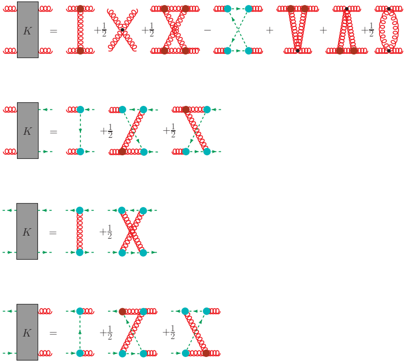

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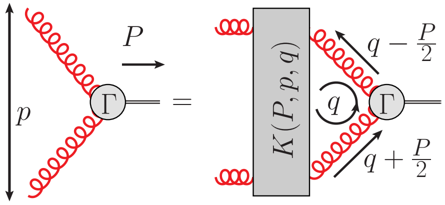
Solving a BSE



(pseudoscalar glueball)

$$\boldsymbol{\lambda(P)}\Gamma(P) = \mathcal{K}\cdot\Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:

1. Solve for $\lambda(P)$.
2. Find P with $\lambda(P) = 1$.
   $\Rightarrow M^2 = -P^2$

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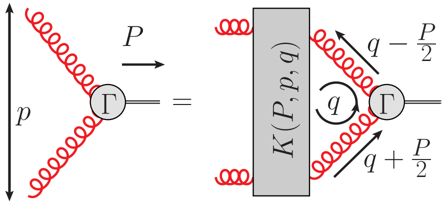

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However:

Propagators are probed at $\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\,q^2}\cos\theta = -\frac{M^2}{4} + q^2 \pm \boldsymbol{i}\,M\sqrt{q^2}\,\cos\theta$

$\rightarrow$ Complex for $P^2 < 0$!

Time-like quantities ($P^2 < 0$) $\rightarrow$ Correlation functions for complex arguments.

Potential from Maris-Tandy

$$V(\vec{r}) = -C_f 4\pi \int đ^3 k \, e^{-i\vec{k}\cdot\vec{r}} \frac{\alpha(\vec{k}^2)}{\vec{k}^2}.$$