Finite-Range Effects and Coulomb Interactions in Femtoscopic Correlation Functions

Pablo Encarnación

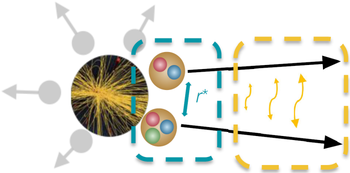
Institut de Física Corpuscular (IFIC), Valencia

Collaborators:
Juan Nieves, Miguel Albaladejo, Albert Feijoo, Isaac Vidaña, Amador García-Lorenzo

The correlation function can be expressed in the Koonin-Pratt formalism:

$$\boxed{C(\vec{k}) = \mathcal{N}\frac{N_{SE}(\vec{k})}{N_{ME}(\vec{k})} = \int d^3\vec{r}\, S(\vec{r})\, |\psi(\vec{k},\vec{r})|^2}$$

- $S(\vec{r})$: probability distribution of the relative separation of the particles.
- $\psi(\vec{k},\vec{r})$: relative wave function, containing the final-state interaction



We consider throughout this work a Gaussian source profile:

$$S(r) = \frac{1}{(4\pi R^2)^{3/2}} \exp\left(-\frac{r^2}{4R^2}\right)$$

INSTITUT DE FÍSICA CORPUSCULAR

The correlation function can be expressed in the Koonin-Pratt formalism:

$$C(\vec{k}) = \int d^3\vec{r}\, S(\vec{r})\, |\psi(\vec{k},\vec{r})|^2$$

For an S-wave interaction,

$$\psi(\vec{k},\vec{r}) = \begin{cases} \underbrace{e^{i\vec{k}\vec{r}} - j_0(kr)} + \overbrace{\psi_S^{\ell=0}(k,r)}^{\text{Interacting S-wave WF}} & \text{(No Coulomb)} \\ \underbrace{\phi_C(\vec{k},\vec{r}) - \phi_C^{\ell=0}(k,r)}_{\text{Non-interacting WF (or Coulomb) w/o S-wave}} + \psi_{SC}^{\ell=0}(k,r) & \text{(Coulomb)} \end{cases}$$

The correlation function can be written as

$$C(k) = \begin{cases} 1 + 4\pi \int_0^\infty r^2 dr\, S(r) \left[|\psi_S^{\ell=0}(k,r)|^2 - j_0(kr)^2 \right] & \text{(No Coulomb)} \\ C_C(k) + 4\pi \int_0^\infty r^2 dr\, S(r) \left[|\psi_{SC}^{\ell=0}(k,r)|^2 - |\phi_C^{\ell=0}(k,r)|^2 \right] & \text{(Coulomb)} \end{cases}$$

We need the strong (in the presence of Coulomb) interacting wave function

INSTITUT DE FÍSICA CORPUSCULAR

Why $\overline{K}\Omega$?

- **High strangeness:** $S = -4$
- Scarcely explored meson–baryon interaction
- Single-channel system
- **Repulsive** interaction and clean access to **Coulomb** effects

Two charge sectors: $\overline{K}^0\Omega^-$ $(Q = -1)$, $K^-\Omega^-$ $(Q = -2)$

Chiral interaction [S. Sarkar, E. Oset, and M. J. Vicente Vacas, Nucl. Phys. A 750 294 (2005)]

The lowest-order chiral Lagrangian for Goldstone bosons and decuplet baryons is

$$\mathcal{L} = -i\overline{T}^{\mu}_{abc}\gamma^{\nu}\mathcal{D}_{\nu}T^{abc}_{\mu}.$$

In the non-relativistic limit, the S-wave interaction reduces to the contact term

$$\boxed{\mathcal{V}_{\bar{K}\Omega}(p', p) = \frac{3}{4f^2}\left(2m_K + \frac{p^2 + p'^2}{2m_K}\right)} \qquad f \sim 93\,\mathrm{MeV}$$

Unitarization & Wave function

Unitarity is achieved by solving the off-shell Lippmann-Schwinger equation

$$\boxed{T(p',p) = V(p',p) + \frac{\mu}{\pi^2}\int_0^\infty dq\, \frac{q^2\, V(p',q)\, T(q,p)}{k^2 - q^2 + i\epsilon}}$$

with $V(p',p) = V_S(p',p) + V_C(p',p)$

Strong interaction (UV regulator):

$$V_S(p',p) = \frac{\mathcal{V}_{\bar{K}\Omega}(p',p)}{2m_K}\, \mathbf{exp}\left[-\frac{\boldsymbol{p^2 - k^2}}{\boldsymbol{\Lambda^2}}\right] \mathbf{exp}\left[-\frac{\boldsymbol{p'^2 - k^2}}{\boldsymbol{\Lambda^2}}\right] \qquad \Lambda = 800\,\mathrm{MeV}$$

$$\mathcal{V}_{\bar{K}\Omega}(p',p) = \frac{3}{4f^2}\left(2m_K + \frac{p^2 + p'^2}{2m_K}\right)$$

The exact S-wave wave function can be obtained from the half-off-shell T-matrix

$$\boxed{[\psi^{\ell=0}(k,r)]^* = j_0(kr) + \frac{\mu}{\pi^2}\int_0^\infty dq\, \frac{q^2\, j_0(qr)\, T(k,q)}{k^2 - q^2 + i\epsilon}}$$

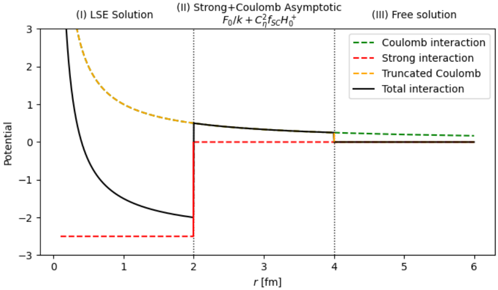
$T(k,q)$: half-off-shell amplitude $\qquad$ k on shell, $\quad$ q off shell

How can we implement the **Coulomb** interaction?

The Coulomb interaction is long-ranged and cannot be included directly in the momentum-space Lippmann–Schwinger equation, $V(p', p) = V_S(p', p) + V_C(p', p)$

Vincent-Phatak method: M. Vincent and S. C. Phatak, Phys. Rev. C 10, 391 (1974)

Auxiliary potential $V_C(r) = \frac{\alpha}{r}\theta(R - r)$



Problem: truncated Coulomb is oscillatory in momentum space

We follow a modified Vincent–Phatak construction with a Woods–Saxon regulator:

$$V_C(r) = \frac{Z_1 Z_2 \alpha}{r} g_{\mathrm{WS}}(r) \qquad g_{\mathrm{WS}}(r) = \frac{1 + e^{-R_{\mathrm{WS}}/b}}{1 + e^{(r - R_{\mathrm{WS}})/b}}$$

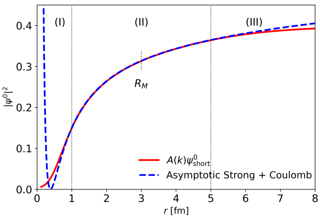
P. Encarnación, A. García-Lorenzo, M. Albaladejo, A. Feijoo, J. Nieves, I. Vidaña, e-Print: 2607.11321

R_M in region II, only Coulomb interaction remains

Full exact wave function:

$$[\psi^{\ell=0}(k,r)]^* = \begin{cases} A(k)\,[\psi^{\ell=0}_{\rm short}(k,r)]^*\,, & r \leq R_{\rm M}, \\ e^{i\sigma_0}\left[\dfrac{F_0(\eta,kr)}{kr} + f_{SC}(k)\,C_\eta^2\,\dfrac{H_0^+(\eta,kr)}{r}\right], & r > R_{\rm M} \end{cases}$$

F_0: regular Coulomb wave $H_0^+ = G_0 + iF_0$: outgoing Coulomb wave $f_{SC}(k)$: Coulomb-modified strong amplitude



- $\psi_{\rm short}(k,r)$: solution of the LSE with the regularized strong+Coulomb potential
- Normalization $A(k)$ and amplitude $f_{SC}(k)$ fixed by matching the wave function and its derivative at $r = R_M$

With this method:

- Full wave function, at short and long distances
- On-shell strong scattering amplitude modified by Coulomb

P. Encarnación, A. García-Lorenzo, M. Albaladejo, A. Feijoo, J. Nieves, I. Vidaña, e-Print: 2607.11321

Full exact wave function:

$$[\psi^{\ell=0}(k,r)]^* = \begin{cases} A(k)\,[\psi^{\ell=0}_{\mathrm{short}}(k,r)]^*\,, & r \leq R_{\mathrm{M}}, \\ e^{i\sigma_0}\left[\dfrac{F_0(\eta,kr)}{kr} + f_{SC}(k)\,C_\eta^2\,\dfrac{H_0^+(\eta,kr)}{r}\right], & r > R_{\mathrm{M}} \end{cases}$$

F_0: regular Coulomb wave $\quad H_0^+ = G_0 + iF_0$: outgoing Coulomb wave $\quad f_{SC}(k)$: Coulomb-modified strong amplitude

Continuous and differentiable at $r = R_M \implies$ Two equations for $A(k), f_{SC}(k)$
Solution:

$$A(k) = \frac{e^{i\sigma_0}}{\left[H_0^+(\eta,kr),\, u^0_{\mathrm{short}}(k,r)^*\right]_{R_{\mathrm{M}}}}\,,$$

$$f_{SC}^{-1}(k) = -\boldsymbol{C_\eta^2 k}\left(\boldsymbol{i} + \frac{[G_0(\eta,kr),\, u^0_{\mathrm{short}}(k,r)^*]_{R_{\mathrm{M}}}}{[F_0(\eta,kr),\, u^0_{\mathrm{short}}(k,r)^*]_{R_{\mathrm{M}}}}\right)\,,$$

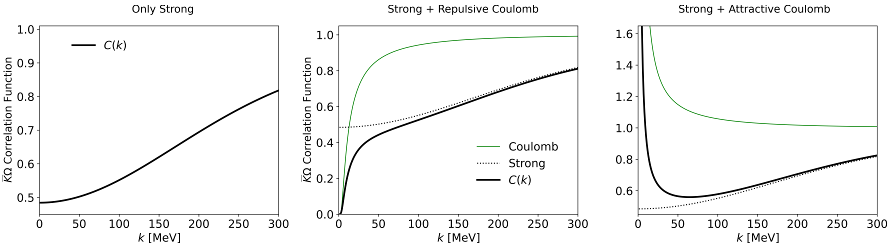
where

$$\left[\mathcal{F}(r),\mathcal{G}(r)\right]_{R_{\mathrm{M}}} = \left(\mathcal{F}(r)\frac{d\mathcal{G}(r)}{dr} - \mathcal{G}(r)\frac{d\mathcal{F}(r)}{dr}\right)_{r=R_{\mathrm{M}}},$$

P. Encarnación, A. García-Lorenzo, M. Albaladejo, A. Feijoo, J. Nieves, I. Vidaña, e-Print: 2607.11321

Implementation of Coulomb Interaction IV

- Source size: $R = 0.5$ fm
- Strong interaction only ($Z_1 Z_2 = 0$) and Strong+Coulomb interactions ($Z_1 Z_2 = \pm 1$)



- The modified Vincent-Phatak implementation of Coulomb reproduces the expected behavior
- Repulsive Coulomb: $C(k) \xrightarrow[k \to 0]{} 0$
- Attractive Coulomb: $C(k) \xrightarrow[k \to 0]{} \infty$
- At $k \to 0$, Coulomb dominates
- At $k \to \infty$, Strong dominates

Finite-Range effects

Recall the correlation function:

$$C(k) = \int d^3r\, S(r)|\psi(k,r)|^2 = \begin{cases} 1+4\pi \int_0^\infty r^2 dr\, S(r)\left[|\psi^{\ell=0}(k,r)|^2 - j_0(kr)^2\right] & \text{(No Coulomb)} \\ C_C(k)+4\pi \int_0^\infty r^2 dr\, S(r)\left[|\psi^{\ell=0}(k,r)|^2 - |\phi_C^{\ell=0}(k,r)|^2\right] & \text{(Coulomb)} \end{cases}$$

The wave function requires the half-off-shell scattering amplitude:

$$[\psi^{\ell=0}(k,r)]^* = j_0(kr) + \frac{\mu}{\pi^2}\int_0^\infty dq\, \frac{q^2 j_0(qr)\, T(k,q)}{k^2 - q^2 + i\epsilon} \qquad q \text{ off-shell}$$

Generally we have no access to the off-shell amplitude: it is only constrained on-shell

$$T(k,k) \quad \longleftrightarrow \quad f(k),\ \delta(k),\ \ldots$$

Two different off-shell extensions can reproduce the same on-shell physics while modifying the short-distance wave function, and therefore the femtoscopic correlation function.

What can we calculate from the on-shell amplitude?

Asymptotic (LL) Approximation

On-shell physics determine the asymptotic strong+Coulomb wave function:

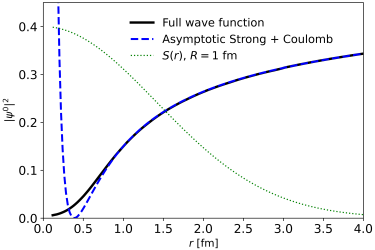
$$[\psi^{\ell=0}_{\rm asy}(k,r)]^* = \begin{cases} \dfrac{\sin(kr)}{kr} + f(k)\dfrac{e^{ikr}}{r} & \text{(No Coulomb)} \\ e^{i\sigma_0}\left[\dfrac{F_0(\eta,kr)}{kr} + f_{SC}(k)\, C_\eta^2\, \dfrac{H_0^+(\eta,kr)}{r}\right] & \text{(Coulomb)} \end{cases}$$

Using the asymptotic wave function in the Koonin–Pratt formula defines the Lednicky–Lyuboshitz (LL) approximation:

$$C_{\rm LL}(k) = \begin{cases} 1+4\pi\displaystyle\int_0^\infty r^2 dr\, S(r)\left[|\psi^{\ell=0}_{\rm asy}(k,r)|^2 - j_0(kr)^2\right] & \text{(No Coulomb)} \\ C_C(k)+4\pi\displaystyle\int_0^\infty r^2 dr\, S(r)\left[|\psi^{\ell=0}_{\rm asy}(k,r)|^2 - |\phi^{\ell=0}_C(k,r)|^2\right] & \text{(Coulomb)} \end{cases}$$

Finite-Range Correction

- The exact wave function is not equal to its asymptotic form in the region probed by the source.
- The short-distance behavior of the wave function is dependent on the off-shell model adopted.



FRP (Finite-range potential) correction:

$$C(k) = C_{\mathrm{LL}}(k) + \delta C_{\mathrm{FRP}}(k)$$

$$\delta C_{\mathrm{FRP}}(k) = \int d^3r\, S(r) \left[|\psi^{\ell=0}(k,r)|^2 - |\psi^{\ell=0}_{\mathrm{asy}}(k,r)|^2 \right].$$

For a:

- **Local** interaction, $V = V(r)$ (closely related to off-shellness)
- **Extensive source**: $S(r) \simeq S(0)$ in the interaction range

$$\delta C_{\mathrm{FRP}}(k) \simeq -4\pi\, S(0)\, C_\eta^2 |f_{SC}(k)|^2 \frac{d}{dk^2} \left[\mathrm{Re}\, f_{SC}^{-1} + 2\eta k h^\lambda(\eta) \right]$$

Phenomenological Parameter β

For a:

- **Local** interaction, $V = V(r)$ (closely related to off-shellness)
- **Extensive source**: $S(r) \simeq S(0)$ in the interaction range

$$\delta C_{\mathrm{FRP}}(k) \simeq -4\pi\, S(0)\, C_\eta^2 |f_{SC}(k)|^2 \frac{d}{dk^2}\left[\mathrm{Re}\, f_{SC}^{-1} + 2\eta k h^\lambda(\eta)\right]$$

We introduce a **phenomenological parameter** $\boldsymbol{\beta}$ to absorb off-shell and source size effects:

$$\boxed{\delta C_{\mathrm{FRP}}(k) \simeq -4\pi\, \boldsymbol{\beta}\, S(0)\, C_\eta^2 |f_{SC}(k)|^2 \frac{d}{dk^2}\left[\mathrm{Re}\, f_{SC}^{-1} + 2\eta k h^\lambda(\eta)\right]}$$

- $\beta \simeq 1$
- $\beta \neq 1$: accounts effectively for source-size and off-shell effects.

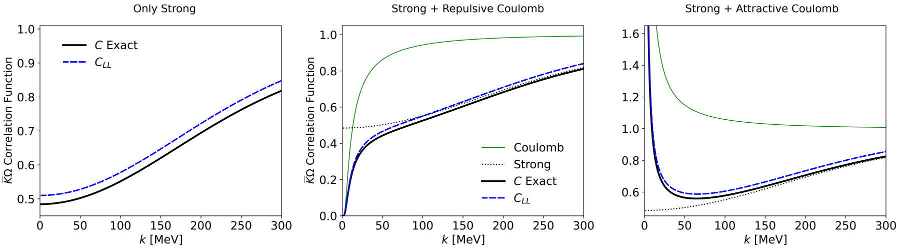
Therefore,

$$\begin{aligned} C(k) \quad \simeq \quad & C_C(k) + \int_0^\infty r^2 dr S(r) \left[|\psi_{\mathrm{asy}}^{\ell=0}(k,r)|^2 - |\phi_C^{\ell=0}(k,r)|^2\right] \\ & -4\pi \boldsymbol{\beta} S(0) C_\eta^2 |f_{SC}(k)|^2 \frac{d}{dk^2}\left[\mathrm{Re}\, f_{SC}{}^{-1} + 2\eta k h^\lambda(\eta)\right] \end{aligned}$$

Dependent on: On-shell physics, Coulomb parameters, and an additional parameter $\boldsymbol{\beta}$

Results: Exact vs. Asymptotic Wave Function

- Source size: $R = 0.5$ fm to enhance short-distance effects

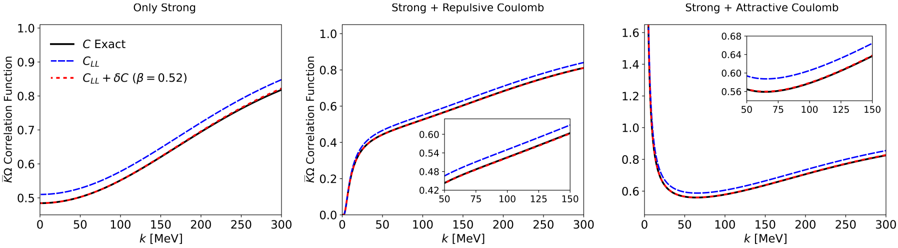


- $\boldsymbol{C(k)} \leftrightarrow$ exact wave function from the momentum-space LSE.
- $C_{\mathrm{LL}}(k) \leftrightarrow$ asymptotic wave function using the extracted $f_{SC}(k)$.

The difference $C(k) - C_{\mathrm{LL}}(k)$ is due to the short-distance region where $\psi_{\mathrm{exact}} \neq \psi_{\mathrm{asy}}$.

Results: The FRP Correction

- The FRP correction accounts for the difference between the exact and asymptotic CFs



$$C(k) = C_{\rm LL}(k) + \delta C(k),$$
$$\delta C_{\rm FRP} \simeq -4\pi \beta S(0) C_\eta^2 |f_{SC}|^2 \frac{d}{dk^2}\left[\mathrm{Re}\, f_{SC}^{-1} + 2\eta k h^\lambda\right].$$

- β effectively absorbs the finite source and short-distance interaction effects.
- For the present interaction and source: $\boxed{\beta = 0.52}$

In a realistic analysis, the on-shell scattering amplitude f_{SC} plus the phenomenological parameter β provide an effective description of the full correlation function

Summary & Outlook

- **Coulomb** effects can be incorporated in momentum space via the WS–VP method
- The correlation function is sensitive to the wave function at the **short distances** covered by the source.
- The short-distance wave function depends on the **half-off-shell** interaction, which is not fixed by on-shell scattering observables.
- The usual asymptotic (LL) approximation contains only the **on-shell** scattering amplitude and can therefore miss finite-range effects.
- The FRP correction, parametrized by β, provides an **effective way to account for the missing short-distance information**.