Kinematic power corrections for TMD factorization theorem

Based on **S.Piloñeta** and **A.Vladimirov** JHEP12(2024)059 and JHEP03(2026)049

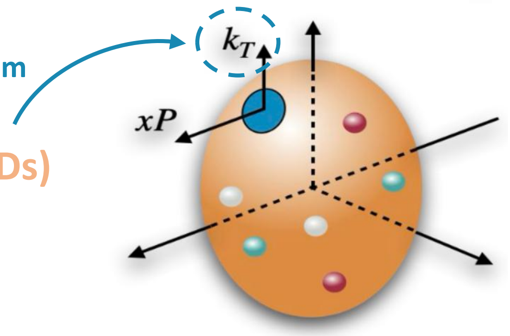
NSTAR26

Sara Piloñeta. September 10th 2026

PID2022-136510NB-C31

The need for power corrections

- What do we do? We are interested in studying the structure of nucleons
  - TMD factorization theorem
    - Transverse Momentum Dependent Distributions (TMDs)
    - Very powerful tool but has some limitations



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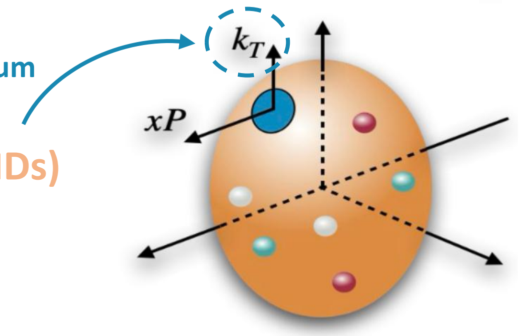
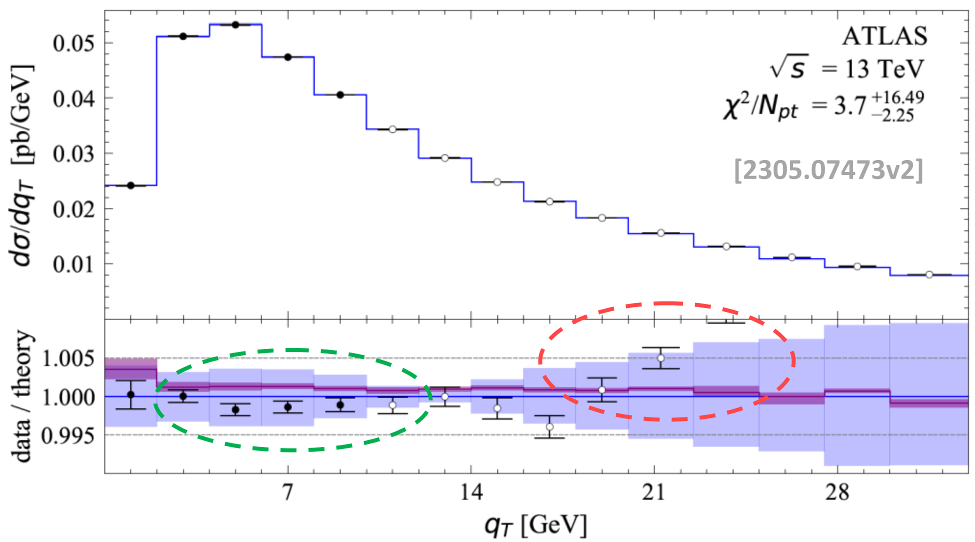


Z-boson production ATLAS measurement comparison



→ Very precise description at low q_T

→ Prediction systematically lower than the data at larger q_T

Signal of the need for power corrections

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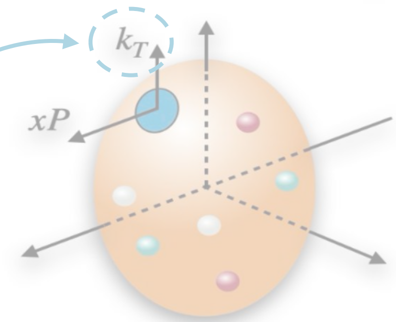
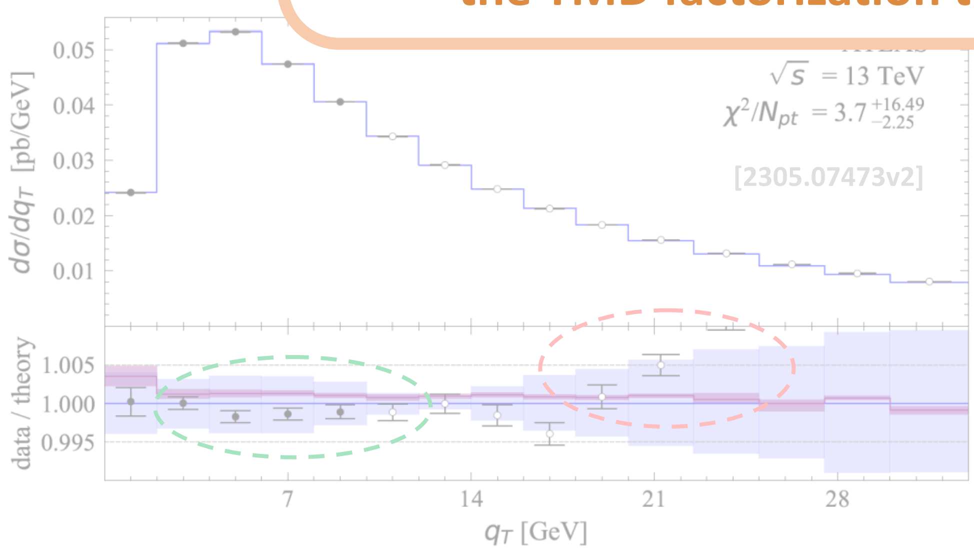
Parton's transverse momentum



This has motivated us to explore the structure of the TMD factorization theorem deeper

Z-boson p



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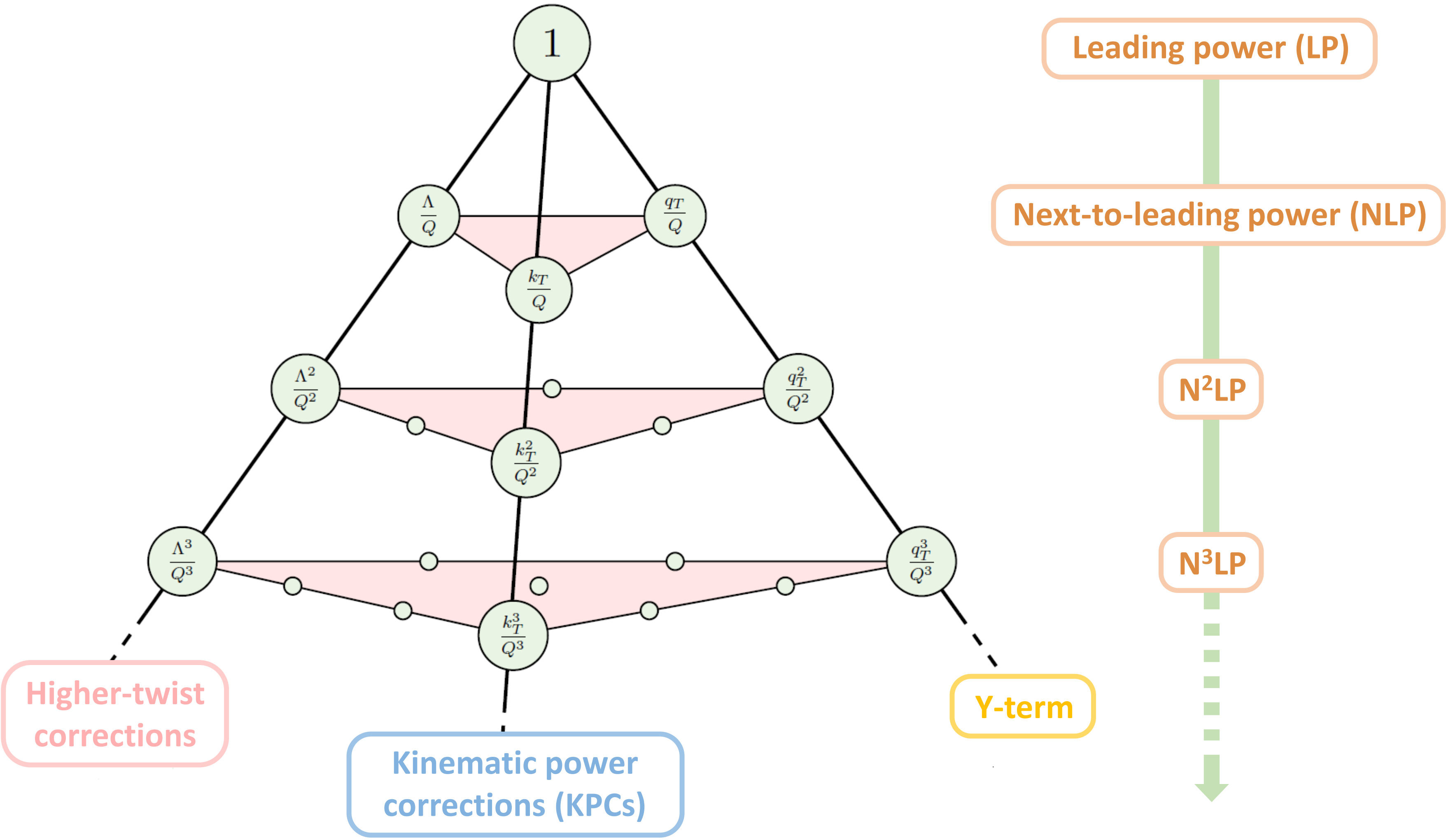
↓

Signal of the need for power corrections

The hierarchy of the TMD factorization theorem

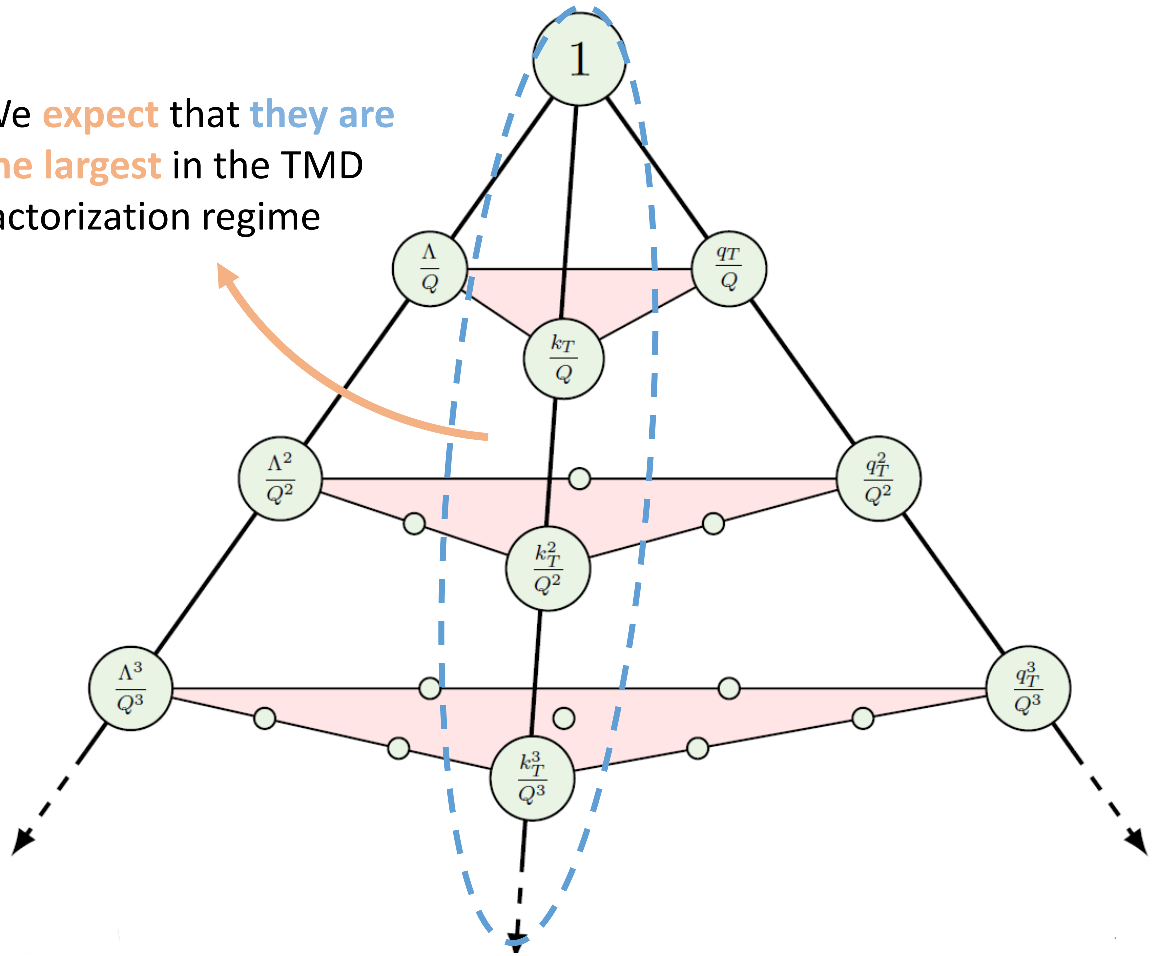
- There is a whole **"pyramid"** of **power corrections**



The TMD-with-KPCs factorization theorem

- We focus on the kinematic power corrections (KPCs) that follow the LP term

We expect that they are the largest in the TMD factorization regime



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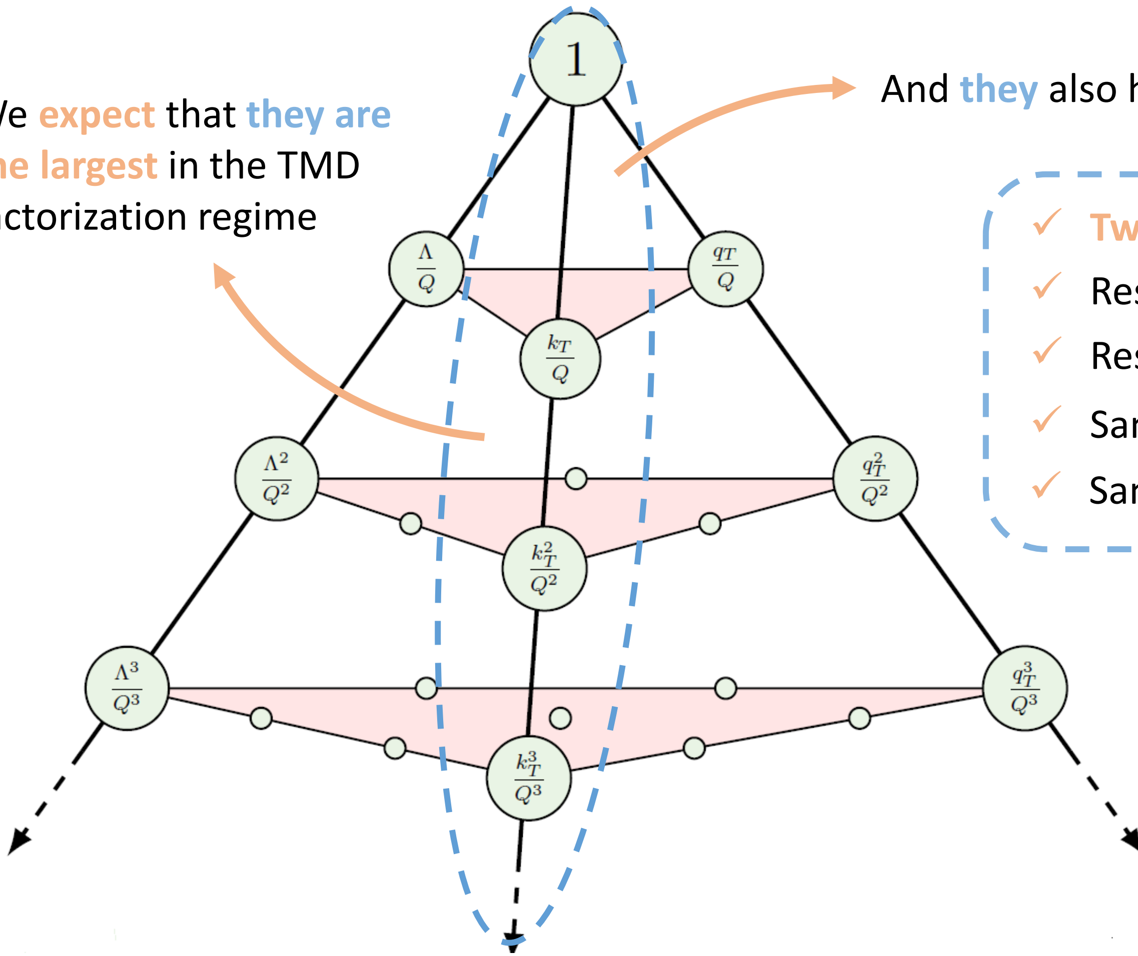

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And they also have nice properties!

- ✓ Twist-2 terms at $q_T \ll Q$
- ✓ Restore EM-gauge invariance
- ✓ Restore frame-invariance
- ✓ Same TMD distributions as LP
- ✓ Same coefficient function as LP

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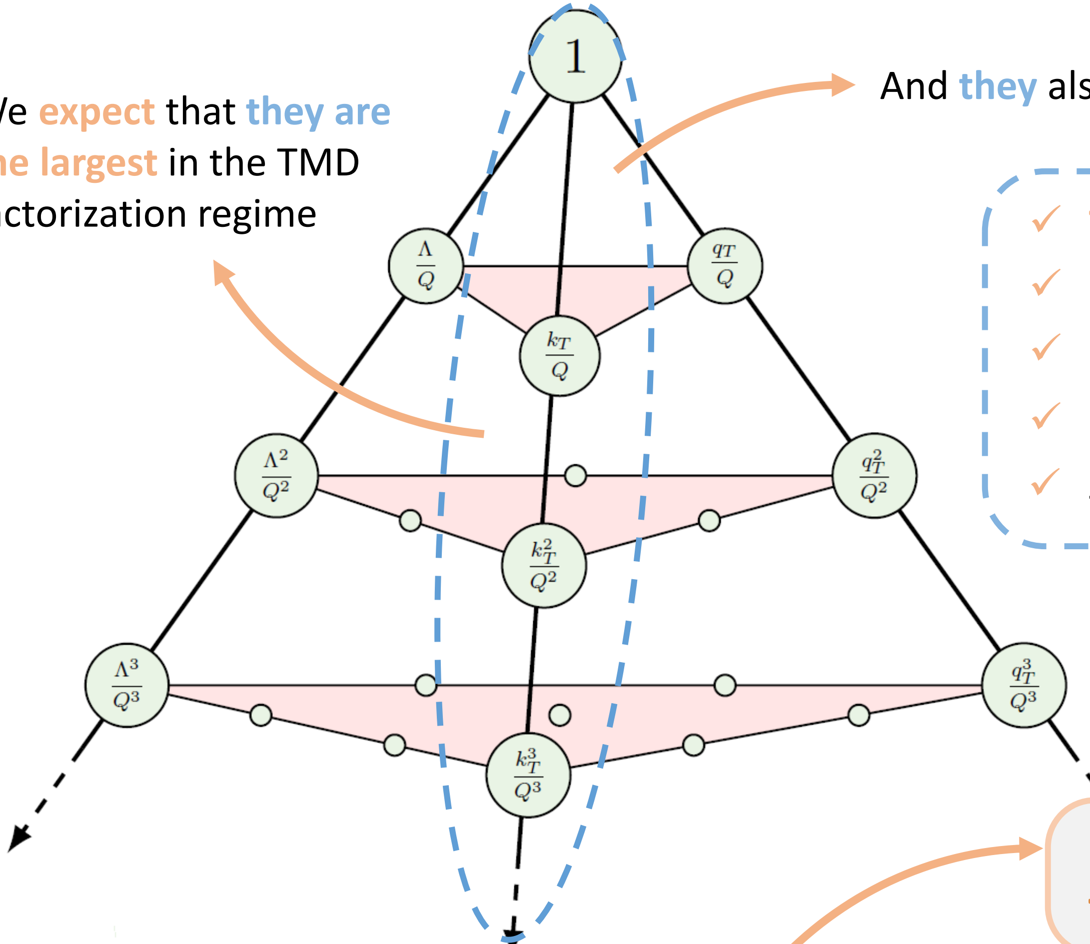

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Theoretical framework including them

TMD-with-KPCs factorization theorem

For details, see [AV, 2307.13054v2]

Angular distributions of Drell-Yan leptons

- **Feasibility** of using the **TMD-with-KPCs factorization** ⟹ unpolarized Drell-Yan as a starting point

    ↳ ***Angular distributions*** *of Drell-Yan leptons in the TMD factorization approach*

        ↳ First practical application of this framework **[S. Piloñeta, AV, 2407.06277]**

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Angular decomposition of the cross-section

$$\frac{d\sigma}{dp_T dy dQ d\Omega} = \frac{3}{16\pi}\boxed{\frac{d\sigma^{U+L}}{dp_T dy dQ}}\Big[(1+\cos^2\theta) + \frac{1-3\cos^2\theta}{2}A_0 + \sin 2\theta\cos\phi A_1 + \frac{\sin^2\theta\cos 2\phi}{2}A_2 + \sin\theta\cos\phi A_3 + \cos\theta A_4 + \sin^2\theta\sin 2\phi A_5 + \sin 2\theta\sin\phi A_6 + \sin\theta\sin\phi A_7\Big]$$

A_U → Unpolarized angle-integrated cross-section

$A_{0,\dots,7}$ → Only four at LP $\{A_U, A_2, A_4, A_5\}$

→ The rest are power-suppressed

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Angular decomposition of the cross-section

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$$\frac{\sin^2\theta\cos 2\phi}{2} A_2 + \sin\theta\cos\phi A_3 + \cos\theta A_4 +$$

$$\sin^2\theta \sin 2\phi A_5 + \sin 2\theta \sin\phi A_6 + \sin\theta\sin\phi A_7 \Big]$$

$A_{0,\ldots,7}$ → Only four at LP $\{A_U, A_2, A_4, A_5\}$

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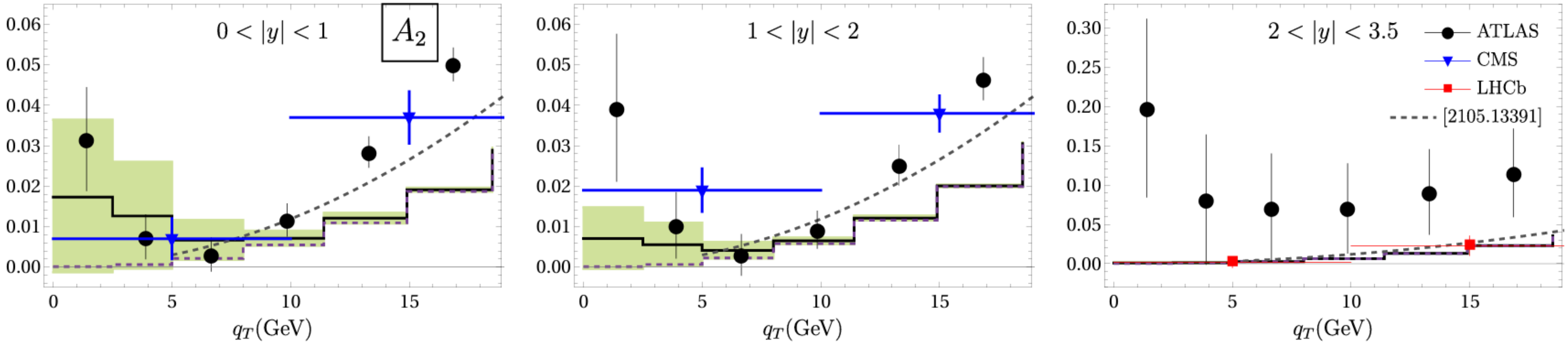
When we incorporate the summed KPCs we can describe all of them!

Angular distribution A_2



$$A_2 \sim \left[(v^2 - a^2)_q \frac{h_1^\perp h_1^\perp}{M^2} + (v^2 + a^2)_q \frac{f_1 f_1}{Q^2}\right]$$

- Contains both Boer-Mulders and unpolarized distributions

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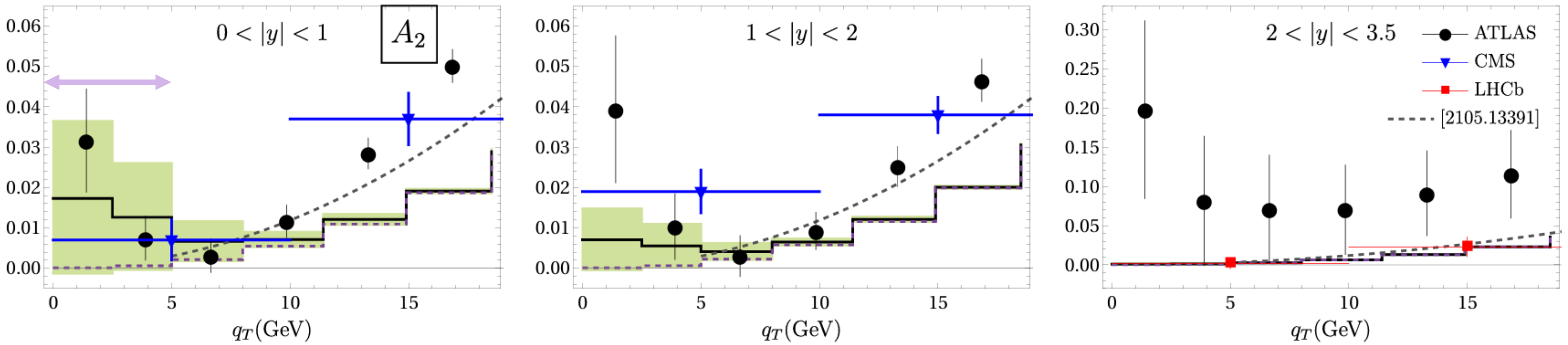

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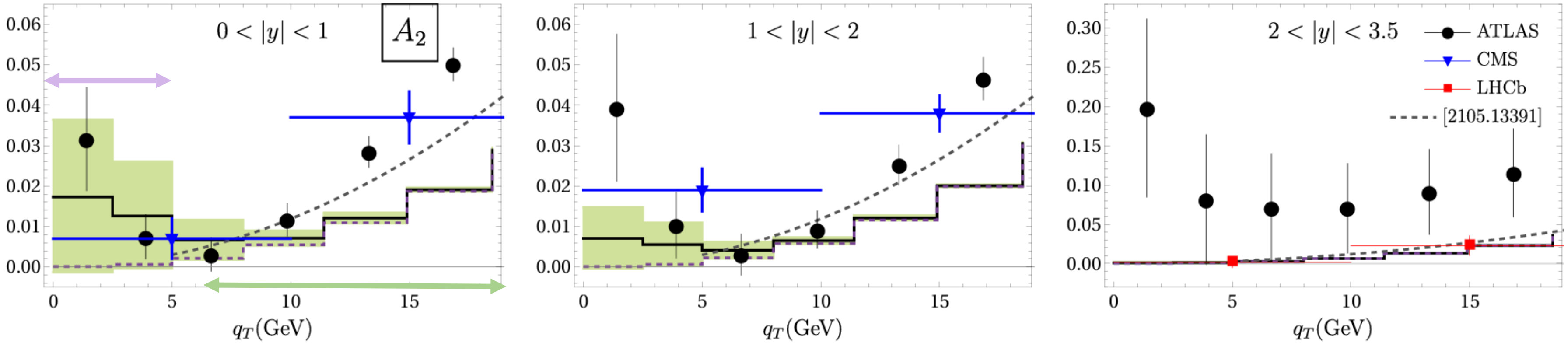

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Dominant at larger q_T

Dominant at $q_T \to 0$

- Contains both Boer-Mulders and unpolarized distributions

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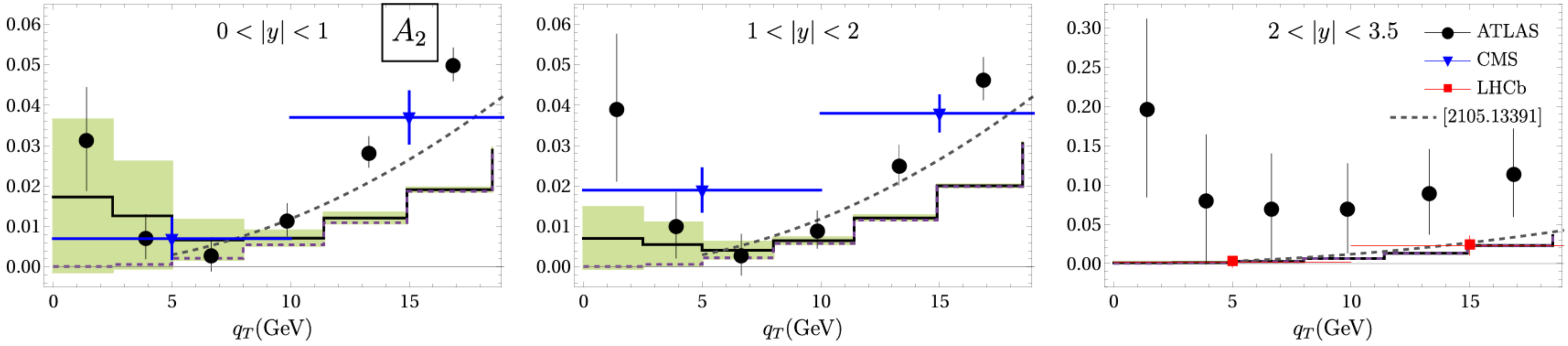

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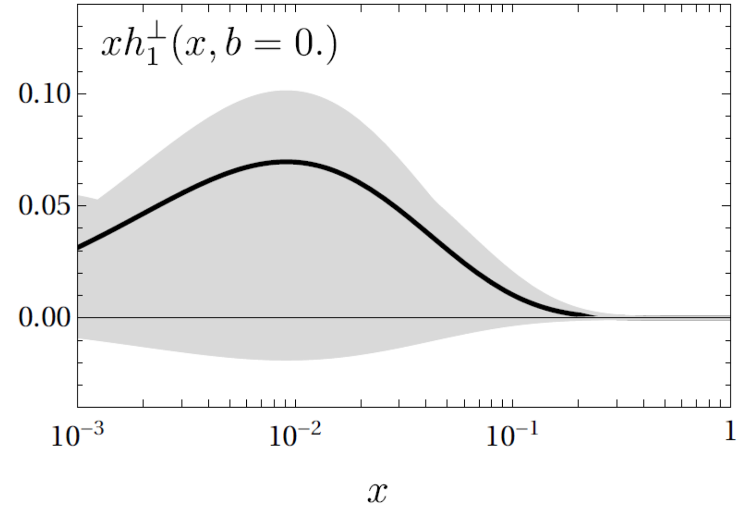
- Boer-Mulders distribution extraction from A_2
  - Very primitive model
    - New fit is required
  - First evidence (?) of twist-3 effects at LHC

$$x|E(-x,0,x)|$$



Lam-Tung relation ($A_0 - A_2$) description

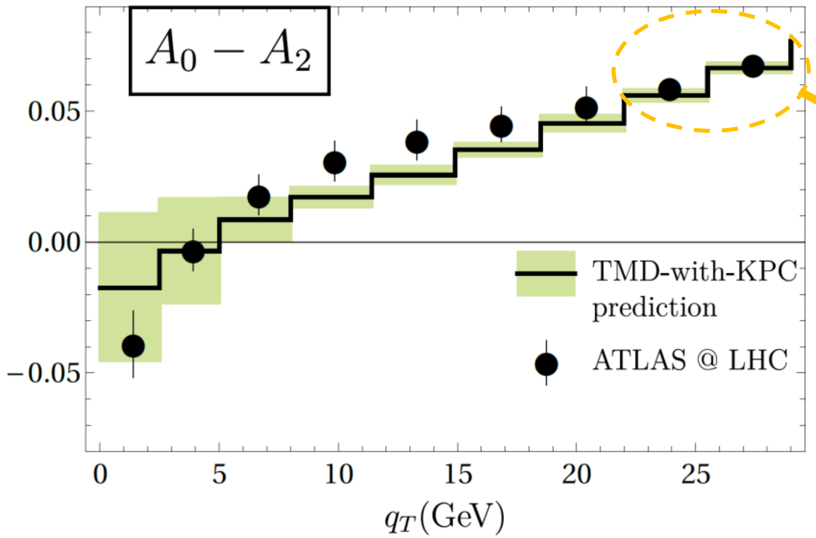
- At leading power, the TMD factorization theorem can not describe it

$$(A_0 - A_2)_{LP} \sim \boldsymbol{k}^2/M^2\, h_1^\perp h_1^\perp$$

The Boer-Mulders is very small!

- If we include KPCs the theoretical expression also contains the unpolarized f_1

This allows us to make a prediction for the Lam-Tung relation



Y-term suppressed by an extra α_S factor

Excellent agreement up to $q_T \sim 30\,\mathrm{GeV}$!

Good description only possible due to KPCs inclusion!

Moving to SIDIS: structure functions computation using KPCs

- Now, let's shift our focus to SIDIS → essential process for probing hadron structure

Cross-section decomposition in terms of structure functions

$$\frac{d\sigma}{dx\,dy\,d\psi\,dz\,d\phi_h\,d\boldsymbol{p}_\perp^2} = \frac{\alpha_{\rm em}^2}{xyQ^2}\frac{y^2}{2(1-\varepsilon)}\Big\{F_{UU,T} + \varepsilon F_{UU,L} + \sqrt{2\varepsilon(1+\varepsilon)}\cos\phi_h F_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h) F_{UU}^{\cos 2\phi_h} + \lambda_e\sqrt{2\varepsilon(1-\varepsilon)}\sin\phi_h F_{LU}^{\sin\phi_h} + \ldots.$$

We want to compute them including KPCs

$$\Big\{F_{UU,T} + \ldots\Big\} = \frac{x}{4z}\frac{1-\varepsilon}{Q^2}L_{\mu\nu}W^{\mu\nu}$$

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1 Lepton tensor conveniently decomposed via the tensors $P^\mu,\ q^\mu,\ p_\perp^\mu$

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2 Hadron tensor computed using the TMD-with-KPCs factorization theorem

- Same coefficient function as LP $\quad q_\mu W^{\mu\nu} = 0$
- Main difference with LP → Convolution integral $\mathcal{C}_{KPC}$

A closer look at the convolution integral

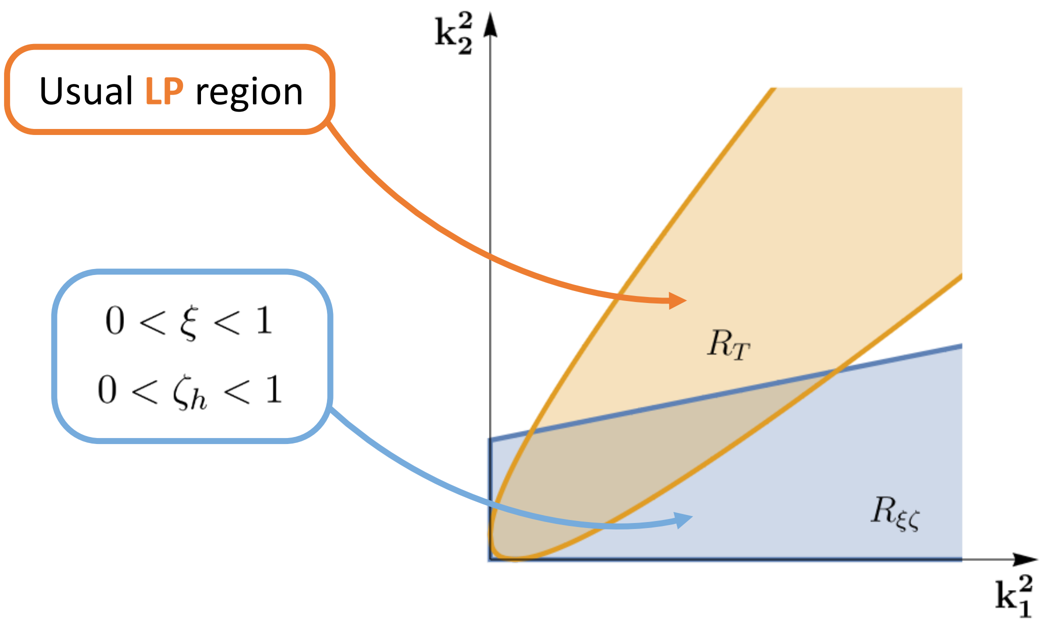
- The convolution integral is more complicated now

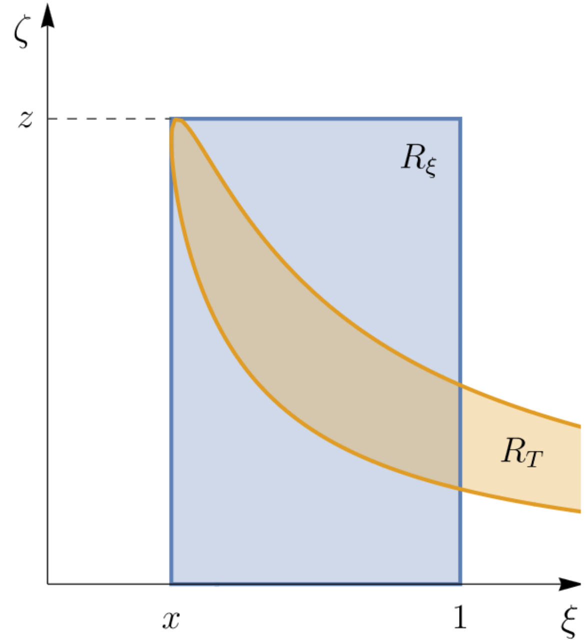
LP case $\quad \mathcal{C}_{LP}[A, f_1 D_1] \sim \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2\, \delta(\boldsymbol{q_T} + \boldsymbol{k_1} - \boldsymbol{k_2})\, f_1(x_1, \boldsymbol{k}_1^2) D_1(z_1, \boldsymbol{k}_2^2)$

KPCs case $\quad \mathcal{C}_{KPC}[A, f_1 D_1] \sim \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2\, \delta(\boldsymbol{q_T} + \boldsymbol{k_1} - \boldsymbol{k_2})\, f_1(\underbrace{\xi(x_1, \boldsymbol{k}_{1,2}^2), \boldsymbol{k}_1^2) D_1(\zeta_h(z_1, \boldsymbol{k}_{1,2}^2)}, \boldsymbol{k}_2^2)$

Additional dependance
coming from extra δ-functions

- The integration domain also changes





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KPCs case $$\mathcal{C}_{KPC}[A, f_1 D_1] \sim \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2\, \delta(\boldsymbol{q_T} + \boldsymbol{k_1} - \boldsymbol{k_2})\, f_1(\xi(x_1, \boldsymbol{k}_{1,2}^2), \boldsymbol{k}_1^2) D_1(\zeta_h(z_1, \boldsymbol{k}_{1,2}^2), \boldsymbol{k}_2^2)$$

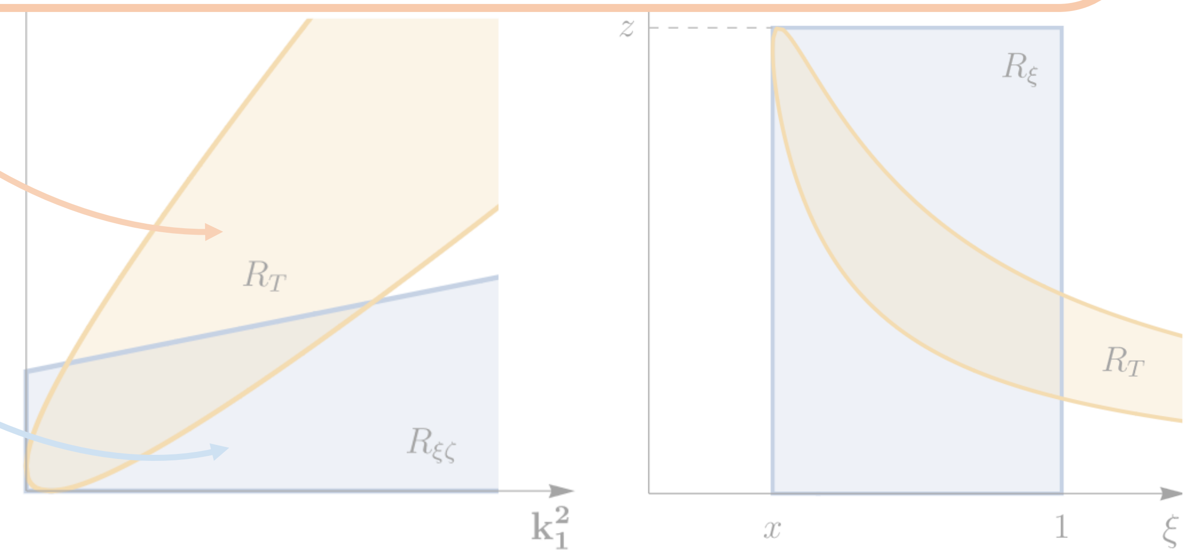
- The i

The convolution integral and the KPCs have been implemented in *arTeMiDe*, github.com/VladimirovAlexey/artemide-development

Usual LP region

$0 < \xi < 1$

$0 < \zeta_h < 1$



Focusing on $F_{UU,T}$ and $F_{UU,L}$

- I have obtained the **theoretical expressions** for all of them using the **TMD-with-KPCs factorization**

  But in **this talk** I am going to **focus on** $F_{UU,T}$ and $F_{UU,L}$

$$F_{UU,T} = \frac{x}{4z} F_0 (\mathcal{S}_1^{\mu\nu} - \mathcal{S}_0^{\mu\nu}) W_U^{\mu\nu} \qquad F_{UU,L} = \frac{x}{4z} F_0 (2\mathcal{S}_1^{\mu\nu}) W_U^{\mu\nu}$$

$$\mathcal{S}_0^{\mu\nu} = g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2} \qquad \mathcal{S}_1^{\mu\nu} = \frac{(2xP^\mu + q^\mu)(2xP^\nu + q^\nu)}{(1+\gamma^2)Q^2}$$

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The theoretical expression for $F_{UU,T}$ including KPCs is

$$F_{UU,T} = \frac{x}{2z} Q^2 \, \mathcal{C}_{KPC}[1, f_1 D_1] + \frac{1}{2} F_{UU,L}$$

The inclusion of the KPCs implies that $F_{UU,T}$ is not exactly 1 as it was in the LP case!

$$\underbrace{\mathcal{C}_{LP}[1, f_1 D_1]}_{(F_{UU,T})_{LP}} + \varepsilon F_{UU,L} \quad \longrightarrow \quad \text{Cross-section gets modified} \quad \longrightarrow \quad \mathcal{C}_{KPC}[1, f_1 D_1] + \left(\frac{1}{2} + \varepsilon\right) F_{UU,L}$$

$$F_{UU,T} + \varepsilon F_{UU,L}$$

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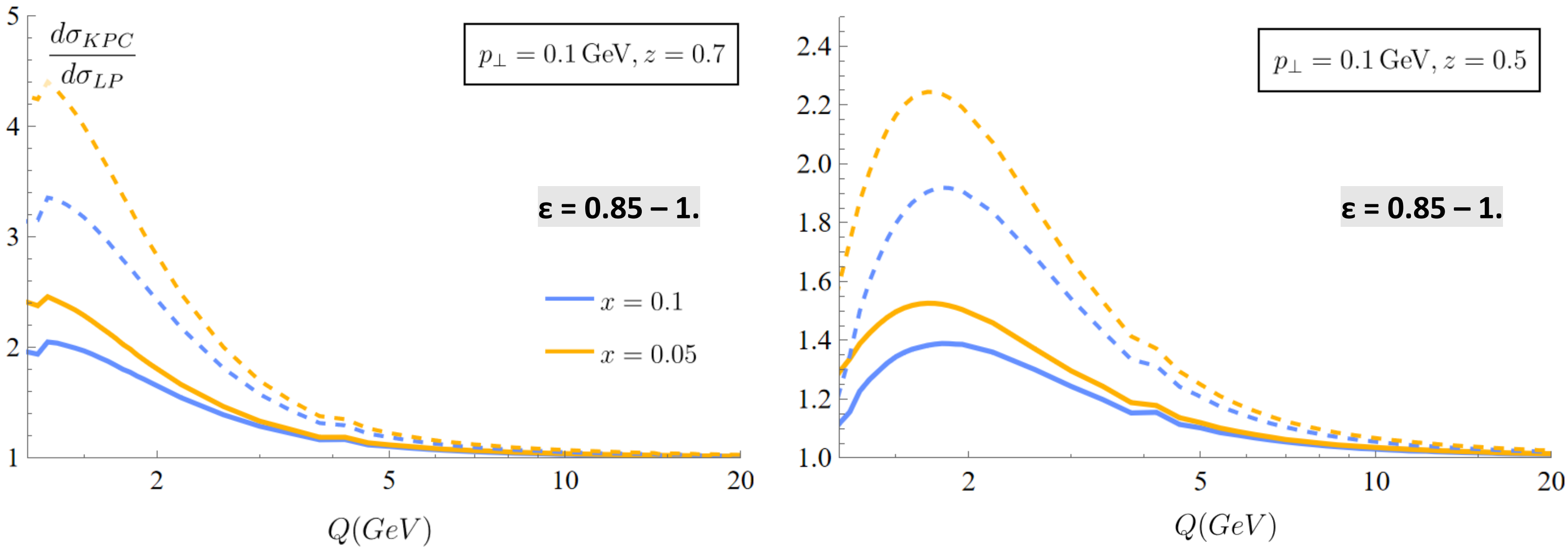
$$F_{UU,T} + \varepsilon F_{UU,L}$$

- I have studied the KPCs impact on
  - The unpolarized SIDIS cross-section $F_{UU,T} + \varepsilon F_{UU,L}$
  - The structure functions $F_{UU,T}$ and $F_{UU,L}$

Ratio of KPCs-summed to LP cross-sections

- I study the ratio $d\sigma_{KPC}/d\sigma_{LP}$ in two different scenarios

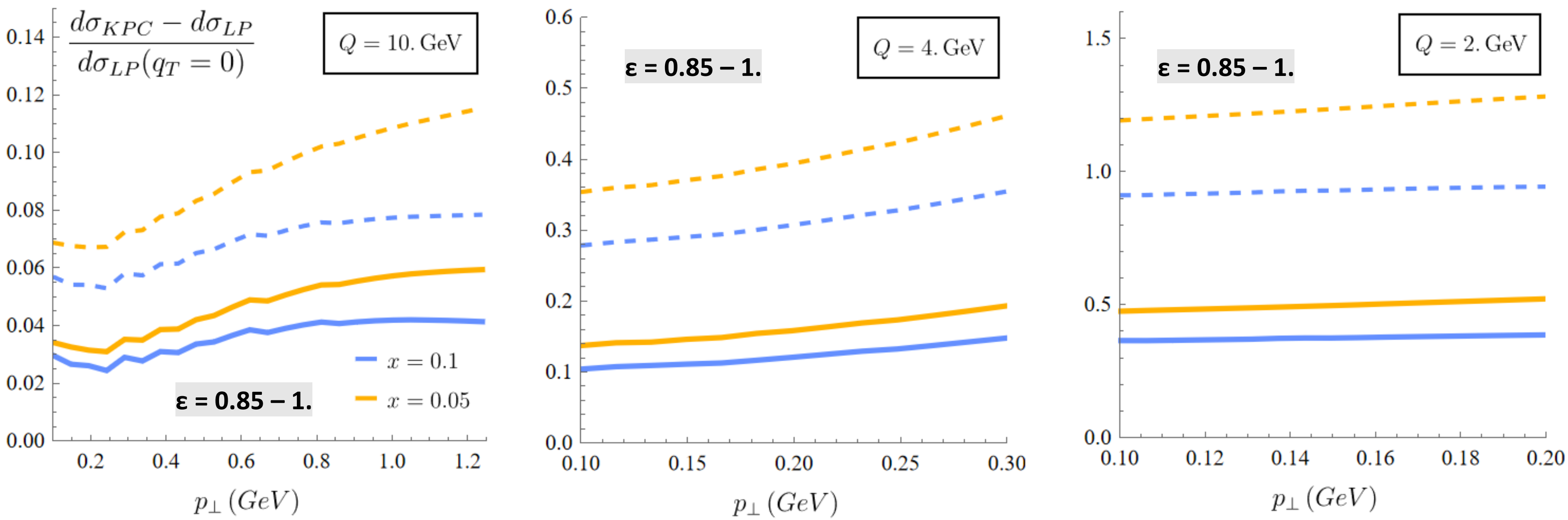
(i) —— $d\sigma_{KPC} \sim F_{UU,T}$ (ii) - - - $d\sigma_{KPC} \sim F_{UU,T} + \varepsilon F_{UU,L}$



Q = 2 GeV → Corrections comparable to $(F_{UU,T})_{LP}$ both for $F_{UU,T}$ and $F_{UU,L}$

→ As Q grows the corrections become smaller and $\lim_{Q\to\infty} d\sigma_{KPC} = d\sigma_{LP}$

Cross-sections difference relative to the $q_T = 0$ LP cross-section



✓ KPCs very important at low energies

Q = 2 GeV → About 40-50% $\xrightarrow{+\,\varepsilon F_{UU,L}}$ Several tens of percents!

✓ Almost flat increase of the cross-section → The inclusion of KPCs mostly changes the normalization

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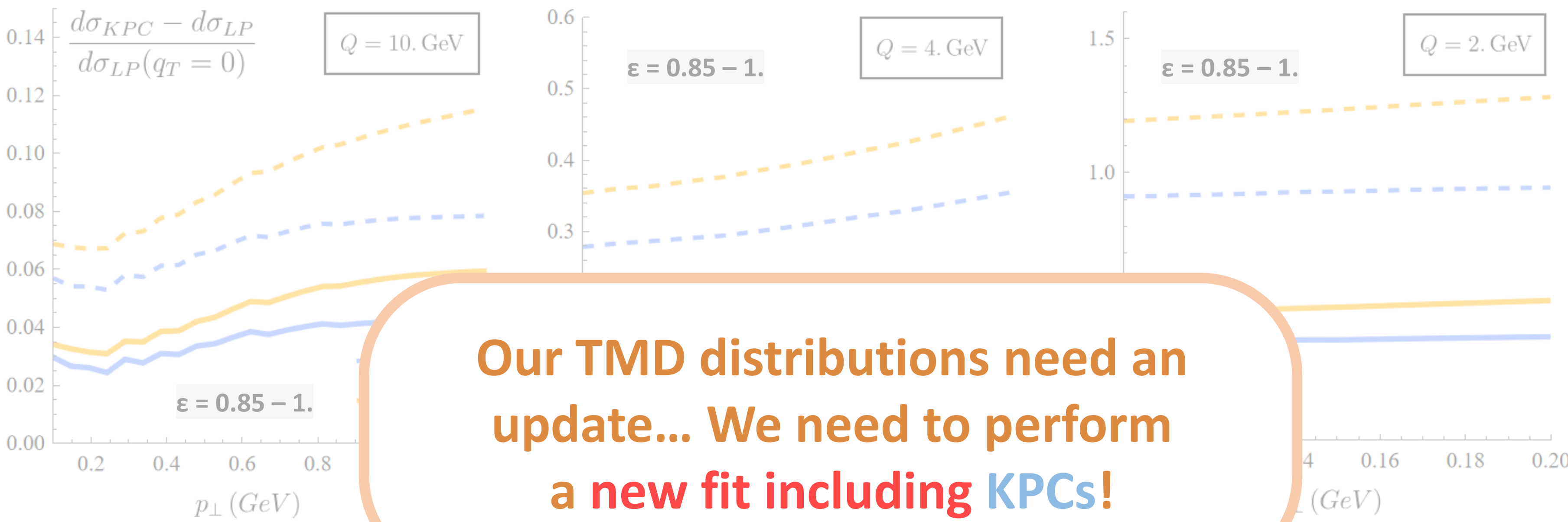

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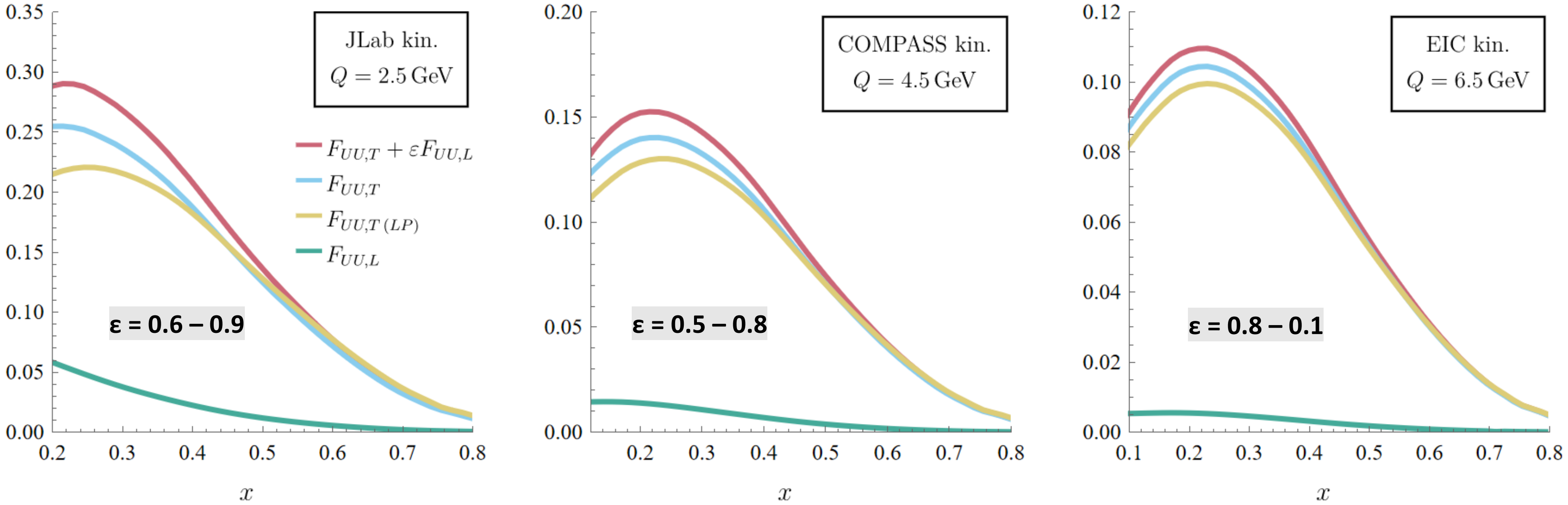
Longitudinal photons effects in different kinematics

- What about the contribution of longitudinal photons for different experiments?



✓ At low energies, the $F_{UU,L}$ contribution is clearly not negligible

JLab → significant difference between $(F_{UU,T})_{LP}$ and $F_{UU,T} + \varepsilon F_{UU,L}$

✓ Less important at higher energies → EIC → smaller but still visible

Conclusions

- The TMD-with-KPCs factorization theorem [2307.13054v2] has been tested
    - The angular distributions of Drell-Yan leptons can be satisfactorily described
    - It gives a nice description of the Lam-Tung relation

[2407.06277]

- The subleading $F_{UU,T}$ and $F_{UU,L}$ SIDIS structure functions have been computed using it
    - The SIDIS cross-section grows when including KPCs
    - The $F_{UU,L}$ contribution is not negligible at low energies like 2 – 5 GeV

[2510.14496]

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  - The $F_{UU,L}$ …

Thank you for your attention! ☺

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BACKUP

Types of power corrections

- The power corrections that appear in the "pyramid" are

Higher-twist corrections $\sim (\Lambda/Q)^{n-2}$ → TMDs of larger twist ($n = D - S$)

q_T/Q corrections

Position space

$$\sim \frac{1}{bQ}$$

↔

Momentum space

$$\sim \frac{q_T}{Q}$$

k_T/Q corrections → kinematic power corrections (KPCs)

Position space

$$\sim F_1(x_1, b)\,\frac{\partial}{\partial b^\mu}\,F_2(x_2, b)$$

↔

Momentum space

$$\sim \frac{k_{2T}^\mu}{Q} F_1(x_1, k_{1T}) F_2(x_2, k_{2T})$$

Transverse derivatives of TMD distributions

The hierarchy of the TMD factorization theorem (II)

$$W^{\mu\nu} = \frac{1}{N_c}\int \frac{d^2b}{(2\pi)^2} e^{-i(q_T b)} \Big\{ \boxed{\Phi_2 \times \Phi_2} \quad \text{LP}$$

$$+\frac{1}{Q}\boxed{\Big(D\ \Phi_2 \times \Phi_2 + \quad \Phi_2 \times \Phi_3\Big)} \quad \text{NLP}$$

$$+\frac{1}{Q^2}\boxed{\Big(D^2\Phi_2 \times \Phi_2 + D\ \Phi_2 \times \Phi_3 + \quad \Phi_3 \times \Phi_3 + \quad \Phi_2 \times \Phi_4 + \frac{\Phi_2 \times \Phi_2}{b^2}\Big)} \quad \text{NNLP}$$

$$+\frac{1}{Q^3}\Big(D^3\Phi_2 \times \Phi_2 + D^2\Phi_2 \times \Phi_3 + D\Phi_3 \times \Phi_3 + D\Phi_2 \times \Phi_4 + \frac{D\Phi_2 \times \Phi_2}{b^2} + \Phi_3 \times \Phi_4 + \ldots\Big)$$

$$+\cdots \Big\}^{\mu\nu}$$

Notation

$\Phi_n(b, x_i)$ → TMD of twist-*n*

$D \sim \frac{\partial}{\partial b^\mu}$ → transverse derivative

Separating power corrections is not trivial

→ Organize operators by evolution properties

TMD-twist-*n*

Φ_n and Φ_m with $n \neq m$ do not mix under TMD evolution

The hierarchy of the TMD factorization theorem (III)

$$W^{\mu\nu} = \frac{1}{N_c}\int \frac{d^2b}{(2\pi)^2} e^{-i(q_T b)} \Big\{ \boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2}$$

$$+\frac{1}{Q}\Big(D\ \boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2} + \quad \boldsymbol{\Phi_2} \times \boldsymbol{\Phi_3}\Big)$$

$$+\frac{1}{Q^2}\Big(D^2\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2} + D\ \boldsymbol{\Phi_2} \times \boldsymbol{\Phi_3} + \quad \boldsymbol{\Phi_3} \times \boldsymbol{\Phi_3} + \quad \boldsymbol{\Phi_2} \times \boldsymbol{\Phi_4} + \frac{\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2}}{b^2}\Big)$$

$$+\frac{1}{Q^3}\Big(D^3\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2} + D^2\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_3} + D\boldsymbol{\Phi_3} \times \boldsymbol{\Phi_3} + D\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_4} + \frac{D\boldsymbol{\Phi_2} \times \boldsymbol{\Phi_2}}{b^2} + \boldsymbol{\Phi_3} \times \boldsymbol{\Phi_4} + \ldots\Big)$$

$$+\cdots \Big\}^{\mu\nu}$$

Notation

$\Phi_n(b, x_i)$ → TMD of twist-*n*

$D \sim \frac{\partial}{\partial b^\mu}$ → transverse derivative

KPCs that follow the LP term

Only involve twist-2 distributions

We extract this part from the whole set

Definition of TMD-twist

- At any power in the TMD factorization theorem, TMDs have the same structure

$$\widetilde{\Phi}_{AB}(\{z\}_A, \{z\}_B; b) \sim \langle P | U_A(\{z\}_A; b) U_B(\{z\}_B; 0) | P \rangle$$

- The renormalization of a TMD distribution consists of two renormalization factors

$$\mu^2 \frac{d}{d\mu^2} \widetilde{\Phi}_{AB}(\{z\}_A, \{z\}_B; b) = \Big(\tilde{\gamma}_A(\{z\}_A) + \tilde{\gamma}_B(\{z\}_B)\Big) \otimes \widetilde{\Phi}_{AB}(\{z\}_A, \{z\}_B; b)$$

U_A and U_B independent $\tilde{\gamma}$ if different geometrical twist → dimension-spin

TMDs $\widetilde{\Phi}_{AB}$ and $\widetilde{\Phi}_{A'B'}$ do not mix if geometrical twist of U_A or U_B is different from $U_{A'}$ or $U_{B'}$

TMD-twist → pair of integers (N, M) — U_A U_B

TMDs with different TMD-twist evolve independently

Vital for systematization of power corrections

Definition of TMD-twist

- At any power in the TMD factorization theorem, TMDs have the same structure

$$\widetilde{\Phi}_{AB}(\{z\}_A,\{z\}_B;b) \sim \langle P|U_A(\{z\}_A;b)U_B(\{z\}_B;0)|P\rangle$$

- The reno

Comment on notation

TMD-twist as a single number (N+M)

$$\Phi_n \longrightarrow n = N + M$$

- Twist-2 TMD → TMD-twist (1,1)
- Twist-3 TMD → TMD-twist (1,2) or (2,1)
- Twist-4 TMD → TMD-twist (1,3), (3,1) or (2,2)

TMDs with different TMD-twist evolve independently

Vital for systematization of power corrections

Twist-2 part of a TMDPDF correlator

- Focusing on the following TMDPDF correlator

$$\widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y) = \langle P|\bar{q}_{\bar{n}}(y)[y, -\infty n + y_T + y^+\bar{n}]\frac{\Gamma}{2}[-\infty n, 0]q_{\bar{n}}(0)|P\rangle$$

Sum of contributions with different twist

First step in the extraction of the twist-2 part

Expand the fields around the coordinate *y*

$$\widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y) = \sum_{n=0}^{\infty}\frac{(y^+)^n}{n!}\underbrace{\left[\partial_-^n \widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y^- n + y_T)\right]}$$

Generating function for a series of power corrections

$$\partial_-^n \widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y^- n + y_T) = \langle P|\bar{q}_{\bar{n}}(y^- n + y_T)[y^- n + y_T, -\infty n + y_T]\overleftarrow{\partial_-^n}\frac{\Gamma}{2}[-\infty n, 0]q_{\bar{n}}(0)|P\rangle$$

Twist-2 part of a TMDPDF correlator

- We have an infinite series of contributions with different twist

$$\partial_-^n \widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y^- n + y_T) = \langle P|\bar{q}_{\bar{n}}(y^- n + y_T)[y^- n + y_T, -\infty n + y_T]\,\overleftarrow{\partial}_-^n\, \frac{\Gamma}{2}\,[-\infty n, 0] q_{\bar{n}}(0)|P\rangle$$

Next step in the extraction of the twist-2 part

Split quark fields into good and bad components and rewrite the bad ones using equations of motion

Drop gluons (), they increase the twist!

$$\widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y)\Big|_{\text{tw-2}} = \sum_{n=0}^{\infty} \frac{(y^+)^n}{n!}\left(-\frac{\partial_T^2}{2\partial_+}\right)^n \Bigg\{ \widetilde{\Psi}_{\bar{n}11}^{[\![\Gamma]\!]}(y^- n + y_T) - \frac{\partial_{T\alpha}}{2\partial_+}\,\widetilde{\Psi}_{\bar{n}11}^{[\![\gamma^\alpha\gamma^+\Gamma + \Gamma\gamma^+\gamma^\alpha]\!]}(y^- n + y_T) + \\ + \frac{\partial_{T\alpha}\partial_{T\beta}}{4\partial_+^2}\,\widetilde{\Psi}_{\bar{n}11}^{[\![\gamma^\alpha\gamma^+\Gamma\gamma^+\gamma^\beta]\!]}(y^- n + y_T)\Bigg\}.$$

Now the correlator only contains twist-2 contributions

Twist-2 part of a TMDPDF correlator

- The summation of the series is straightforward in the momentum space

$$\widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y)\Big|_{\text{tw-2}} = \sum_{n=0}^{\infty} \frac{p^+}{n!} \int dx\, d^2k_T\, e^{ixp^+y^- + i(ky)_T} \left(\frac{-iy^+k_T^2}{2xp^+}\right)^n \Bigg\{$$

$$\Phi_{\bar{n}11}^{[\![\Gamma]\!]}(x,k_T) - \frac{k_{T\alpha}}{2xp^+}\, \Phi_{\bar{n}11}^{[\![\gamma^\alpha\gamma^+\Gamma+\Gamma\gamma^+\gamma^\alpha]\!]}(x,k_T) + \frac{k_{T\alpha}k_{T\beta}}{4(xp^+)^2}\, \Phi_{\bar{n}11}^{[\![\gamma^\beta\gamma^+\Gamma\gamma^+\gamma^\alpha]\!]}(x,k_T) \Bigg\}$$

- The final expression for the twist-2 correlator after some manipulations is

$$\widetilde{\Psi}_{\bar{n}}^{[\Gamma]}(y)\Big|_{\text{tw-2}} = p^+ \int d^4k\, \delta(k^2)\, (2k^+)\, e^{i(ky)} \int dx\, \delta(k^+ - xp^+) \Bigg\{$$

$$\Phi_{\bar{n}11}^{[\![\Gamma]\!]}(x,k_T) - \frac{k_{T\alpha}}{2xp^+}\, \Phi_{\bar{n}11}^{[\![\gamma^\alpha\gamma^+\Gamma+\Gamma\gamma^+\gamma^\alpha]\!]}(x,k_T) + \frac{k_{T\alpha}k_{T\beta}}{4(xp^+)^2}\, \Phi_{\bar{n}11}^{[\![\gamma^\alpha\gamma^+\Gamma\gamma^+\gamma^\beta]\!]}(x,k_T) \Bigg\}$$

Twist-2 part of a TMDPDF correlator

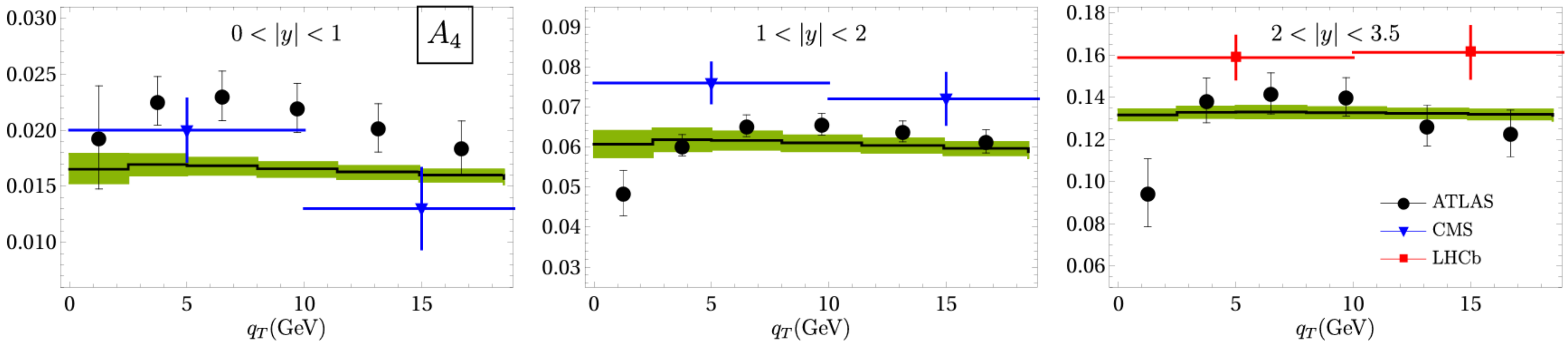
- Inserting the elements of the Dirac basis $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, i\sigma^{\mu\nu}\gamma^5\}$ in the correlator allows the TMDPDFs to appear explicitly

$$
\begin{aligned}
\widetilde{\Psi}_{\bar{n}}^{[1]}(y)\Big|_{\text{tw-2}} &= \widetilde{\Psi}_{\bar{n}}^{[\gamma^5]}(y)\Big|_{\text{tw-2}} = 0, \\
\widetilde{\Psi}_{\bar{n}}^{[\gamma^\mu]}(y)\Big|_{\text{tw-2}} &= 2p^+ \int d^4k\,\delta(k^2)\,e^{i(ky)} \int dx\,\delta(k^+ - xp^+)\,k^\mu\,\Phi_{\bar{n}11}^{[\gamma^+]}(x, k_T), \\
\widetilde{\Psi}_{\bar{n}}^{[\gamma^\mu\gamma^5]}(y)\Big|_{\text{tw-2}} &= 2p^+ \int d^4k\,\delta(k^2)\,e^{i(ky)} \int dx\,\delta(k^+ - xp^+)\,k^\mu\,\Phi_{\bar{n}11}^{[\gamma^+\gamma^5]}(x, k_T), \\
\widetilde{\Psi}_{\bar{n}}^{[i\sigma^{\mu\nu}\gamma^5]}(y)\Big|_{\text{tw-2}} &= 2p^+ \int d^4k\,\delta(k^2)\,e^{i(ky)} \int dx\,\delta(k^+ - xp^+)\Big(\\
& g_T^{\mu\alpha}k^\nu - g_T^{\nu\alpha}k^\mu + \frac{k^\mu n^\nu - n^\mu k^\nu}{k^+}k_T^\alpha\Big)\,\Phi_{\bar{n}11}^{[i\sigma^{\alpha+}\gamma^5]}(x, k_T).
\end{aligned}
$$

Angular distribution A_4

- Leading power
- Proportional to the difference between quark and anti-quark distributions (anti-symmetric flavor)
- Inclusion in standard f_1 extractions

➢ Comparison as a function of q_T with **ATLAS**, CMS and **LHCb** measurements

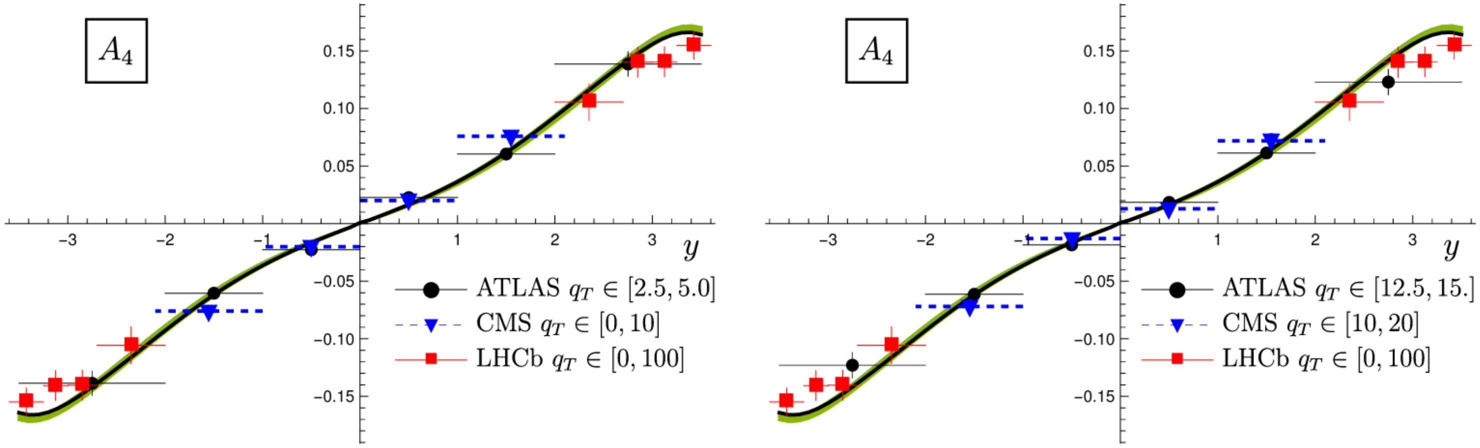


- Theory prediction agrees very well with the measurements

Angular distribution A_4

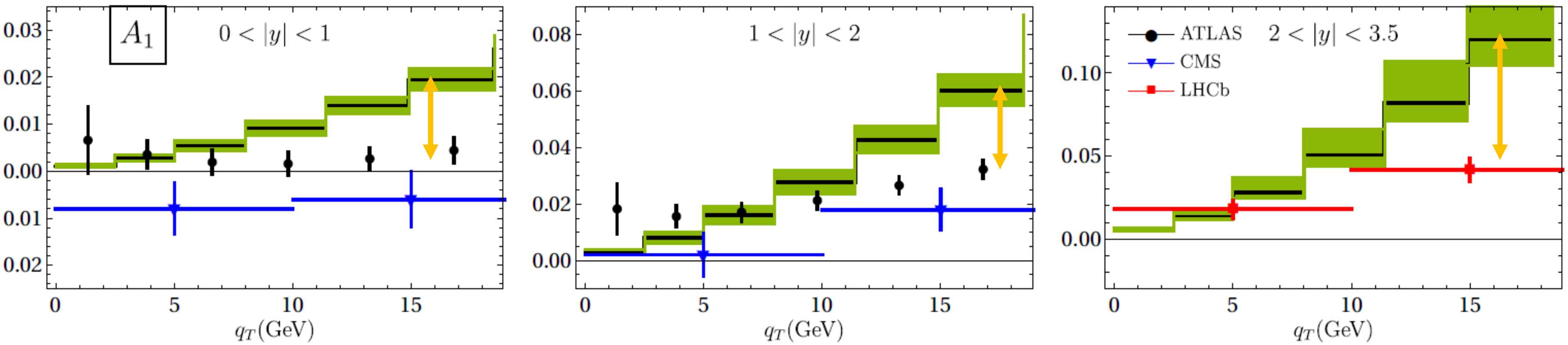
- **Leading power**
- Proportional to the **difference between quark and anti-quark** distributions (**anti-symmetric flavor**)
- **Inclusion** in standard f_1 **extractions**

➢ Comparison as a **function** of **y** with **ATLAS**, **CMS** and **LHCb** measurements



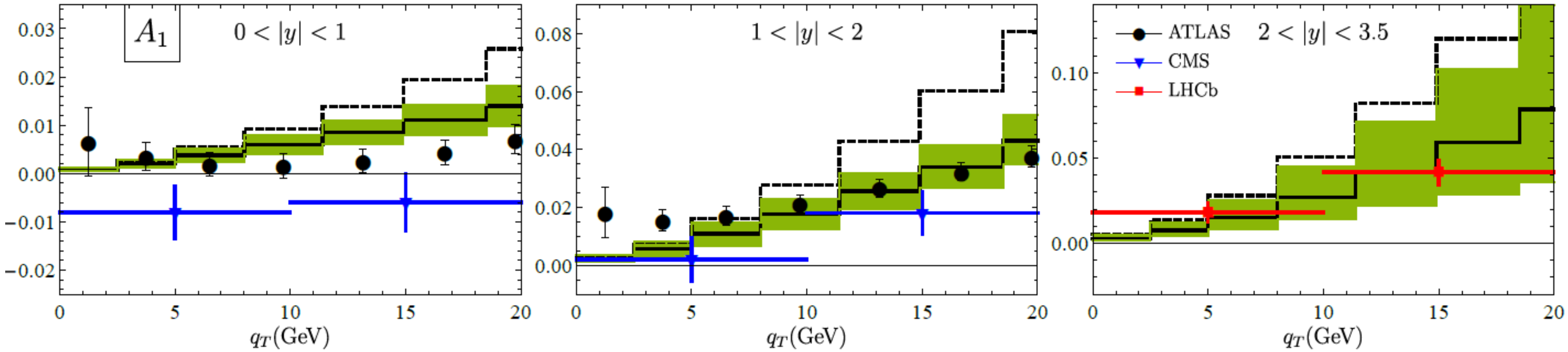
- The **agreement** is **even more transparent**

Angular distribution A_1



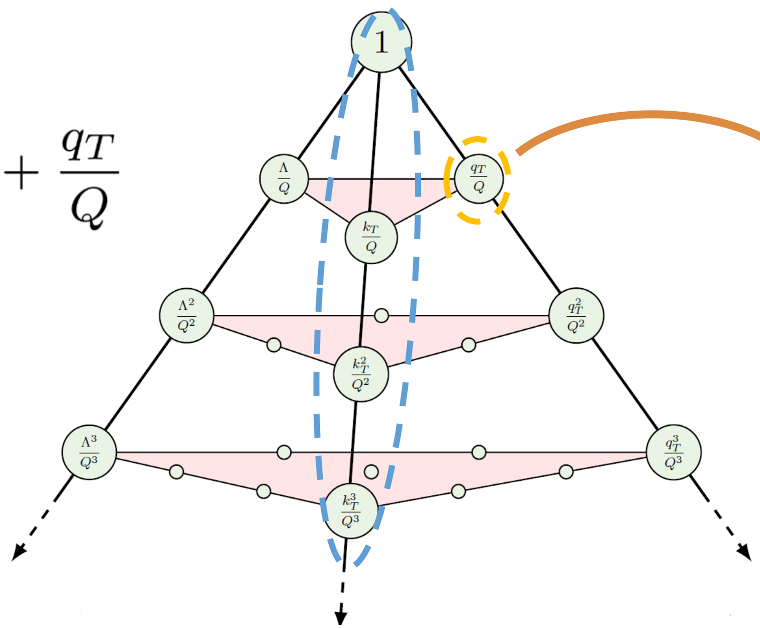
- Problems with the A_1 data description at larger values of q_T

Angular distribution A_1



- **Problems** with the A_1 data description at **larger values of q_T**

KPCs $+ \frac{q_T}{Q}$



Including the computation of the **leading q_T/Q correction** fixes the jump!

[A.Arroyo, I.Scimemi, AV, 2503.24336]

Twist-2 part of a TMDFF correlator

- The TMDFF correlator reads

$$\widetilde{\Theta}_n^{[\Gamma]}(y) = \sum_X \mathrm{Tr}\,\langle 0| \frac{\Gamma}{2}\,[-\infty\bar{n} + y_T + y^- n, y] q_n(y) |p_h, X\rangle\langle p_h, X| \bar{q}_n(0)[0, -\infty\bar{n}]|0\rangle$$

- Follow the same strategy as before to extract its twist-2 part

1) Multipole expansion of the quark fields
2) Split into "good" and "bad" components
3) Rewrite the "bad" ones using equations of motion

Drop gluons (), they increase the twist!

A slight difference: the produced hadron is aligned with the *n*-direction

- The final expression for the twist-2 TMDFF correlator is

$$\widetilde{\Theta}_n^{[\Gamma]}(y)\Big|_{\mathrm{tw2}} = 2\,p_h^- \int d^4k\,\delta(k^2)(2k^-)e^{-i(yk)} \int \frac{d\zeta}{\zeta}\delta\left(k^- - \frac{p_h^-}{\zeta}\right)\Bigg[$$

$$\Delta_{n11}^{[\![\Gamma]\!]}(\zeta, k_T) - \frac{k_{T\alpha}}{2p_h^-}\Delta_{n11}^{[\![\gamma^\alpha\gamma^-\Gamma + \Gamma\gamma^-\gamma^\alpha]\!]}(\zeta, k_T) + \zeta\frac{k_{T\alpha}k_{T\beta}}{4p_h^-}\Delta_{n11}^{[\![\gamma^\beta\gamma^-\Gamma\gamma^-\gamma^\alpha]\!]}(\zeta, k_T)\Bigg]$$

Twist-2 part of a TMDFF correlator

- Inserting the elements of the Dirac basis $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, i\sigma^{\mu\nu}\gamma^5\}$ in the correlator allows the TMDFFs to appear explicitly

$$
\begin{aligned}
\widetilde{\Theta}_n^{[1]}(y)\Big|_{\rm tw2} &= \widetilde{\Theta}_n^{[\gamma^5]}(y)\Big|_{\rm tw2} = 0, \\
\widetilde{\Theta}_n^{[\gamma^\mu]}(y)\Big|_{\rm tw2} &= 4\,p_h^- \int d^4k\, \delta(k^2) e^{-i(ky)} \int \frac{d\zeta}{\zeta} \delta\left(k^- - \frac{p_h^-}{\zeta}\right) k^\mu\, \Delta_{n11}^{[\gamma^-]}(\zeta, k_T), \\
\widetilde{\Theta}_n^{[\gamma^\mu\gamma^5]}(y)\Big|_{\rm tw2} &= 4\,p_h^- \int d^4k\, \delta(k^2) e^{-i(ky)} \int \frac{d\zeta}{\zeta} \delta\left(k^- - \frac{p_h^-}{\zeta}\right) k^\mu\, \Delta_{n11}^{[\gamma^-\gamma^5]}(\zeta, k_T), \\
\widetilde{\Theta}_n^{[i\sigma^{\mu\nu}\gamma^5]}(y)\Big|_{\rm tw2} &= 4\,p_h^- \int d^4k\, \delta(k^2) e^{-i(ky)} \int \frac{d\zeta}{\zeta} \delta\left(k^- - \frac{p_h^-}{\zeta}\right) \Big(\\
& g_T^{\mu\alpha} k^\nu - g_T^{\nu\alpha} k^\mu + \frac{k^\mu \bar{n}^\nu - \bar{n}^\mu k^\nu}{k^-} k_T^\alpha \Big) \Delta_{n11}^{[i\sigma^{\alpha -}\gamma^5]}(\zeta, k_T).
\end{aligned}
$$

SIDIS hadron tensor with summed KPCs

- In SIDIS, the hadron tensor obtained a tree level after collecting the fields into matrix elements is

$$W^{\mu\nu}_{\text{tree}} \sim \sum_{a,b} \left\{ \mathrm{Tr}(\gamma^\mu \overline{\Gamma}_b \gamma^\nu \overline{\Gamma}_a) \widetilde{\Psi}^{[\Gamma_a]}(y) \widetilde{\Theta}^{[\Gamma_b]}(y) + \mathrm{Tr}(\gamma^\mu \overline{\Gamma}_a \gamma^\nu \overline{\Gamma}_b) \overline{\widetilde{\Psi}}^{[\Gamma_a]}(y) \overline{\widetilde{\Theta}}^{[\Gamma_b]}(y) \right\} + \ldots$$

Contains contributions of various twists

diagrams with additional 

- Keeping only the first part and inserting the expressions for the twist-2 correlators we obtain the SIDIS hadron tensor with summed KPCs

$$\begin{aligned} W^{\mu\nu}_{\text{KPC}} \quad \sim \quad & \sum_{a,b} \left\{ \mathrm{Tr}(\gamma^\mu \overline{\Gamma}_b \gamma^\nu \overline{\Gamma}_a) \widetilde{\Psi}^{[\Gamma_a]}(y)\Big|_{\text{tw2}} \widetilde{\Theta}^{[\Gamma_b]}(y)\Big|_{\text{tw2}} \right. \\ & \left. + \mathrm{Tr}(\gamma^\mu \overline{\Gamma}_a \gamma^\nu \overline{\Gamma}_b) \overline{\widetilde{\Psi}}^{[\Gamma_a]}(y)\Big|_{\text{tw2}} \overline{\widetilde{\Theta}}^{[\Gamma_b]}(y)\Big|_{\text{tw2}} \right\} , \end{aligned}$$

SIDIS hadron tensor with summed KPCs

KPCs convolution integral

$$
\begin{aligned}
W^{\mu\nu}_{\mathrm{KPC}} \;=\;& 8\,e^2\,P^+p_h^-C_{0,\mathrm{DIS}}\left(\frac{Q^2}{\mu^2}\right)\boxed{\int d^4k_1d^4k_2\delta(k_1^2)\delta(k_2^2)\delta^{(4)}(q+k_1-k_2)}\\
&\boxed{\times\int d\xi\frac{d\zeta}{\zeta}\delta(k_1^+-\xi P^+)\delta\left(k_2^- - \frac{p_h^-}{\zeta}\right)}\Bigg\{\\
&-(k_1k_2)H_0^{\mu\nu}\left(\Phi_{11}^{[\gamma^+]}\Delta_{11}^{[\gamma^-]}+\overline{\Phi}_{11}^{[\gamma^+]}\overline{\Delta}_{11}^{[\gamma^-]}+\Phi_{11}^{[\gamma^+\gamma^5]}\Delta_{11}^{[\gamma^-\gamma^5]}+\overline{\Phi}_{11}^{[\gamma^+\gamma^5]}\overline{\Delta}_{11}^{[\gamma^-\gamma^5]}\right)\\
&+i\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\left(\Phi_{11}^{[\gamma^+]}\Delta_{11}^{[\gamma^-\gamma^5]}-\overline{\Phi}_{11}^{[\gamma^+]}\overline{\Delta}_{11}^{[\gamma^-\gamma^5]}+\Phi_{11}^{[\gamma^+\gamma^5]}\Delta_{11}^{[\gamma^-]}-\overline{\Phi}_{11}^{[\gamma^+\gamma^5]}\overline{\Delta}_{11}^{[\gamma^-]}\right)\\
&+(k_1k_2)\Bigg[H_0^{\mu\nu}\left(g^{\alpha\beta}+\frac{k_1^\alpha k_2^\beta}{k_1^+k_2^-}-\frac{(k_1^\alpha k_2^+-k_2^\alpha k_1^+)(k_2^\beta k_1^- - k_1^\beta k_2^-)}{(k_1k_2)k_1^+k_2^-}\right)\\
&\quad -H_0^{\mu\alpha}H_0^{\nu\beta}-H_0^{\mu\beta}H_0^{\nu\alpha}+\frac{N^\mu H_0^{\nu\beta}k_1^\alpha+N^\nu H_0^{\mu\beta}k_1^\alpha}{k_1^+}+\frac{\bar{N}^\mu H_0^{\nu\alpha}k_2^\beta+\bar{N}^\nu H_0^{\mu\alpha}k_2^\beta}{k_2^-}\\
&\quad -\frac{(N^\mu\bar{N}^\nu+\bar{N}^\mu N^\nu)k_1^\alpha k_2^\beta}{k_1^+k_2^-}\Bigg]\left(\Phi_{11}^{[i\sigma^{\alpha+}\gamma^5]}\Delta_{11}^{[i\sigma^{\beta-}\gamma^5]}+\overline{\Phi}_{11}^{[i\sigma^{\alpha+}\gamma^5]}\overline{\Delta}_{11}^{[i\sigma^{\beta-}\gamma^5]}\right)\Bigg\},
\end{aligned}
$$

The KPCs convolution integral

- The convolution integral can be written in a schematic way as

$$
\begin{aligned}
\mathcal{I} \;&=\; 4P^+p_h^- \int d^4k_1 d^4k_2 \delta^{(4)}(q+k_1-k_2)\delta(k_1^2)\delta(k_2^2) \int d\xi \frac{d\zeta}{\zeta}\delta\left(k_1^+ - \xi P^+\right)\delta\left(k_2^- - \frac{p_h^-}{\zeta}\right) \\
&\quad \times (f_{1,f}(\xi,\boldsymbol{k}_1)f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2))A.
\end{aligned}
$$

We can rewrite it in a more beautiful form using the δ-functions

The KPCs convolution integral

- Integrating over k_1^- and k_2^+ using $\delta(k_1^2)$ and $\delta(k_2^2)$

$$\mathcal{I} = 4P^+p_h^- \int d\xi \frac{d\zeta}{\zeta} \int \frac{dk_1^+ dk_2^-}{4k_1^+ k_2^-} \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \delta^2(\boldsymbol{q}_T + \boldsymbol{k}_1 - \boldsymbol{k}_2) \delta\left(q^+ + k_1^+ - \frac{\boldsymbol{k}_2^2}{2k_2^-}\right) \delta\left(q^- + \frac{\boldsymbol{k}_1^2}{2k_1^+} - k_2^-\right)$$
$$\times \delta\left(k_1^+ - \xi P^+\right) \delta\left(k_2^- - \frac{p_h^-}{\zeta}\right) (f_{1,f}(\xi,\boldsymbol{k}_1) f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2))A.$$

- Integrating over k_1^+ and k_2^- using $\delta(k_1^+ - \xi P^+)$ and $\delta(k_2^- - p_h^-/\zeta)$

$$\mathcal{I} = 4P^+p_h^- \int d\xi \frac{d\zeta}{\zeta} \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \delta^2(\boldsymbol{q}_T + \boldsymbol{k}_1 - \boldsymbol{k}_2) \delta\left(q^+ + \xi P^+ - \frac{\boldsymbol{k}_2^2}{2p_h^-}\zeta\right) \delta\left(q^- + \frac{\boldsymbol{k}_1^2}{2\xi P^+} - \frac{p_h^-}{\zeta}\right)$$
$$\times \frac{\zeta}{4\xi P^+ p_h^-} (f_{1,f}(\xi,\boldsymbol{k}_1) f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2))A.$$

The KPCs convolution integral

- Rewriting it using the LP expressions for the collinear momentum fractions x_1 and z_1

$$\mathcal{I} = \frac{1}{P^+ p_h^-} \int \frac{d\xi\, d\zeta}{\xi\zeta}\zeta \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \delta^2(\boldsymbol{q}_T + \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \delta\left(x_1 - \xi + \frac{x_1}{z_1}\zeta\frac{\boldsymbol{k}_2^2}{\tau^2}\right) \delta\left(\frac{1}{z_1} - \frac{1}{\zeta} + \frac{x_1}{z_1}\frac{\boldsymbol{k}_1^2}{\xi\tau^2}\right)$$
$$\times \left(f_{1,f}(\xi,\boldsymbol{k}_1) f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2)\right) A.$$

- To integrate over ξ and ζ , we need to solve the δ-functions

$$\xi = \frac{x_1}{2}\left(1 - \frac{\boldsymbol{k}_1^2}{\tau^2} + \frac{\boldsymbol{k}_2^2}{\tau^2} \pm \frac{\sqrt{\lambda(\boldsymbol{k}_1^2, \boldsymbol{k}_2^2, -\tau^2)}}{\tau^2}\right) \qquad \zeta = \frac{z_1}{2}\left(1 - \frac{\boldsymbol{k}_1^2}{\boldsymbol{k}_2^2} - \frac{\tau^2}{\boldsymbol{k}_2^2} \pm \frac{\sqrt{\lambda(\boldsymbol{k}_1^2, \boldsymbol{k}_2^2, -\tau^2)}}{\boldsymbol{k}_2^2}\right)$$

Two solutions, identify the fake ones taking the large τ^2 limit

The KPCs convolution integral

- The correct solutions are the ones with the "+" sign

$$\xi = \frac{x_1}{2}\left(1 - \frac{\boldsymbol{k}_1^2}{\tau^2} + \frac{\boldsymbol{k}_2^2}{\tau^2} + \frac{\sqrt{\lambda(\boldsymbol{k}_1^2, \boldsymbol{k}_2^2, -\tau^2)}}{\tau^2}\right) \qquad \zeta = \frac{z_1}{2}\left(1 - \frac{\boldsymbol{k}_1^2}{\boldsymbol{k}_2^2} - \frac{\tau^2}{\boldsymbol{k}_2^2} + \frac{\sqrt{\lambda(\boldsymbol{k}_1^2, \boldsymbol{k}_2^2, -\tau^2)}}{\boldsymbol{k}_2^2}\right)$$

- After performing the integration, the integral reads

$$\mathcal{I} = \frac{1}{P^+ p_h^-}\int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \delta^2(\boldsymbol{q}_T + \boldsymbol{k}_1 - \boldsymbol{k}_2)\frac{J_{\xi,\zeta}}{\xi\zeta}\,\zeta\left(f_{1,f}(\xi,\boldsymbol{k}_1)f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2)\right)A$$

- After some simplifications, we arrive to the final form

$$\begin{aligned}\mathcal{I} &= \int_{R_{\xi\zeta}\cap R_T} d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \delta^{(2)}(\boldsymbol{q}_T + \boldsymbol{k}_1 - \boldsymbol{k}_2)\\ &\quad\times\frac{2\zeta}{\sqrt{\lambda(\boldsymbol{k}_1^2, \boldsymbol{k}_2^2, -\tau^2)}}\left(f_{1,f}(\xi,\boldsymbol{k}_1)f_{2,f}(\zeta,\boldsymbol{k}_2) + \overline{f}_{1,f}(\xi,\boldsymbol{k}_1)\overline{f}_{2,f}(\zeta,\boldsymbol{k}_2)\right)A,\end{aligned}$$