Exact renormalization of the quark-sector Schwinger-Dyson equations

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Multiplicative Renormalization of Schwinger-Dyson equations

- The dressed quark propagator and quark-gluon vertex are key ingredients for hadron physics and much effort has been dedicated to understand their nonperturbative structure through continuum methods.

G. Eichmann, R. Alkofer, I. C. Cloet, A. Krassnigg and C. D. Roberts, Phys. Rev. C **77**, 042202 (2008).
L. Chang and C. D. Roberts, Phys. Rev. Lett. **103**, 081601 (2009).
A. C. Aguilar and J. Papavassiliou, Phys. Rev. D **83**, 014013 (2011).
S. X. Qin, L. Chang, Y. X. Liu, C. D. Roberts and S. M. Schmidt, Phys. Lett. B **722**, 384-388 (2013).
A. C. Aguilar, D. Binosi, D. Ibañez and J. Papavassiliou, Phys. Rev. D **90**, no.6, 065027 (2014).
R. Williams, C. S. Fischer and W. Heupel, Phys. Rev. D **93**, no.3, 034026 (2016).
L. Chang and M. Ding, Phys. Rev. D **103**, no.7, 074001 (2021).
F. Gao, J. Papavassiliou and J. M. Pawlowski, Phys. Rev. D **103**, no.9, 094013 (2021).
A. C. Aguilar, M.N.F., B. M. Oliveira, J. Papavassiliou and G. T. Linhares, Eur. Phys. J. C **84**, no.11, 1231 (2024).

- A self consistent framework is to determine ***both*** from the ***coupled system*** of Schwinger-Dyson equations (SDEs).

- Approach recently successfully applied to meson masses (see Miramontes' talk) and to the study of the analytic structure of the propagator and vertex (see Morgado's and Wessely's talks).

M.N.F., A. S. Miramontes, J. M. Morgado, J. Papavassiliou and J. M. Pawlowski, Eur. Phys. J. C **86**, no.4, 325 (2026).
M.N.F., A. S. Miramontes, J. M. Morgado and J. Papavassiliou, Eur. Phys. J. C **86**, no.7, 858 (2026). Eur. Phys. J. C **86**, no.8, 954 (2026).
R. Alkofer, M.N.F., A. S. Miramontes, J. M. Morgado and J. Papavassiliou, Phys. Lett. B **880**, 140856 (2026).

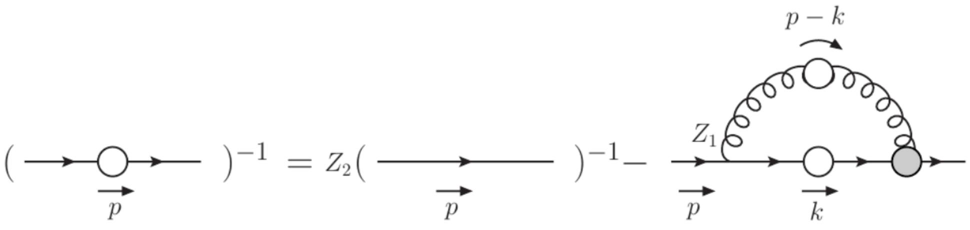
- In this context, Multiplicative Renormalization (MR) poses a fundamental challenge.

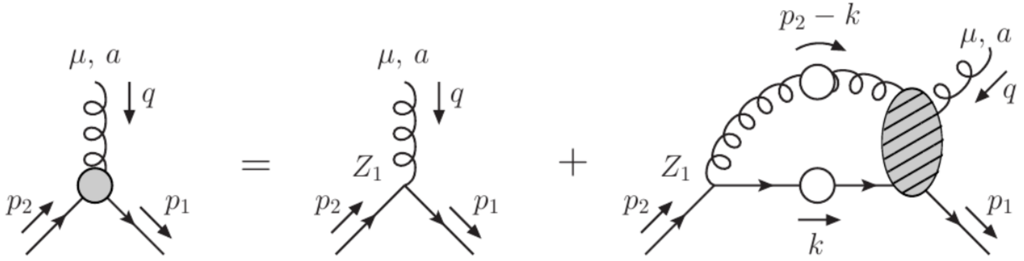
- Dressed vertices introduce divergences not seen in Rainbow Ladder, and casual truncation can spoil key cancellations.

- We will show how to implement MR ***exactly*** in the quark sector SDEs.
- And derive a sufficient condition for truncations to preserve it.

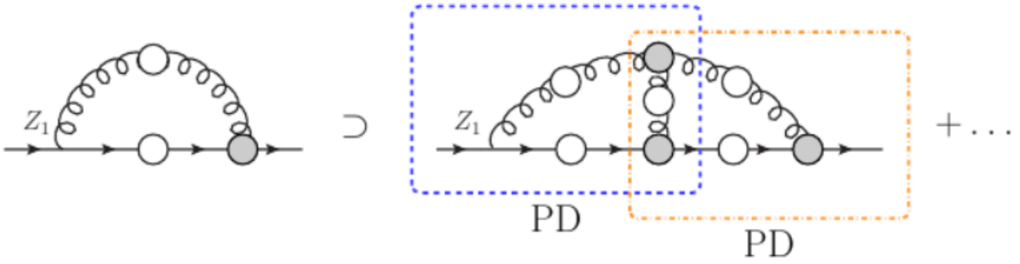
M.N.F. and J. Papavassiliou, in preparation.

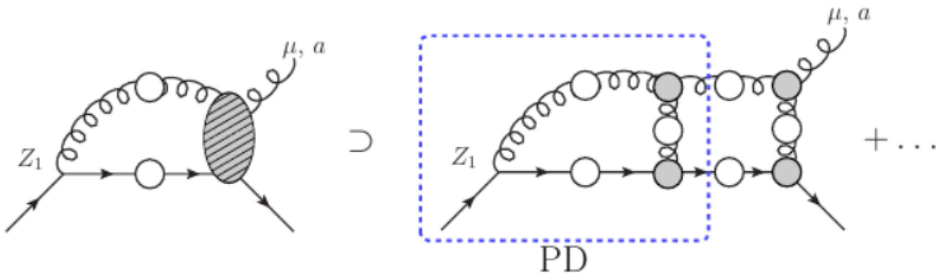
Divergences beyond RL





- The renormalization constants in the tree level terms provide subtractions.
- But these cannot eliminate certain subdivergences, which result from the dressed vertices and SDE kernel.





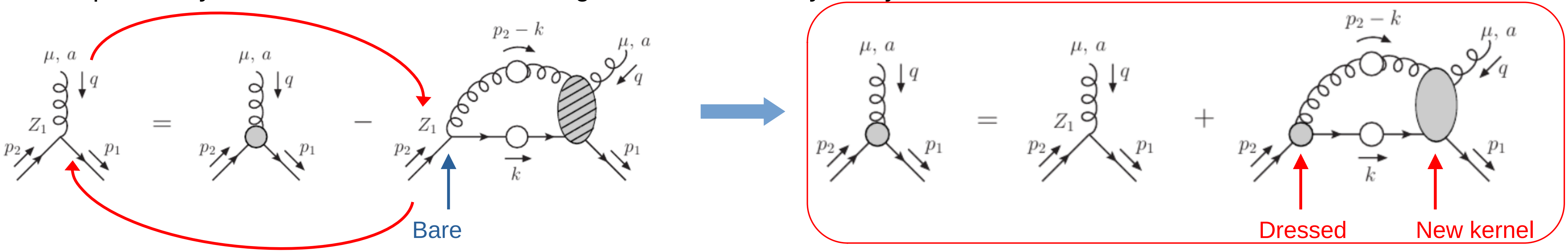
- Instead, the subdivergences must be canceled by the Z_1 multiplying the quantum diagrams.
- The challenge is to implement this cancellation nonperturbatively.
- And determine which truncations respect multiplicative renormalizability.

MR of the quark-gluon vertex

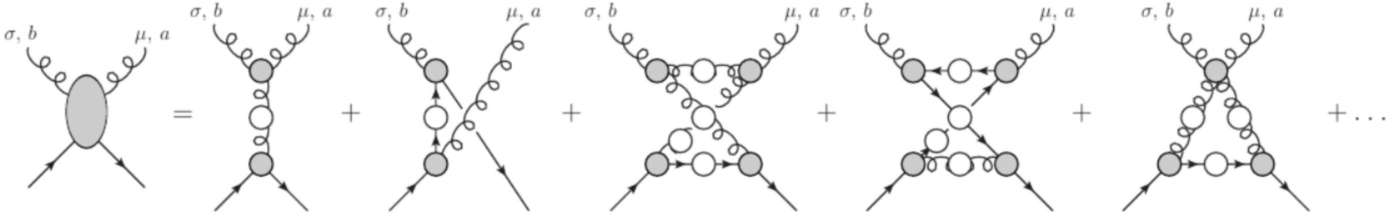
- The MR of the vertex proceeds through the textbook procedure of converting its SDE into Bethe-Salpeter (BS) form.

J. D. Bjorken & S. D. Drell, McGraw-Hill, (1965).

- Specifically, solve the SDE for Z_1, iterating it on itself infinitely many times, to obtain the BS form:



- The BS kernel is 2-particle-irreducible (2PI) in quark-gluon scattering channel.

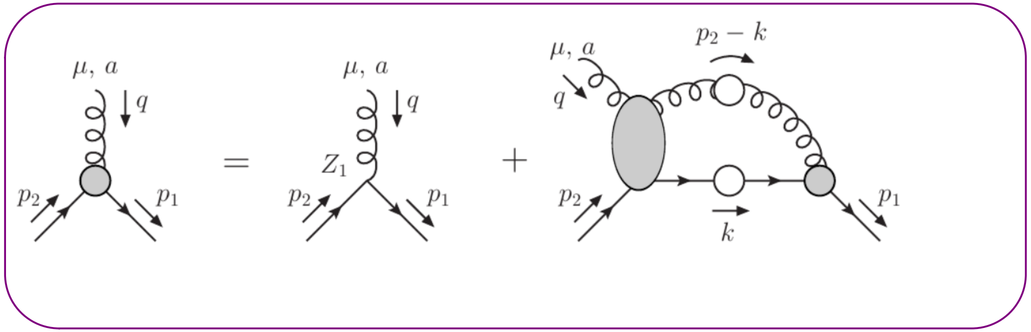


- Since it does not have 2 particle reducible diagrams, it does not generate subdivergences in the SDE...
- ...and does not cause overcounting when combined with the dressed vertex.

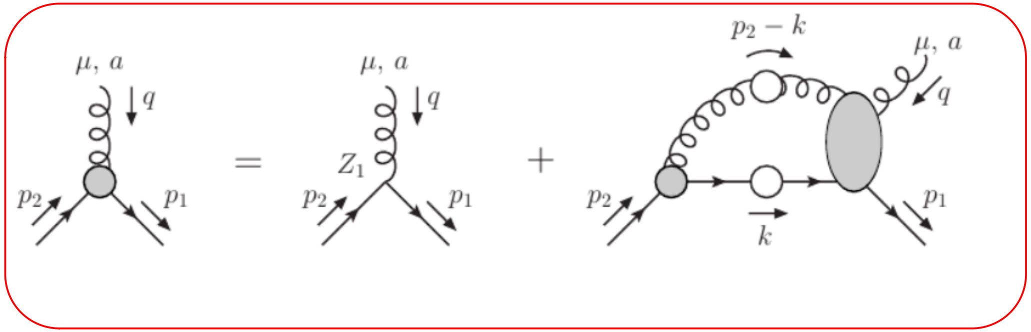
MR of the quark-gluon vertex

- We can repeat the procedure, starting with the bare vertex on the antiquark to obtain:

antiquark perspective



quark perspective



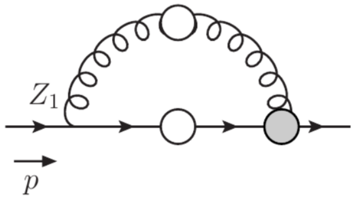
- As we will see, the **key condition for the MR of the quark propagator** is that the vertex satisfies **BOTH** of these equations with same functional kernel.

MR of the quark dressing function

Now the quark propagator. With the standard decomposition, $S^{-1}(p) = A(p^2)\not{p} - B(p^2)$, we begin by the renormalization of $A(p^2)$.

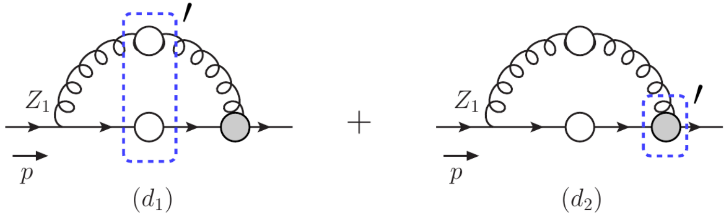
- Using Lorentz invariance and taking the appropriate trace, the gap equation yields

$$A(p^2) = Z_2 - \frac{i}{12} P^{\mu\nu}(p) \mathrm{Tr}\left[\gamma_\nu \frac{\partial \Sigma(p)}{\partial p^\mu}\right] , \qquad P_{\mu\nu}(p) := g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} , \qquad \Sigma(p) :=$$



- Differentiating the SDE for $\Sigma(p)$ furnishes

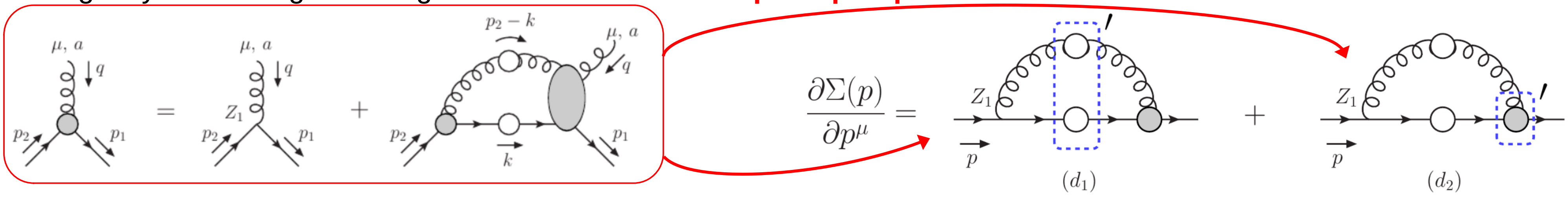
$$\frac{\partial \Sigma(p)}{\partial p^\mu} =$$



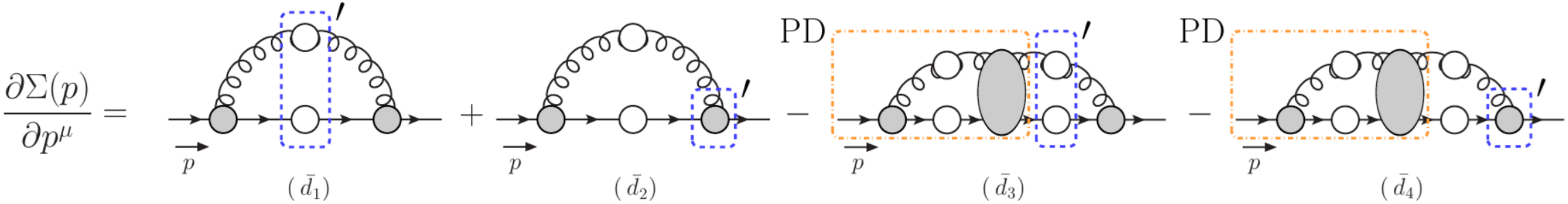
- The dressed vertex causes subdvergences beyond one loop. → <u>We have to cancel them.</u>

MR of the quark dressing function

We begin by eliminating Z_1 through the vertex SDE in the **quark perspective**:

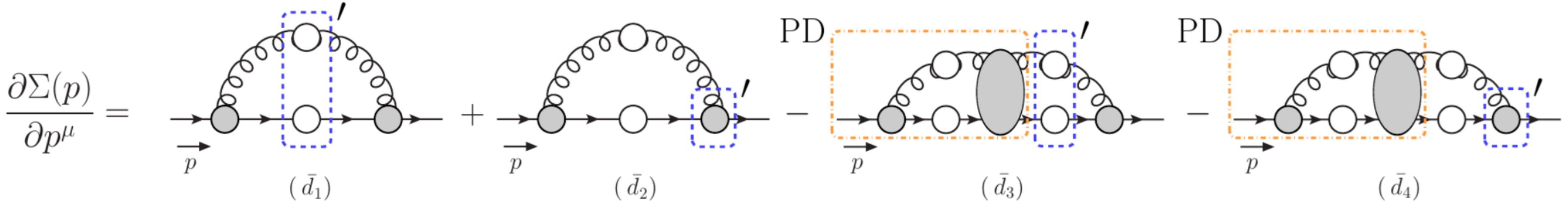


- Leads to an equation with explicit subdivergences, plus subdivergences hidden in the dressed vertices.

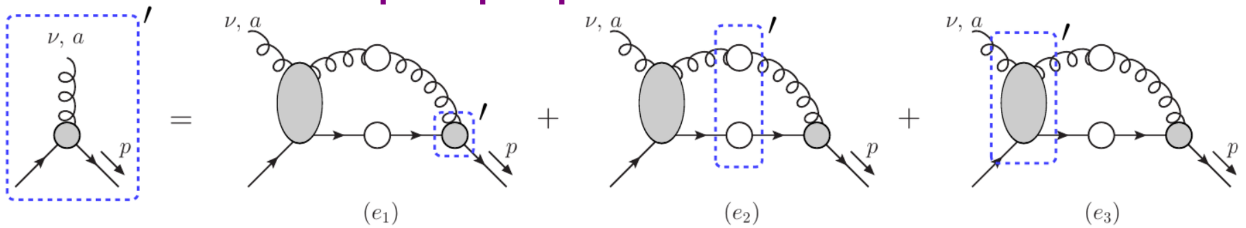


- How do we cancel these?

MR of the quark dressing function



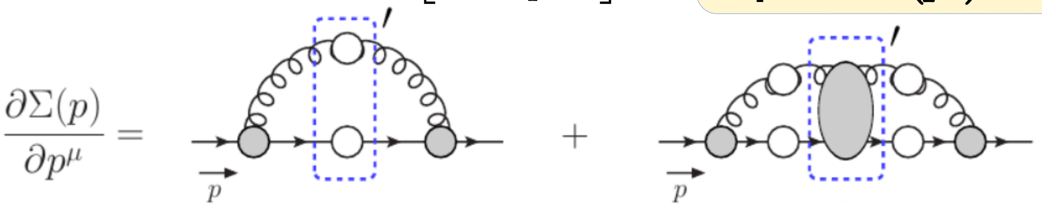
Differentiate the SDE for the vertex in **antiquark perspective**:



Substituting in the the SDE for $\partial\Sigma(p)/\partial p^\mu$ the subdivergences cancel:

$$A(p^2) = Z_2 - \frac{i}{12} P^{\mu\nu}(p) \left[\gamma_\nu \frac{\partial \Sigma(p)}{\partial p^\mu} \right],$$

Manifestly finite eq. for $A(p^2)$



M.N.F. and J. Papavassiliou, in preparation.

- No overcounting, thanks to 2PI kernel.
- Has no multiplicative Z_1;
- Power counting shows that the subdiagrams are finite;
- Weinberg's theorem guarantees $A(p^2)$ is finite.

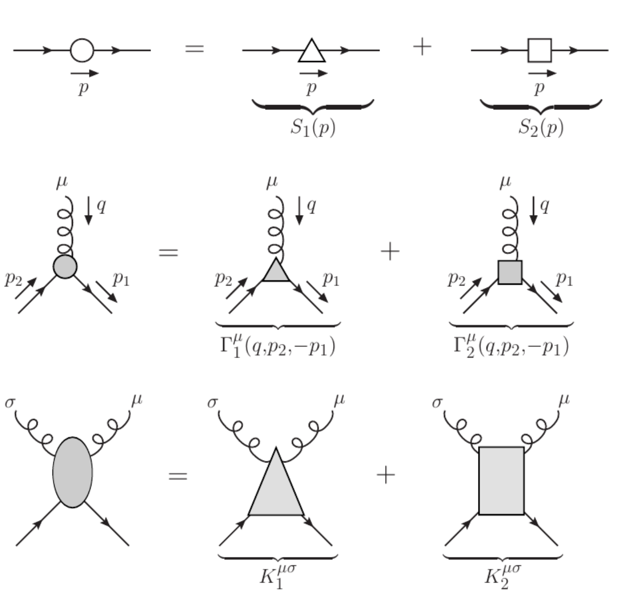
MR of the constituent mass function

Similarly for $B(p^2)$, related to the RGI constituent mass by $\mathcal{M}(p^2) := B(p^2)/A(p^2)$. The "algorithm" is the same:

- Eliminate Z_1 through the vertex SDE in the **quark perspective** and expose subdivergences.
- Cancel the subdivergences with the vertex SDE in the **antiquark perspective**.

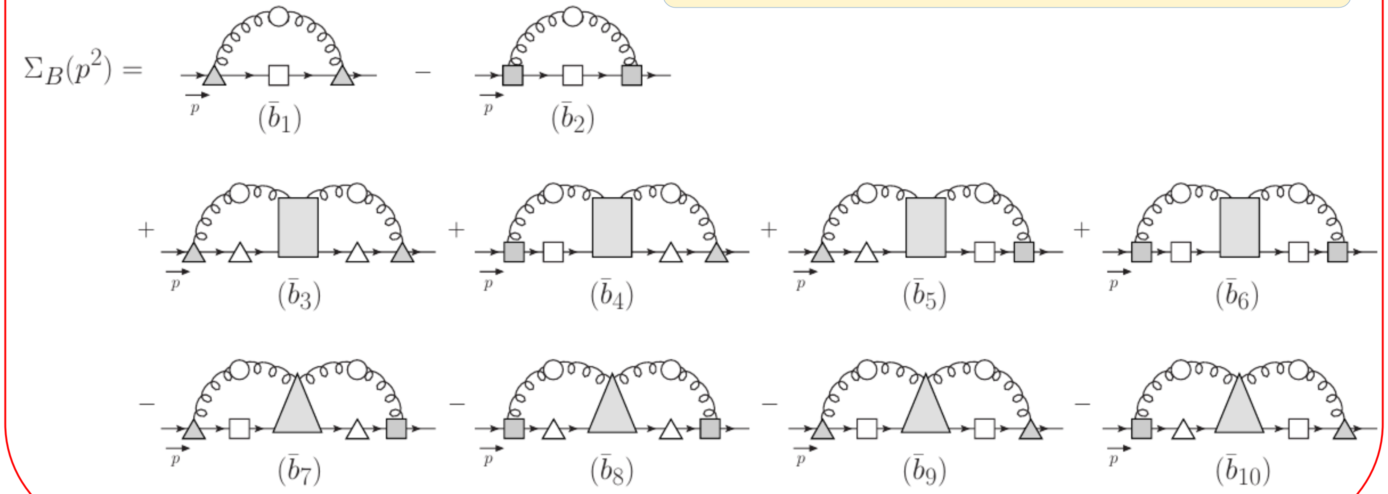
The key in the case of $B(p^2)$ is to separate the Green's function into chiral even and odd parts (# of γ matrices):

$\triangle, \blacktriangle = \text{odd} \# \text{ in } \gamma\,, \qquad \square, \blacksquare = \text{even} \# \text{ in } \gamma\,.$



$$B(p^2) = Z_4 m(\mu^2) + \Sigma_B(p^2)$$

Manifestly finite eq. for $B(p^2)$



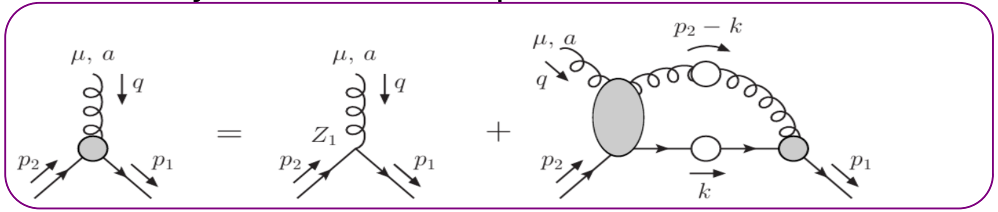
M.N.F. and J. Papavassiliou, in preparation.

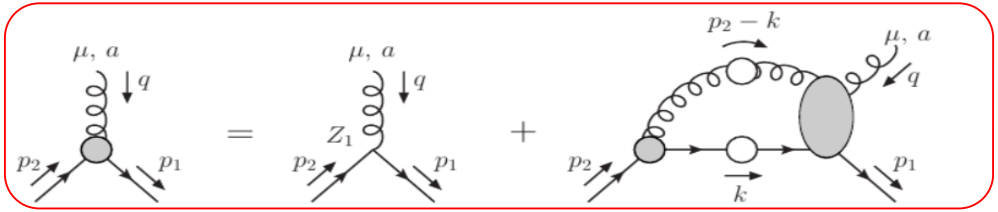
MR of the constituent mass function

- Crucially, **we never used Slavnov-Taylor identities** to cancel divergences. This had been shown explicitly before in QED: M. Baker and C. k. Lee, Phys. Rev. D **15**, 2201 (1977).

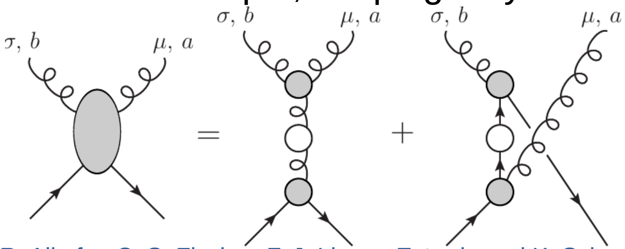
- That is assuming the gauge sector is somehow renormalized - STIs are key there – and provided externally.

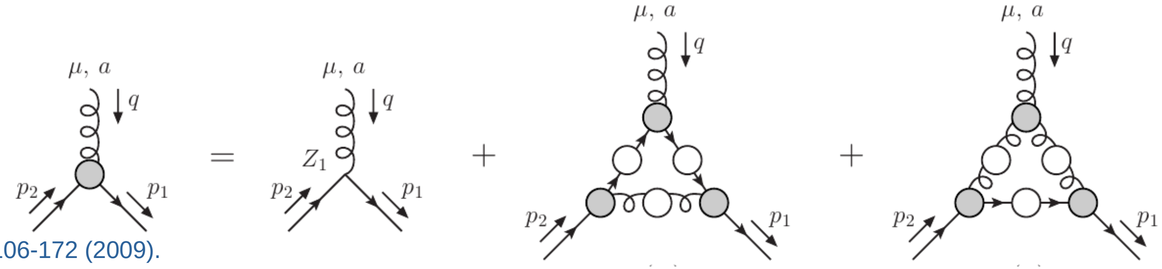
- The only condition in the quark sector is that Z_1 is determined from the vertex SDE and that **BOTH** forms are satisfied.





- For example, keeping only the one-particle exchange kernel in either form, leads to the same (3PI) vertex SDE:





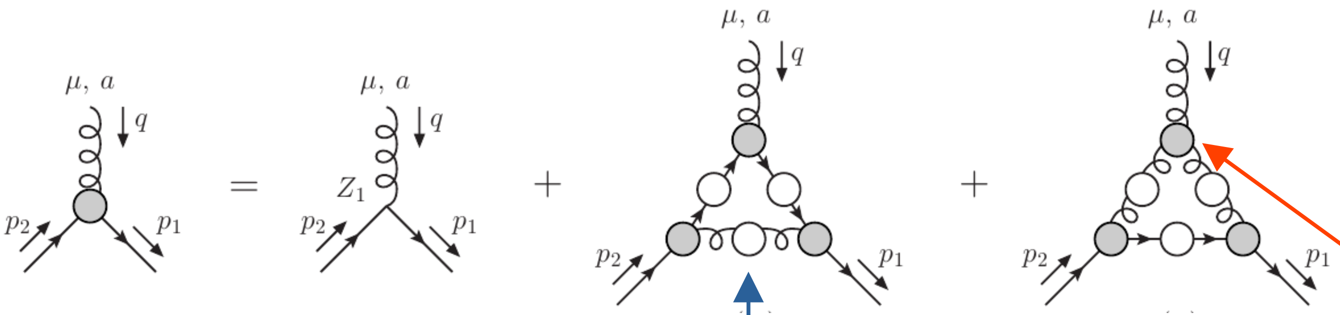
R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada and K. Schwenzer, Annals Phys. **324**, 106-172 (2009).
R. Williams, C. S. Fischer and W. Heupel, Phys. Rev. D **93**, no.3, 034026 (2016).
A. L. Blum, R. Alkofer, M. Q. Huber and A. Windisch, EPJ Web Conf. **137**, 03001 (2017).
A. C. Aguilar, M.N.F., B. M. Oliveira, J. Papavassiliou and G. T. Linhares, Eur. Phys. J. C **84**, no.11, 1231 (2024).
G. Wieland and R. Alkofer, arXiv:2604.20235.
More on 3PI: M. Q. Huber, Phys. Rept. **879**, 1-92 (2020).

Preserves MR!

Numerical analysis

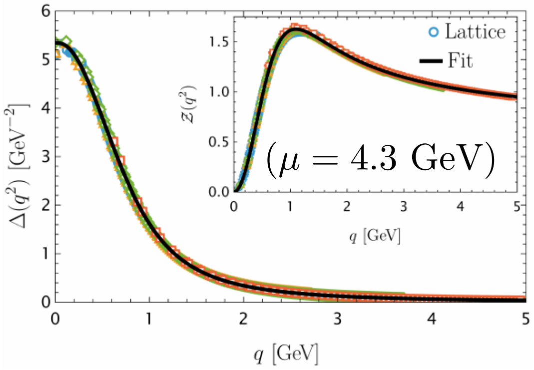
Let us test it numerically:

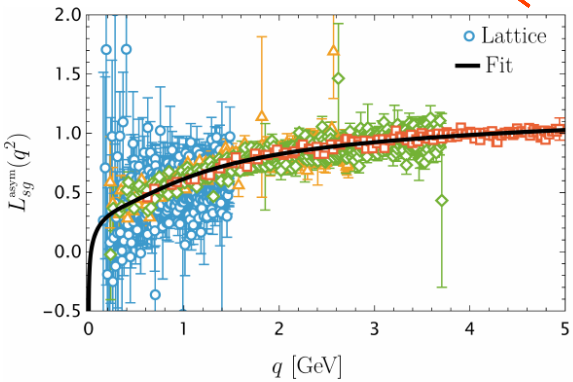
- We truncate the equation precisely by using the 3PI form (one particle exchange kernel):



- For the gauge sector inputs, we use $N_f = 2 + 1$ lattice data:

S. Zafeiropoulos, P. Boucaud, F. De Soto, J. Rodríguez-Quintero and J. Segovia, Phys. Rev. Lett. **122**, no.16, 162002 (2019).
A. C. Aguilar, F. De Soto, M.N.F., J. Papavassiliou, J. Rodríguez-Quintero and S. Zafeiropoulos, Eur. Phys. J. C **80**, no.2, 154 (2020).





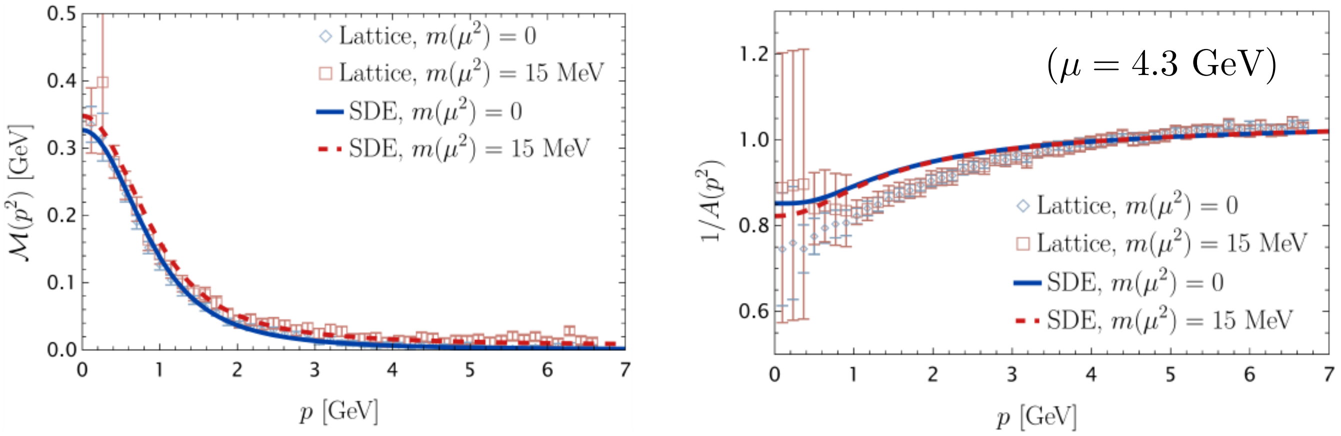
- Saturation of the gluon propagator transmits the effect of the dynamical gluon mass (see Papavassiliou's and Comitini's talks).

M.N.F. and J. Papavassiliou, Prog. Part. Nucl. Phys. **144**, 104186 (2025).

Numerical analysis

We set the scale by adjusting the constituent mass to the $N_f = 2 + 1$ lattice data from:

A. Virgili, W. Kamleh and D. B. Leinweber,
Phys. Lett. B **840**, 137865 (2023).



- We find also very good agreement for $A(p^2)$, even though it was not used to set our parameters [$\alpha_s(\mu^2)$, $m(\mu^2)$].

- $A(p^2)$ is monotonic as seen on the lattice. Some previous studies with dressed vertices had a minimum around $p = 1$ GeV.

A. C. Aguilar, J. C. Cardona, M.N.F. and J. Papavassiliou,
Phys. Rev. D **96**, no.1, 014029 (2017).
A. C. Aguilar, J. C. Cardona, M.N.F. and J. Papavassiliou,
Phys. Rev. D **98**, no.1, 014002 (2018).
F. Gao, J. Papavassiliou and J. M. Pawlowski,
Phys. Rev. D **103**, no.9, 094013 (2021).

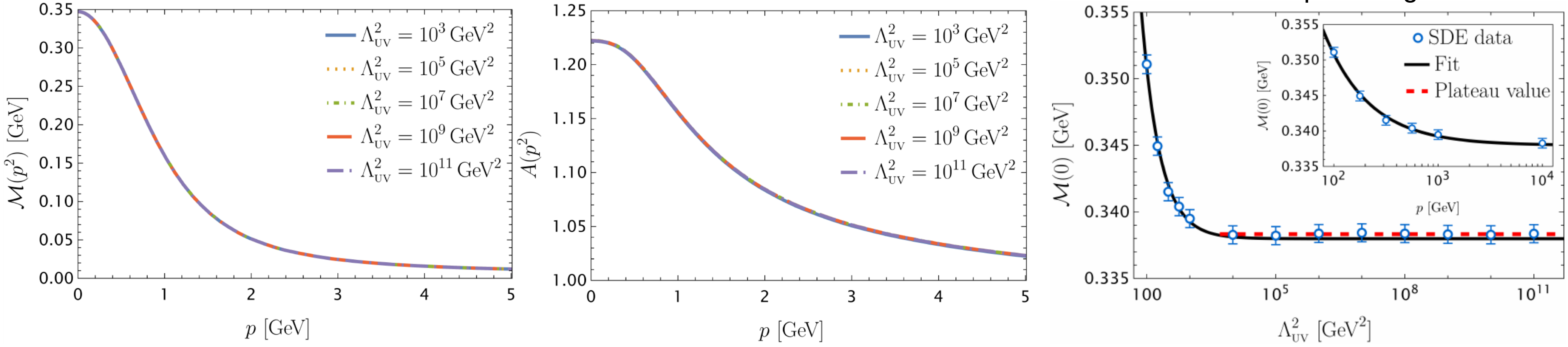
Numerical analysis

The finiteness of the renormalized system is tested by varying the cutoff.

- But small variations can fail to see sub-logarithmic divergences, resulting from failed cancellations

- So we make BIG variations:

Expected power law dependence on cutoff and plateuing



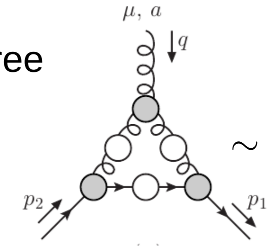
Varying the cutoff $\Lambda^2_{\rm UV}$ by 9 orders of magnitude does not change the result.

Numerical verification of finiteness

Finiteness of Z_1

The 1 loop RG behavior can be shown analytically to be preserved. It follows a surprising conclusion: $\boxed{\lim_{\Lambda_{\rm UV}\to\infty} Z_1 = 0\,.}$

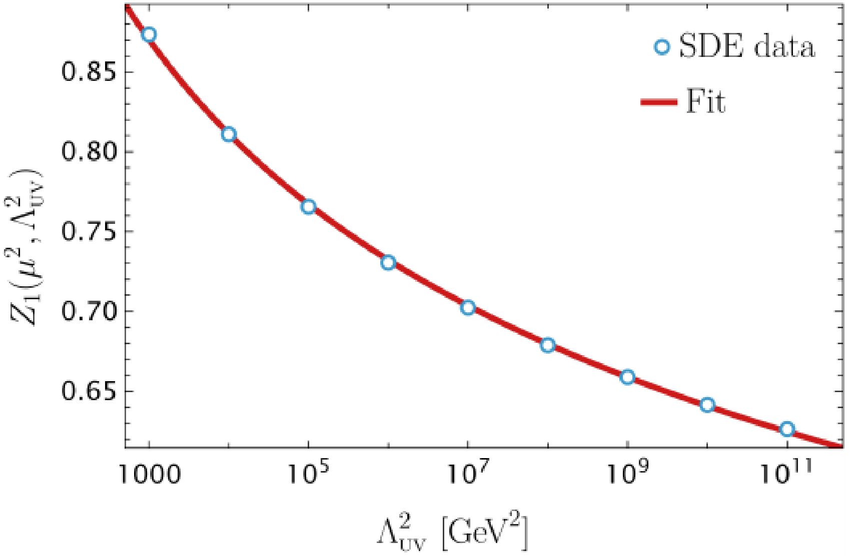
- This is because although the vertex superficial degree of divergence is 0 (log. divergence), the anomalous dimensions add up to a power < -1 in the logs.



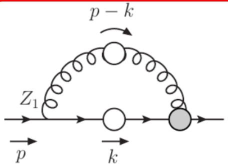
$$\sim \int^{\Lambda_{\rm UV}} \frac{dk}{k\,[\ln(k^2/\Lambda^2_{\rm MOM})]^{1+9/(4\beta_0)}} \sim \frac{1}{[\ln(\Lambda^2_{\rm UV}/\Lambda^2_{\rm MOM})]^{9/(4\beta_0)}}$$

- The theoretical prediction for the dependence of Z_1 on the cutoff fits the numerical data perfectly:

$$Z_1(\mu^2,\Lambda^2_{\rm UV}) \sim \frac{c}{\left[\ln\left(\frac{\Lambda^2_{\rm UV}}{\Lambda^2_{\rm MOM}}\right)\right]^{\frac{9}{4\beta_0}}}\,, \qquad c = \left[\ln\left(\frac{\mu^2}{\Lambda^2_{\rm MOM}}\right)\right]^{\frac{9}{4\beta_0}}\,.$$



- The standard SDE contain delicate limits, that our manifestly finite forms essentially resolve.



$$= 0 \times \infty$$

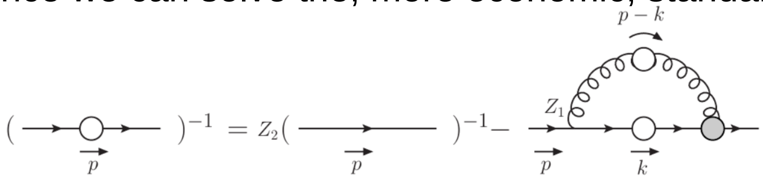
Conclusions

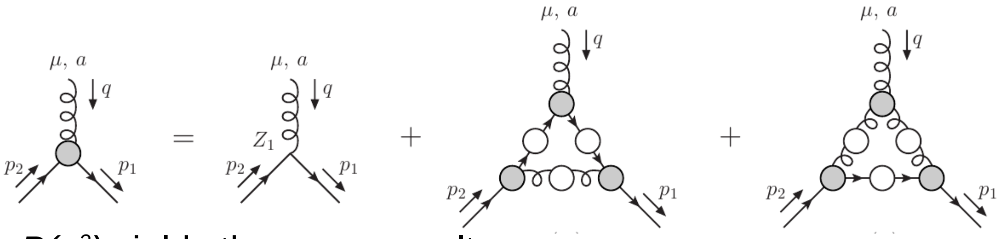
- The exact multiplicative renormalization of the quark propagator and quark-gluon vertex SDEs was performed to obtain manifestly finite equations.
- The finiteness of these equations does not depend on Slavnov-Taylor identities (as long as gauge sector is externally renormalized).
- Instead, only the symmetry of the quark-gluon scattering BS kernel under permutation of the quark and antiquark legs needs to be preserved.
- This cavalier condition is easily preserved under truncation.
- We explicitly verified numerically and analytically that the 3PI truncation preserves MR and the correct UV behavior of the solution.
- Leads to a vanishing quark-gluon renormalization constant.
- 3PI truncation leads to good agreement with recent lattice results for the propagator, which is not trivial for $A(p^2)$ with dressed vertices.

Backup slides

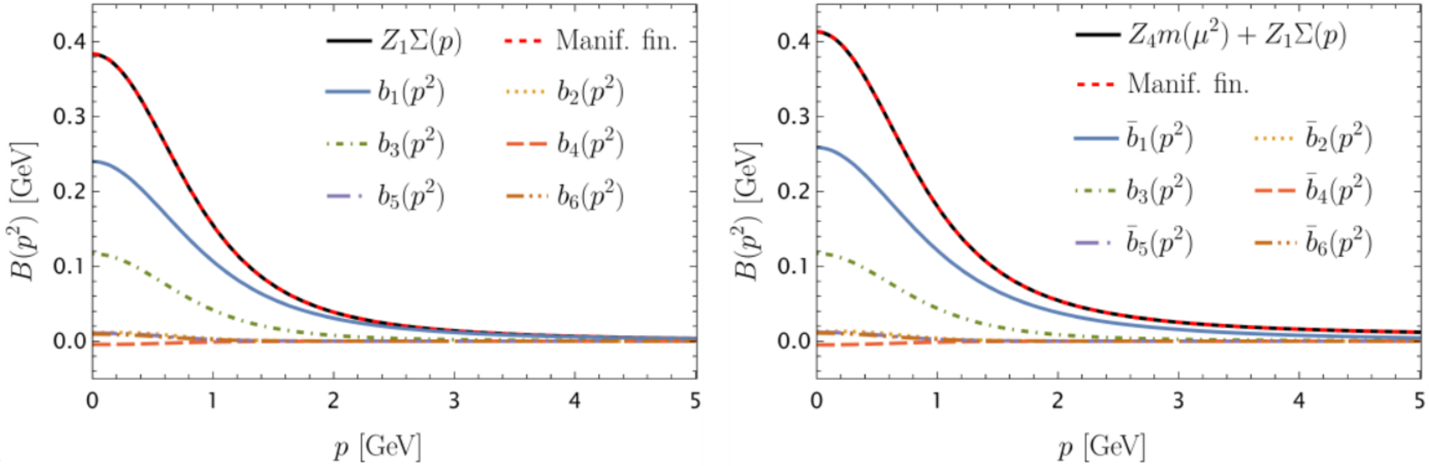
Numerical analysis

- The manifestly finite form of the gap equation looks complicated.
- But it is equivalent to the original form, provided the truncation preserves the MR and the (regulated) numerical value of Z_1 is determined from the vertex SDE.
- Hence we can solve the, more economic, standard form:





- We explicitly checked that the manifestly finite equation for $B(p^2)$ yields the same result:



Ultraviolet behavior

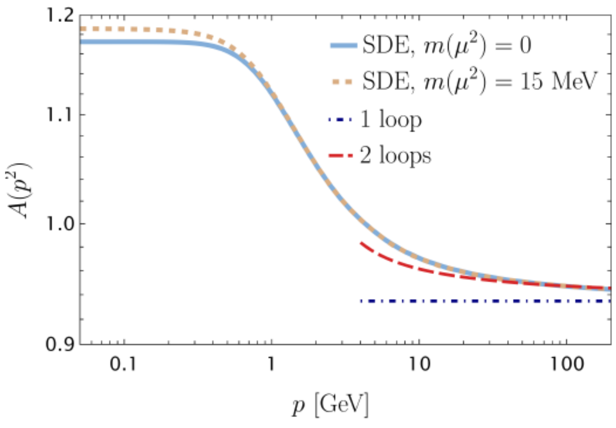
Since we are focused on the renormalization, it is important to look closely at the ultraviolet behavior.

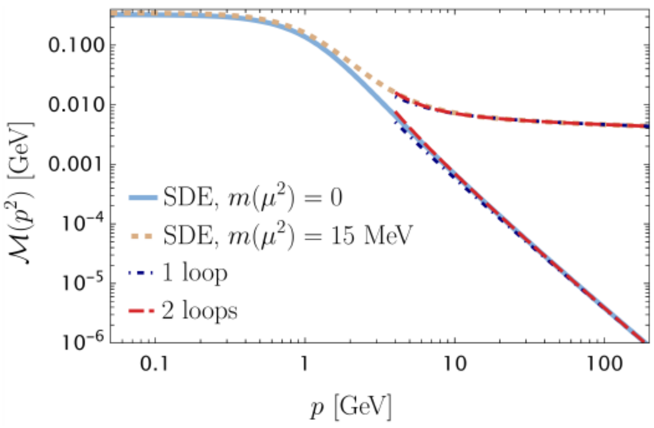
- We can solve solve the 3PI system analytic at leading log in the ultraviolet. For example, the SDE for the classical form factor of the quark-gluon vertex, $\lambda_1(p^2)$, at large *p* can be converted into 5[th] order nonlinear differential equation:

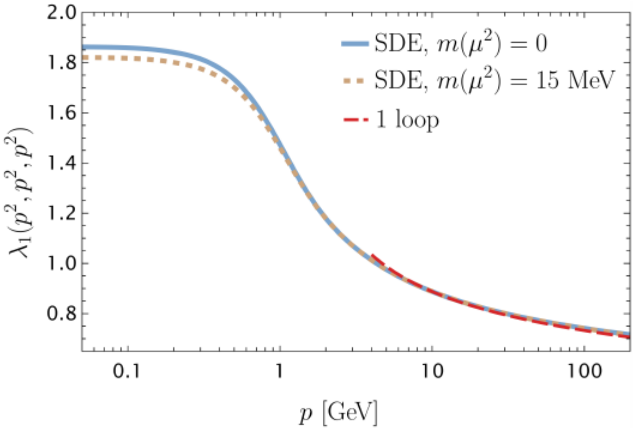
$$x^3 \frac{d^5}{dx^5}\left[x^2\lambda_1(x)\right] - 6xC\frac{d}{dx}\left[\lambda_1^2(x)x^2\Delta^2(x)L_{sg}(x)\right] + 36C\lambda_1^2(x)x^2\Delta^2(x)L_{sg}(x) = 0\,, \qquad C := \frac{\alpha_s C_A}{48\pi}\,.$$

- Can be solved asymptotically to show that the Green's functions have the correct RG resummed behaviors:

$$A(p^2) \sim 1\,, \qquad M(p^2) \sim \hat{m}\left[\frac{1}{2}\ln(p^2/\Lambda^2_{\text{MOM}})\right]^{-4/\beta_0} - \frac{8\pi^2}{3}\frac{\langle\bar{q}q\rangle}{p^2\left[\frac{1}{2}\ln(p^2/\Lambda^2_{\text{MOM}})\right]^{1-4/\beta_0}}\,, \qquad \lambda_1(p^2) \sim \left[\frac{\ln(p^2/\Lambda^2_{\text{MOM}})}{\ln(\mu^2/\Lambda^2_{\text{MOM}})}\right]^{-9/(4\beta_0)}.$$

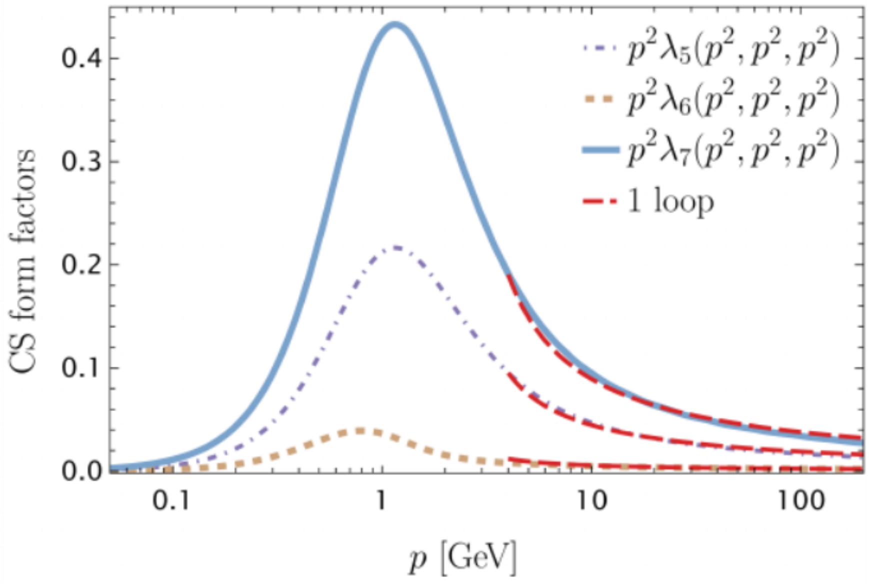


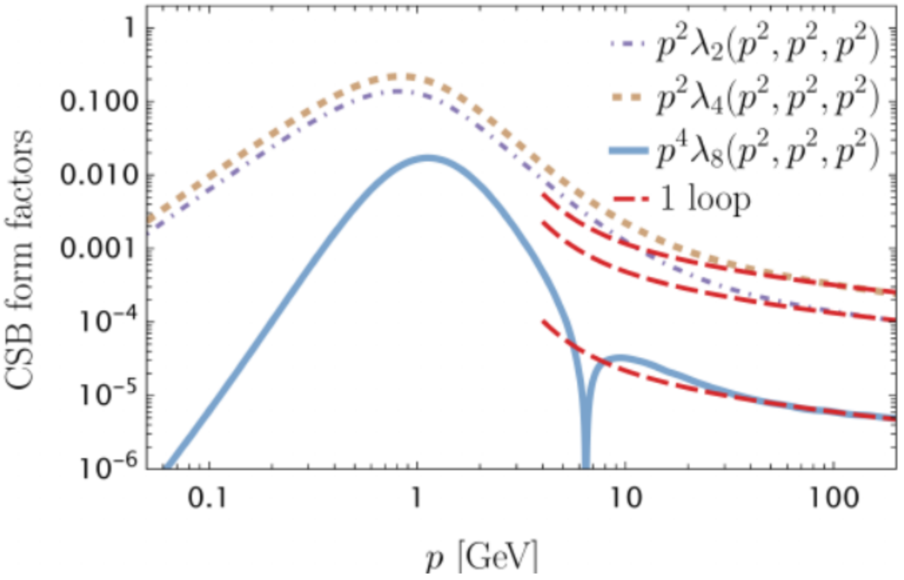




Ultraviolet behavior

Also for the non tree-level tensor structures:

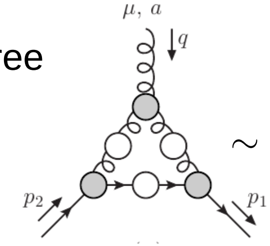




Finiteness of Z_1

The RG behavior can be shown to be preserved. It follows a surprising conclusion: $\boxed{\lim_{\Lambda_{\rm UV}\to\infty} Z_1 = 0\,.}$

- This is because although the vertex superficial degree of divergence is 0 (log. divergence), the anomalous dimensions add up to a power < -1 in the logs.



$$\sim \int^{\Lambda_{\rm UV}} \frac{dk}{k\,[\ln(k^2/\Lambda^2_{\rm MOM})]^{1+9/(4\beta_0)}} \sim \frac{1}{[\ln(\Lambda^2_{\rm UV}/\Lambda^2_{\rm MOM})]^{9/(4\beta_0)}}$$

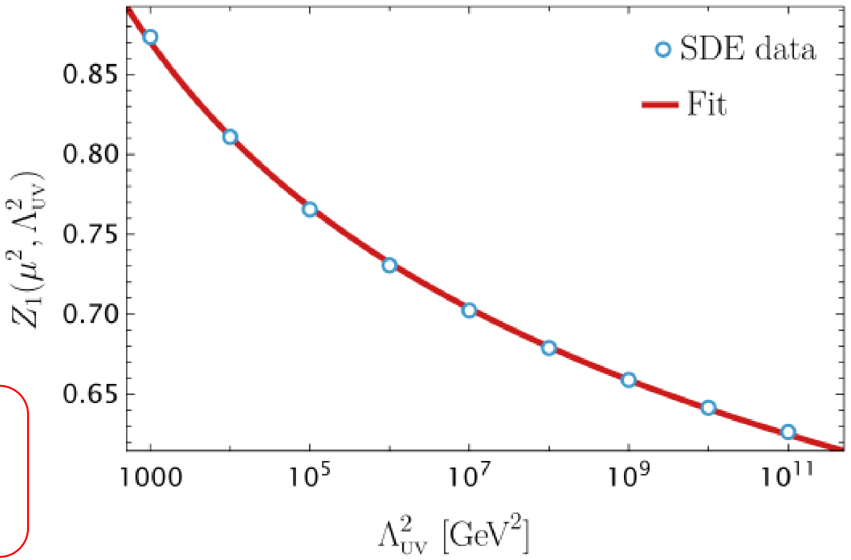
- Note that the sum of the perturbative series can add up to a finite value:

$$Z_1 = 1 + ag^2\ln(\Lambda^2_{\rm UV}/M^2) + \ldots = \frac{1}{[1 - g^2(a/b)\ln(\Lambda^2_{\rm UV}/M^2)]^b}\,.$$

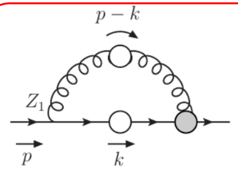
J. C. Collins, Cambridge University Press, 1984.

- The theoretical prediction for the dependence of Z_1 on the cutoff fits the numerical data perfectly:

$$Z_1(\mu^2,\Lambda^2_{\rm UV}) \sim \frac{c}{\left[\ln\left(\frac{\Lambda^2_{\rm UV}}{\Lambda^2_{\rm MOM}}\right)\right]^{\frac{9}{4\beta_0}}}\,, \qquad c = \left[\ln\left(\frac{\mu^2}{\Lambda^2_{\rm MOM}}\right)\right]^{\frac{9}{4\beta_0}}\,.$$



- The standard SDE contain delicate limits, that our manifestly finite forms essentially resolve.



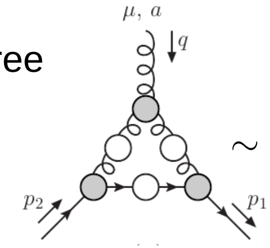
$$= 0 \times \infty$$

Finiteness of Z_1

The RG behavior can be shown to be preserved. It follows a surprising conclusion:

$$\boxed{\lim_{\Lambda_{\rm UV}\to\infty} Z_1 = 0\,.}$$

- This is because although the vertex superficial degree of divergence is 0 (log. divergence), the anomalous dimensions add up to a power < -1 in the logs.



$$\sim \int^{\Lambda_{\rm UV}} \frac{dk}{k\,[\ln(k^2/\Lambda^2_{\rm MOM})]^{1+9/(4\beta_0)}} \sim \frac{1}{[\ln(\Lambda^2_{\rm UV}/\Lambda^2_{\rm MOM})]^{9/(4\beta_0)}}$$

- Note that the sum of the perturbative series can add up to a finite value:

$$Z_1 = 1 + ag^2\ln(\Lambda^2_{\rm UV}/M^2) + \ldots = \frac{1}{[1 - g^2(a/b)\ln(\Lambda^2_{\rm UV}/M^2)]^b}\,.$$

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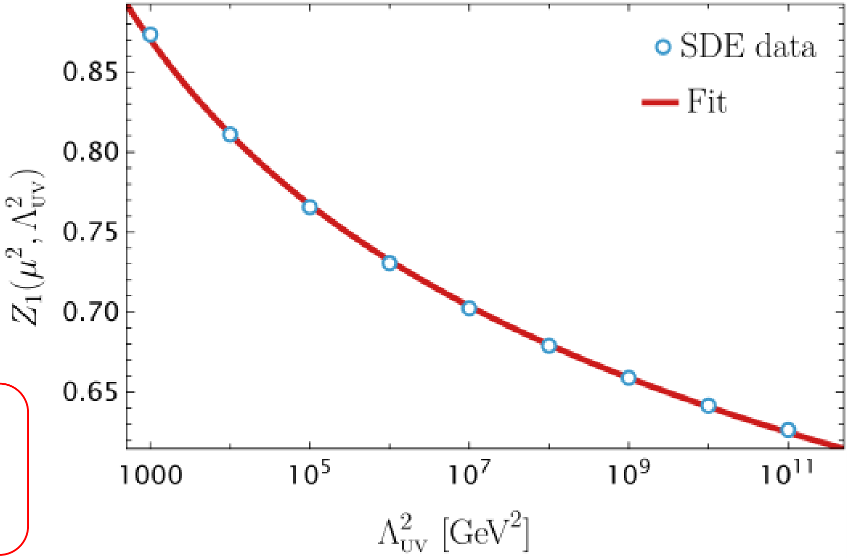
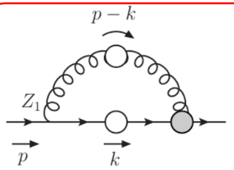


- The standard SDE contain delicate limits, that our manifestly finite forms essentially resolve.



$$= 0 \times \infty$$

Planar degeneracy approximation

For the three-gluon vertex, we employ the *planar degeneracy* approximation,

$$\overline{\Gamma}_{\alpha\mu\nu}(q,r,p) \approx \overline{\Gamma}^{0}_{\alpha\mu\nu}(q,r,p)L_{sg}(s^2) \qquad s^2 = (q^2+r^2+p^2)/2$$

Where L_{sg} is the single transverse tensor structure of the three-gluon vertex in the soft-gluon limit. The accuracy of this approximation has been confirmed in multiple continuum and lattice works:

A. Blum, M. Q. Huber, M. Mitter and L. von Smekal, Phys. Rev. D **89**, 061703 (2014).
G. Eichmann, R. Williams, R. Alkofer and M. Vujinovic, Phys. Rev. D **89**, no.10, 105014 (2014).
F. Pinto-Gómez, F. De Soto, M.N.F., J. Papavassiliou and J. Rodríguez-Quintero, Phys. Lett. B **838**, 137737 (2023)
A. C. Aguilar, M.N.F., J. Papavassiliou and L. R. Santos, Eur. Phys. J. C **83**, no.6, 549 (2023).
F. Pinto-Gómez, F. De Soto and J. Rodríguez-Quintero, Phys. Rev. D **110**, no.1, 014005 (2024).

Using this approximation preserves the key symmetry of the kernel that guarantees multiplicative renormalization.