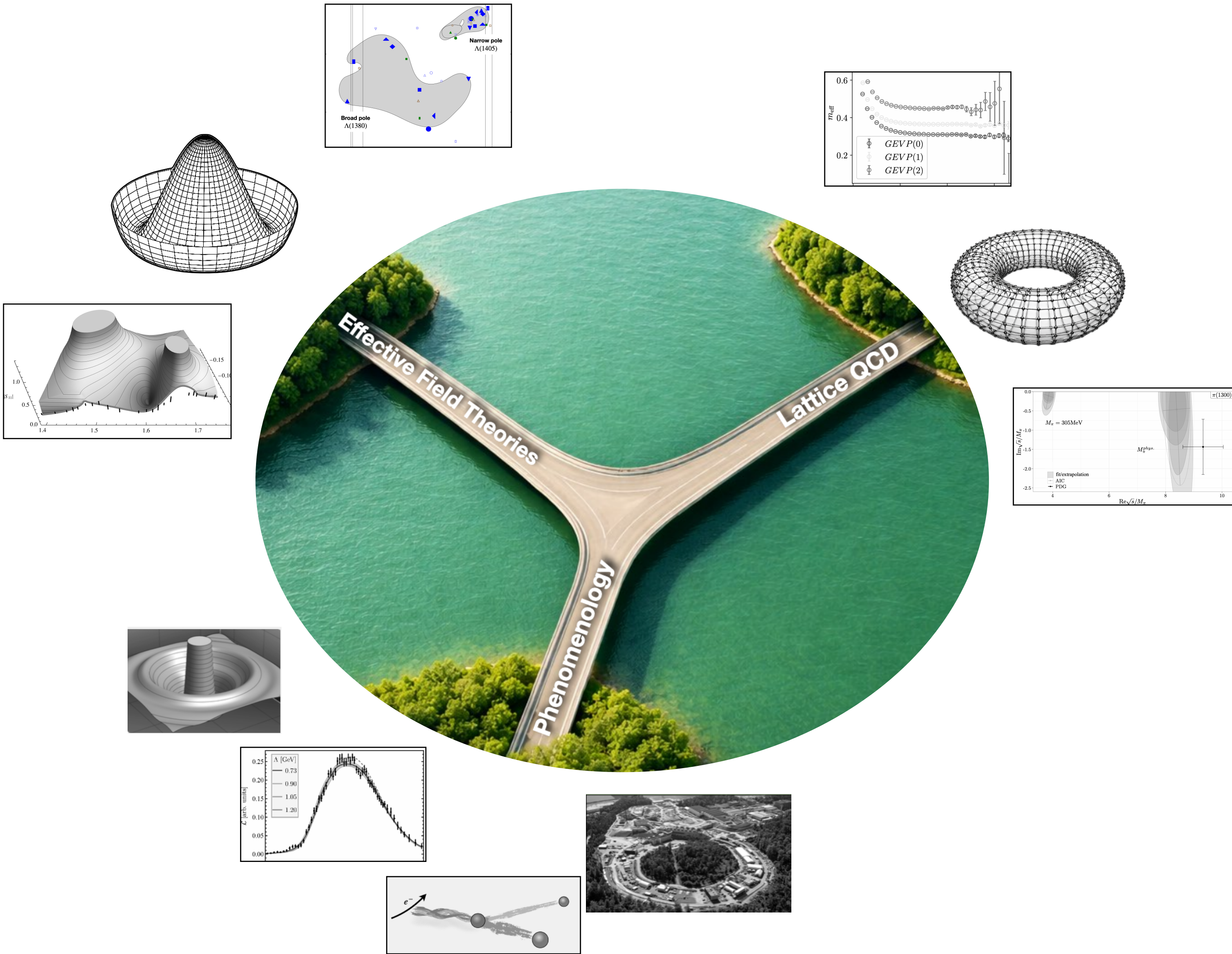


Nstar 2026 — September 2026 — Sevilla, Spain

MAXIM MAI

UNIVERSITY OF GIESSEN

THE GEORGE WASHINGTON UNIVERSITY



HADRON SPECTRUM

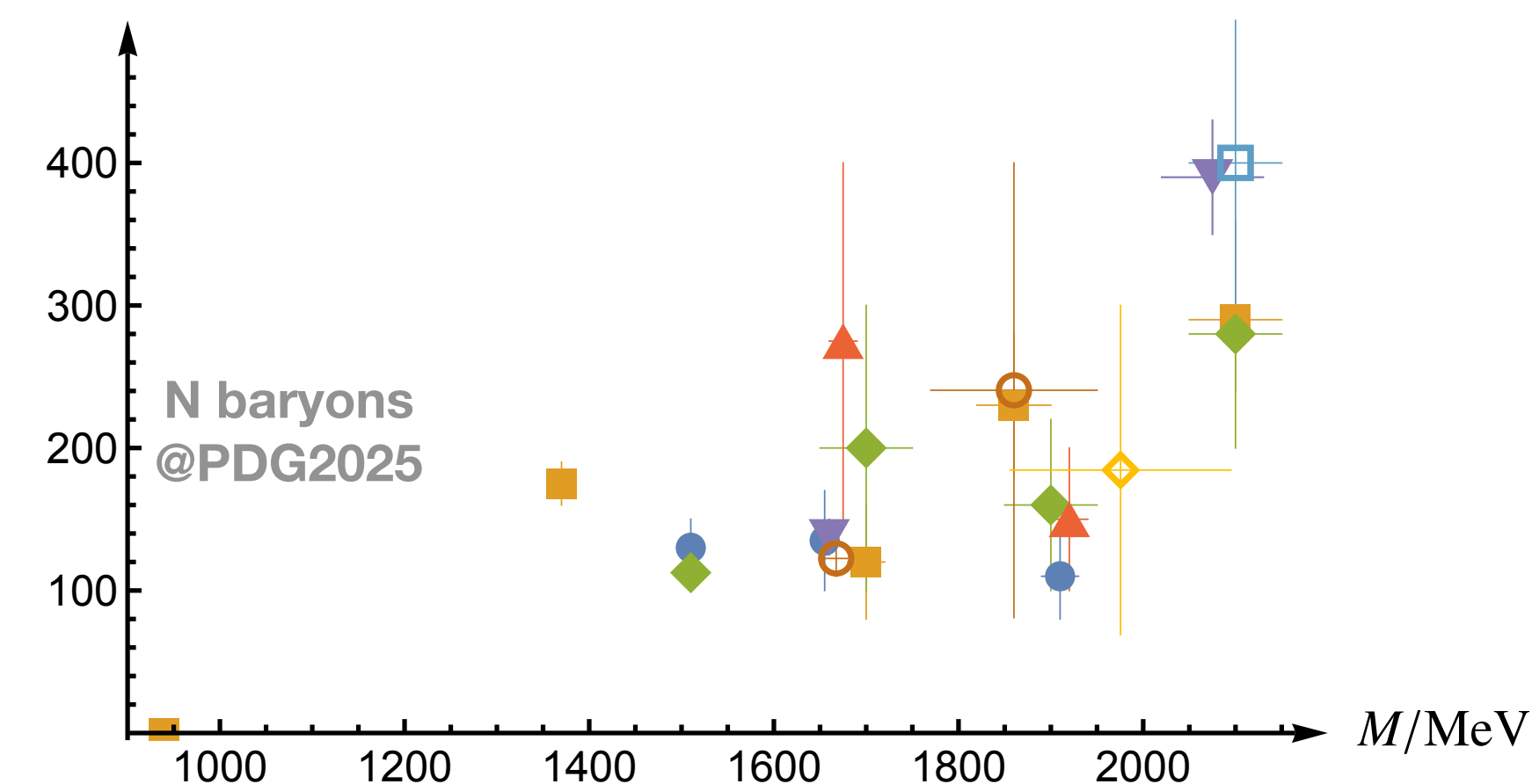
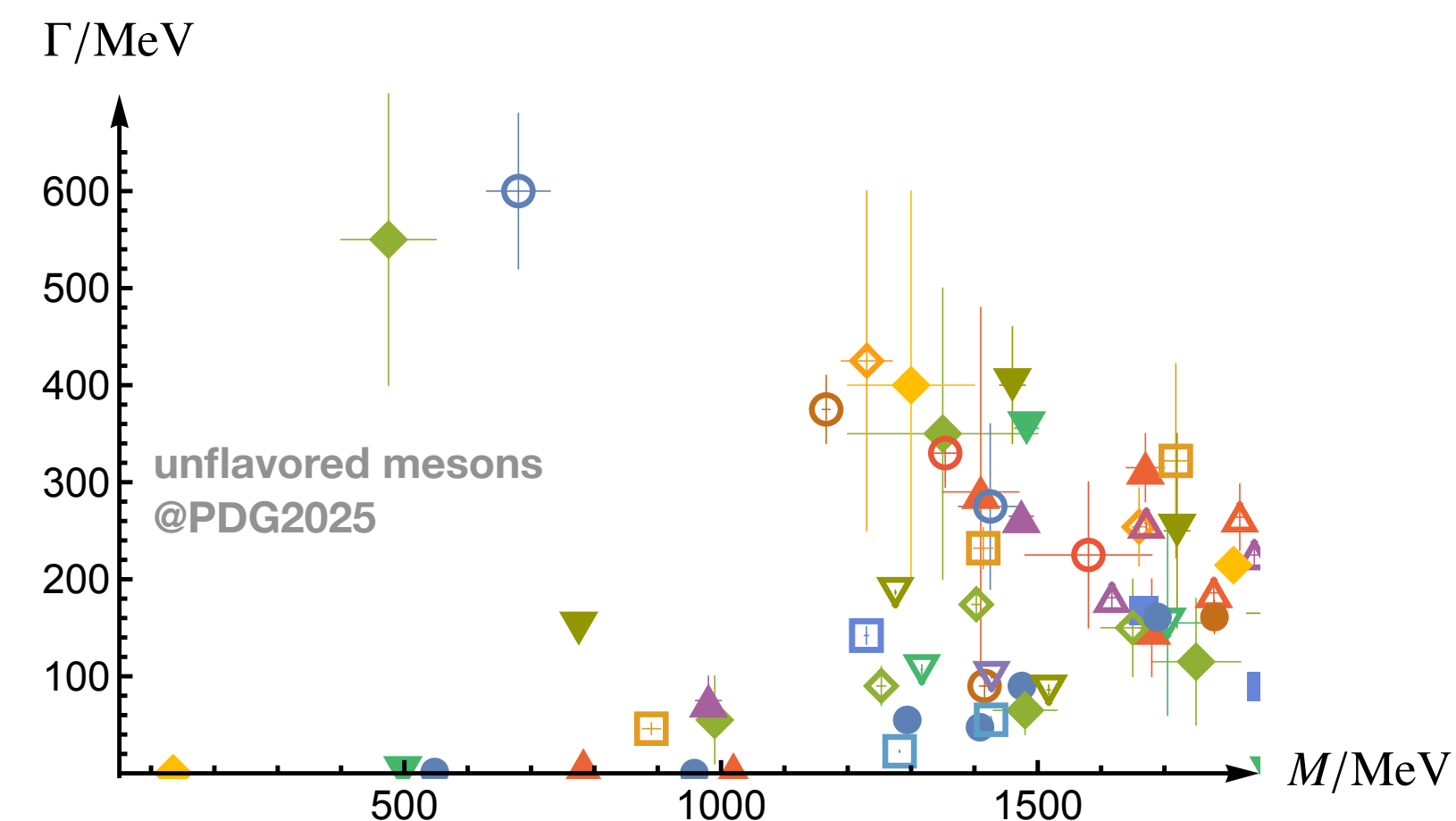
Experimental progress

- 70y research: $\Delta(1232)$, $\rho(770)$, $\omega(782)$, ...
- mostly excited states
 ≈ 100 mesons + 50 baryons (****)
- ongoing experiments @CERN, GSI, JLAB, J-PARC...

Talks: Imoehl, Beck, Schönning, Sako, Thoma Denisov, ...

Theory tools

- Lattice QCD
 - Effective Field Theories
 - Functional Methods
 - Amplitude analyses (machine learning) ...

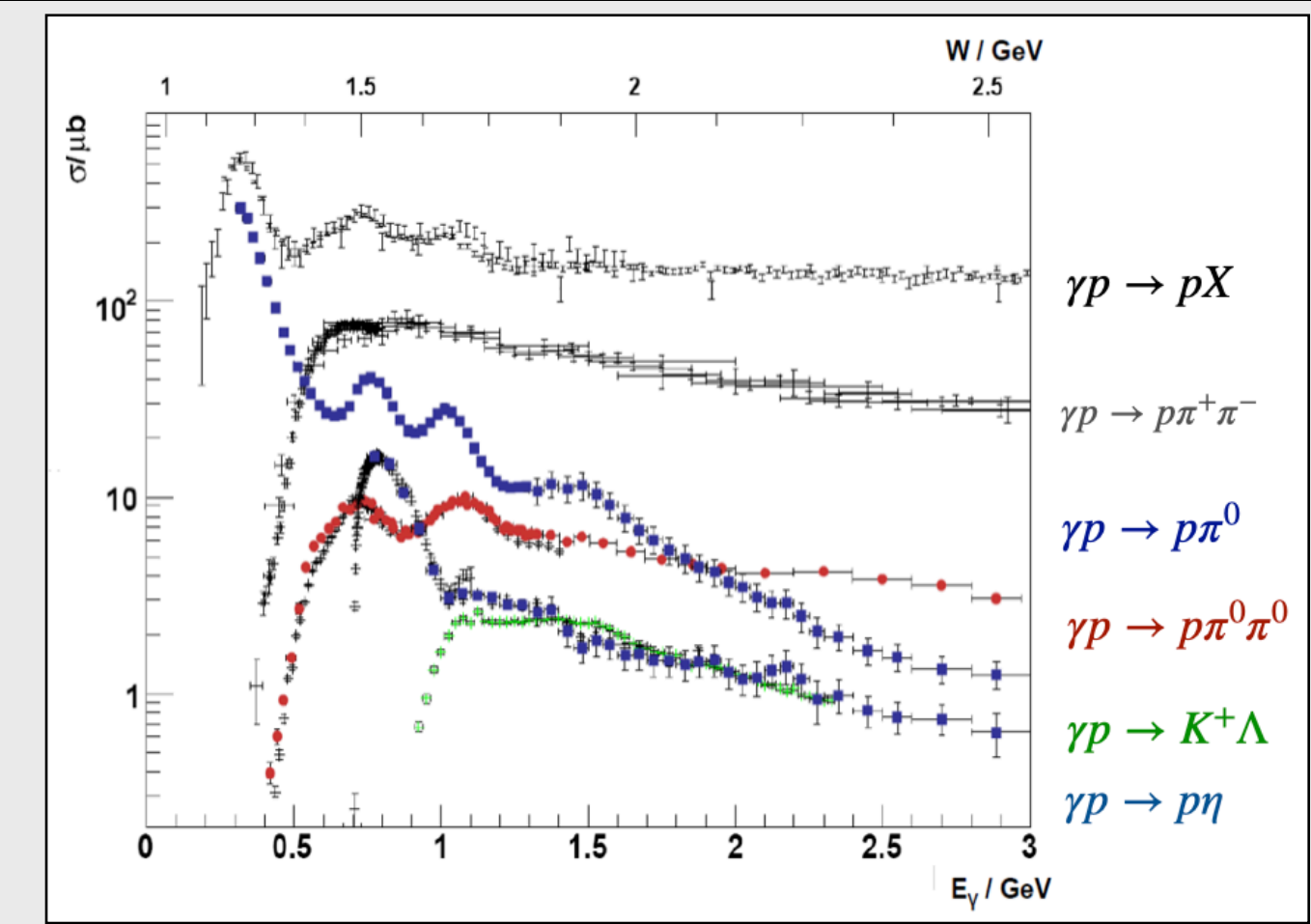


COMPLICATIONS

- overlapping states
- connection to QCD
- large multi (3+)—hadron content
- ...

COMPLICATIONS

- **overlapping states**
- connection to QCD
- large multi (3+)—hadron content
- ...



*see also:
Rabán's talk*

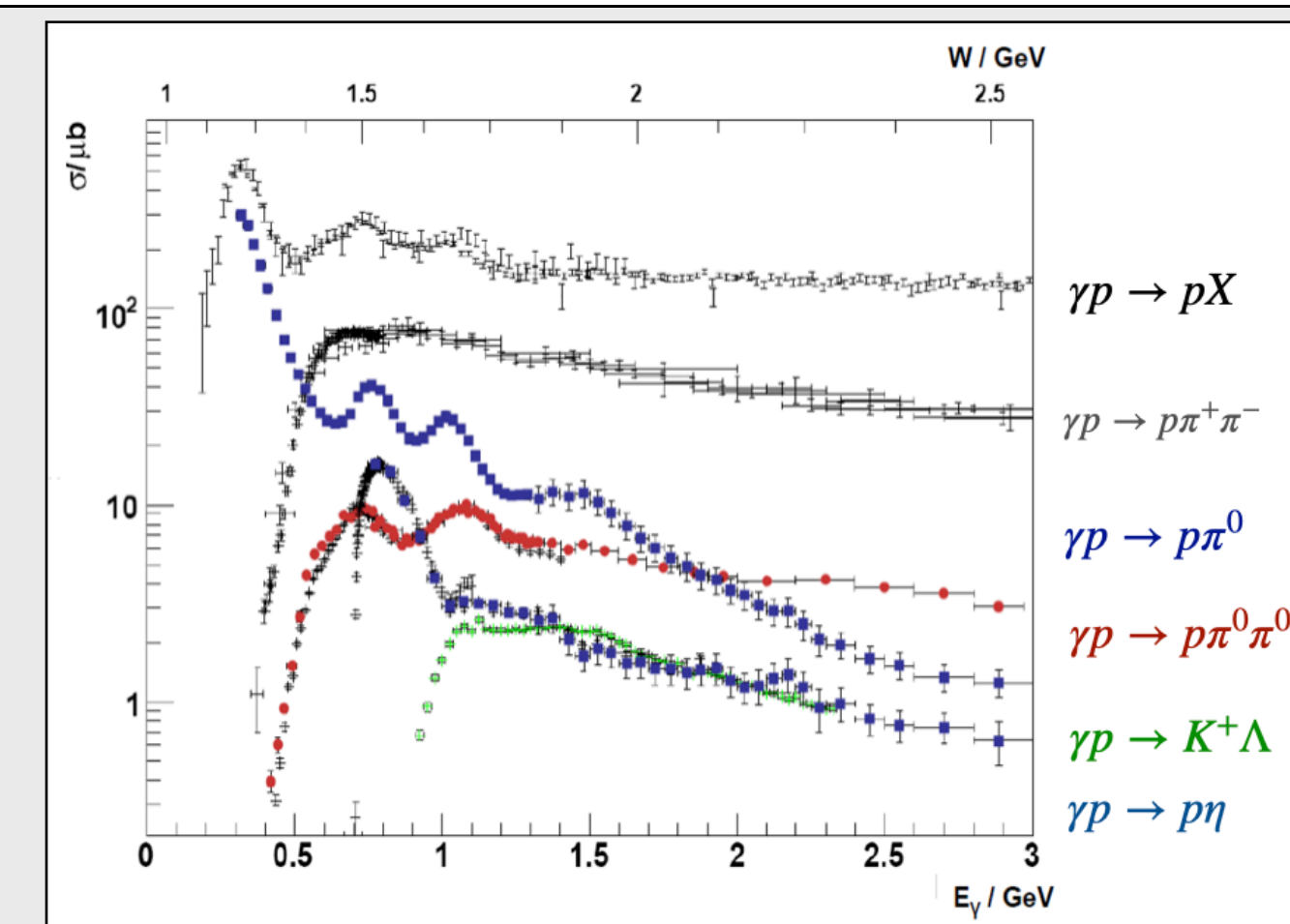
Data

- large amount of data
(10^5 for $\gamma^*p \rightarrow \pi N$)

*Machine learning:
Skoupil,
Pilloni,
Zarei,...*

COMPLICATIONS

- overlapping states
- connection to QCD
- large multi (3+)—hadron content
- ...



see also:
Rabán's talk

Data

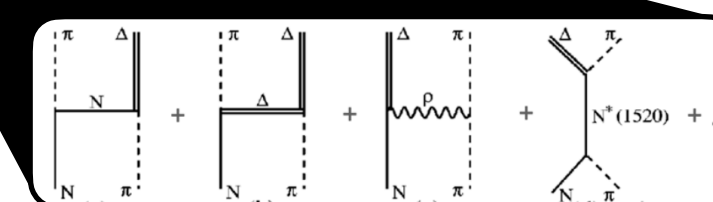
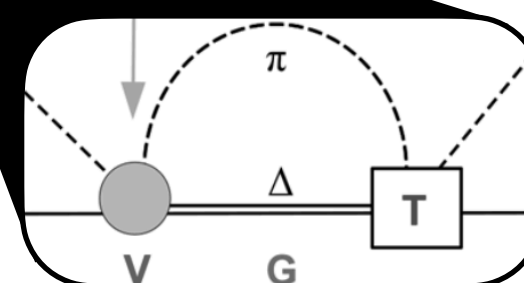
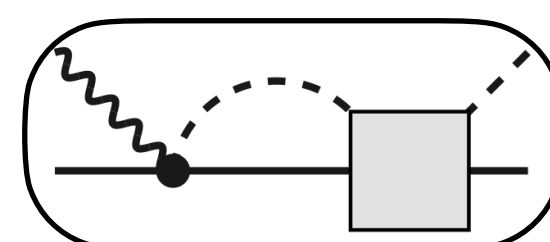
- large amount of data
(10^5 for $\gamma^* p \rightarrow \pi N$)

Machine learning:
Skoupil,
Pilloni,
Zarei,...

Jülich-Bonn-Washington

jbw.phys.gwu.edu

- Phenomenological Lagrangians
- coupled-channel dynamics
- gauge invariance
- Siegert's theorem
- Final-state unitarity
- ...



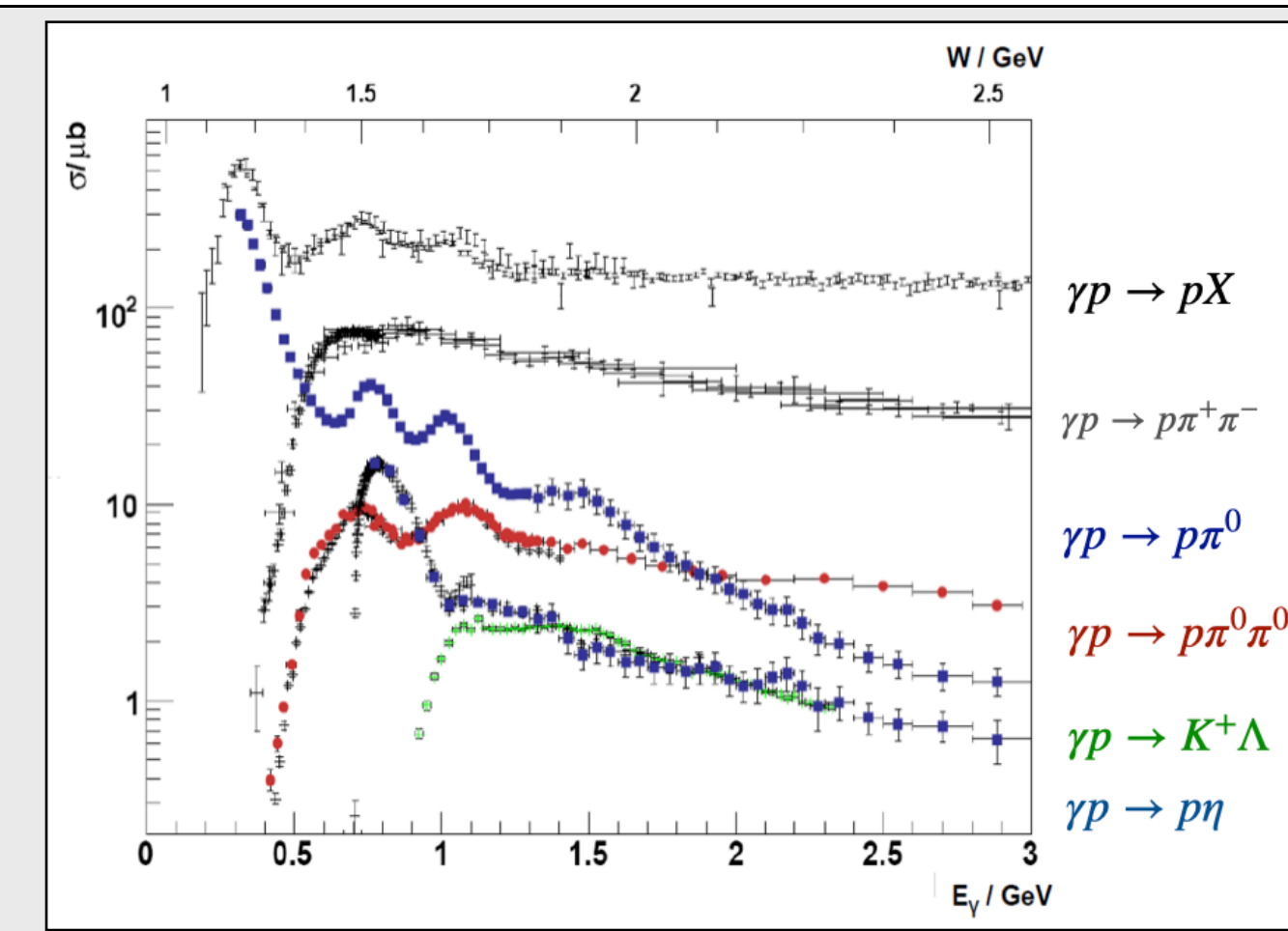
Review: Döring+ PNP(2025)

[1] ANL-Osaka PRC 80(2009), Few-Body Syst. 59(2018),... [2] MAID2007, EPJA 34(2007) EtaMAID2018, EPJA 54(2018)

[3] SAID, PiN Newsletter 16(2002) [4] Aznauryan et al., PRC 80(2009), IJMP(2013),... Gent group PRC 89(2014),...

COMPLICATIONS

- overlapping states
- connection to QCD
- large multi (3+)—hadron content
- ...

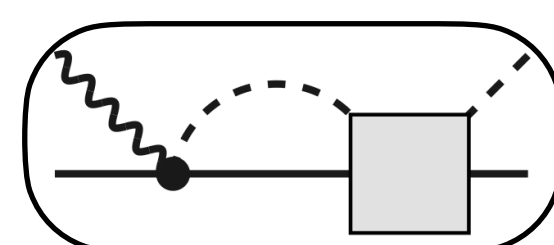


Data $\longrightarrow$ Jülich-Bonn-Washington $\longrightarrow$ Transition FF, charge distribution, ..

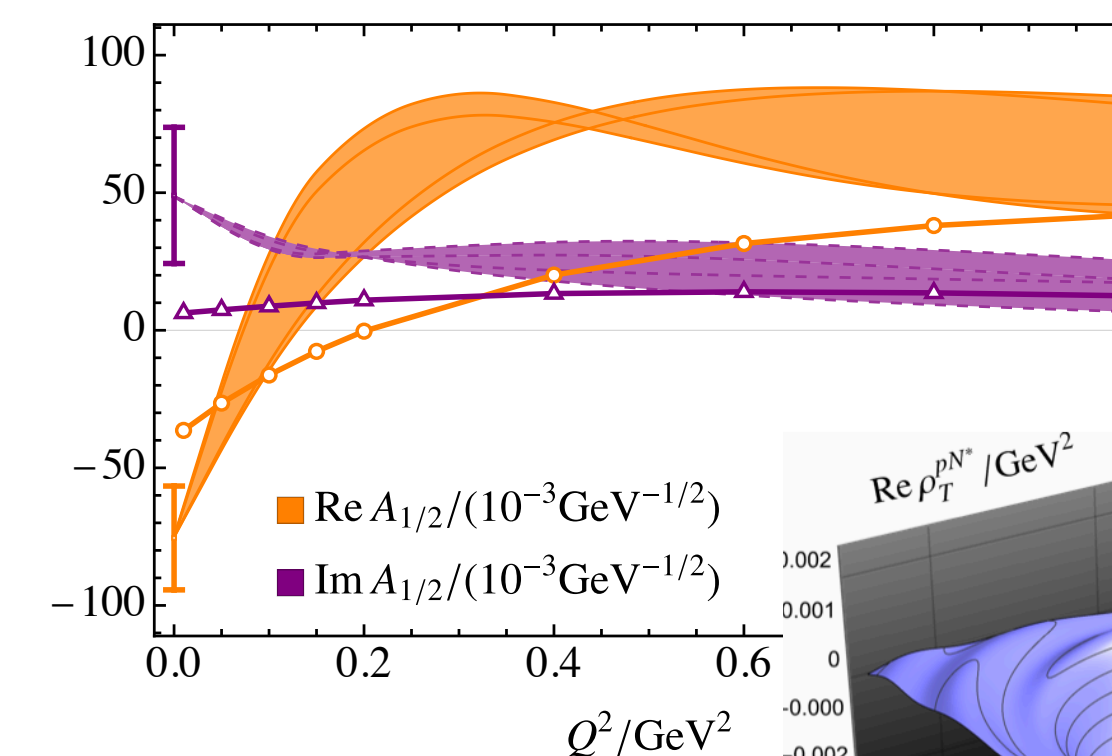
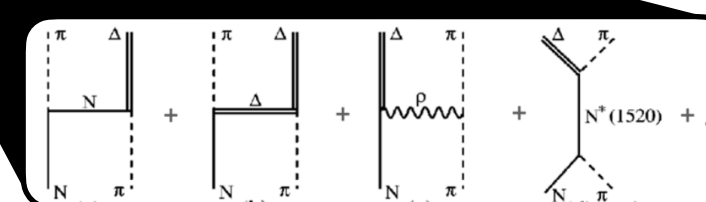
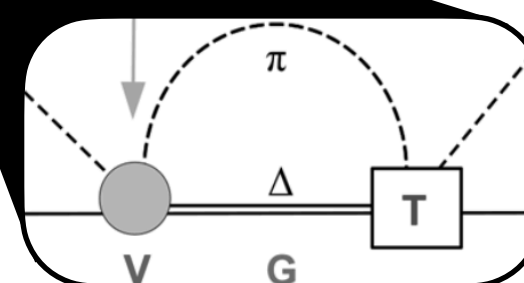
- large amount of data
(10^5 for $\gamma^* p \rightarrow \pi N$)

Machine learning:
Skoupil,
Pilloni,
Zarei,...

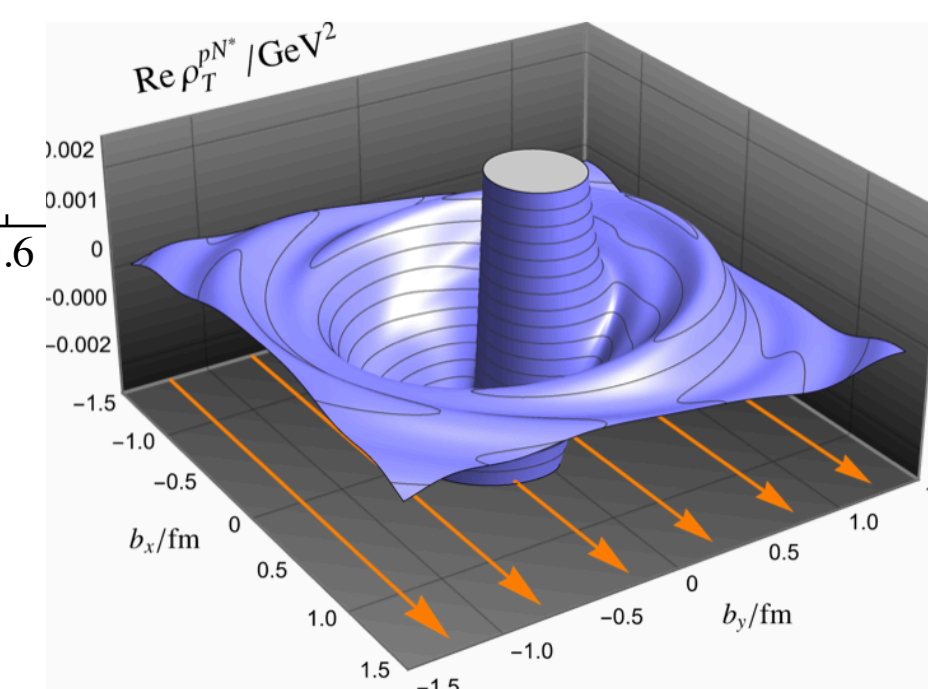
- Phenomenological Lagrangians
- coupled-channel dynamics
- gauge invariance
- Siegert's theorem
- Final-state unitarity
- ...



jbw.phys.gwu.edu



[JBW] Wang et al. PRL
e-Print: 2404.17444



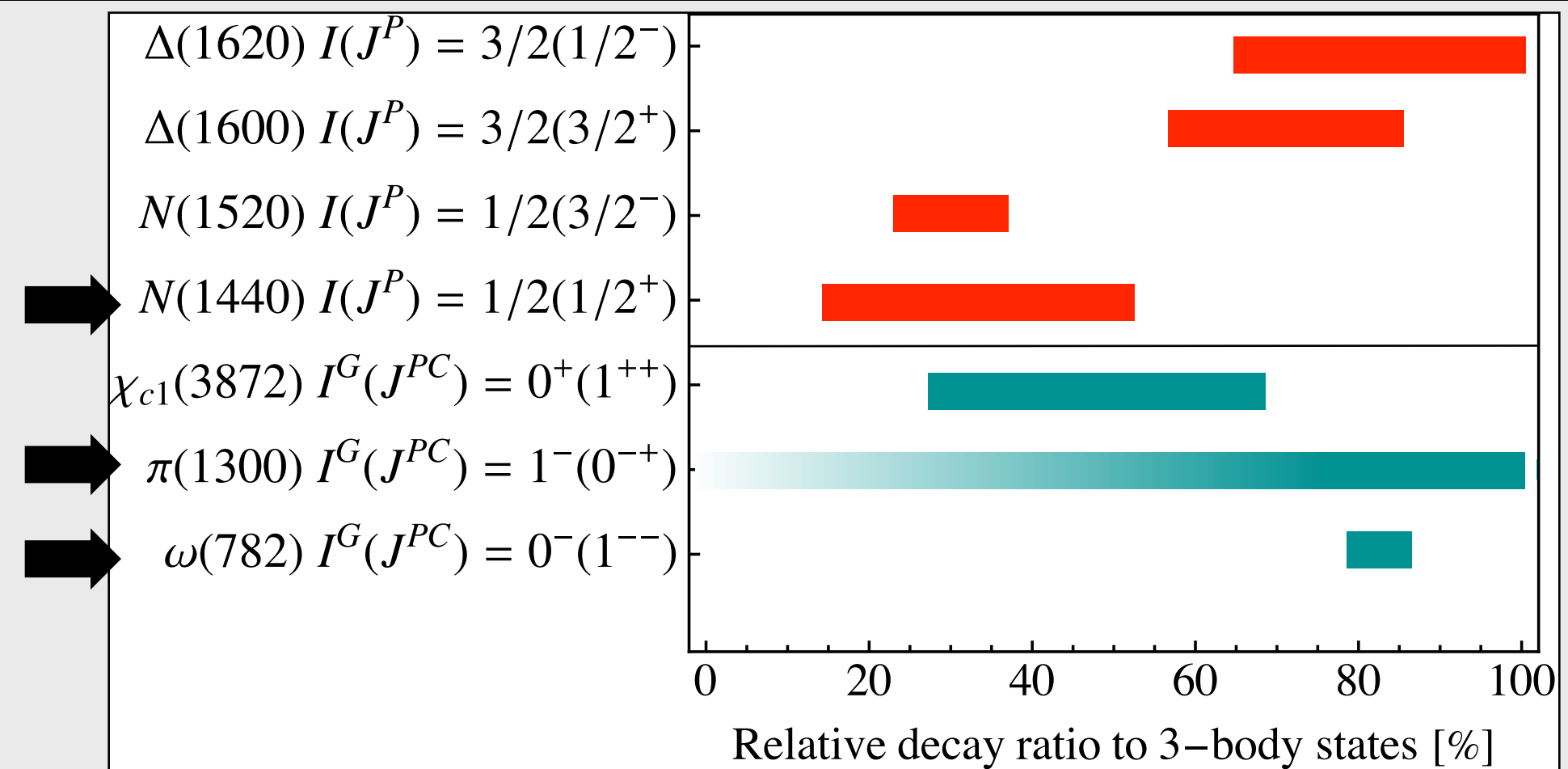
Review: Döring+ PNP(2025)

[1] ANL-Osaka PRC 80(2009), Few-Body Syst. 59(2018),... [2] MAID2007, EPJA 34(2007) EtaMAID2018, EPJA 54(2018)
[3] SAID, PiN Newsletter 16(2002) [4] Aznauryan et al., PRC 80(2009), IJMP(2013),... Gent group PRC 89(2014),...

Talks: Mokeev, Rönchen, Döring, Wang, ...

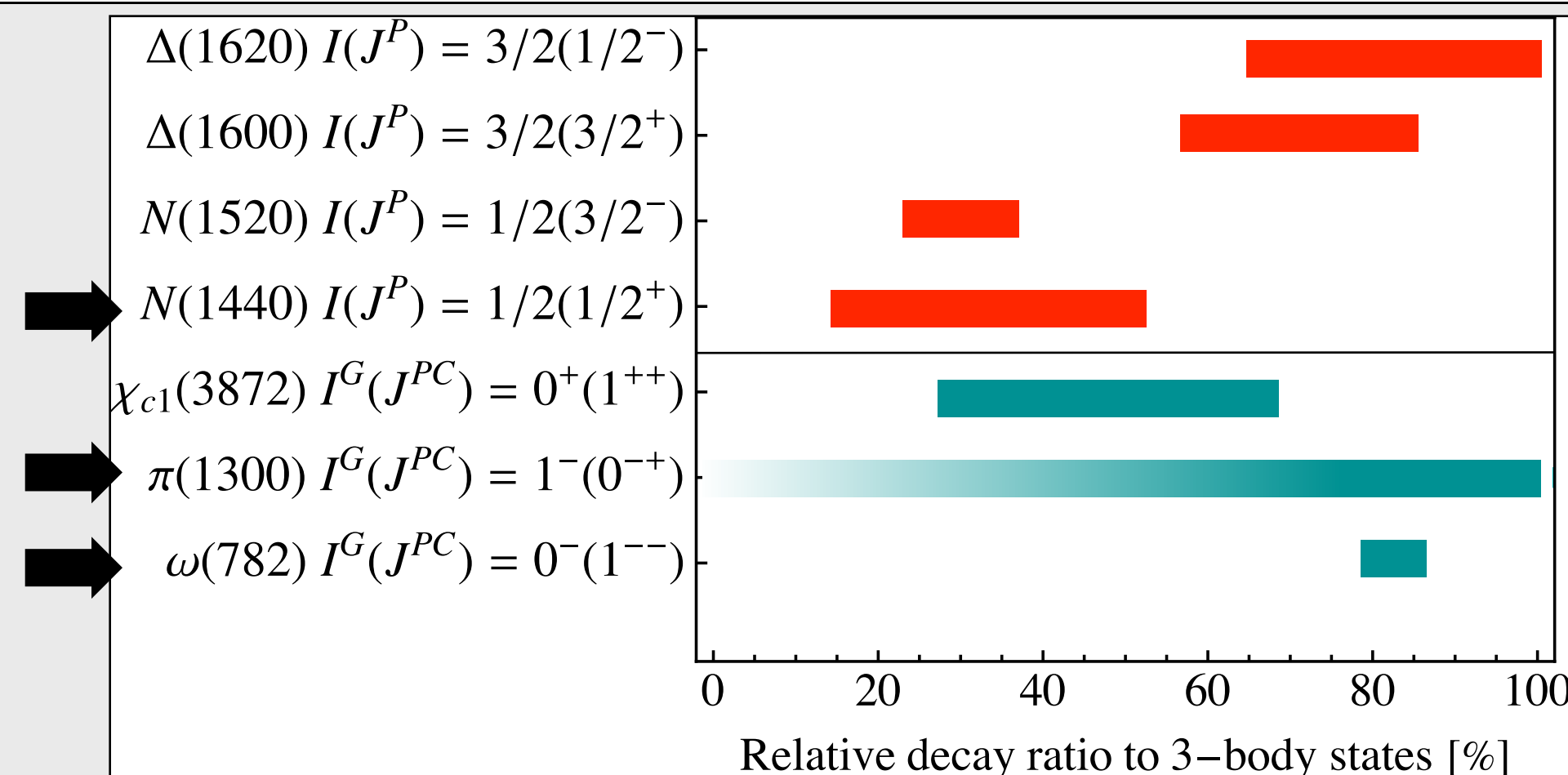
COMPLICATIONS

- overlapping states
- **connection to QCD**
- **large multi (3+)—hadron content**
- ...



COMPLICATIONS

- overlapping states
- **connection to QCD**
- **large multi (3+)—hadron content**
- ...



Discovery frontier

- Spin-exotic states
- Left-hand-cuts: $T_{cc}(3875)$, $N(1440)$, ...
- Triangle singularities: $a_1(1420)$
- ...

Talks:
Sakthivasan, Bulava, Nakamura

Consistency frontier

- Is excited pion so heavy?
- Why is excited nucleon so light?
- ...

Talks:
Rabán, Jinzi Wu, Sako, —this talk—

Precision frontier

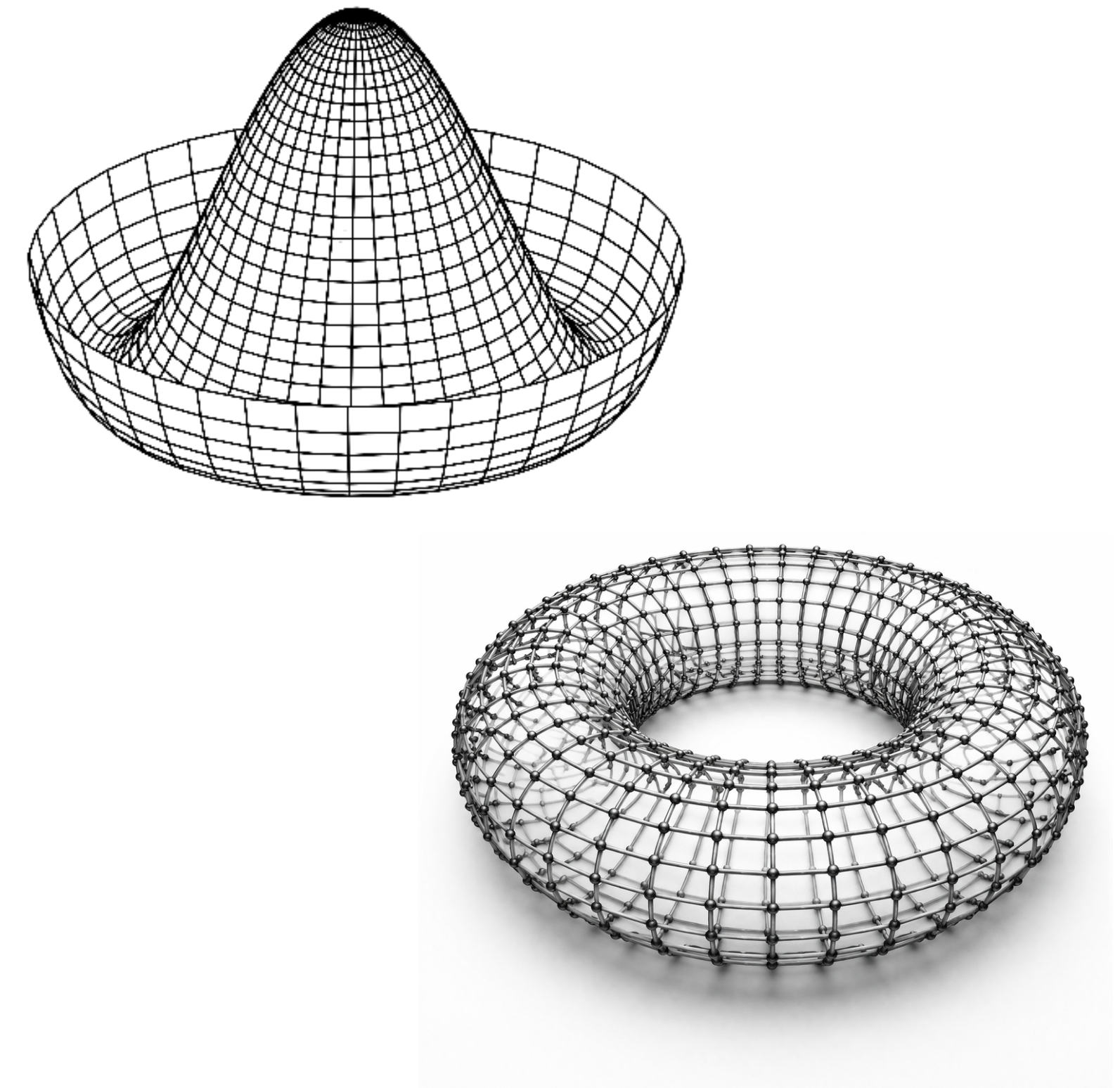
- Hadronic light-by-light
- τ -EDM
- Neutron stars
- ...

THEORY TOOLS

EFFECTIVE FIELD THEORIES

LATTICE QCD

...

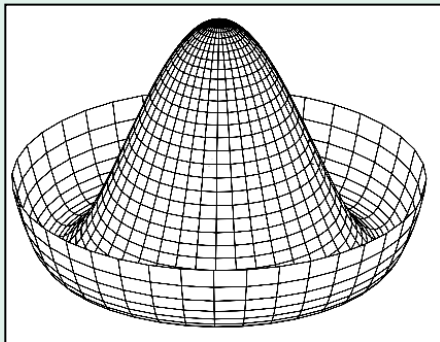


NON-PERTURBATIVE APPROACHES

Effective Field Theory (EFT/CHPT)

Weinberg (1979) Gasser, Leutwyler (1981), ...

Reviews: *Ann. Rev. Nucl. Part. Sci.* 57, 33 (2007), *Adv. Nucl. Phys.* 27, 277 (2003), ...



$$Z[J] = \int [DU] e^{\int d^4x \mathcal{L}_{\text{eff}}(U,v,a,s,p)}$$

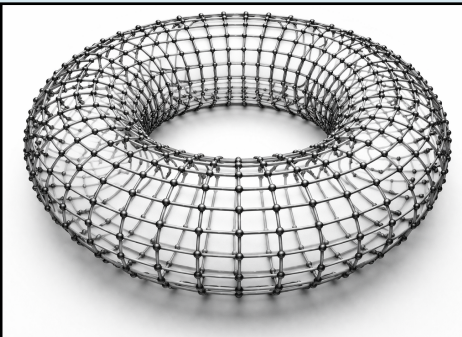


- [+] effective/hadronic degrees of freedom
- [-] unknown low-energy constants
- [+] well-defined power counting/quark mass dependence

Lattice Gauge Theory (LQCD)

Wilson, *Phys. Rev. D* 10 (1974) 2445, ...

Reviews: *hep-lat/9807028* [*hep-lat*] *Rev.Mod.Phys.* 90 (2018) *Rept.Prog.Phys.* 86 (2023)



$$Z[J] = \int [DU] e^{-S_E} \det[M[U]]$$

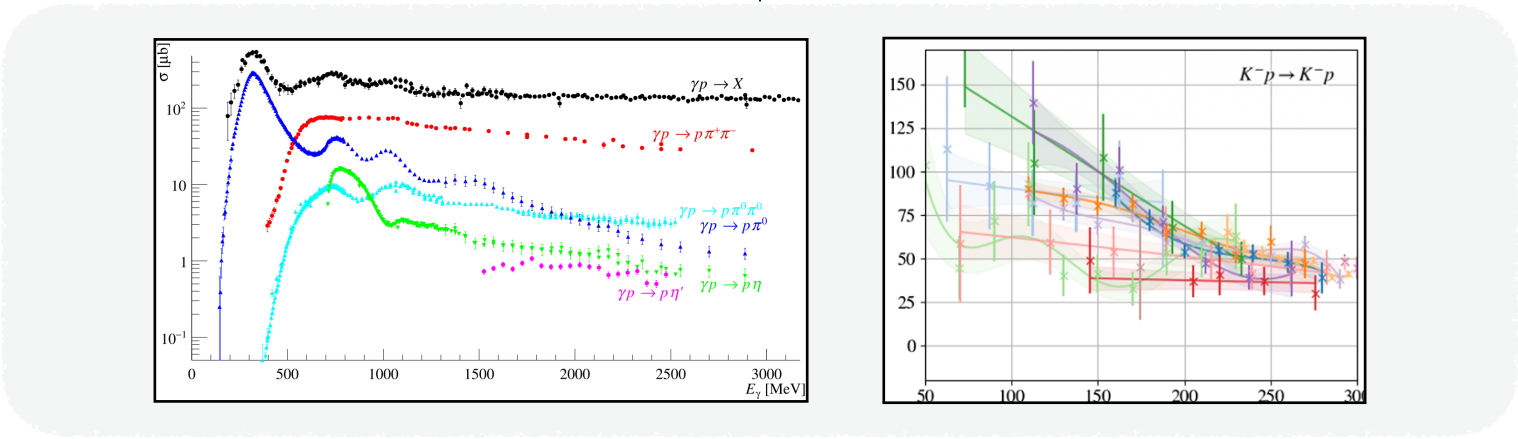


- [+] non-perturbative access to QCD Green's functions
- [-] discretized space-time / Euclidean metric / finite-volume
- [+/-] unphysical quark masses

Collins, Morningstar, Bulava, ...

S-matrix

Crossing symmetry — particle/antiparticle
Unitarity — probability conservation
Analyticity — causality



Further approaches:

- **Functional methods**

Review: Eichmann+ *PPNP* 91 (2016) 1-100

Talks: Roberts, Eichmann, Binosi, Hoffer, ...

• ...

EXAMPLE: 2-BODY CASE $\Lambda(1405)$, $\Lambda(1380)$, $\Lambda(1670)$ *Talks: Morningstar, ...*

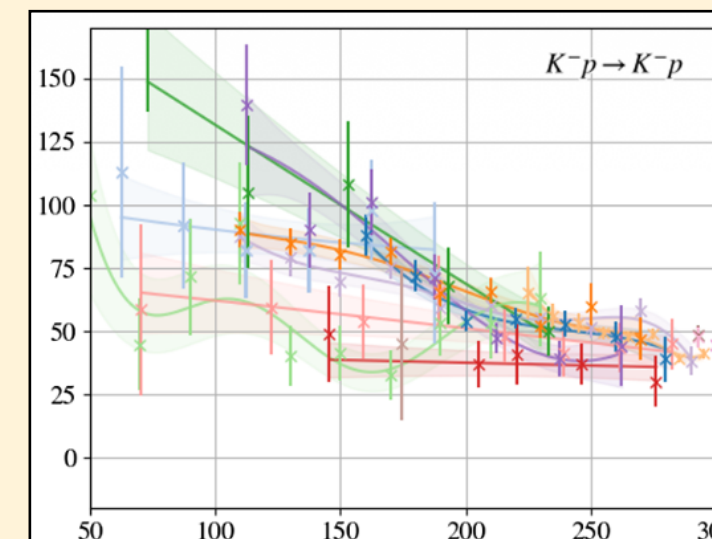
INPUT #1: Physical point

- cross sections, SIDDHARTA(2)

- Amplitude

$$T_l = \frac{1}{K^{-1} - ip_{\text{cm}}}$$

- $\text{Im } T_l^{-1}$ on-shell configurations



INPUT #2: Lattice QCD (unphysical point)

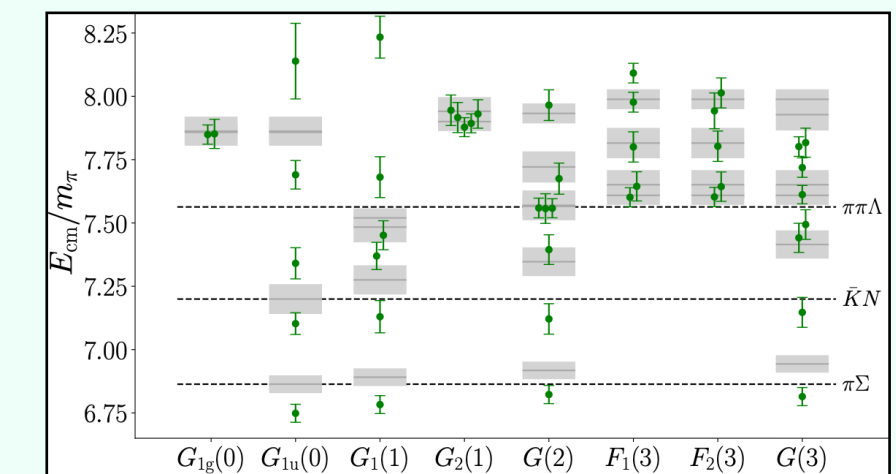
[BaSc] PRL 132 (2024) 5; Sucunza+ arXiv:2605.28447

- discrete spectrum

- 2b Quantization Condition *Lüscher, NPB 354*

$$\frac{1}{K^{-1}(E^*) - B_L(E^*)} = \infty$$

- $B_L = \infty$ on-shell configurations

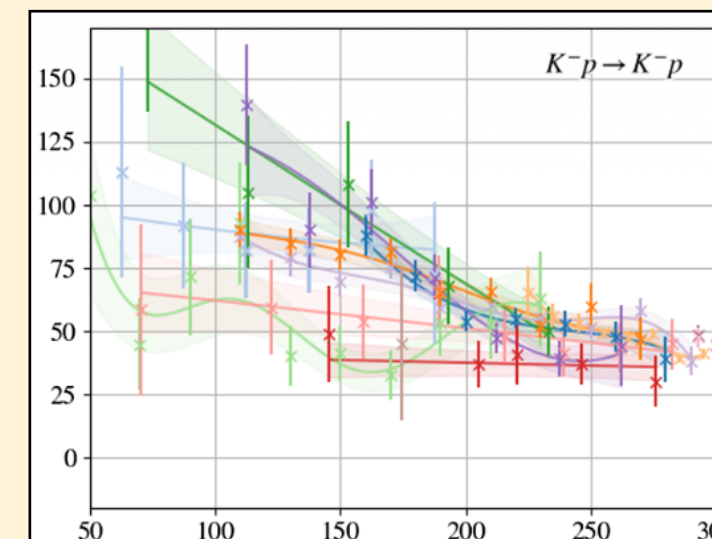


EXAMPLE: 2-BODY CASE $\Lambda(1405)$, $\Lambda(1380)$, $\Lambda(1670)$

Talks: Morningstar, ...

INPUT #1: Physical point

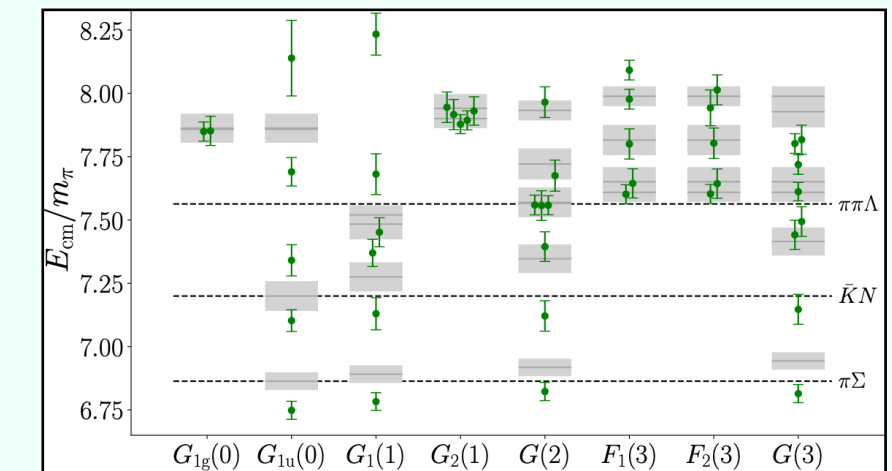
- cross sections, SIDDHARTA(2)
 - Amplitude
- $$T_l = \frac{1}{K^{-1} - ip_{\text{cm}}}$$
- $\text{Im } T_l^{-1}$ on-shell configurations



INPUT #2: Lattice QCD (unphysical point)

[BaSc] PRL 132 (2024) 5; Sucunza+ arXiv:2605.28447

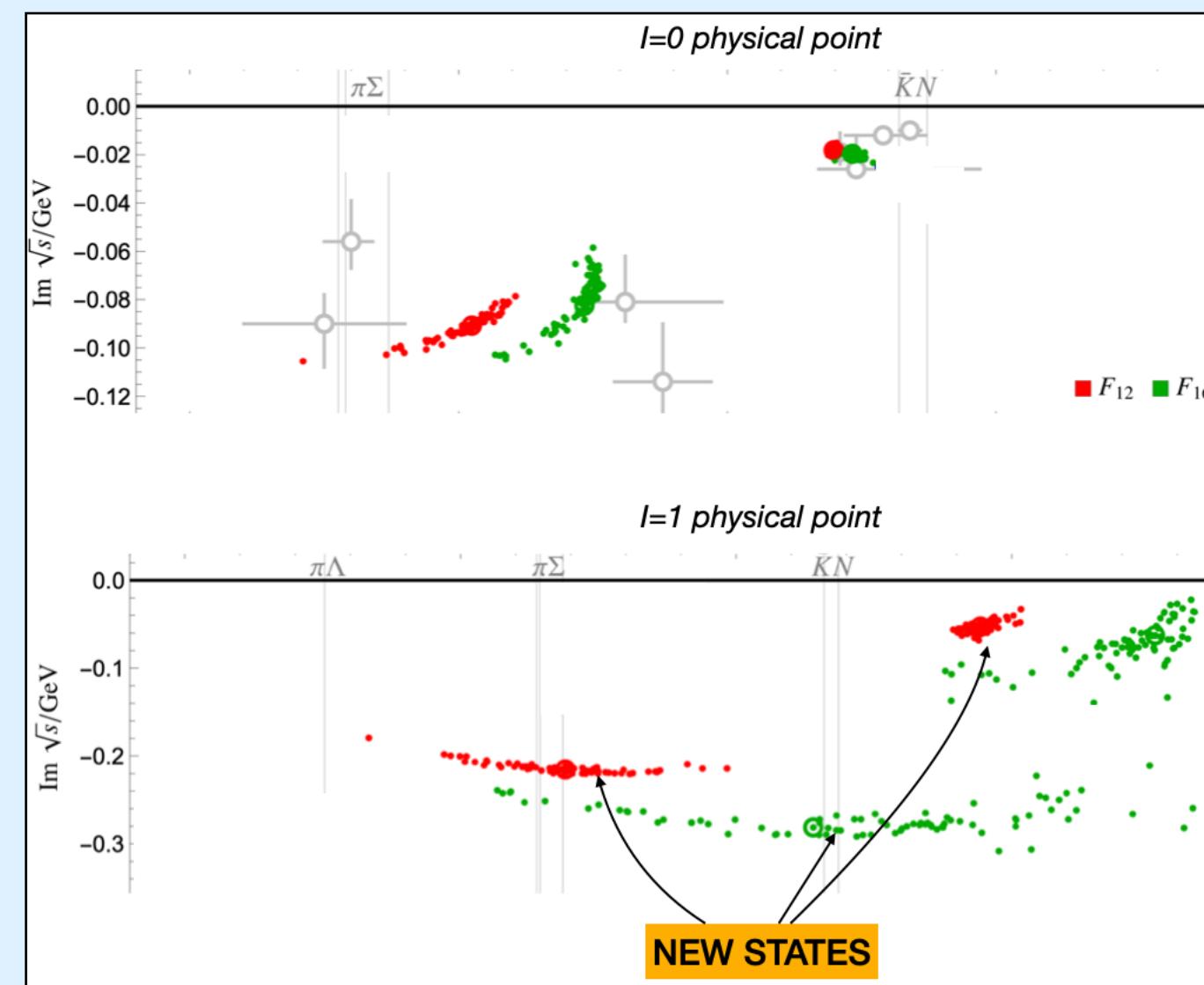
- discrete spectrum
 - 2b Quantization Condition [Lüscher, NPB 354](#)
- $$\frac{1}{K^{-1}(E^*) - B_L(E^*)} = \infty$$
- $B_L = \infty$ on-shell configurations



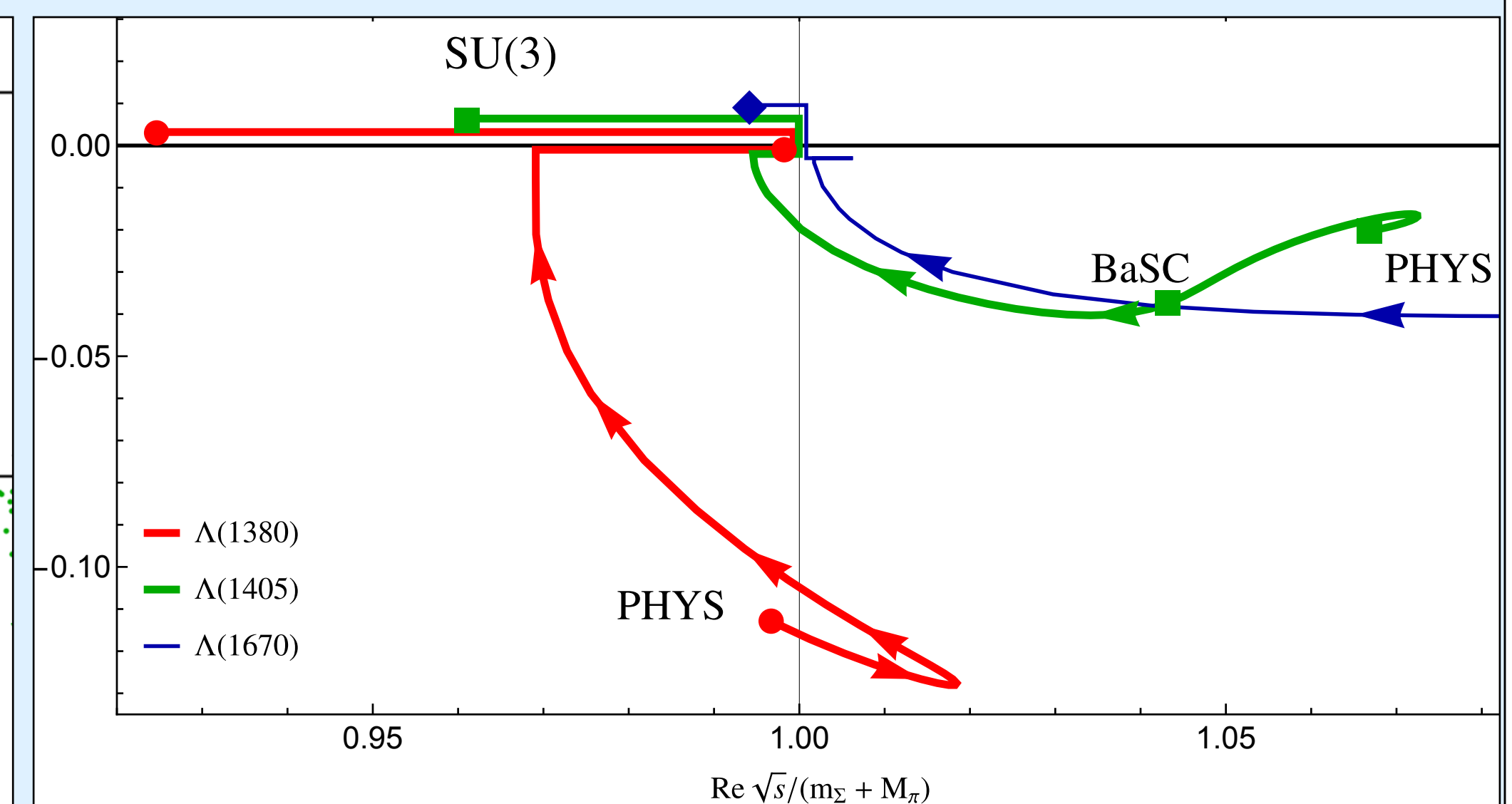
- Unknown quantity K^{-1}
- Quark-mass dep. from CHPT

OUTPUT

- $l=0$ poles
- $l=1$ poles [prediction]
- chiral trajectories...
Exp — Lattice — Lattice (SU3)



Pittler+ Phys.Rev.D 112 (2025) 7, 074037



Guo+ Phys.Lett.B 846 (2023) 138264

Sucunza+ arXiv:2605.28447

3-BODY CASE: FVU APPROACH

Three-body scattering amplitude

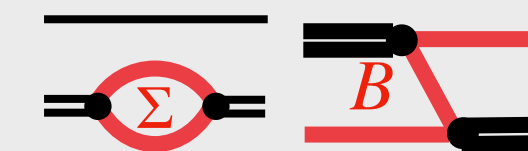
- 3b-Unitarity
- intergal equation (explicit momenta)
- generalized to all channels + strangeness

Feng+ PRD 110 (2024) 094002 2407.08721

“Infinite Volume Unitarity” (IVU)

$$T(s, p, p') = \textcolor{red}{B} + \textcolor{blue}{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(\textcolor{red}{B} + \textcolor{blue}{C})}{2E_l} \frac{1}{\tilde{K}^{-1} - \textcolor{red}{\Sigma}} T$$

MM/Hu/Döring/Pilloni/Szczepaniak *Eur.Phys.J.A* 53 (2017)



on-shell configurations
no free parameters



Real/offshell
“2-/3-body force”

3-BODY CASE: FVU APPROACH

Three-body scattering amplitude

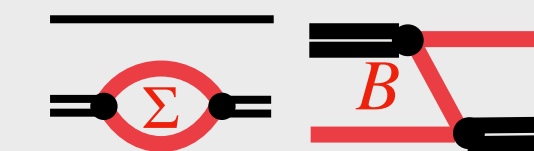
- 3b-Unitarity
- intergal equation (explicit momenta)
- generalized to all channels + strangeness

Feng+ PRD 110 (2024) 094002 2407.08721

“Infinite Volume Unitarity” (IVU)

$$T(s, p, p') = \textcolor{red}{B} + \textcolor{blue}{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(\textcolor{red}{B} + \textcolor{blue}{C})}{2E_l} \frac{1}{\tilde{K}^{-1} - \textcolor{red}{\Sigma}} T$$

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)



on-shell configurations
no free parameters



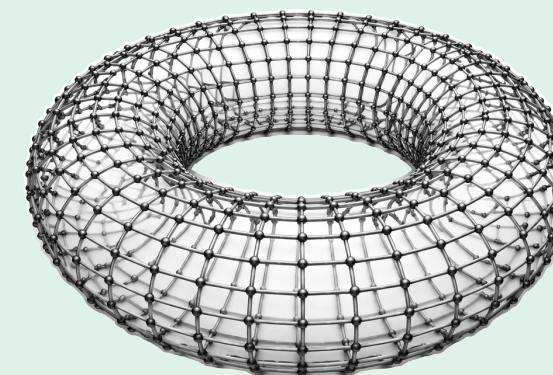
Real/offshell
“2-/3-body force”

$\sim O((ML)^n)$

$\sim O(e^{-ML})$

Three-body quantization condition

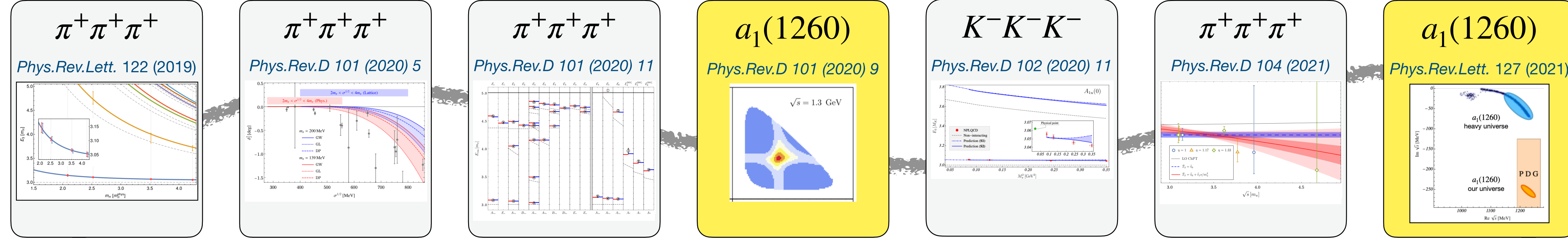
- only discrete momenta are possible $2\pi/L \mathbb{Z}^3$
- determinant equation
- relevant volume-dependence: singularities/onshell states
(+ $O(e^{-ML})$)



“Finite Volume Unitarity” (FVU)

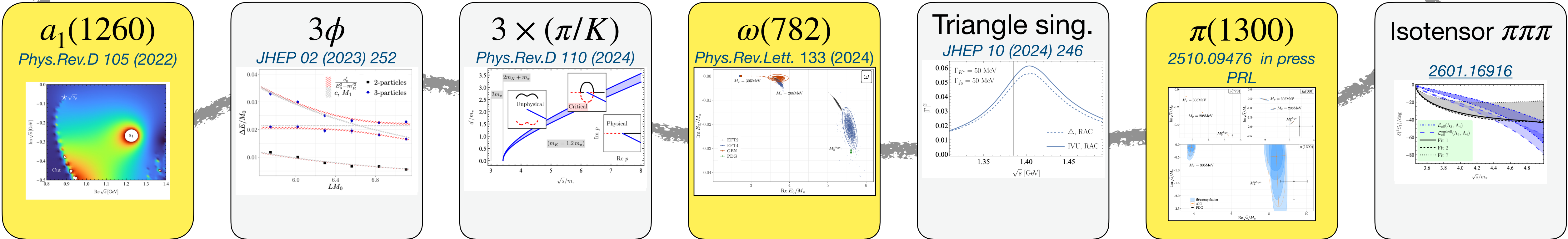
$$\det \left[2L^3 E_p (\tilde{K}^{-1} - \textcolor{red}{\Sigma}^L) - \textcolor{red}{B} - \textcolor{blue}{C} \right]_{\Gamma} \equiv 0$$

MM/Döring Eur.Phys.J.A 53 (2017) 12, 240



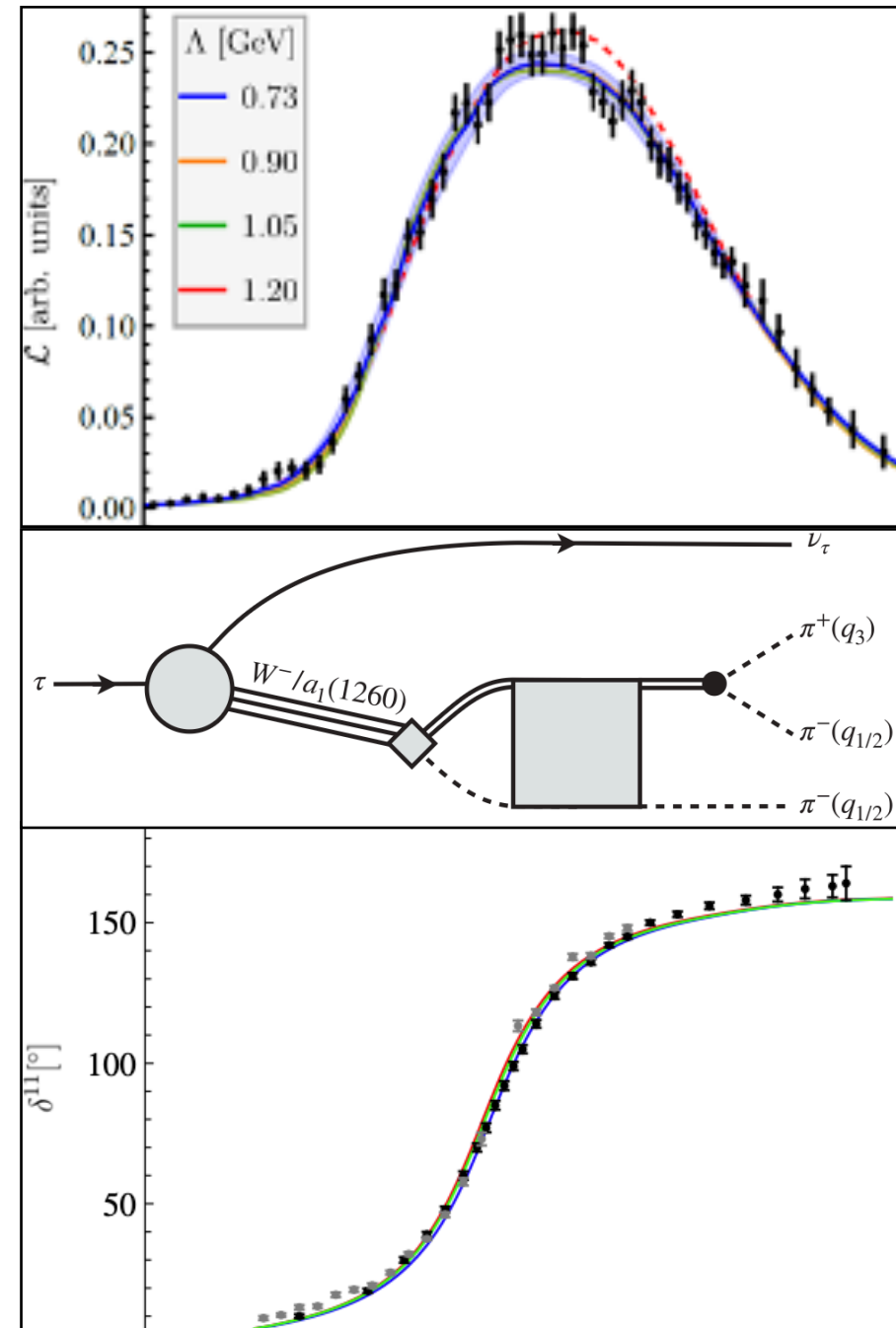
Eur.Phys.J.A 53
(2017) 12, 240

FVU/IVU APPLICATIONS



AXIAL-VECTOR MESON

Sadasivan/Akdag/MM/Culver/Alexandru/Lee/Doring Phys.Rev.D 101 (2020); Phys.Rev.D 105 (2022)



Schael et al. [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7;

“Infinite Volume Unitarity” (IVU)

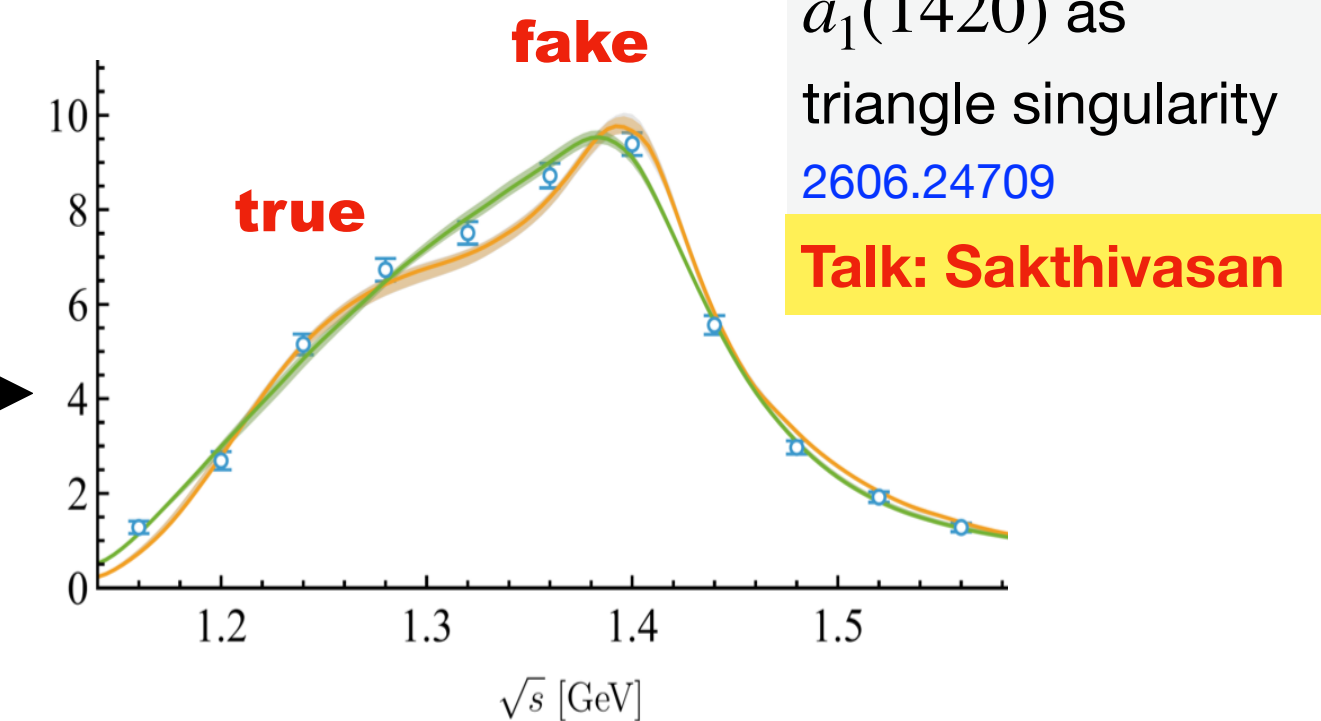
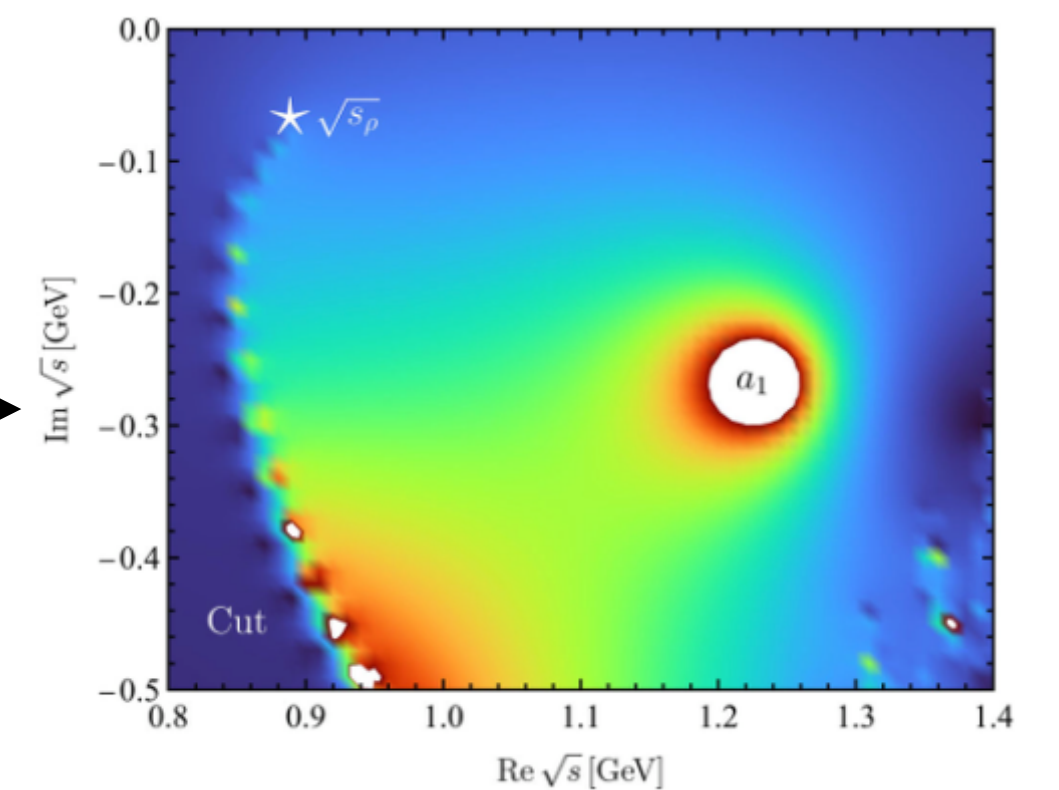
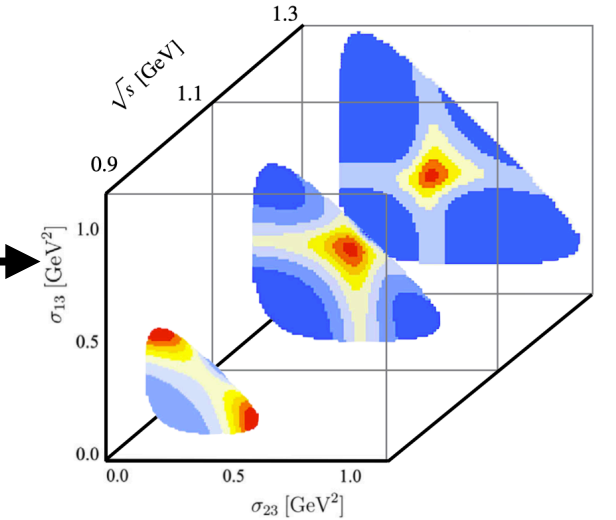
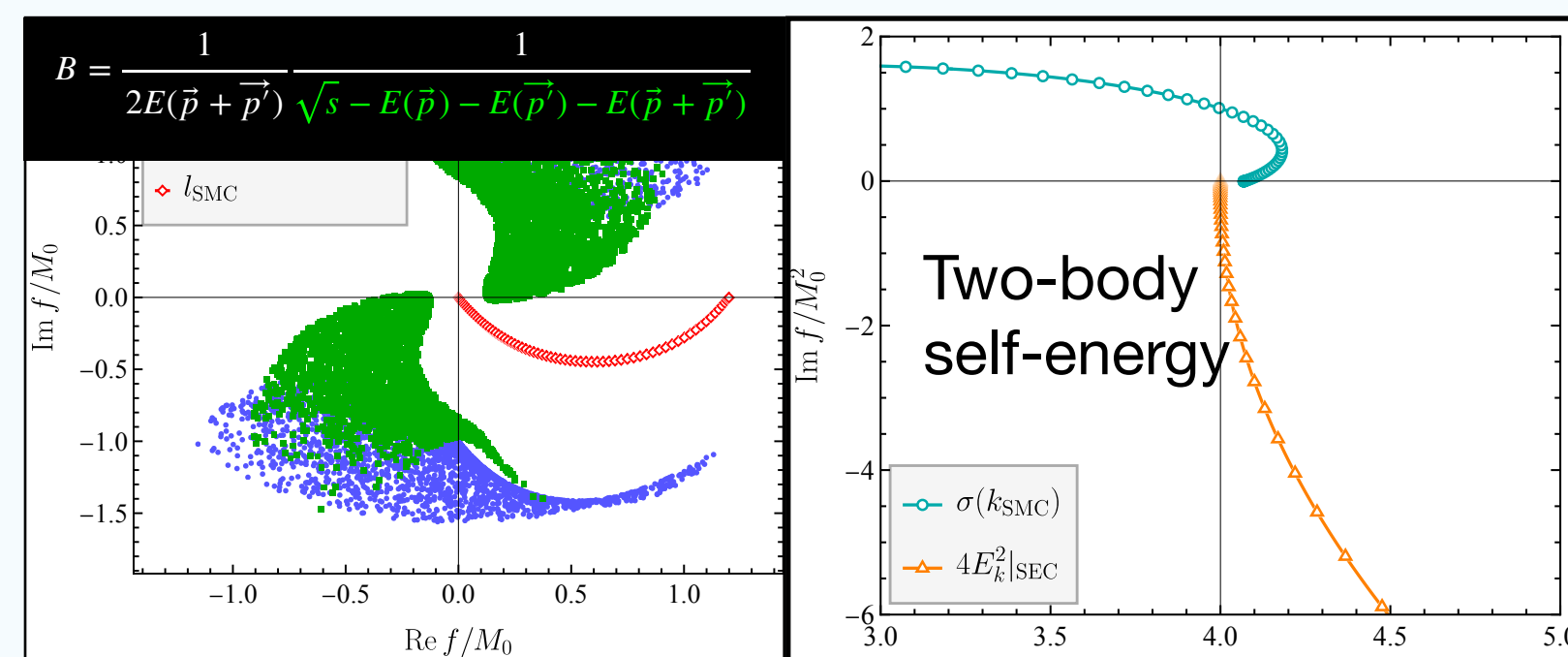
$$T(s, p, p') = B + C + \int \frac{d^3 \ell}{(2\pi)^3} \frac{(B + C)}{2E_\ell} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

Solution techniques

Hetherington/Schick, Phys. Rev. 137, B935–B948 (1965). Cahill/Sloan, Nucl. Phys. A 165, 161–179 (1971) Schmid/Ziegelmann, Adhikari/Amado, Phys. Rev. D 9, 1467–1475 (1974)
Comparison: Sakthivasan et al. JHEP 10 (2024) 246

- Complex contour deformation $|\ell| \in \mathbb{C}$
- Analytic continuation to the real axis



$a_1(1420)$ as triangle singularity
2606.24709

Talk: Sakthivasan

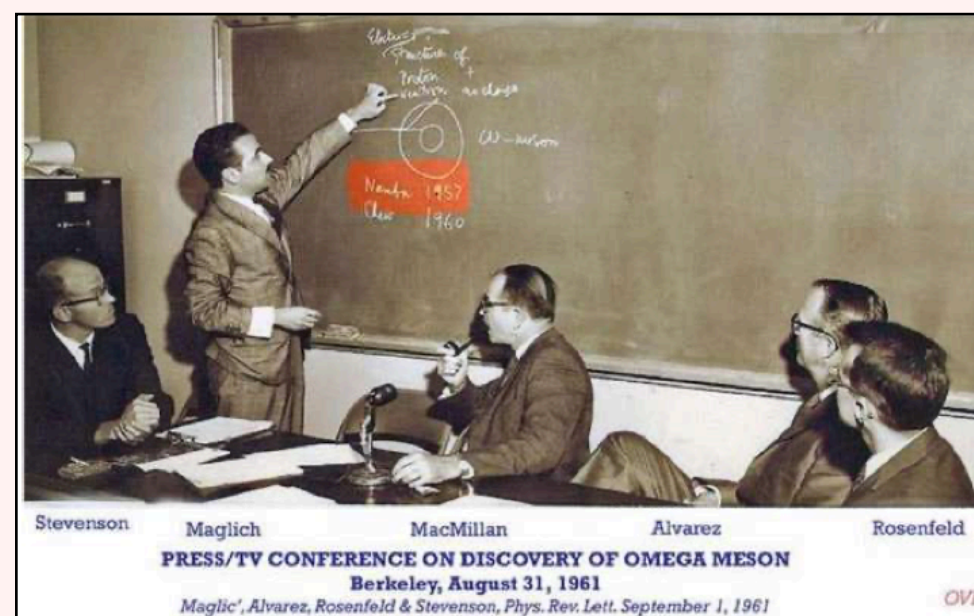
Review on triangle singularities: Guo+ Prog.Part.Nucl.Phys. 112 (2020) 103757

VECTOR MESON

[agemo] Yan+ PRL133 (2024)

$\omega(782)$

- lightest hadron decaying into three particles



- isoscalar response within the VMD of the photon-nucleon interactions
- ~repulsion at < 1 fm in the one-boson-exchange picture of the N–N interaction

Sakurai (1960); Erkelenz (1974); Brown and Jackson (1976); Barkov et al., 1985; Connell et al. (1997); Bazavov et al. (2021)

What can we learn from Lattice QCD?

- two/three-body force
- pion-mass dependence
- KSFR/Universality relations/... in EFT

Gell-Mann/Sharp/Wagner/Fujiwara/Kawarabayashi... Review: Meißner Phys.Rept. 161 (1988) 213

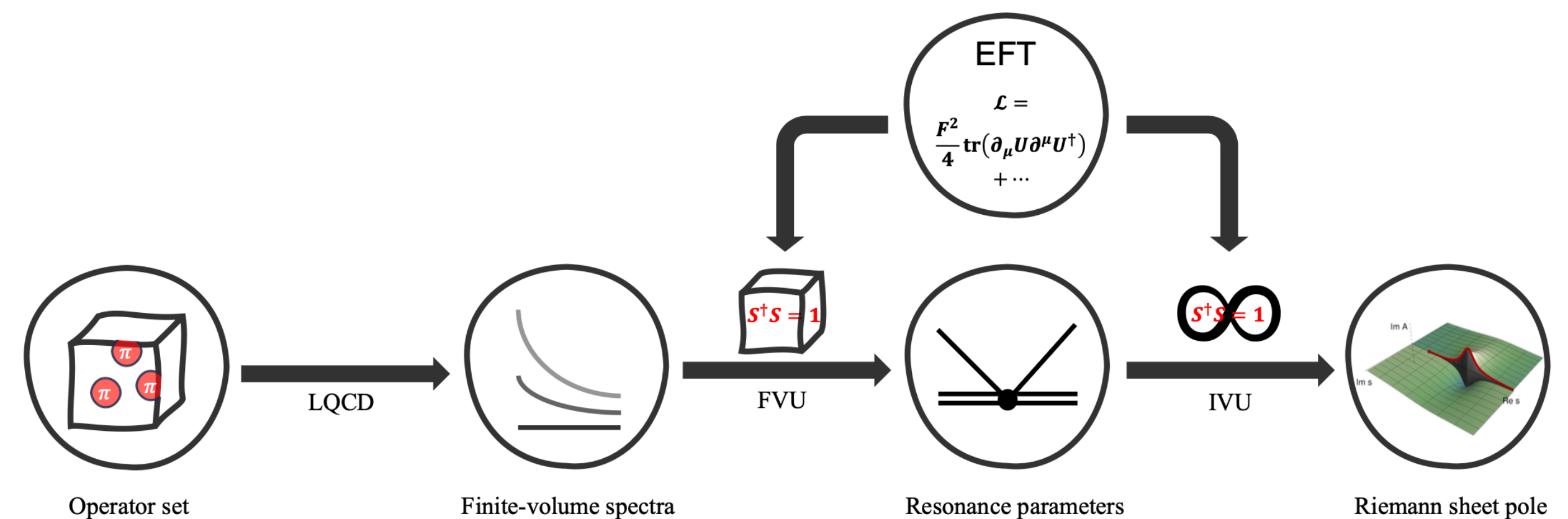


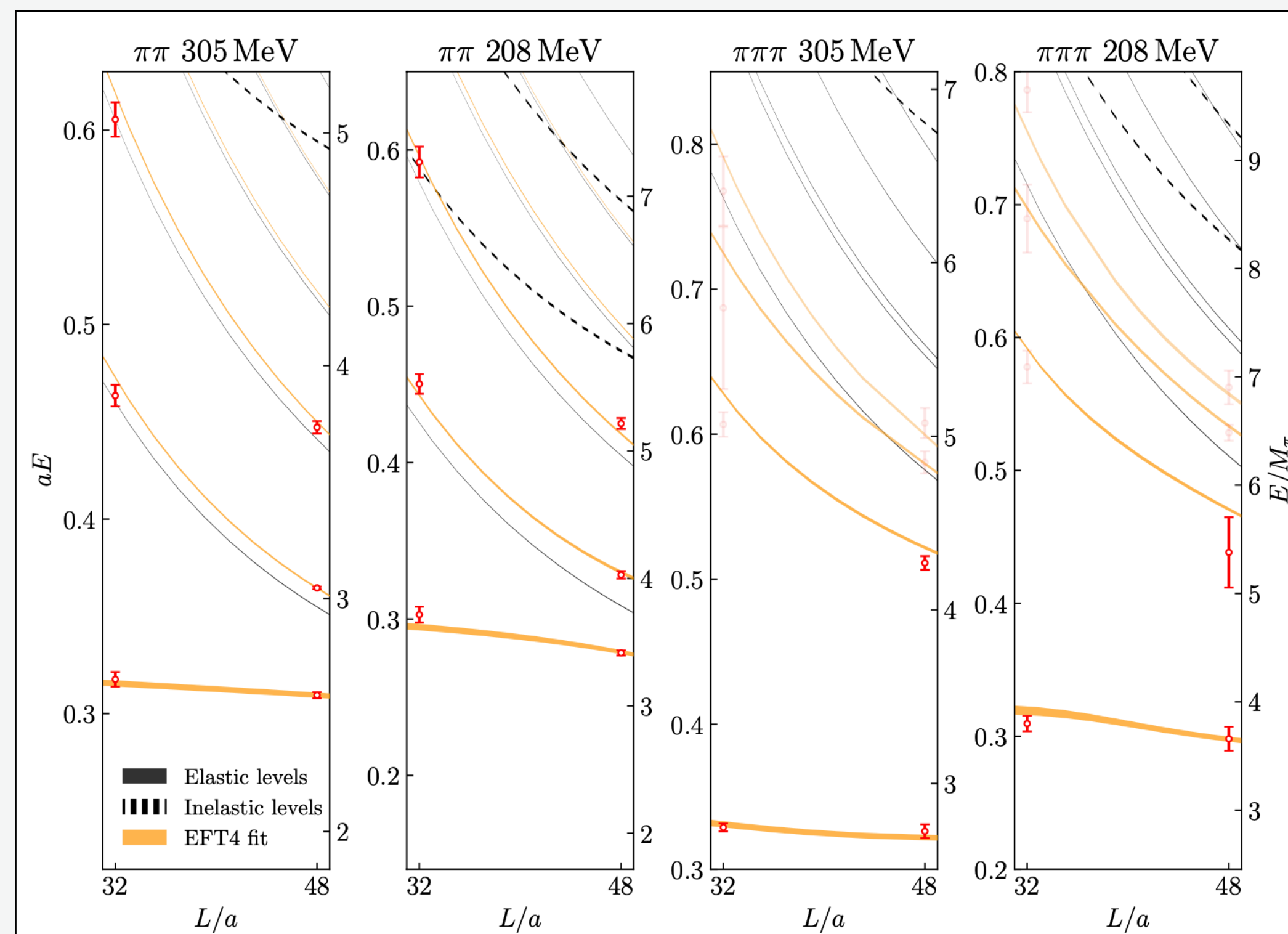
Fig by Haobo Yan (燕浩波)

VECTOR MESON

[agemo] Yan+ PRL133 (2024)

Lattice QCD input

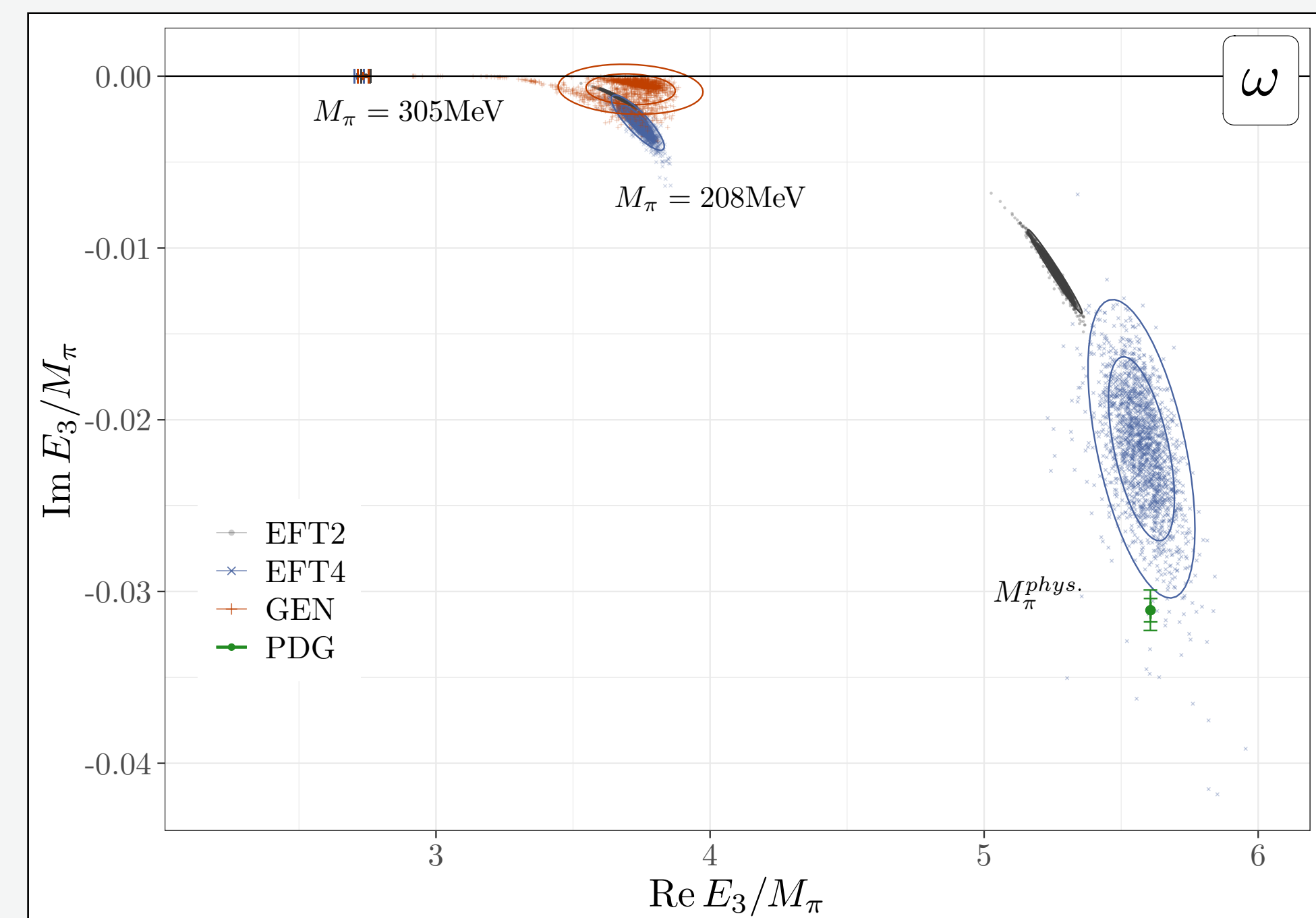
- $N_f = 2 + 1$ Clover fermions
- 2/3 particle operators
- 2 pion masses ($\approx 210, 305$ MeV) 2 volumes ($L^3 = 32^3, 48^3$)



FVU

$$\det \left[2L^3 E_p (\tilde{K}^{-1} - \Sigma^L) - B - C \right] \Gamma \equiv 0$$

- Various EFT based ansatzes
- $\omega(782)$ becomes abound state at ~ 300 MeV
- at the physical point very close to the EXP value



Confirmed in non-perturbative Hamiltonian formalism (NPHF)

Talk JJ Wu

Yu+ 2603.07205 [hep-lat]

$\pi(1300)$ STATE OF THE ART

[agemo] Yan+ PRL136 (2026) no.14, 141901

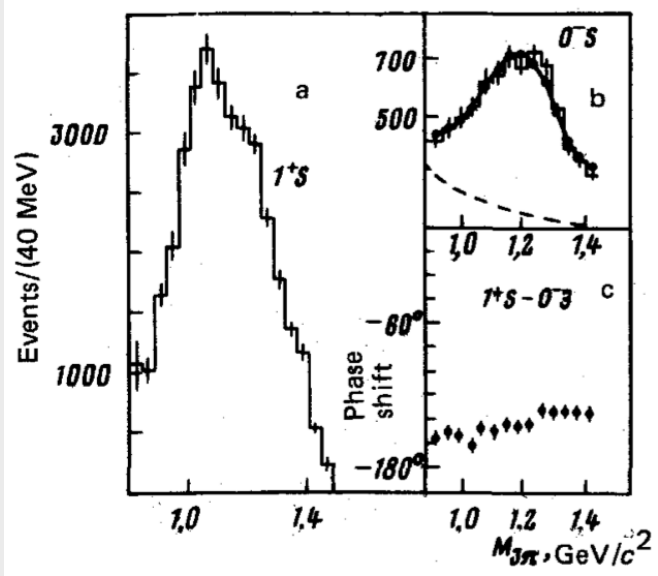
$\pi(1300)$ $I^G(J^{PC}) = 1^-(0^{-+})$

$\pi(1300)$ MASS	[1] 1300 ±100 MeV	▼
$\pi(1300)$ WIDTH	[1] 200 to 600 MeV	▼
$\pi(1300)$ DECAY MODES		
Mode	Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level
Γ_1 $\rho\pi$	seen	404 ▼
Γ_2 $\pi(\pi\pi)_{S\text{-wave}}$	seen	▼

Phenomenology

- Very heavy: $M_{\pi(1300)} \approx 10M_\pi \approx M_{N(1440)}$
- Phenomenologically hard to access:
overlapping effects $a_1(1260), \dots$
- Width very uncertain

Discovery: $\pi Z \rightarrow \pi\pi\pi Z$



Bellini+ [Bologna-Dubna-Milan-Collaboration] JETP Lett. 34 (1981) 488

Most recent analysis $D \rightarrow 4\pi$

- Strong correlations between PWA contributions

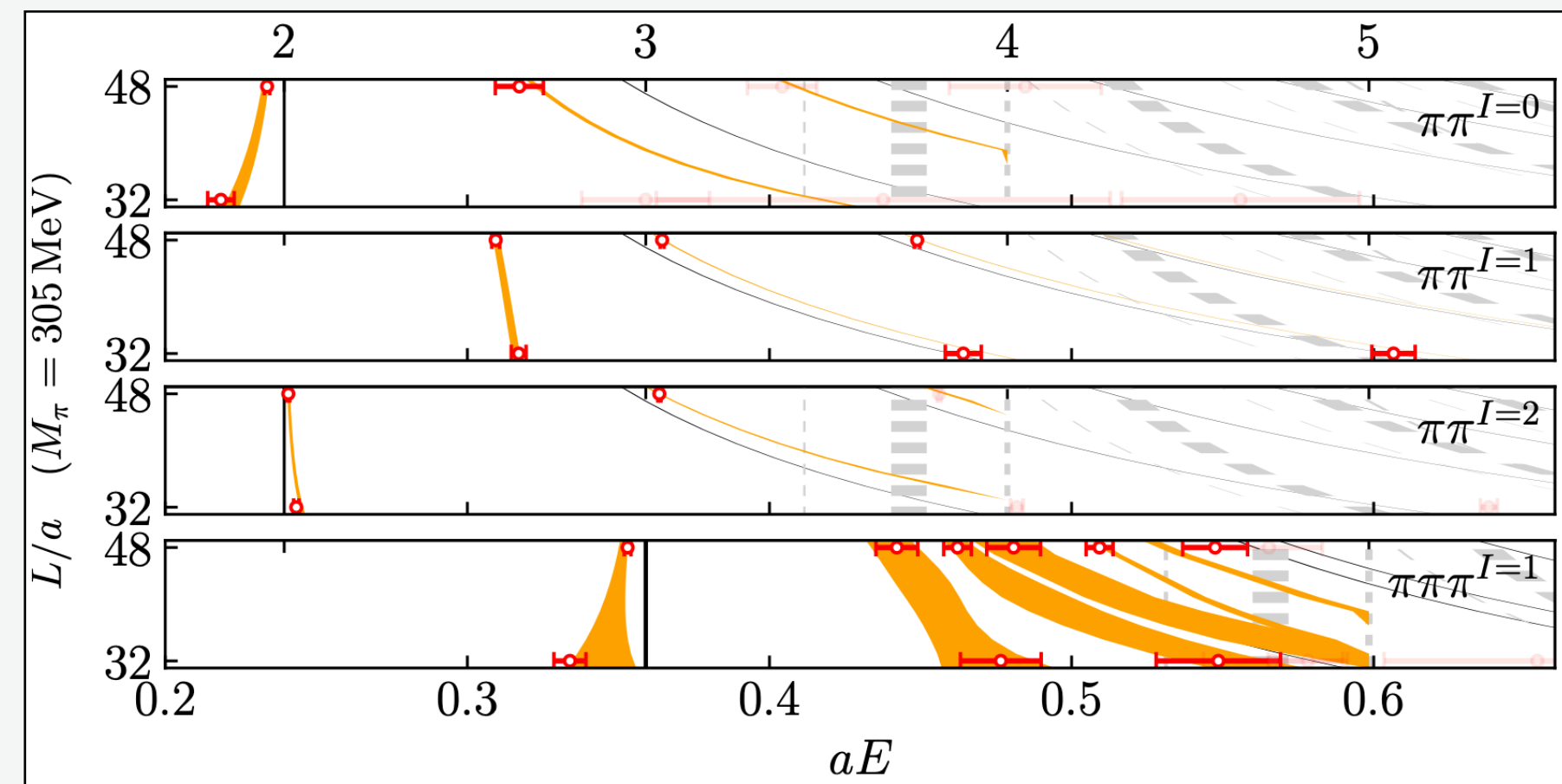
	$m_{a_1(1260)}$	$\Gamma_{a_1(1260)}$	$m_{a_1(1640)}$	$\Gamma_{a_1(1640)}$	$m_{\pi(1300)}$	$\Gamma_{\pi(1300)}$
$m_{a_1(1260)}$	+1.000	+0.689	-0.065	-0.282	+0.116	-0.258
$\Gamma_{a_1(1260)}$		+1.000	-0.114	-0.176	+0.013	-0.004
$m_{a_1(1640)}$			+1.000	-0.335	-0.136	-0.119
$\Gamma_{a_1(1640)}$				+1.000	-0.258	+0.370
$m_{\pi(1300)}$					+1.000	-0.425
$\Gamma_{\pi(1300)}$						+1.000

- Existence required by data?

d'Argent+ JHEP 05 (2017) 143

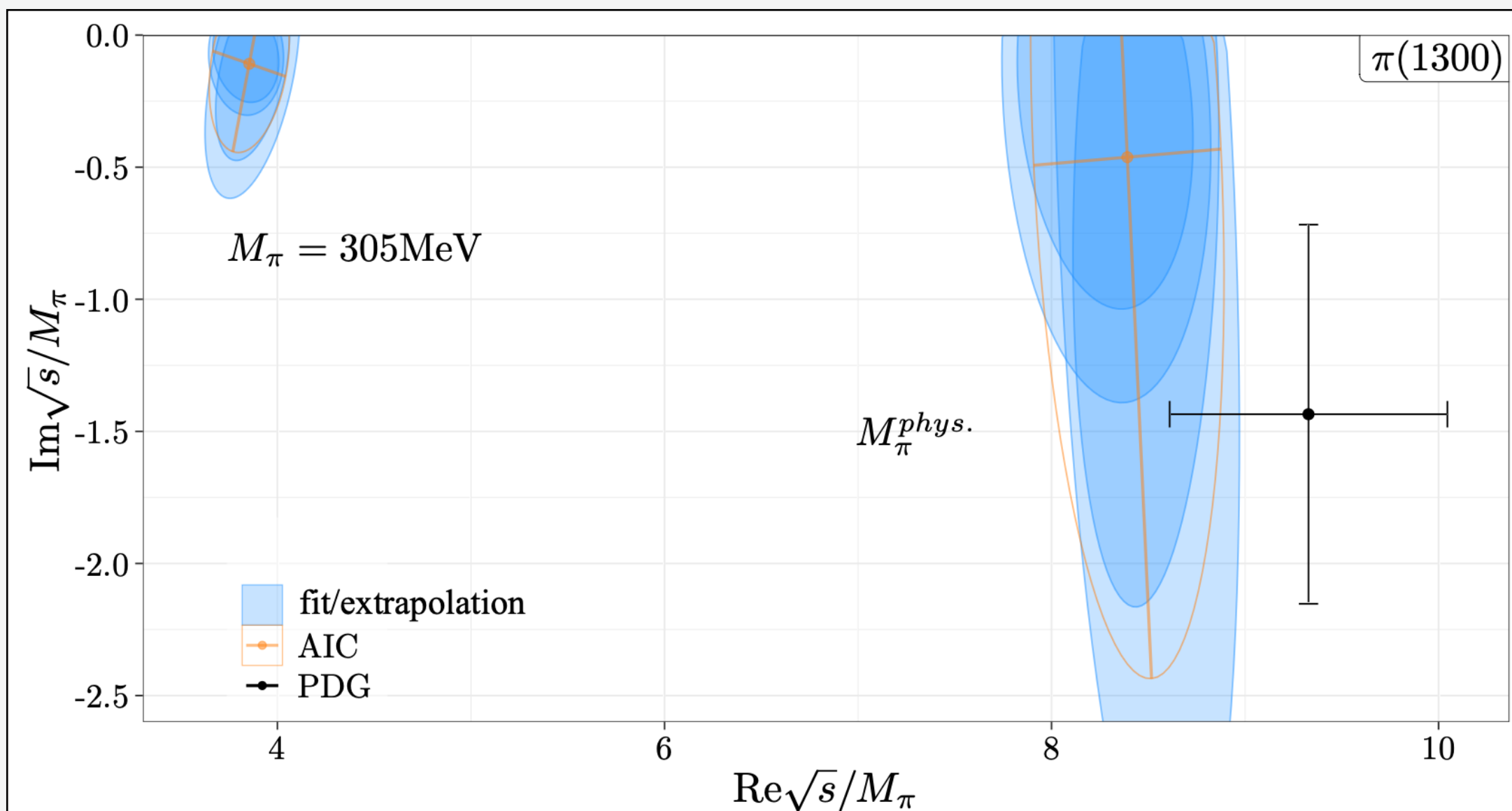
$\pi(1300)$ FROM LATTICE QCD

[agemo] Yan+ PRL136 (2026) no.14, 141901



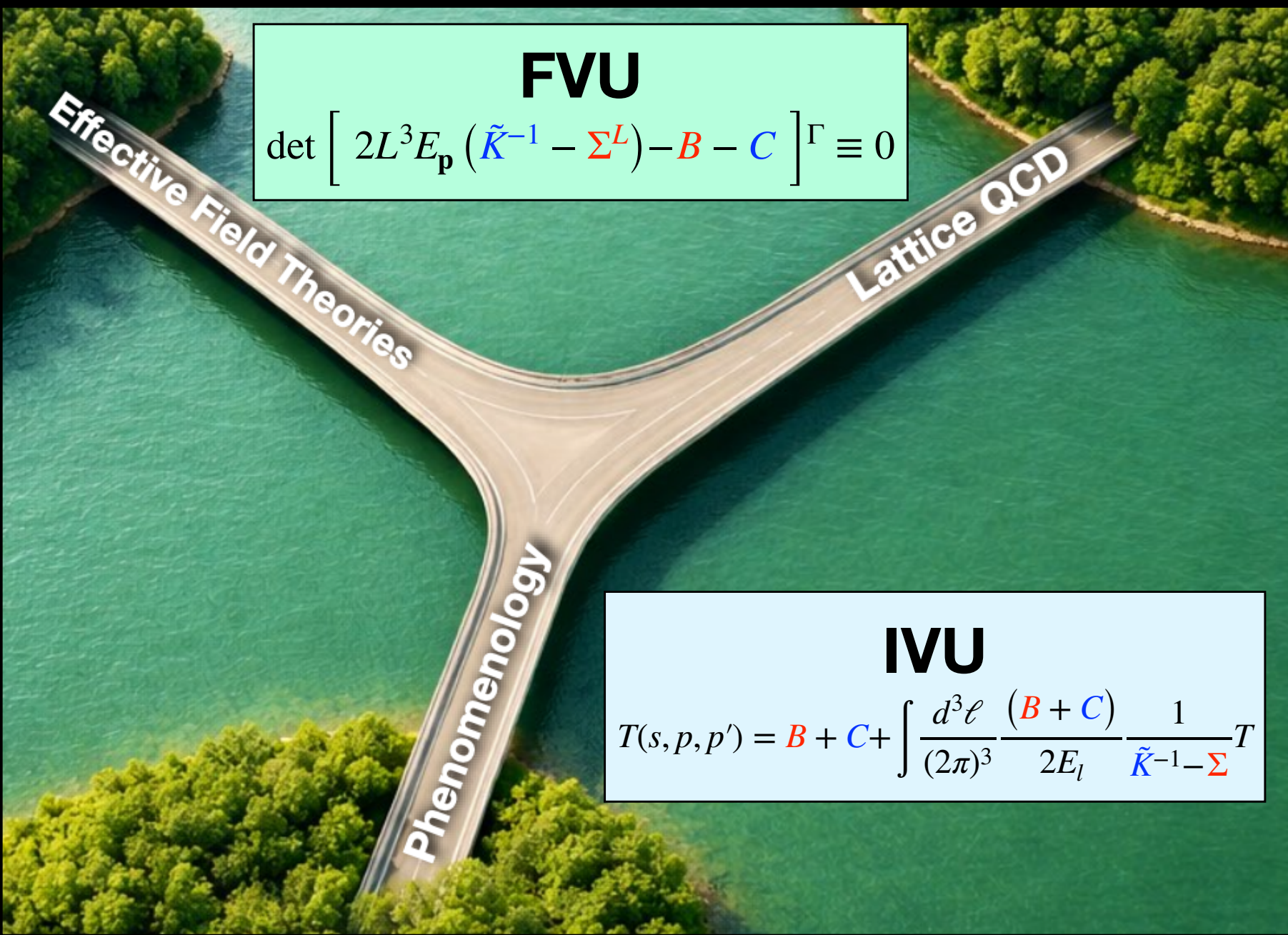
Lattice QCD input

- $N_f = 2 + 1$ Clover fermions
- 2/3 particle operators
- 2 pion masses ($\approx 210, 305$ MeV) 2 volumes ($L^3 = 32^3, 48^3$)
- Contractions (~ 400)



Excited state of the pion

- $M_\pi = 305$ MeV finite-volume ensemble
 - ~~no-resonance scenario~~
 - Includes *dynamically generated scenarios*
- Physical point extrapolation
 - **Mass/Width agree with PDG (1σ)**

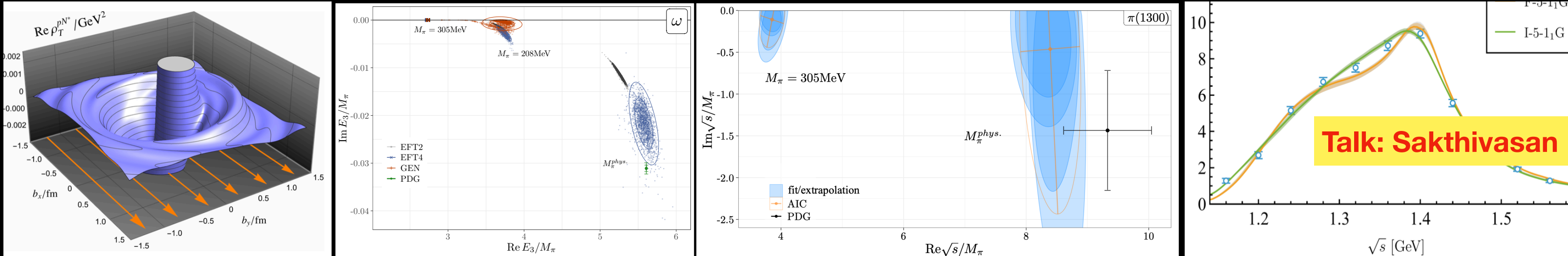


New discoveries from QCD

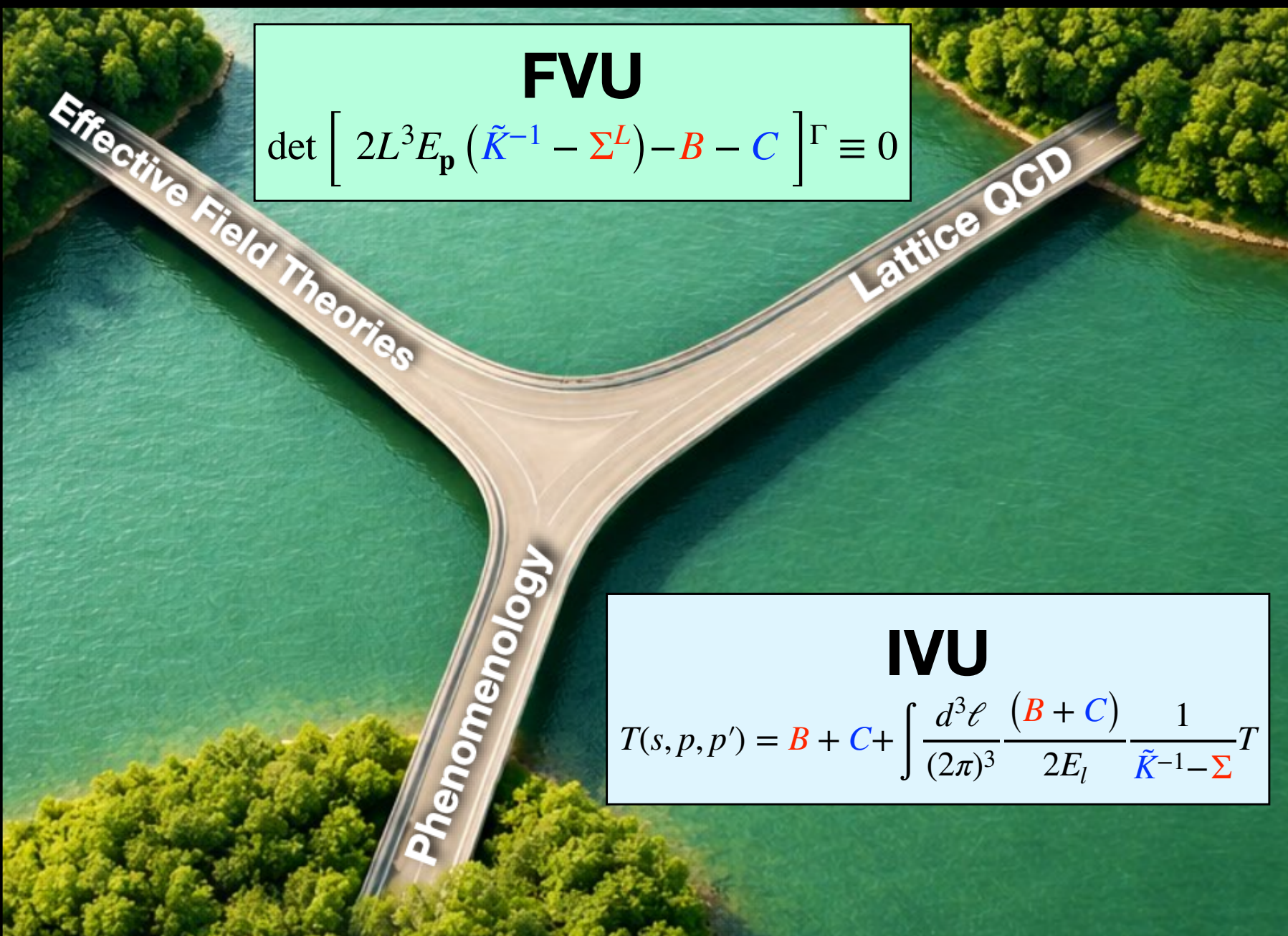
- Ab-initio Lattice input
- quark-mass extrapolations through EFT

S-matrix & 3-body unitarity

- On/off-shell configurations
- **new 3-body quantization condition**

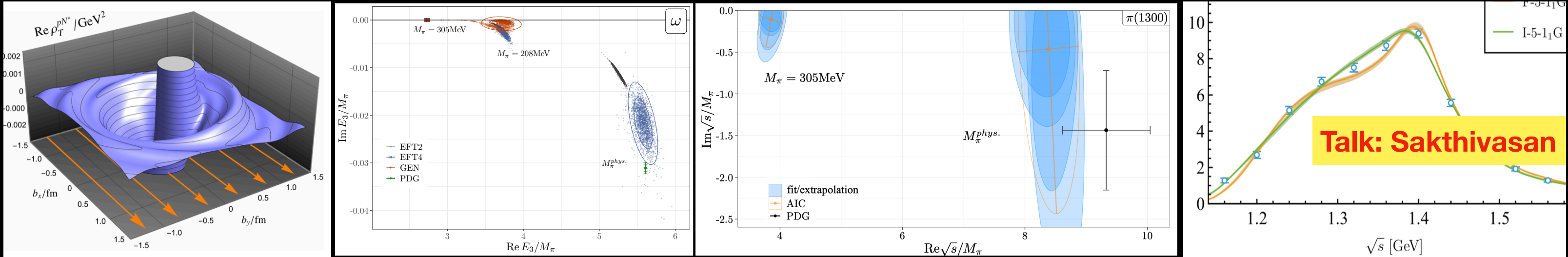


SUMMARY



S-matrix / ETFs / Lattice QCD

- Ab-initio Lattice input
- On/off-shell configurations
- quark-mass extrapolations through EFT
- new 3-body quantization condition



SUMMARY

OUTLOOK

- * FLAG ready: Systematics/statistics improvement, cutoff treatment...
- * $\pi\pi N$ dynamics of $N(1440)$ >> talk: Jinzi Wu
- * Heavy mesons (left-hand-cut)
- * ...

Our measurements are performed on one ensemble (denoted as C103) of gauge configurations generated by the CLS collaboration [44, 45]. This ensemble was generated with a Clover-Wilson action with $N_f = 2 + 1$ dynamical

fermions with $m_u = m_d = m_s$. The lattice spacing is $a \approx 0.086$ fm, $V = 483 \times 96$ and $M\pi L \approx 14.5$ which is very large and allows us to have the exponentially suppressed finite volume effects under control. In total for this work we have used 800 gauge configurations coming from two different replicas with 400 each.

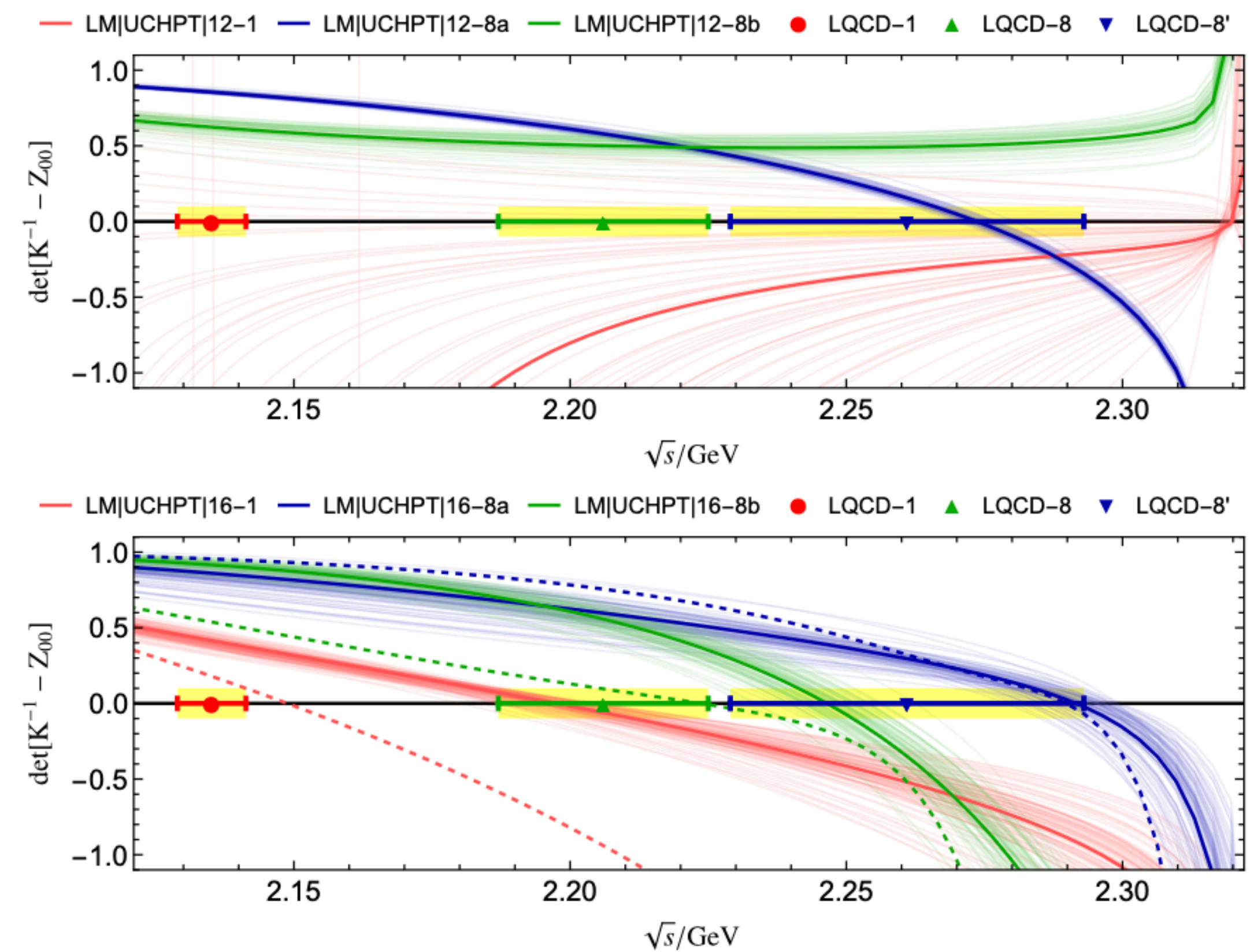


Table I. Numerical values for extracted pole positions (combined fit to lattice and experimental input) for the isoscalar and isovector case (in GeV). Different fits refer to variations of the UCHPT approach as explained in Sect. IV (where M refers to the chosen interaction kernel and S to the method of regularization). The lattice point refers to quark mass setup used in Refs. [20, 21].

	Type	physical point	lattice point
$I = 0 \ S = -1$	M3S1 (F_{17})	$1.342^{+0.009}_{-0.010} - i \ 0.028^{+0.013}_{-0.014}$	$1.359^{+0.010}_{-0.014} - i \ 0.100^{+0.050}_{-0.050}$
		$1.432^{+0.002}_{-0.002} - i \ 0.025^{+0.001}_{-0.001}$	$1.460^{+0.006}_{-0.005} - i \ 0.013^{+0.005}_{-0.004}$
	M3S2 (F_{16})	$1.373^{+0.002}_{-0.005} - i \ 0.082^{+0.011}_{-0.009}$	$1.389^{+0.006}_{-0.007}$
		$1.423^{+0.002}_{-0.001} - i \ 0.020^{+0.001}_{-0.002}$	$1.466^{+0.003}_{-0.004} - i \ 0.020^{+0.001}_{-0.001}$
	M3S3 (F_{12})	$1.352^{+0.004}_{-0.007} - i \ 0.091^{+0.005}_{-0.006}$	$1.401^{+0.003}_{-0.003}$
		$1.420^{+0.001}_{-0.001} - i \ 0.018^{+0.001}_{-0.001}$	$1.478^{+0.005}_{-0.004} - i \ 0.026^{+0.001}_{-0.001}$
$I = 1 \ S = -1$	M3S1 (F_{17})	$1.351^{+0.022}_{-0.015} - i \ 0.112^{+0.010}_{-0.011}$	$1.333^{+0.018}_{-0.024} - i \ 0.144^{+0.008}_{-0.012}$
	M3S2 (F_{16})	$1.356^{+0.005}_{-0.011} - i \ 0.035^{+0.022}_{-0.014}$	
	M3S3 (F_{12})	$1.430^{+0.090}_{-0.040} - i \ 0.281^{+0.170}_{-0.002}$	$1.430^{+0.050}_{-0.060} - i \ 0.208^{+0.090}_{-0.000}$
		$1.338^{+0.021}_{-0.023} - i \ 0.215^{+0.007}_{-0.004}$	$1.335^{+0.025}_{-0.017} - i \ 0.143^{+0.008}_{-0.005}$

AXIAL-VECTOR MESON (LATTICE QCD)

MM/Culver/Alexandru/Doring/Lee/Sadasivan [GWQCD] PRL 127 (2021)

IVU

$$T(s, p, p') = \textcolor{red}{B} + \textcolor{blue}{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(\textcolor{red}{B} + \textcolor{blue}{C})}{2E_l} \frac{1}{\textcolor{blue}{\tilde{K}}^{-1} - \textcolor{red}{\Sigma}} T$$

$\textcolor{blue}{C}, \textcolor{blue}{\tilde{K}}$

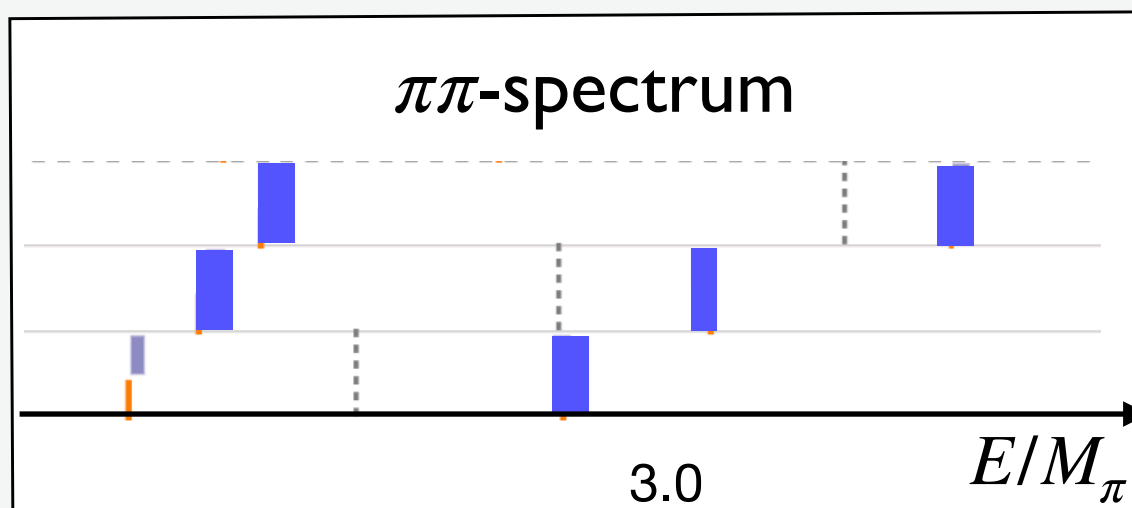
- volume-independent
- generic form

FVU

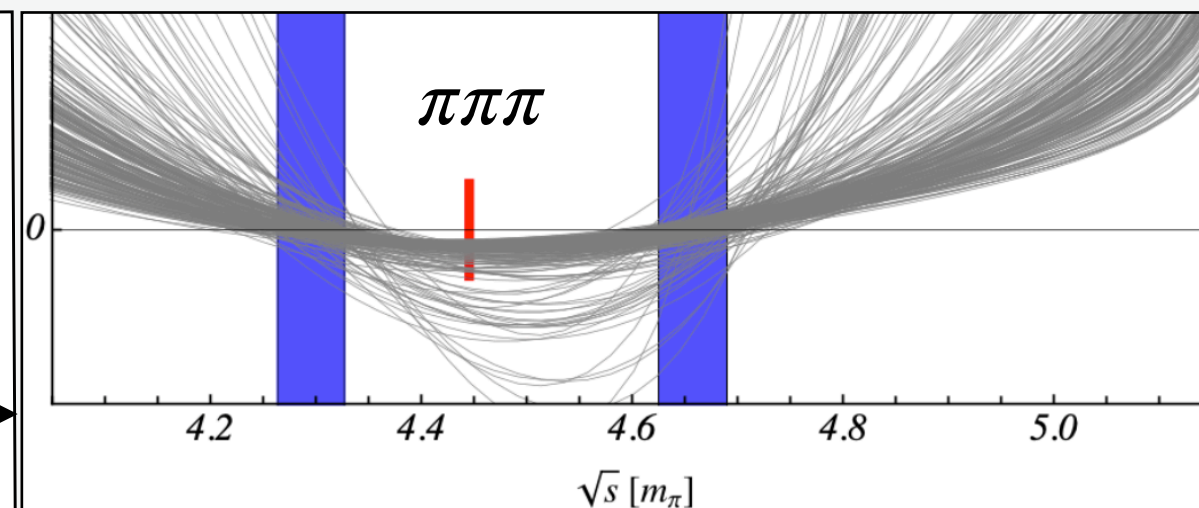
$$\det \left[2L^3 E_p (\textcolor{blue}{\tilde{K}}^{-1} - \textcolor{red}{\Sigma}^L) - \textcolor{red}{B} - \textcolor{blue}{C} \right]^\Gamma \equiv 0$$

Lattice QCD ($M_\pi \approx 315\text{MeV}$, $N_f = 2$)

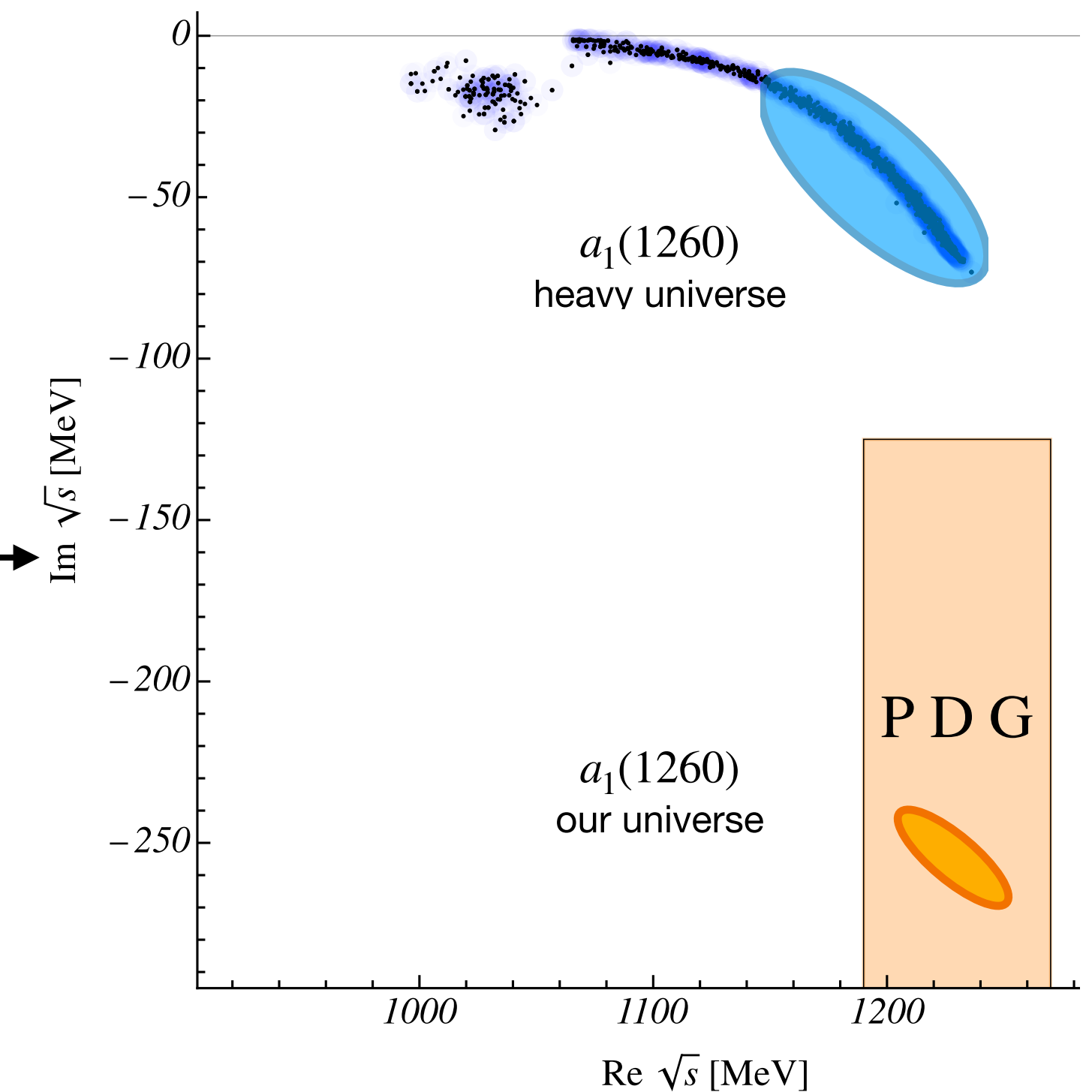
$\pi\pi$ -spectrum



$\pi\pi\pi$



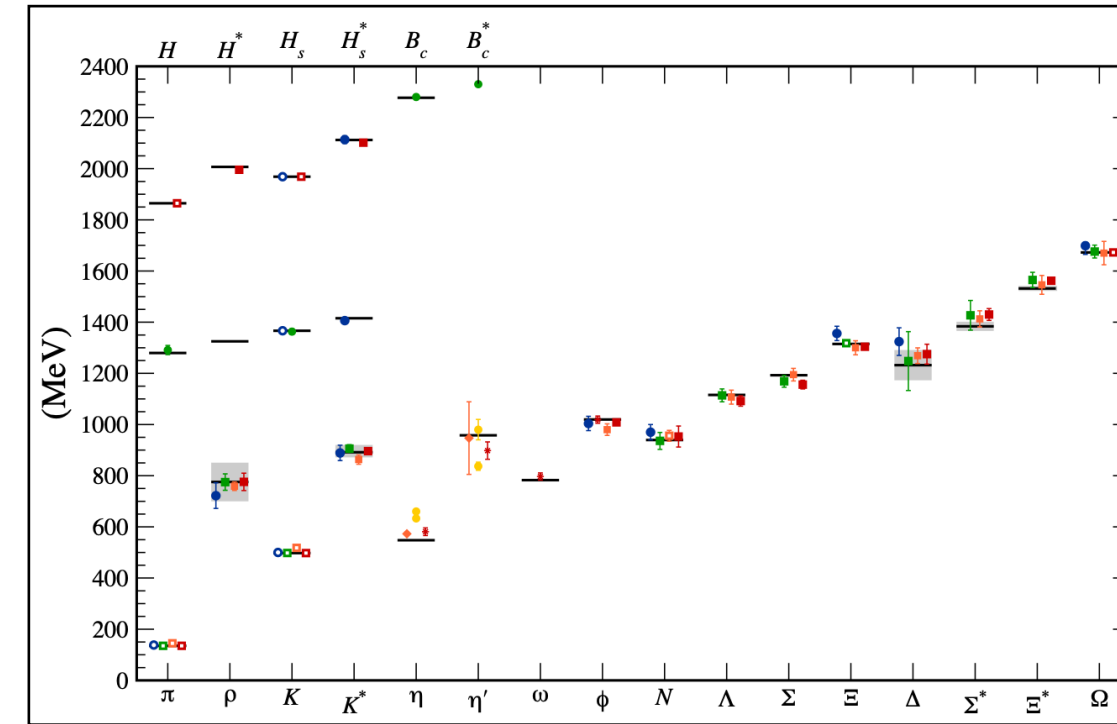
[GWQCD] PRD94(2016) PRD98 (2018) PRD 100(2019)



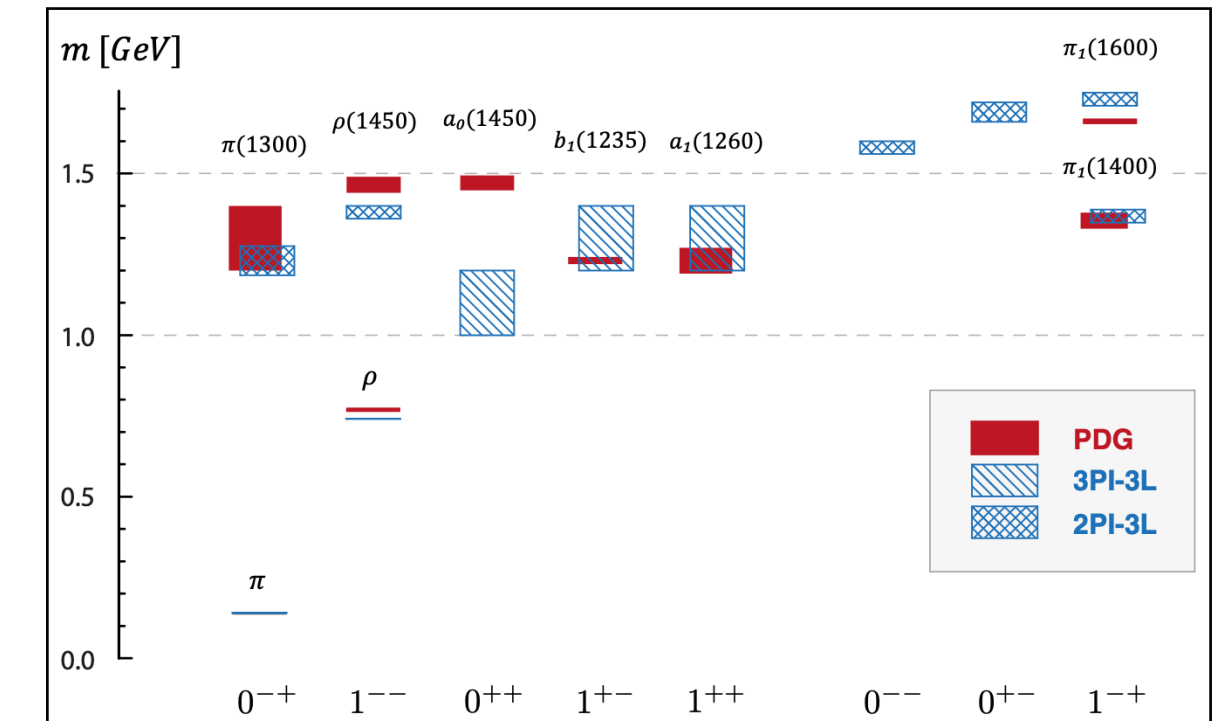
STATE OF THE ART

Theory

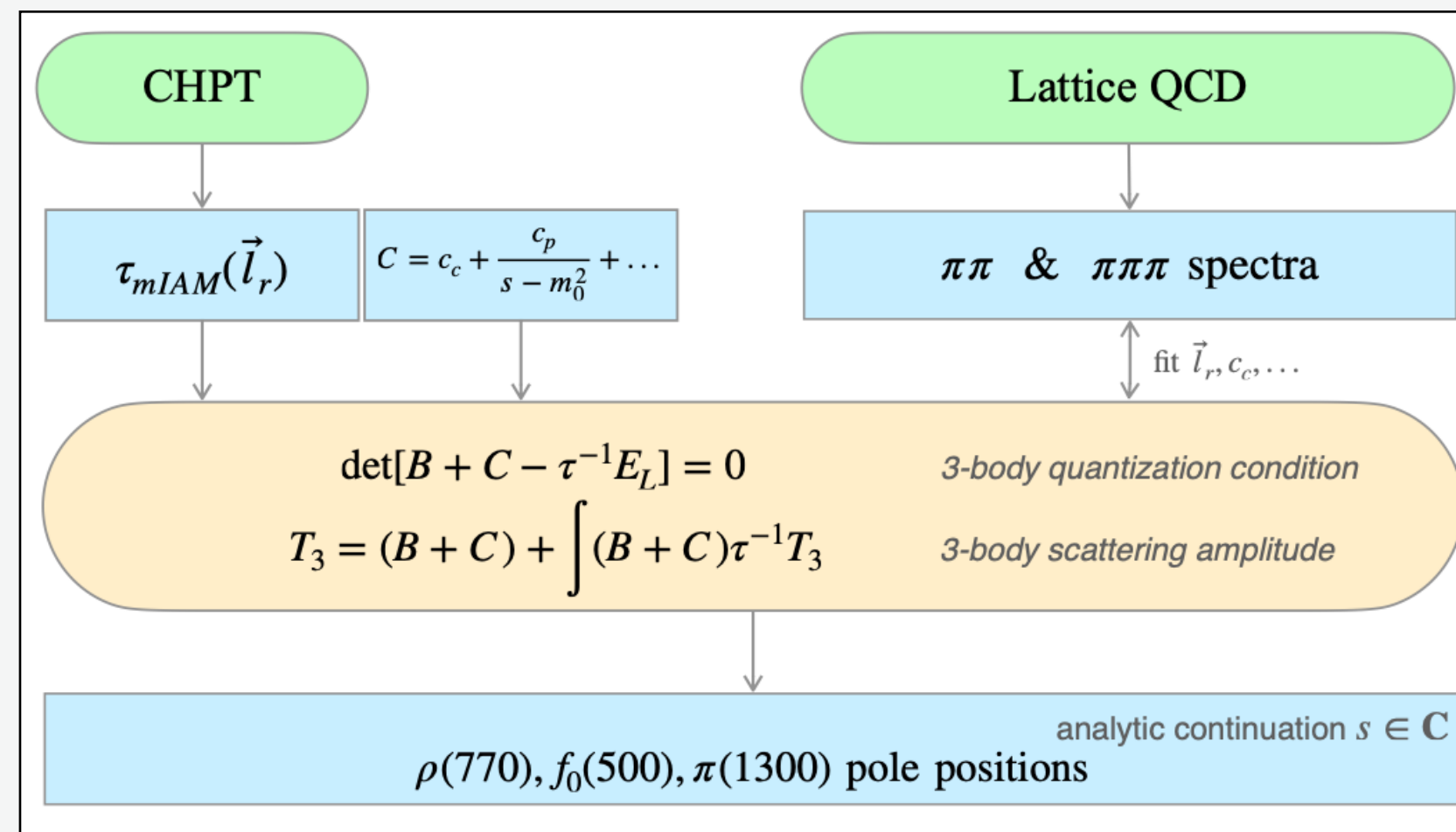
- Constituent quark model *Wang/Ping Chin. Phys. Lett. 24, 1195 (2007)*
- DSE beyond rainbow-ladder cont. *Williams/Fischer Phys.Rev.D 93 (2016) 3, 034026*
- Unitary models $\pi K \bar{K}$ *Phys.Rev.D 84 (2011) 074027*
- Pioneering Lattice QCD (spectrum) calculations *Dudek+ (HadSpec) PRD 82, 034508 (2010); McNeile/Michael (UKQCD) PLB 642, 244 (2006)*



Kronfeld Ann. Rev. Nucl.Part. Sci. 62, 265 (2012)



Eichmann+ Prog.Part.Nucl.Phys. 91 (2016)



This work – workflow

- Three-body operators/scattering states lattice QCD calc.
- Three-body finite-volume formalisms
- Pole positions
- Chiral extrapolations (CHPT) to the physical point

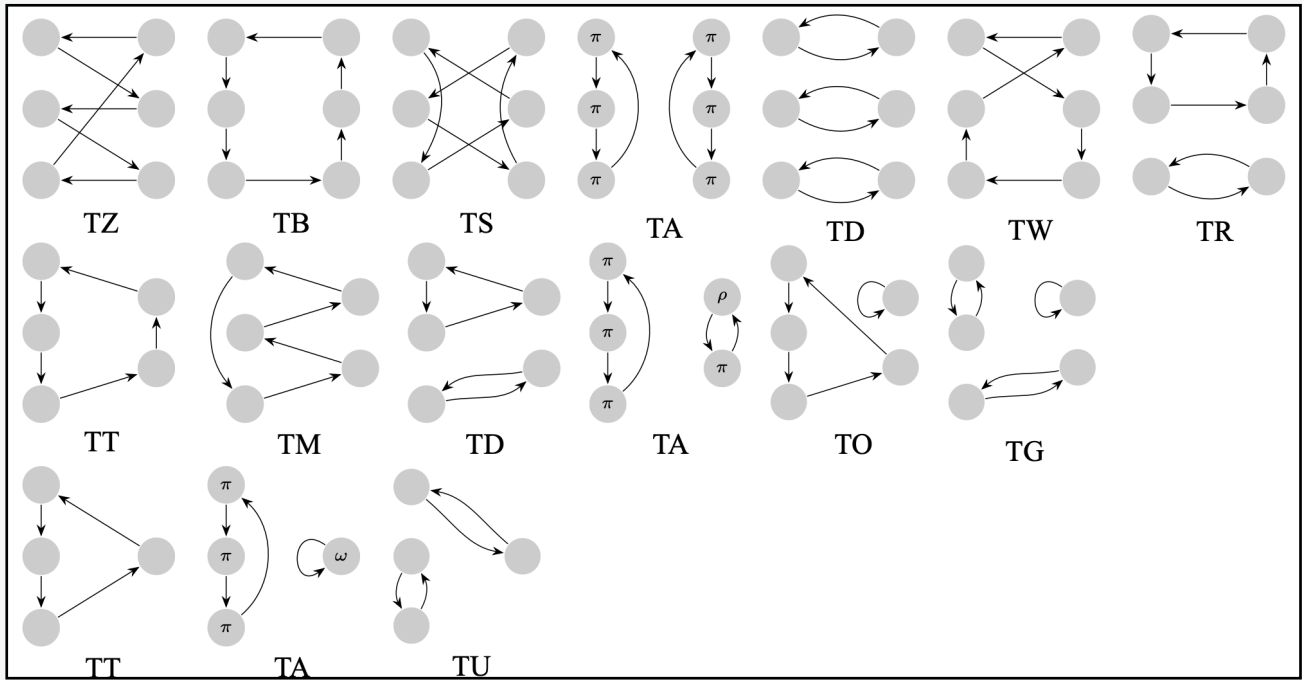
LATTICE SETUP

- 4 CLQCD ensembles with $N=2+1$ Clover fermions [CLQCD 2024]
 - Two volumes
 - Two pion masses (208 MeV, 305MeV)

configuration	volume	a/fm	m_π/MeV	N_{cfgs}
C48P14	$48^3 \times 96$	0.10530(18)	135.5(1.6)	259
F32P21	$32^3 \times 64$	0.07746(18)	210.9(2.2)	459
F48P21	$48^3 \times 96$	0.07746(18)	207.2(1.1)	222
F32P30	$32^3 \times 96$	0.07746(18)	303.2(1.3)	567
F48P30	$48^3 \times 96$	0.07746(18)	303.4(9)	201
H48P32	$48^3 \times 144$	0.05187(26)	317.2(0.9)	274

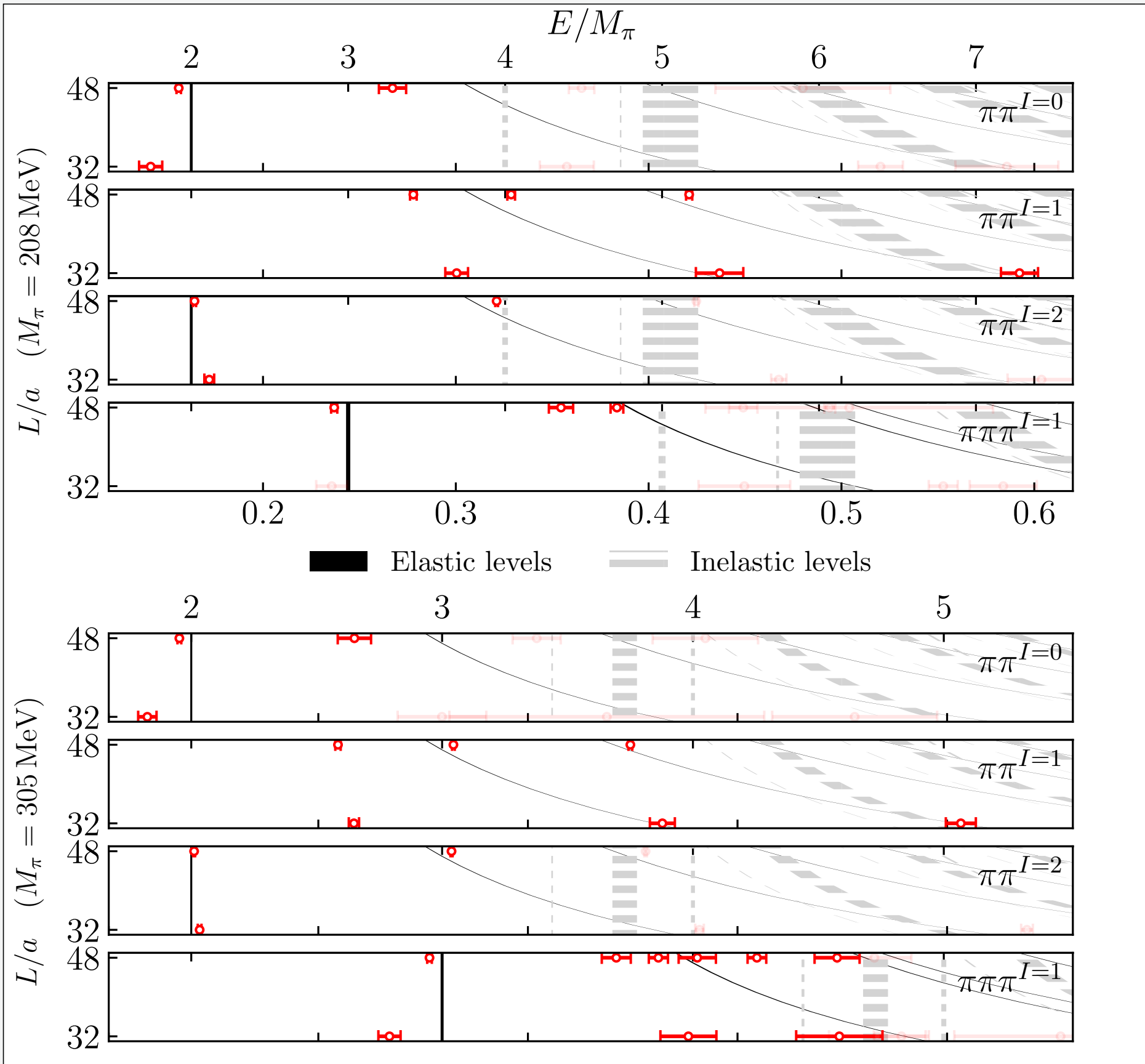
- **OpTion** utilized for 3-hadron operators with arbitrary momenta
Yan+ [arXiv:2507.16070](https://arxiv.org/abs/2507.16070), JHEP (2025) (<https://github.com/wittscien/OpTion>)

- Contractions (~400)



Finite-volume spectrum:

24 ($\pi\pi$) + 12 ($\pi\pi\pi$) interacting levels



QUANTIZATION CONDITIONS

Lüscher QC

$$\sqrt{\sigma} \in \{E_0, E_1, \dots\}_{\pi\pi} \iff \tilde{K}^{-1}(\sigma) - \Sigma^L(\sigma) \equiv 0$$



Döring/Meißner/Rusetsky/Oset *Eur.Phys.J.A* 47 (2011) 139

- Equivalent to 2b-Lüscher QC $\mathcal{O}(e^{-ML})$
- Unknown 2-body interaction — use UCHPT

Hanhart+ *PRL*100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \rightarrow T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

Finite Volume Unitarity (FVU) 3-body quantization condition*

$$\sqrt{s} \in \{E_0, E_1, \dots\}_{\pi\pi\pi} \iff \det \left[2L^3 E_p (\tilde{K}^{-1}(s) - \Sigma^L(s)) - B(s) - C(s) \right] \equiv 0$$



MM/Döring *Eur.Phys.J.A* 53 (2017) 12, 240

- Extended to all isospin/strangeness channels
- Unknown 2-body interaction — use UCHPT

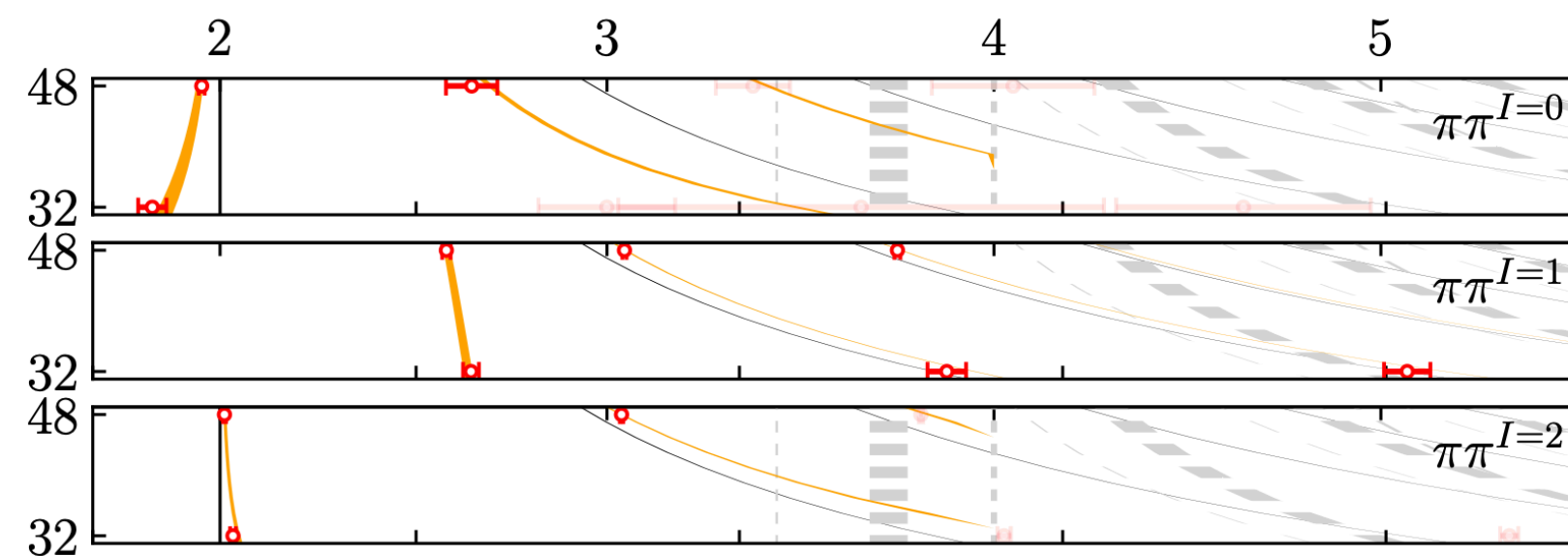
Feng+ *Phys.Rev.D* 110 (2024) 094002
K(1460) 2511.02543

Hanhart+ *PRL*100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \rightarrow T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

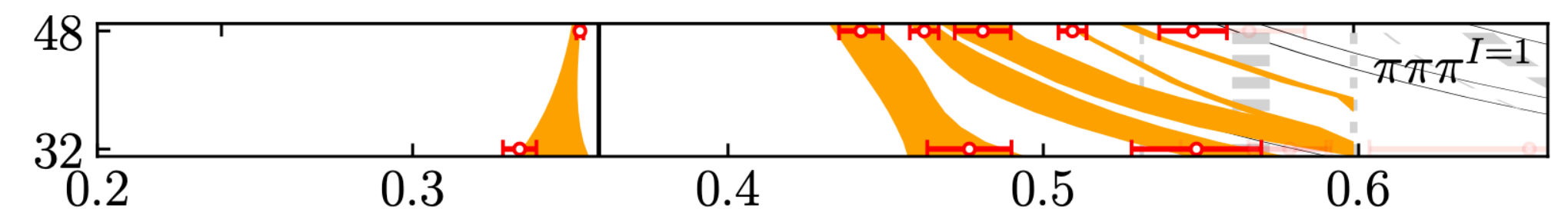
- Unknown 3-body force

$$C(s) \rightarrow c^{\alpha\beta} = c_c^{\alpha\beta} + \frac{c_p^{\alpha\beta}}{s - m_0^2}, \quad \alpha = \beta \in \{\sigma\pi, \rho\pi\}$$



2/3-body fit

$$\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$$



*) **Equivalent/not identical alternatives:** RFT, NREFT, ... (Hansen/Sharpe 2014, Rusetsky/Hammer/Pang 2017, ... Jackura et al. *Phys.Rev.D* 100 (2019) 3, 034508, Garofalo et al. *JHEP* 02 (2023) 252)

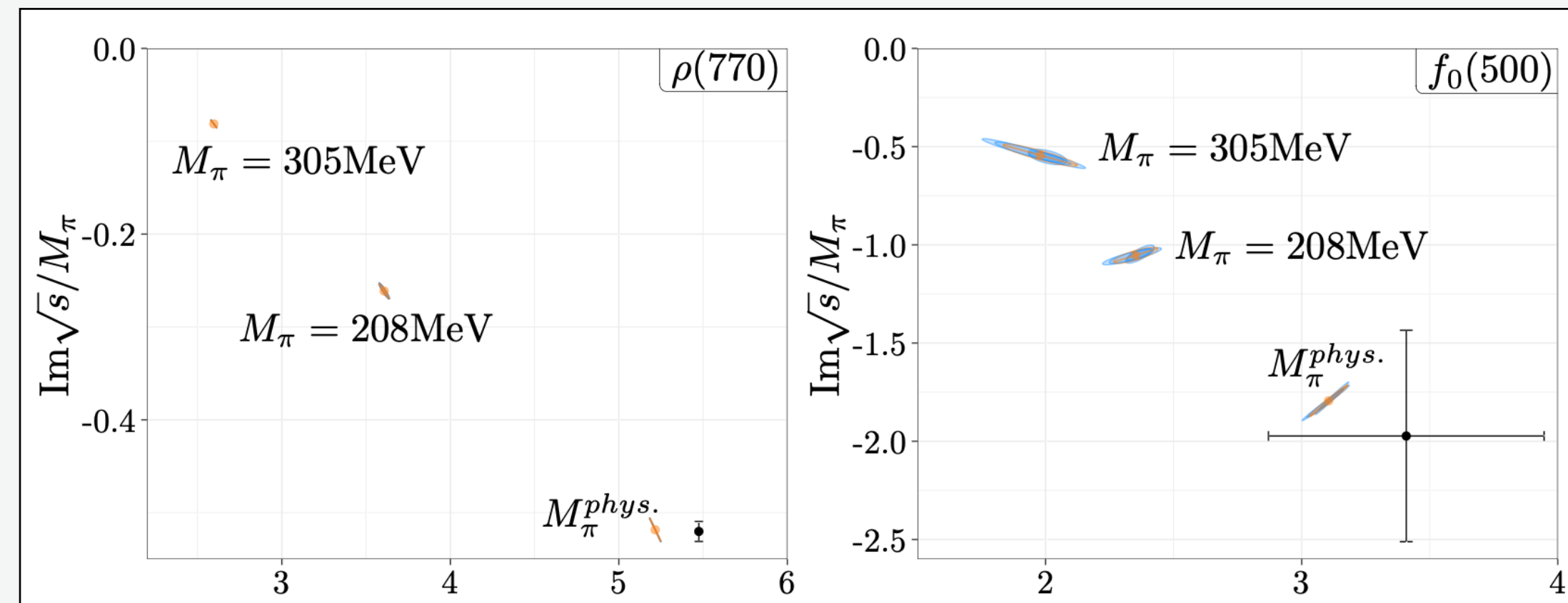
Developments and applications (Blanton, Romero-Lopez, Hansen, Briceño, Dawid, Rusetsky, Davoudi, Guo, MM, Culver, Yan, Garofalo, Urbach, Polejaeva, Raposo, Feng, Döring, ...)

Reviews: Hansen/Sharpe *Ann. Rev. Nucl. Part. Sci.* 69, 65 (2019) MM/Döring/Rusetsky *Eur. Phys. J. ST* 230, 1623 (2021) Romero-Lopez, *PoS LATTICE2022*, 235 (2023)

Related talks: Raposo (Tcc/LHC), Sharpe (Npipi), Romero-Lopez (pipipi)

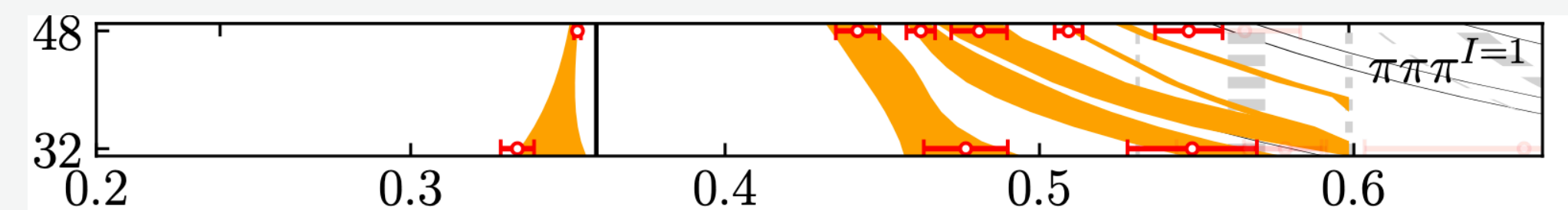
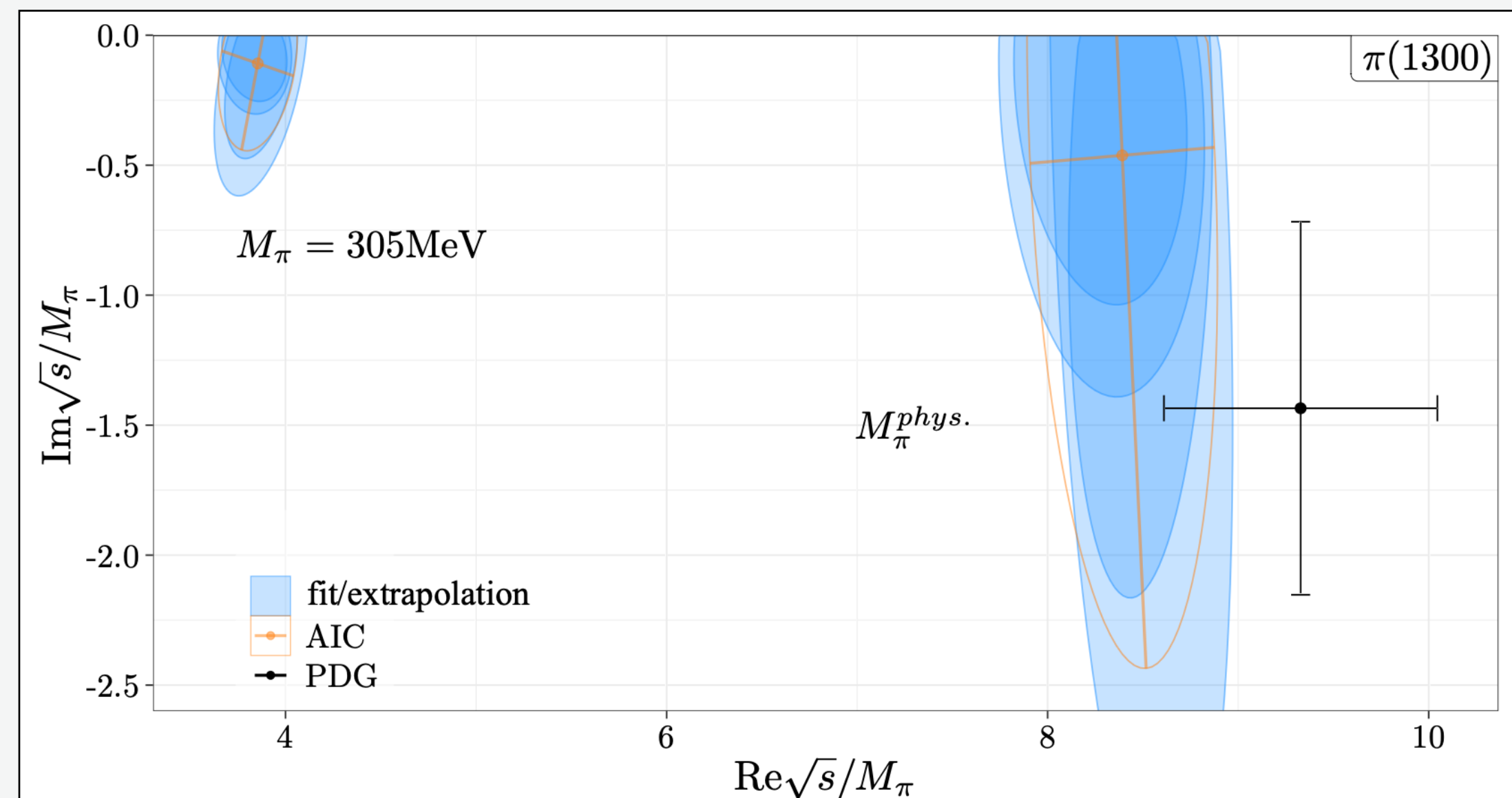
$\pi(1300)$ FROM LATTICE QCD

[agemo] Yan+ PRL 136 (2026) no.14, 141901



2b resonances and chiral trajectories (in passing)

- $f_0(500)$ agrees with PDG
- $\text{Re } M_{\rho(770)}$ slightly too low
 - PDG value improved drastically [Hoferichter+ PL B853 \(2024\)](#)
 - Discretization effects? Over-constrained by mIAM?



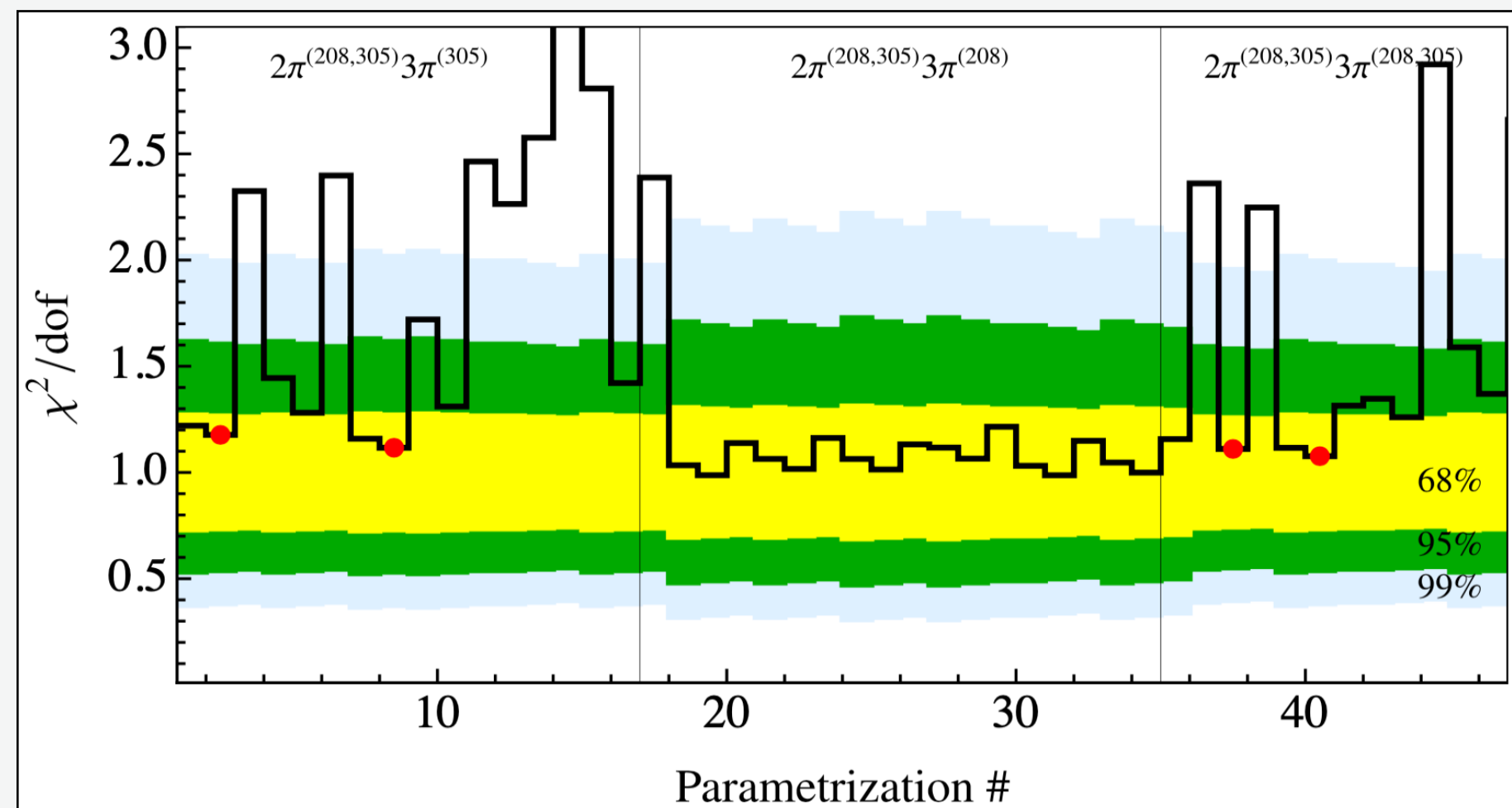
Excited state of the pion

- $M_\pi = 305$ MeV finite-volume ensemble
 - **no-resonance scenario**
 - Includes *dynamically generated* scenarios
- Physical point extrapolation
 - Mass/Width agree with PDG (1σ)

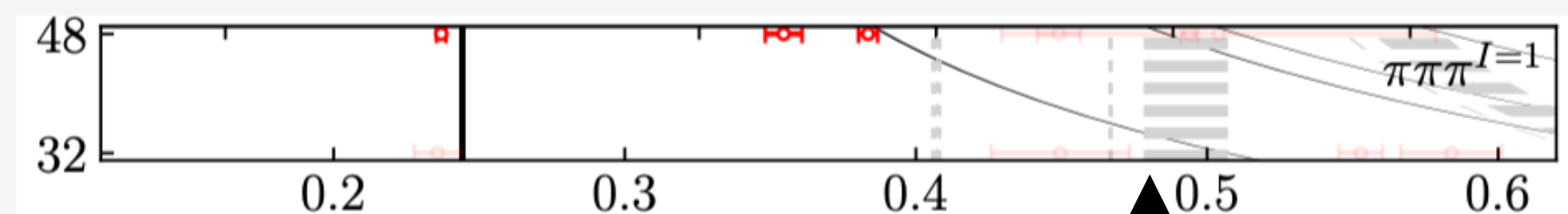
THINKING “OUTSIDE OF THE BOX”

Fits

- $\mathcal{O}(2000)$ fit variations $\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$
- Best $\chi^2/\text{dof} \approx 1.1$



- $\pi\pi\pi^{208}$ does not restrict $\pi(1300)$ region in the elastic regime



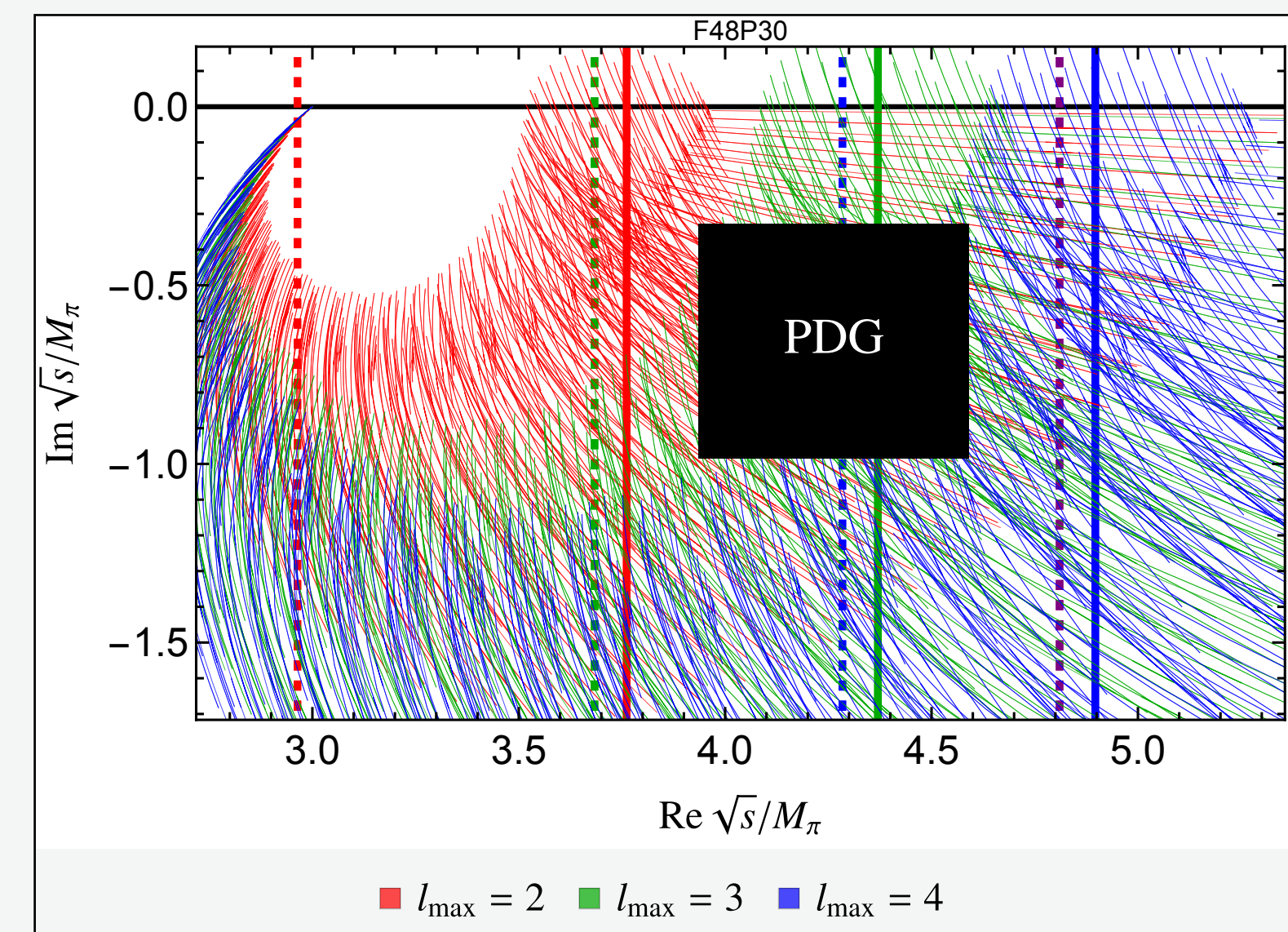
$$M_{\pi(1300)}/M_\pi \approx 1300/208$$

Infinite-volume scattering amplitudes

- $\tilde{K}^{-1}(s), C(s)$ are fixed and **volume independent**

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + C)}{2E_\ell} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

- Maximize the analytically accessible area ($\ell_{\max}, \dots$)



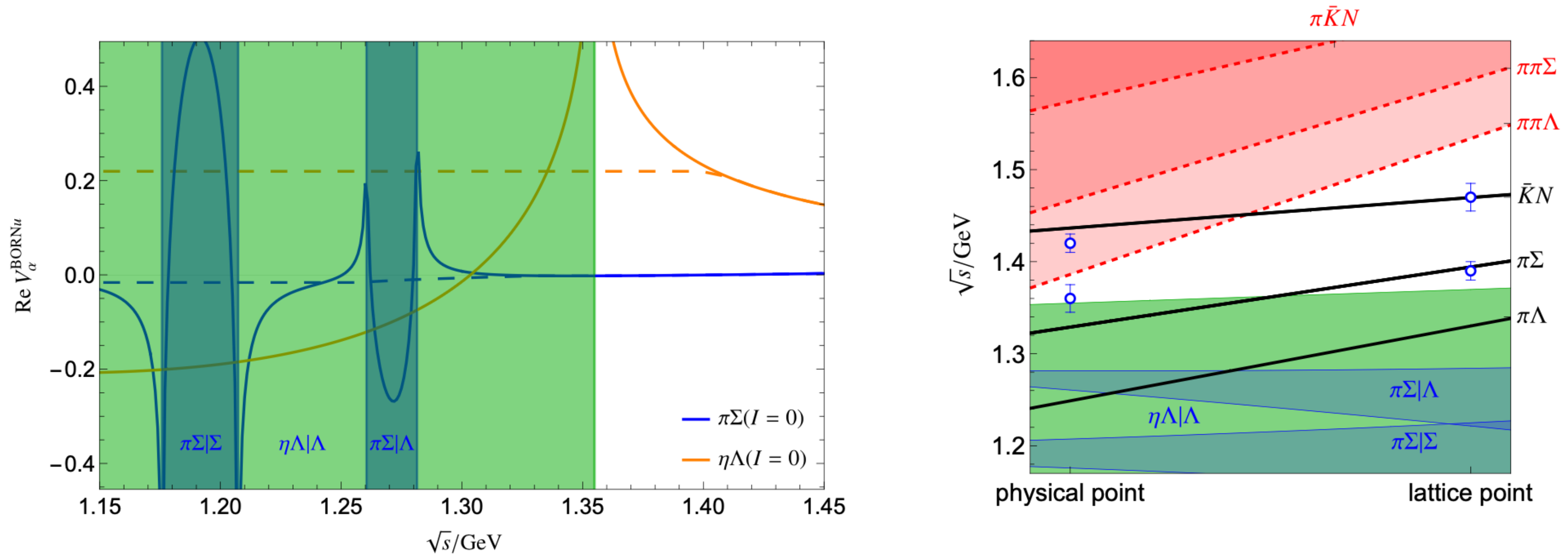


Figure 5. Three-body related singularities for the physical and unphysical quark-mass setups. Left: Singularities of the u -channel Born diagram Eq. (4.10) projected to $I = 0$ for the initial/final states as specified in the legend for physical quark masses. Long-dashed lines represent the potential implemented in the UCHPT amplitude mitigating the appearance of u -channel left-hand cuts. Right: Relevant short left-hand cut associated with the u -channel exchange (green, blue areas), c.f. left figure. Black solid and red dashed lines denote the position of the right-hand cut branching points with respect to two- and three-body states, respectively. Energy region with no allowed three-body onshell states is the remaining white area. Blue dots with error bars denote the averaged result from the global analysis discussed in Sect. VI.

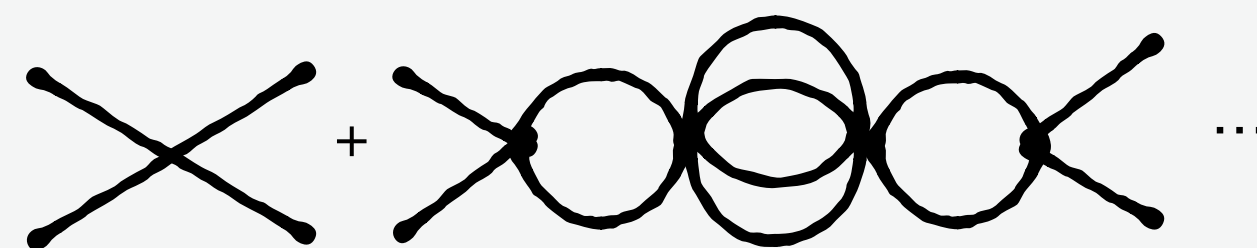
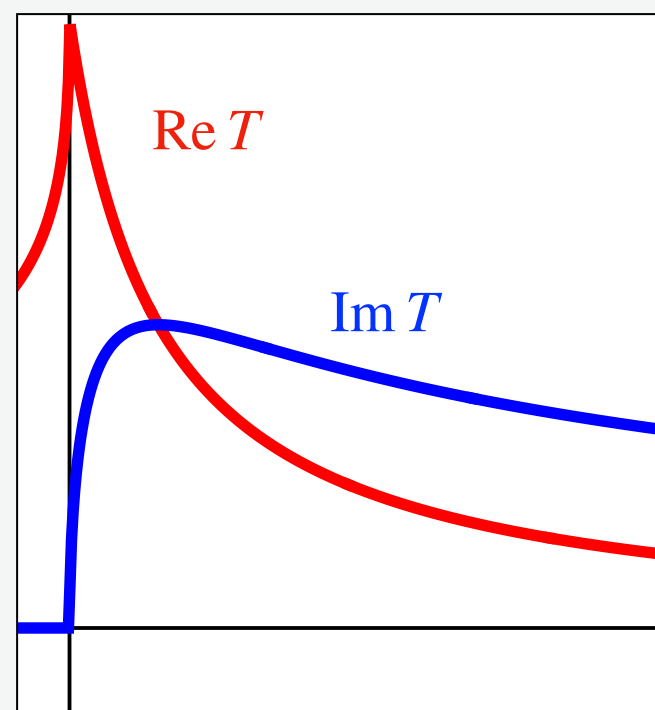
QUANTIZATION CONDITION: 2-BODY EXAMPLE

Review: MM/Doring/Rusetsky *Eur.Phys.J.ST* 230 (2021)

Continuum QFT

- Asymptotic states at $t \rightarrow \pm \infty$
- underlying quantity: $(\sigma \sim |S|^2, \dots)$

$$S = 1 + iT \in \mathbb{C}$$



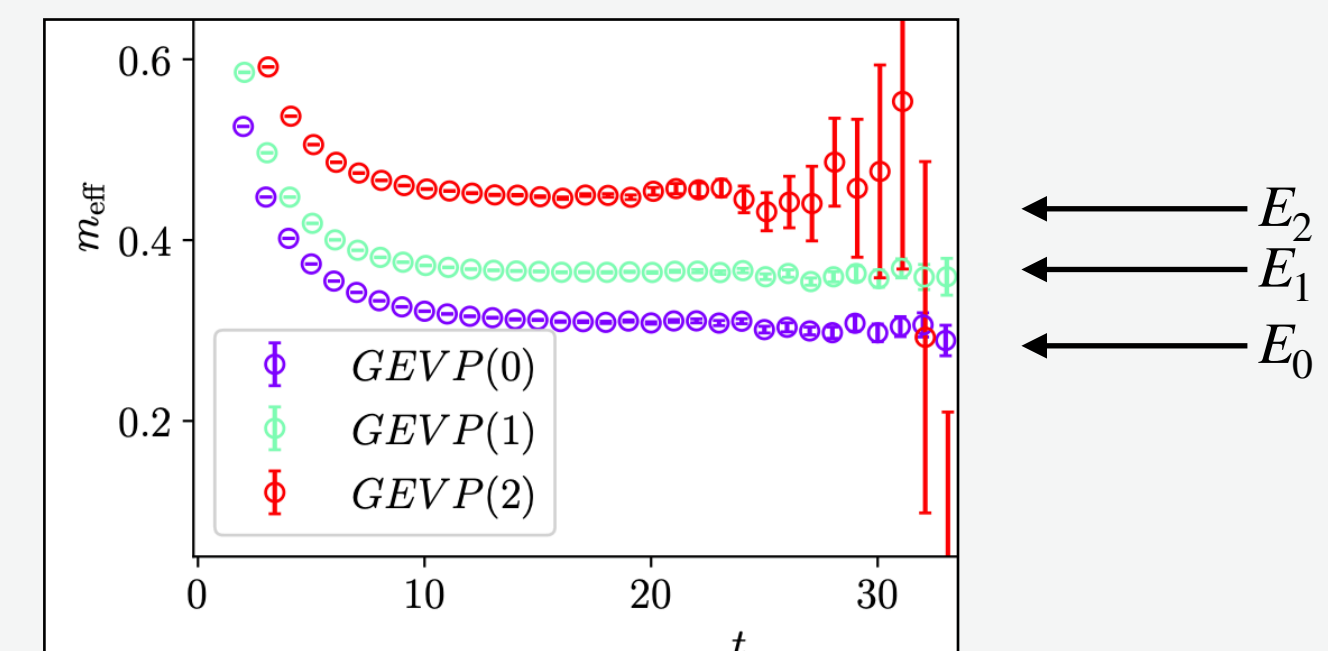
$$\text{disc } T \sim |T|^2 \quad \sqrt{s} < 3m$$

unitarity condition (on-shell-ness)

$$T^{-1} = K^{-1} - \int_l \frac{1}{2E_l} \frac{1}{(s - 4E_l^2 + i\epsilon)} = p \cot \delta - \left(\int \dots - \text{Re} \int \dots \right)$$

Finite-volume setup

- Input: energy eigenvalues $\{E_0, \dots\} \in \mathbb{R}$
- Correlation functions at $it \rightarrow \infty$



- off-shell configurations decay exponentially $\sim O(e^{-ML})$
- on-shell states propagate/feel box-size $\sim O((ML)^n)$

$$T_{FV}^{-1} = p \cot \delta - \left(\frac{1}{L^3} \sum_{\vec{l} \in 2\pi/L \mathbb{Z}} \dots - \text{Re} \int_{\vec{l}} \dots \right) = 0$$

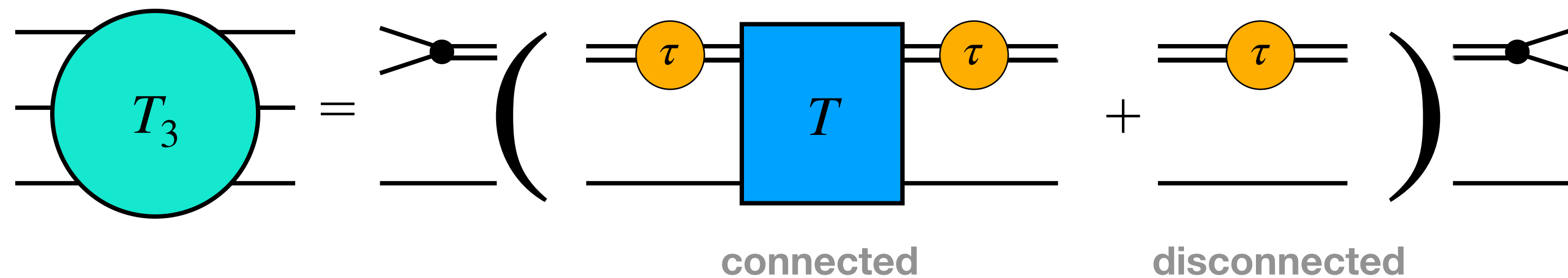
$$p \cot \delta(E_i) = Z_{00}(E_i) + \mathcal{O}(e^{-ML})$$

M. Lüscher, *Nucl. Phys. B* 354, 531 (1991)

3-BODY INTERACTIONS

... same logic: start with a general scattering amplitude

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)



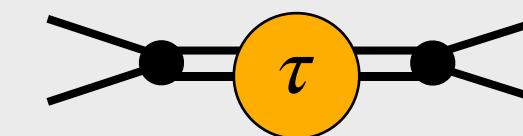
Bethe-Salpeter ansatz

$$T = B + B \tau T$$

... 4D integral equation

... unknown building blocks

isobar, aka dimer, aka 2-body subsystem



- a tower of states for $L = 0, 1, 2, \dots$
- can be repulsive (!)

MM/Döring Phys.Rev.Lett. 122 (2019)

... imposing 3-body unitarity

Olive/Eden/... "The Analytic S-matrix"

Bedaque, Griesshammer (1999)

R. Aaron, R. D. Amado, and J. E. Young, Phys.Rev. 174, 2022 (1968)

3-BODY UNITARITY

... imposing 3-body unitarity

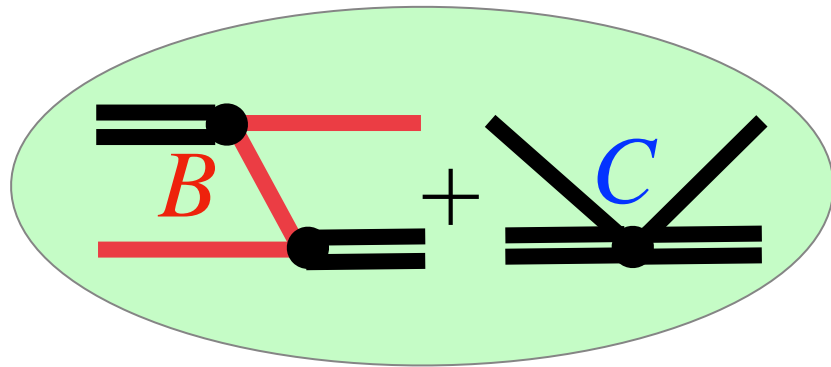
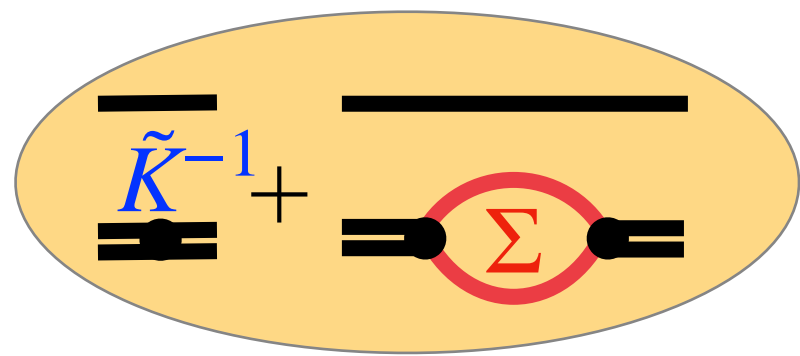
$$\langle \{q\} | (\hat{T}_3 - \hat{T}_3^\dagger) | \{p\} \rangle = i \int \prod_{\ell=1}^3 \frac{d^4 k_\ell}{(2\pi)^4} (2\pi) \delta^+(k_\ell^2 - m^2) (2\pi)^4 \delta^4 \left(P - \sum_{\ell=1}^3 k_\ell \right) \langle \{q\} | \hat{T}_3^\dagger | \{k\} \rangle \langle \{k\} | \hat{T}_3 | \{p\} \rangle$$

... exact solution — constraints on disc B and disc τ^{-1}

MM/Hu/Döring/Pilloni/Szczepaniak *Eur.Phys.J.A* 53 (2017)

$$\tau^{-1}(\sigma(k)) = A + B \sigma(k) + \frac{\sigma(k)^2}{\pi} \int_{4m^2}^{\infty} d\sigma' \frac{\text{Im } \tau^{-1}(\sigma')}{\sigma'^2(\sigma' - \sigma(k) - i\epsilon)}$$

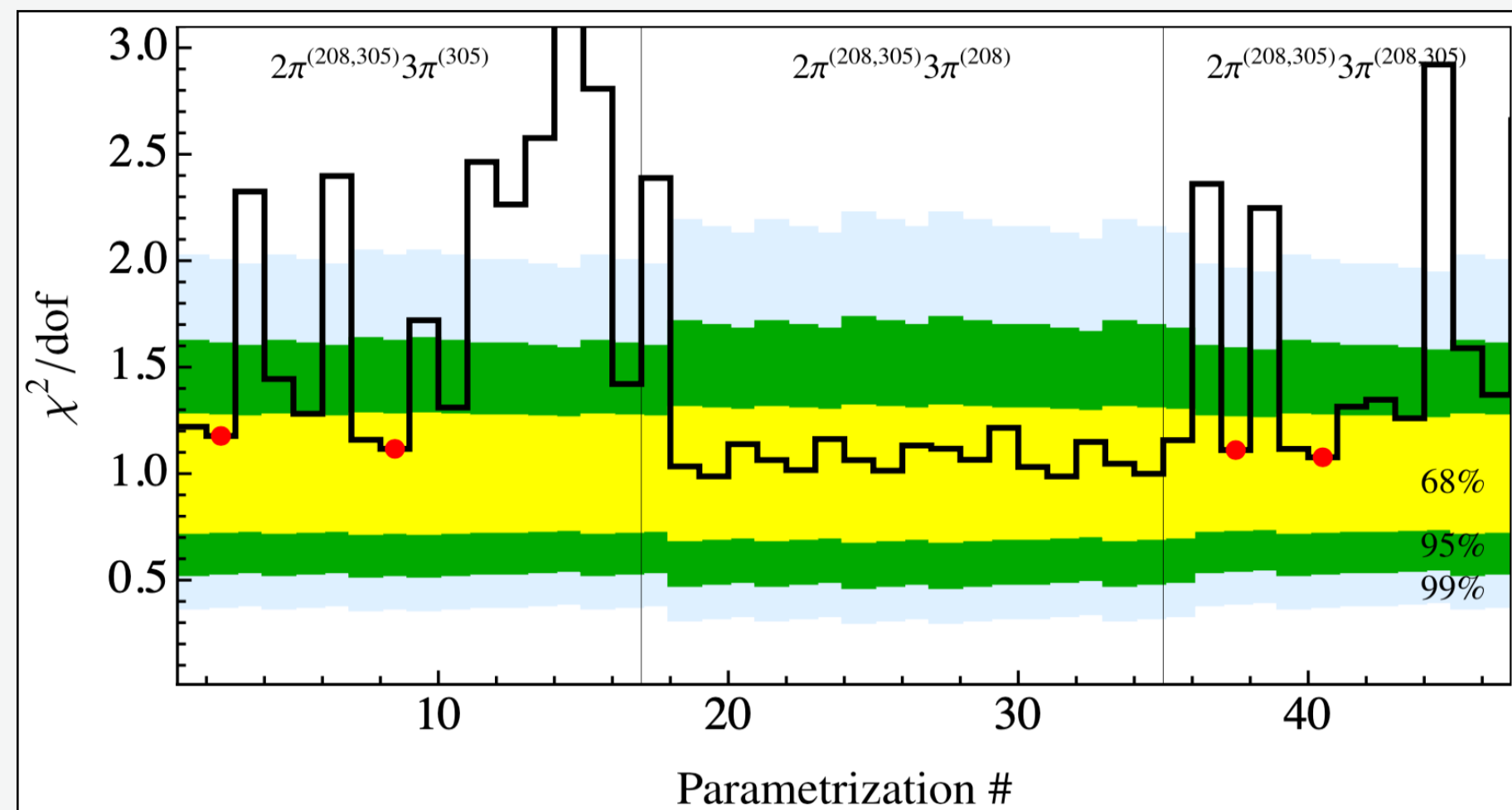
$$B(s,p,p') = - \frac{v(Q,q)v(Q,p)}{2\sqrt{m^2 + \mathbf{Q}^2} \left(E_Q - \sqrt{m^2 + \mathbf{Q}^2} + i\epsilon \right)}$$



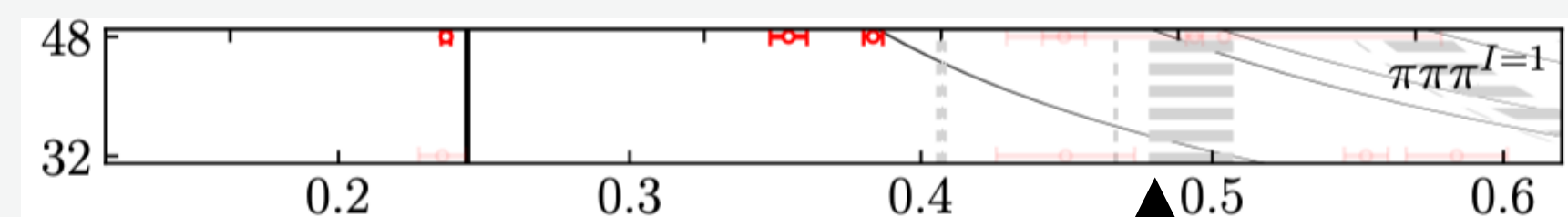
THINKING “OUTSIDE OF THE BOX”

Fits

- $\mathcal{O}(2000)$ fit variations $\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$
- Best $\chi^2/\text{dof} \approx 1.1$



- $\pi\pi\pi^{208}$ does not restrict $\pi(1300)$ region in the elastic regime



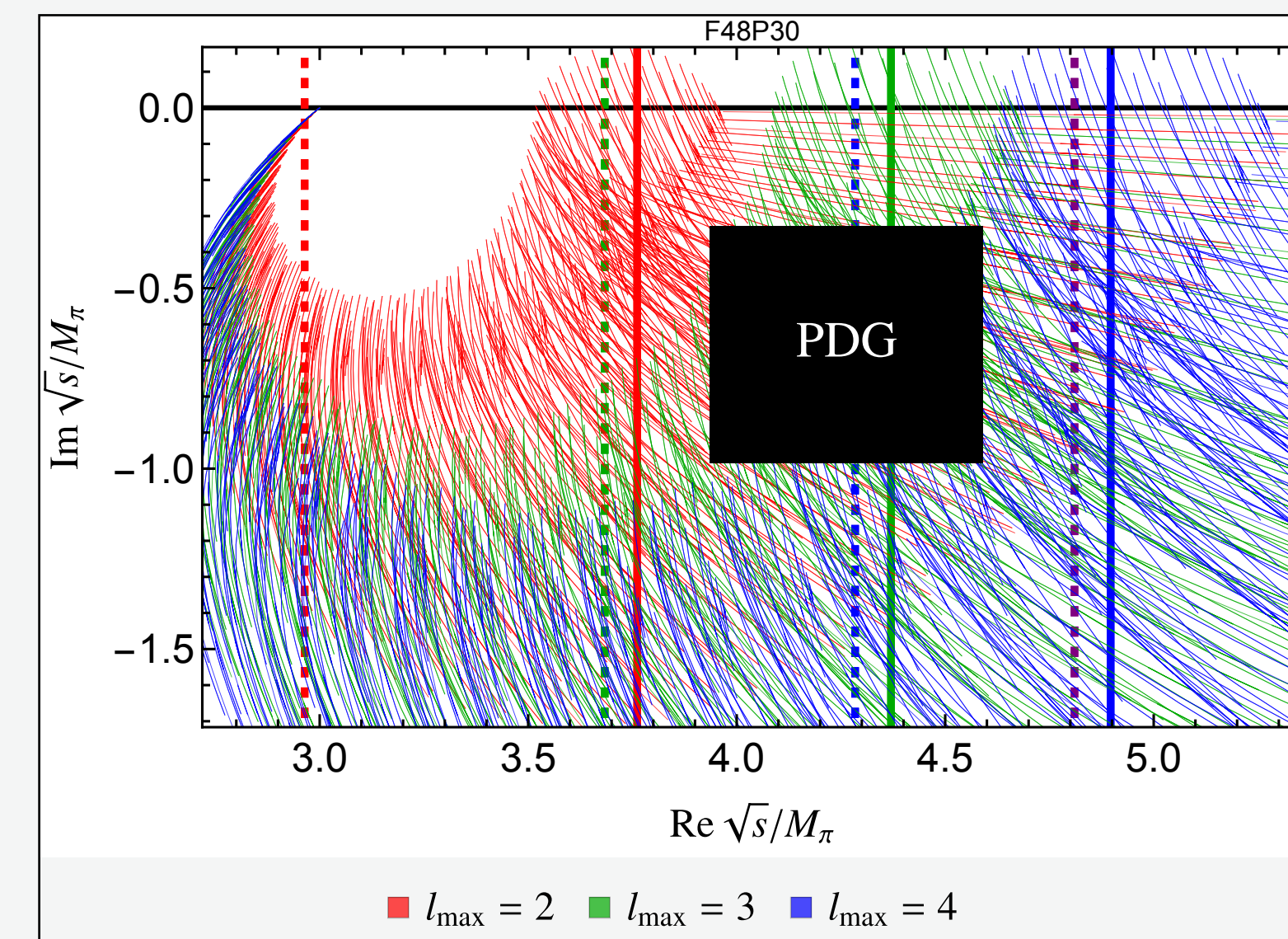
$$M_{\pi(1300)}/M_{\pi} \approx 1300/208$$

Infinite-volume scattering amplitudes

- $\tilde{K}^{-1}(s), C(s)$ are fixed and **volume independent**

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

- Maximize the analytically accessible area ($\ell_{\max}, \dots$)



LATTICE SETUP

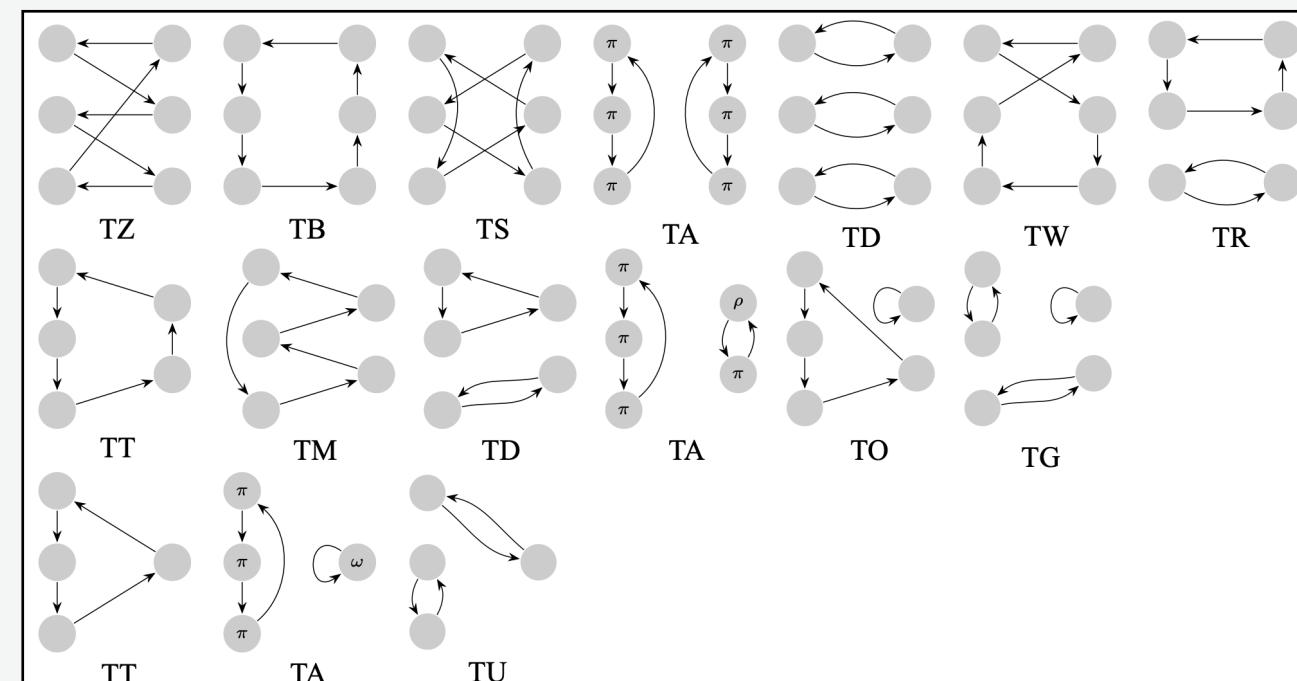
- 4 CLQCD ensembles with $N=2+1$ Clover fermions [CLQCD 2024]

- Two volumes
- Two pion masses (208 MeV, 305 MeV)

configuration	volume	a/fm	m_π/MeV	N_{cfgs}
C48P14	$48^3 \times 96$	0.10530(18)	135.5(1.6)	259
F32P21	$32^3 \times 64$	0.07746(18)	210.9(2.2)	459
F48P21	$48^3 \times 96$	0.07746(18)	207.2(1.1)	222
F32P30	$32^3 \times 96$	0.07746(18)	303.2(1.3)	567
F48P30	$48^3 \times 96$	0.07746(18)	303.4(9)	201
H48P32	$48^3 \times 144$	0.05187(26)	317.2(0.9)	274

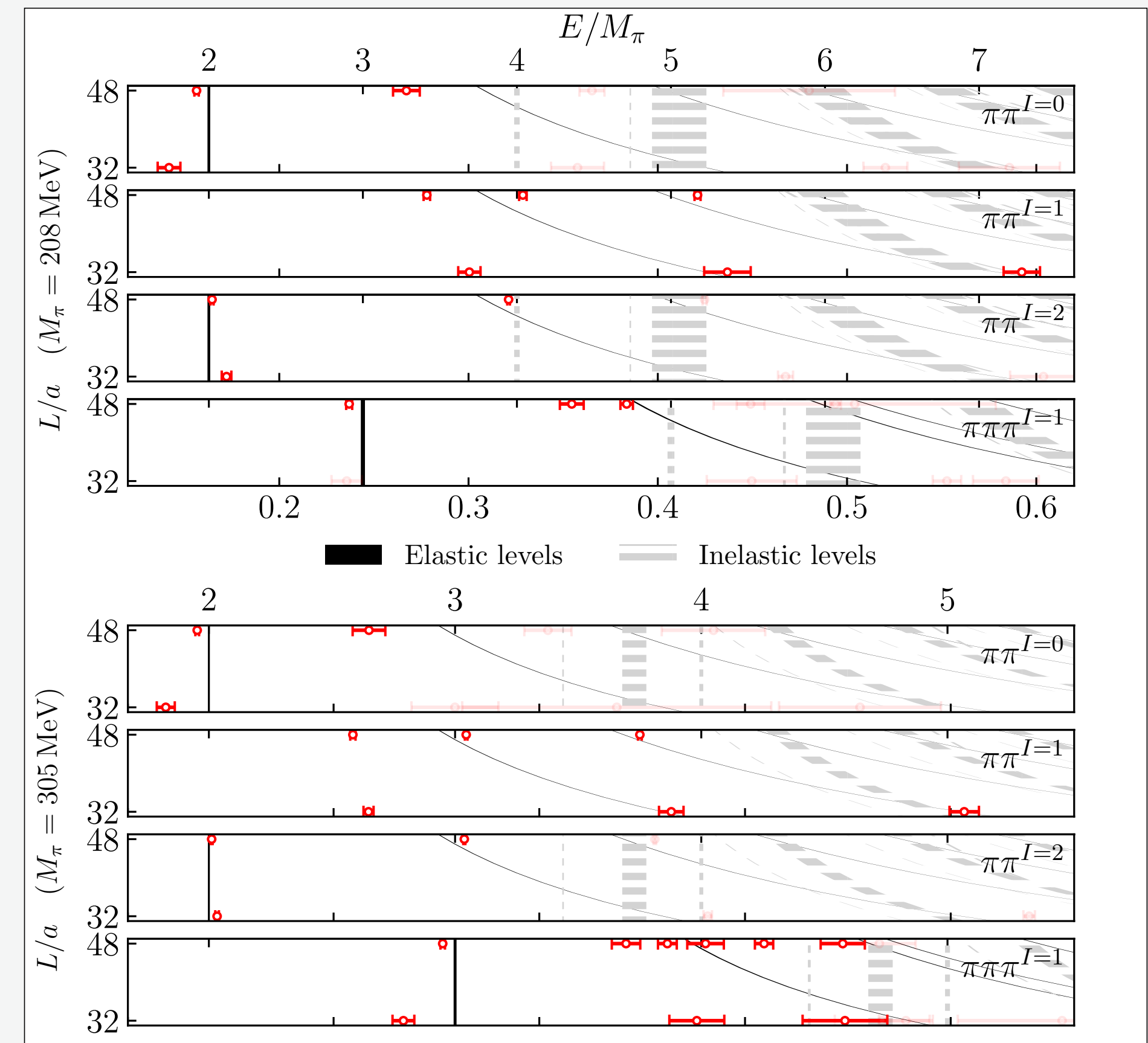
- OpTion** utilized for 3-hadron operators with arbitrary momenta
Yan+ [arXiv:2507.16070](https://arxiv.org/abs/2507.16070), JHEP (2025) (<https://github.com/wittscien/OpTion>)

- Contractions (~400)



Finite-volume spectrum:

24 ($\pi\pi$) + 12 ($\pi\pi\pi$) interacting levels



QUANTIZATION CONDITIONS

Lüscher QC

$$\sqrt{\sigma} \in \{E_0, E_1, \dots\}_{\pi\pi} \iff \tilde{K}^{-1}(\sigma) - \Sigma^L(\sigma) \equiv 0$$



Döring/Meißner/Rusetsky/Oset *Eur.Phys.J.A* 47 (2011) 139

- Equivalent to 2b-Lüscher QC $\mathcal{O}(e^{-ML})$
- Unknown 2-body interaction — use UCHPT

Hanhart+ *PRL*100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \rightarrow T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

Finite Volume Unitarity (FVU) 3-body quantization condition*

$$\sqrt{s} \in \{E_0, E_1, \dots\}_{\pi\pi\pi} \iff \det \left[2L^3 E_p (\tilde{K}^{-1}(s) - \Sigma^L(s)) - B(s) - C(s) \right] \equiv 0$$



MM/Döring *Eur.Phys.J.A* 53 (2017) 12, 240

- Extended to all isospin/strangeness channels
- Unknown 2-body interaction — use UCHPT

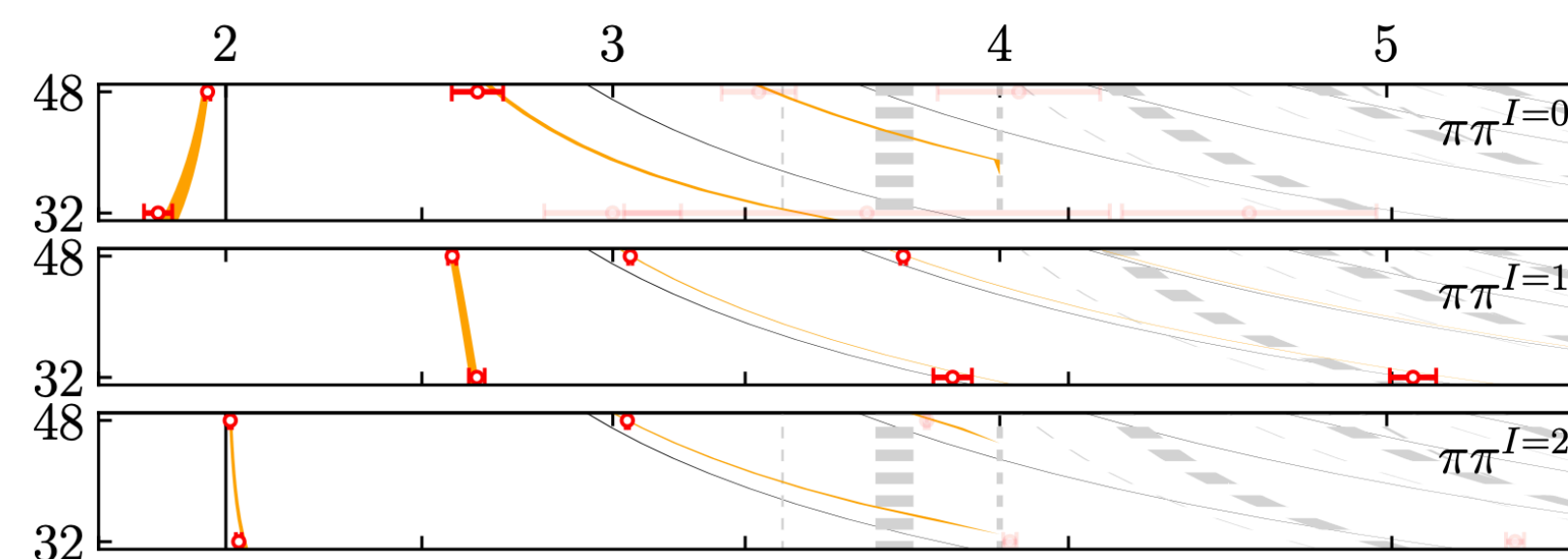
Feng+ *Phys.Rev.D* 110 (2024) 094002
K(1460) 2511.02543

Hanhart+ *PRL*100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \rightarrow T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

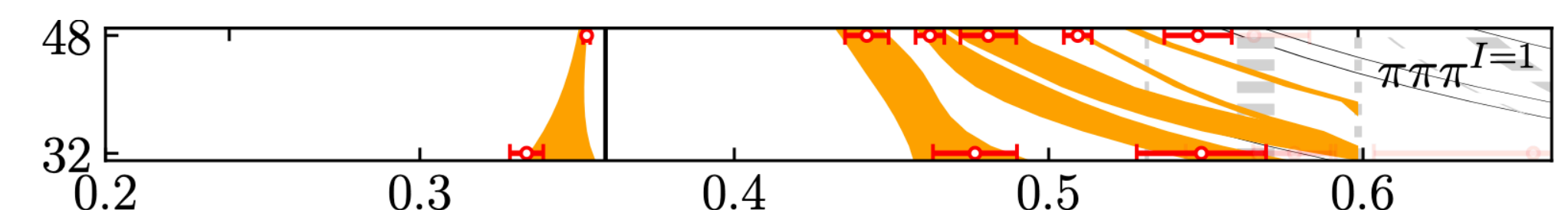
- Unknown 3-body force

$$C(s) \rightarrow c^{\alpha\beta} = c_c^{\alpha\beta} + \frac{c_p^{\alpha\beta}}{s - m_0^2}, \quad \alpha = \beta \in \{\sigma\pi, \rho\pi\}$$



2/3-body fit

$$\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$$



*) **Equivalent/not identical alternatives:** RFT, NREFT, ... (Hansen/Sharpe 2014, Rusetsky/Hammer/Pang 2017, ... Jackura et al. *Phys.Rev.D* 100 (2019) 3, 034508, Garofalo et al. *JHEP* 02 (2023) 252)

Developments and applications (Blanton, Romero-Lopez, Hansen, Briceño, Dawid, Rusetsky, Davoudi, Guo, MM, Culver, Yan, Garofalo, Urbach, Polejaeva, Raposo, Feng, Döring, ...)

Reviews: Hansen/Sharpe *Ann. Rev. Nucl. Part. Sci.* 69, 65 (2019) MM/Döring/Rusetsky *Eur. Phys. J. ST* 230, 1623 (2021) Romero-Lopez, *PoS LATTICE2022*, 235 (2023)

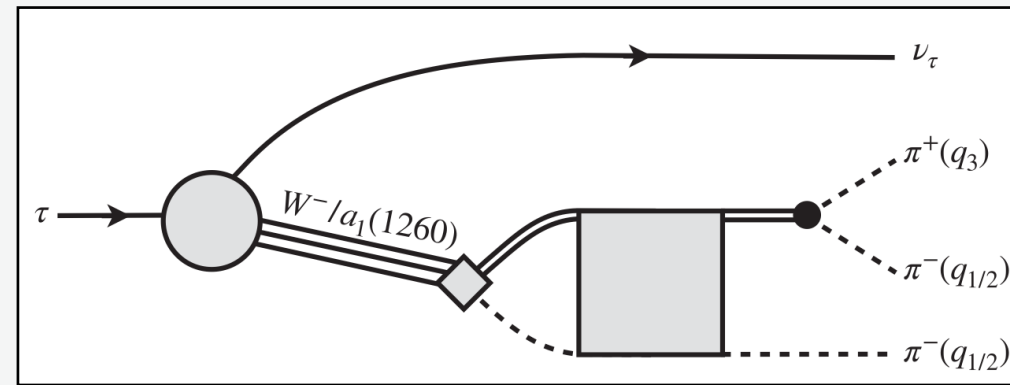
Related talks: Raposo (Tcc/LHC), Sharpe (Npipi), Romero-Lopez (pipipi)

AXIAL-VECTOR MESON

Sadasivan/MM/Culver/Alexandru/Lee/Doring *Phys.Rev.D* 101 (2020);

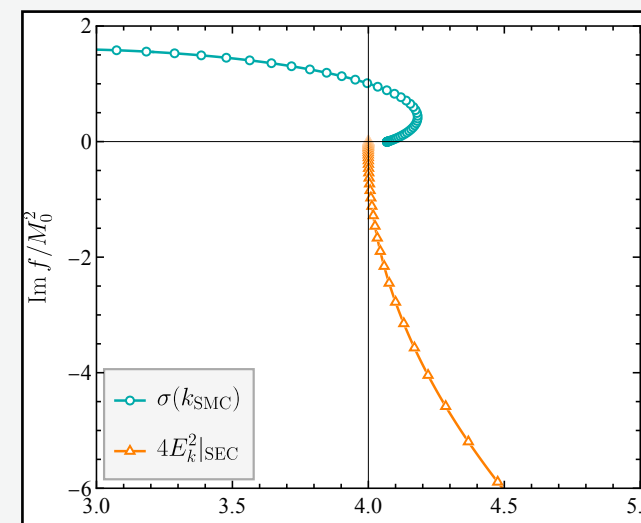
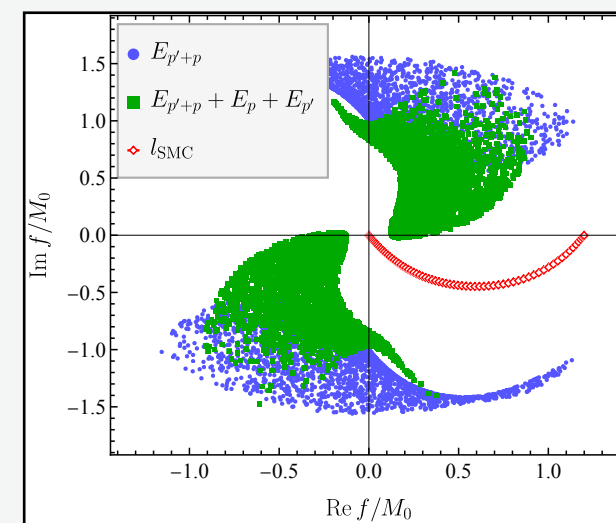
41

Input from τ -decays or hadronic reactions



Solution techniques

- avoid singularities (OPE, self-energy)
- complex contour deformation $|\ell| \in \mathbb{C}$



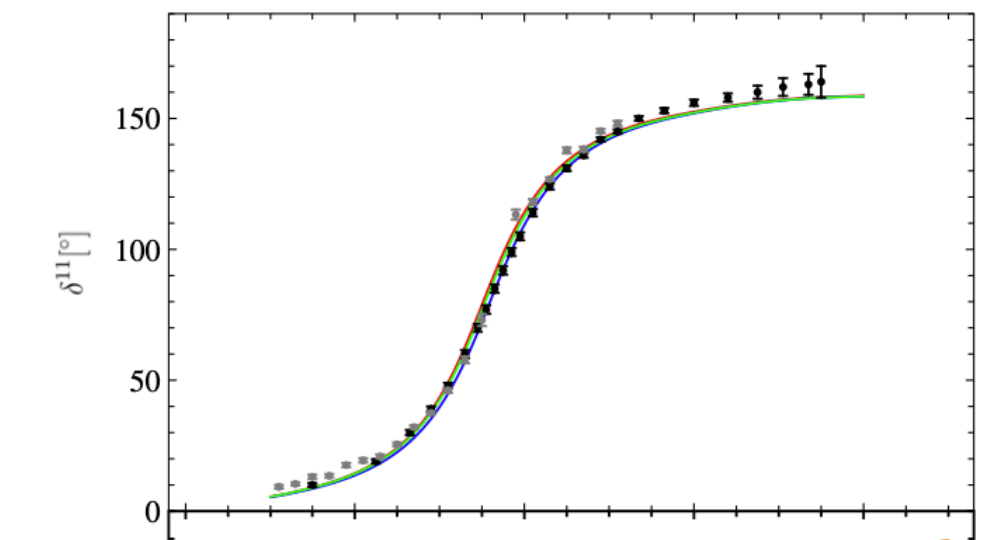
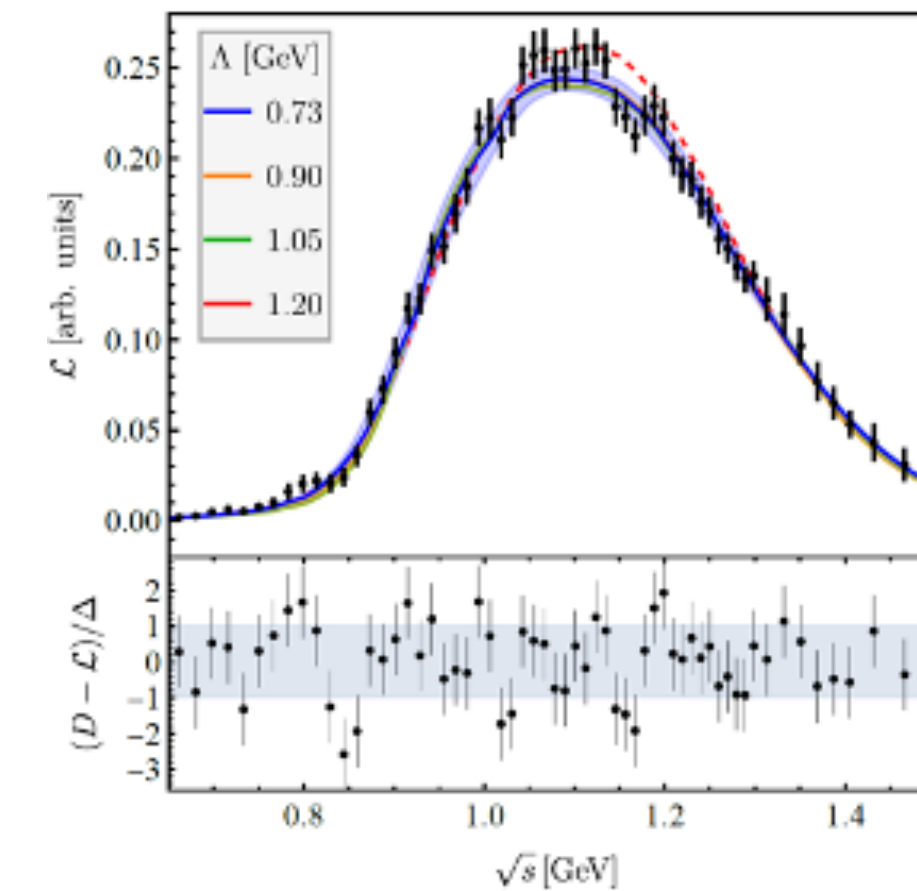
- Analytic continuation to the real axis

J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935–B948 (1965).

R. T. Cahill and I. H. Sloan, Nucl. Phys. A 165, 161–179 (1971)

W. Schmid and Ziegelmann,

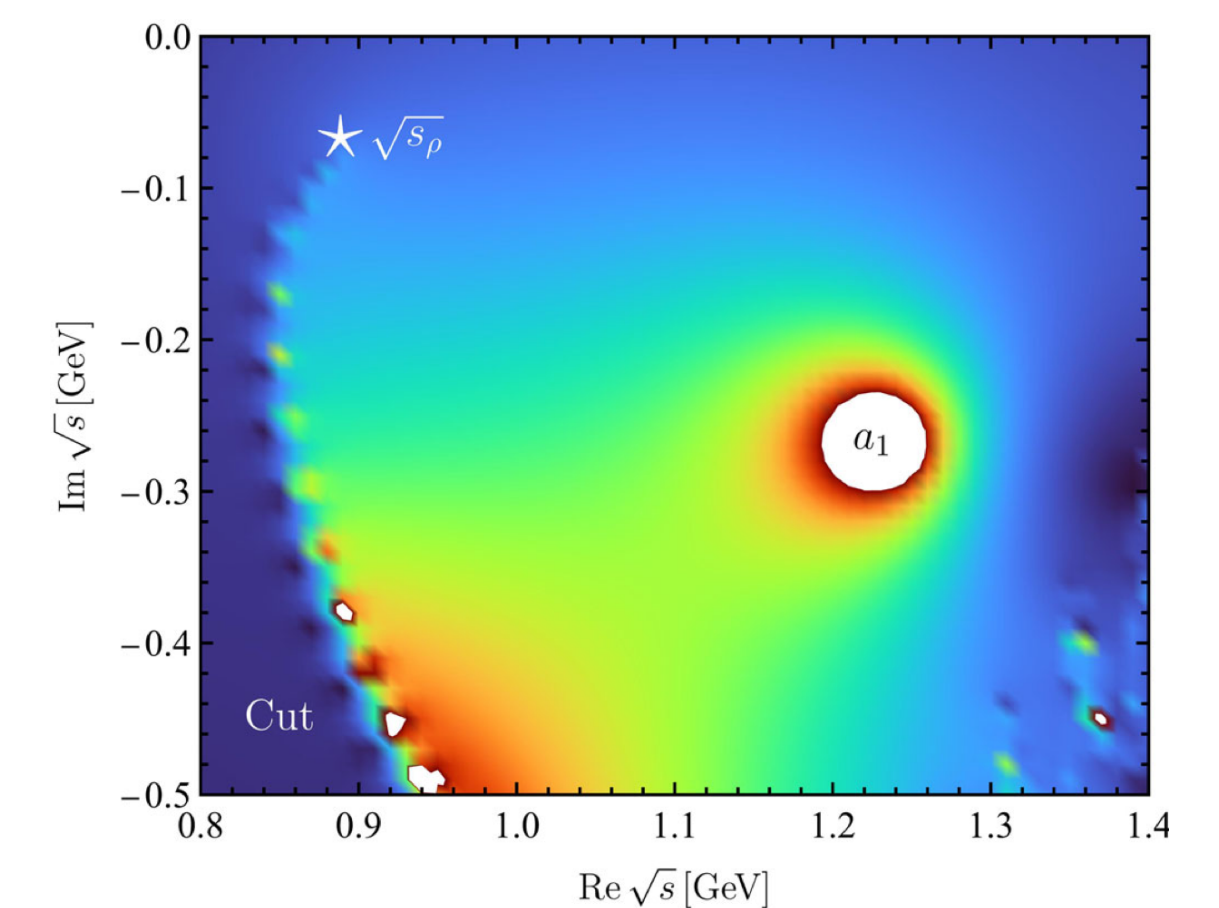
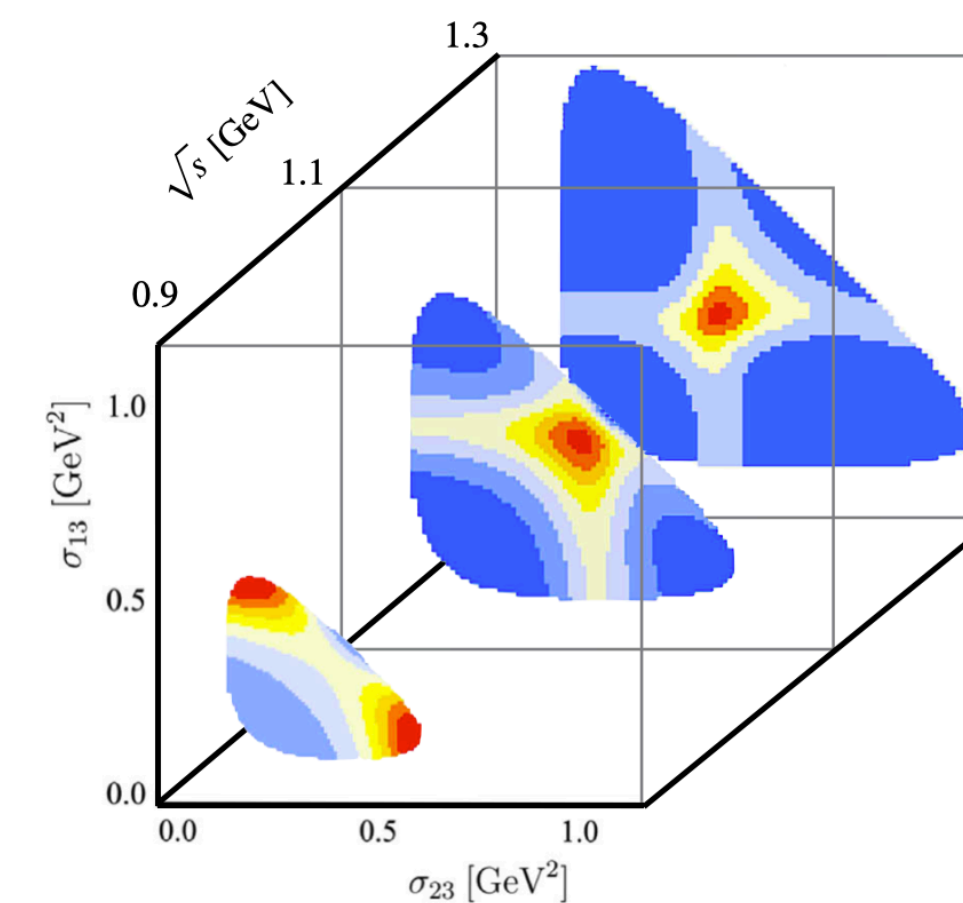
S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467–1475 (1974)



*Schael et al. [ALEPH] Phys.Rept. 421 (2005);
Nucl.Phys.B 79; Phys.Rev.D 7;*

IVU

$$T(s, p, p') = B + C + \int \frac{d^3 \ell}{(2\pi)^3} \frac{(B + C)}{2E_\ell} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

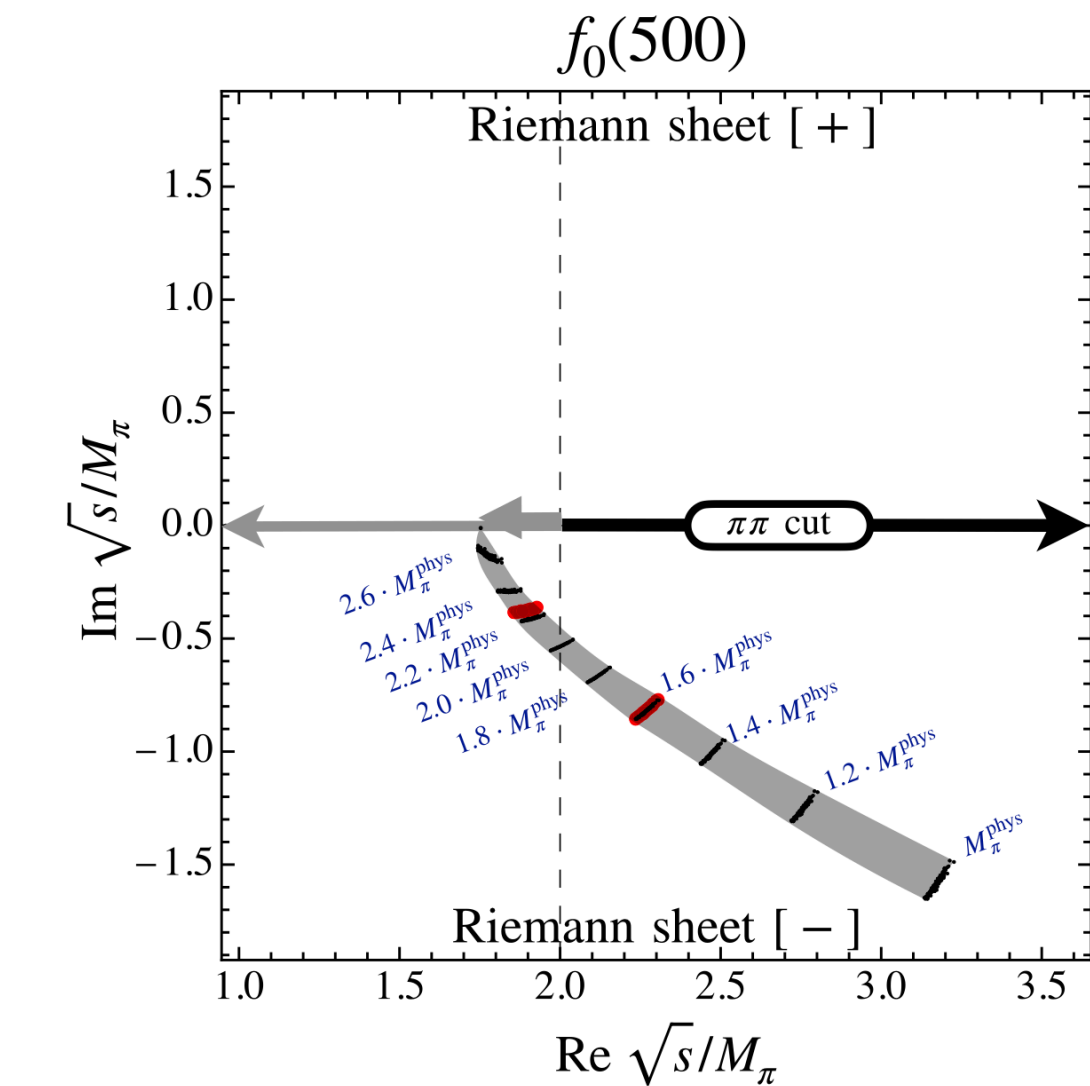


APPLICATION I

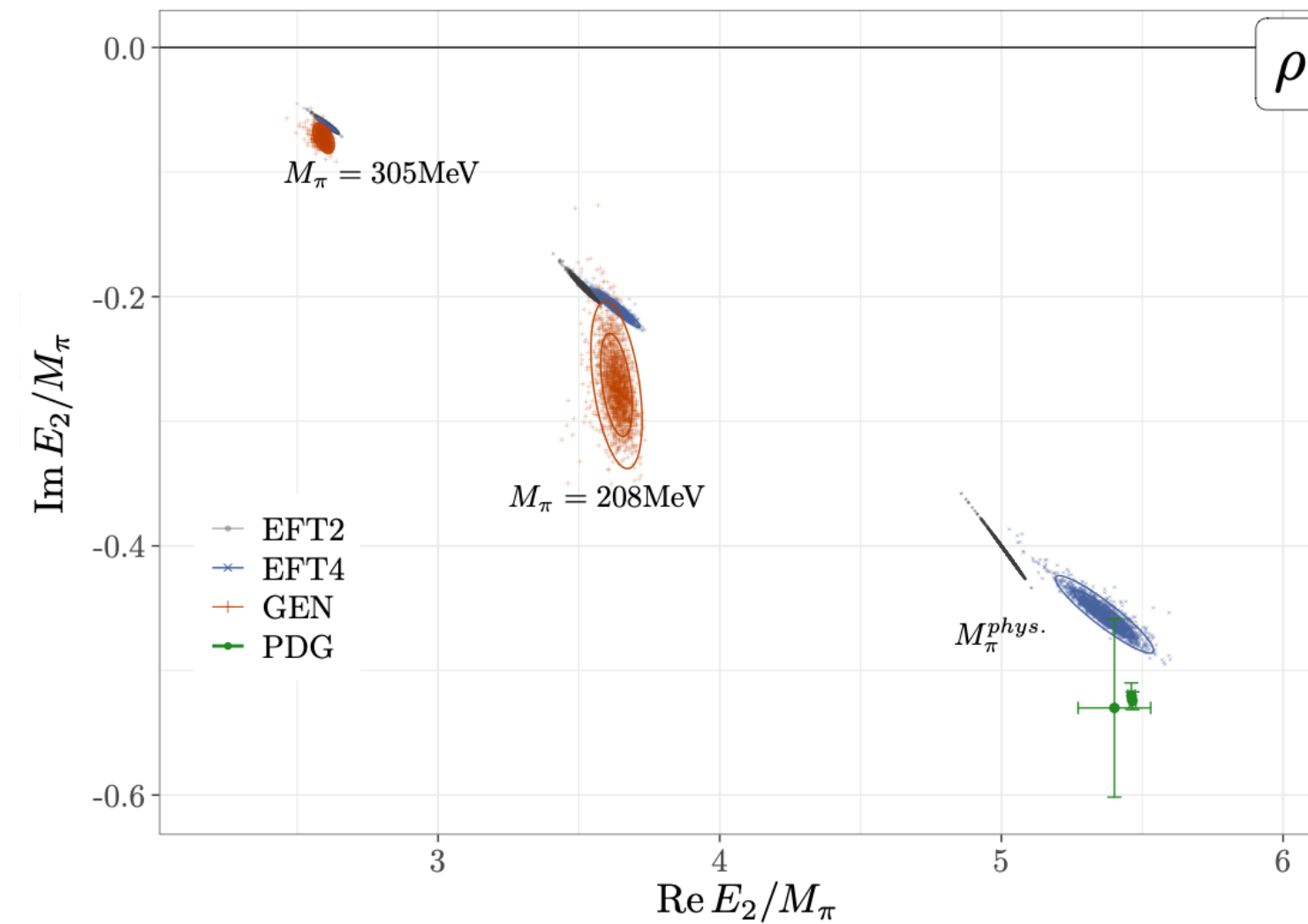
Two pion system

- simplest 2-hadron system
- many LQCD results
NPLQCD; HadSpec; ETMC; GW-lattice; CP-PACS;....
- simultaneous description of all $\pi\pi$ interaction channels through CHPT — UCHPT

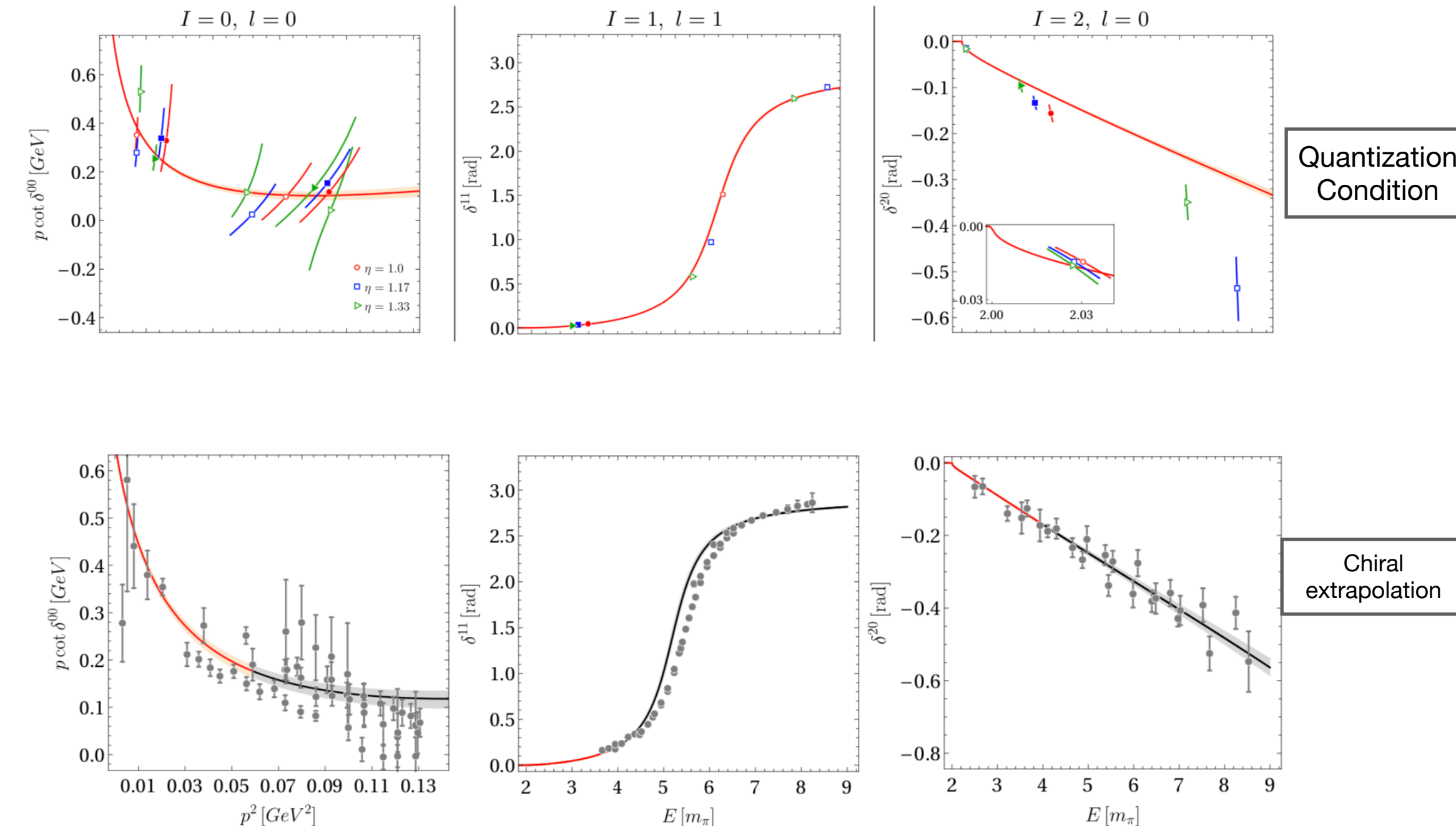
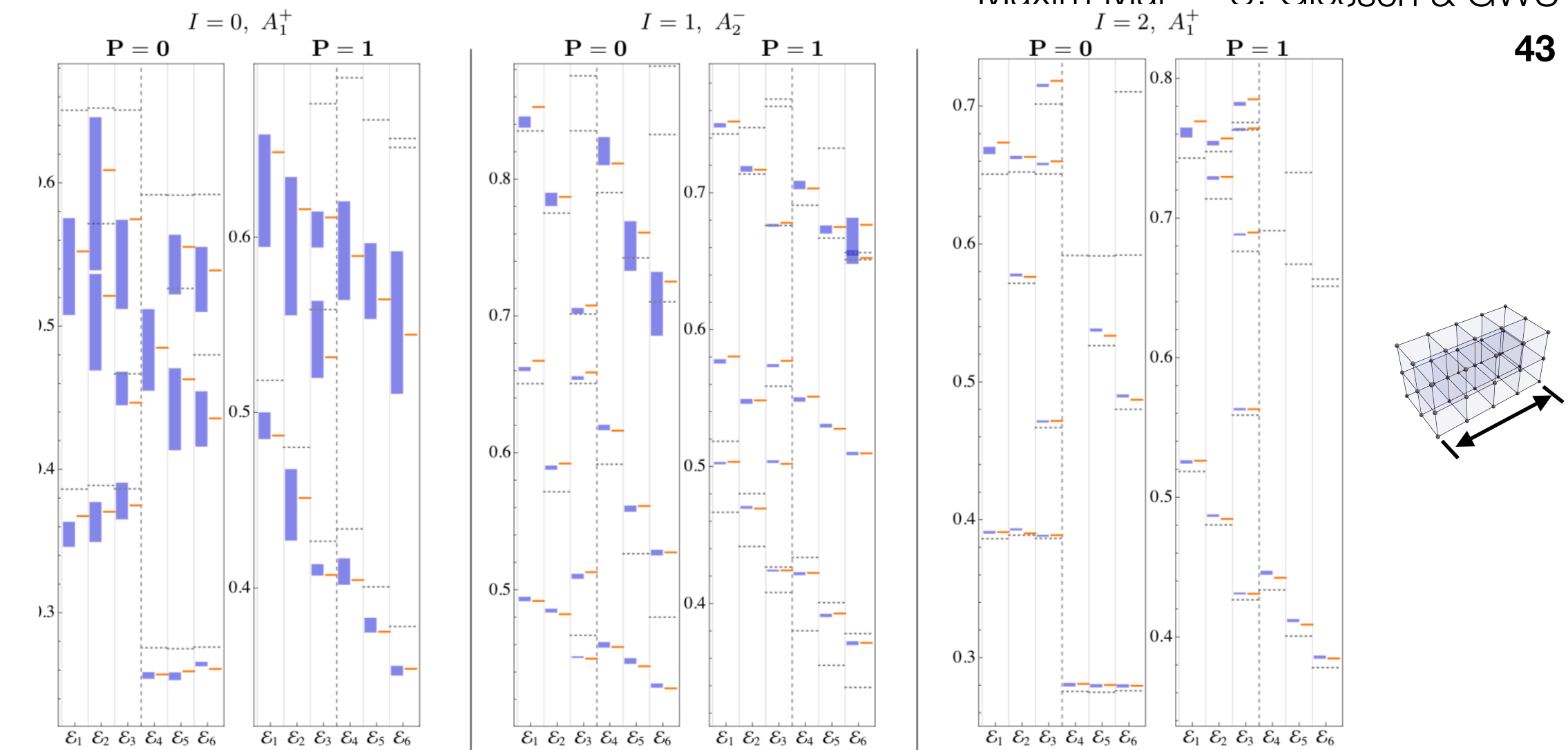
GWQCD: Guo et al. (2016) Guo et al. (2018) Culver et al. (2019) MM et al. (2019)



MM/Urbach/Meißner Phys.Rept. 1001 (2023) 1-66



Yan/MM/... Phys.Rev.Lett. 133 (2024) 21



[GWQCD] MM/Culver/... Phys.Rev.D100(11)(2019) 114514

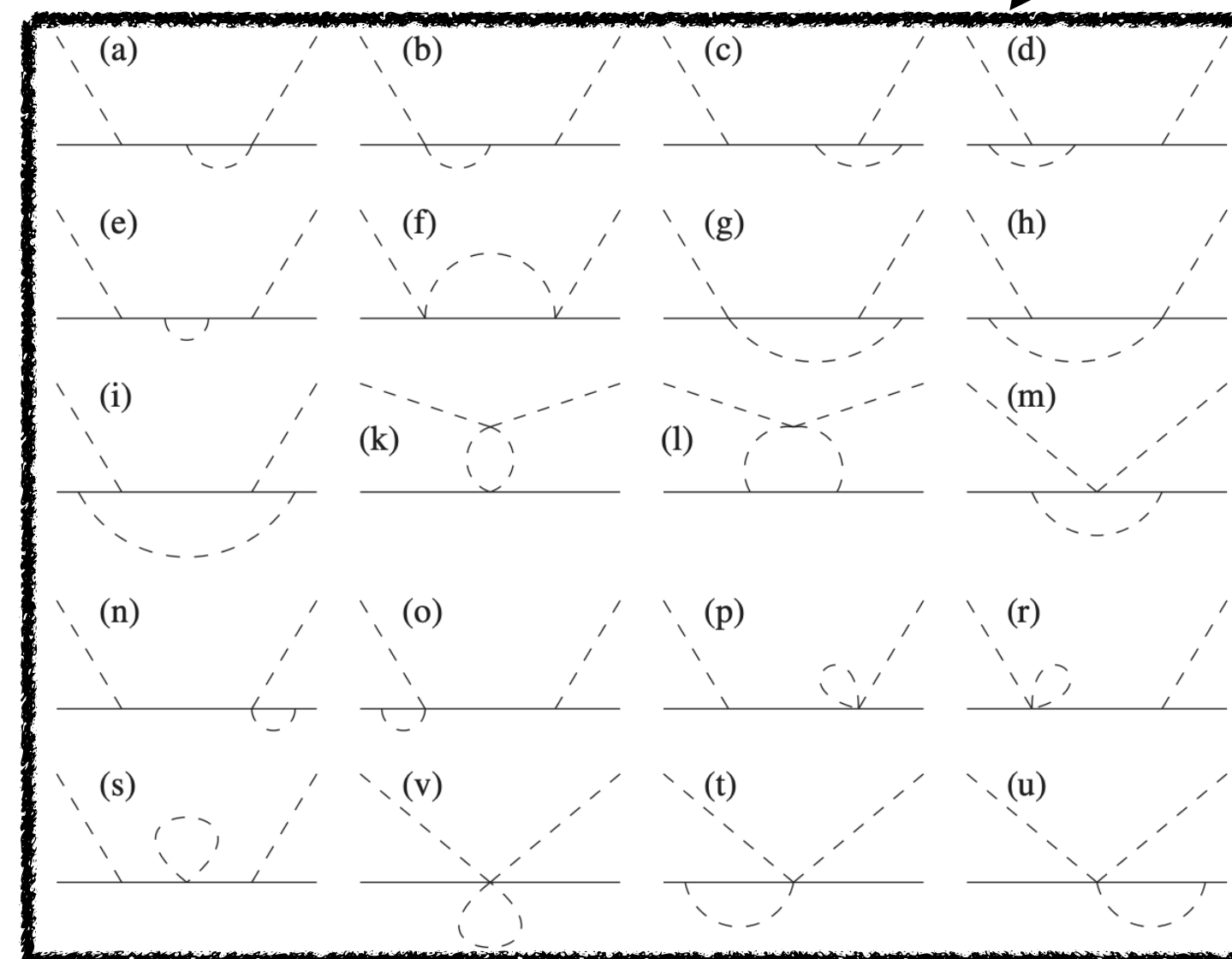
APPLICATIONS I

Meson-baryon scattering from CHPT

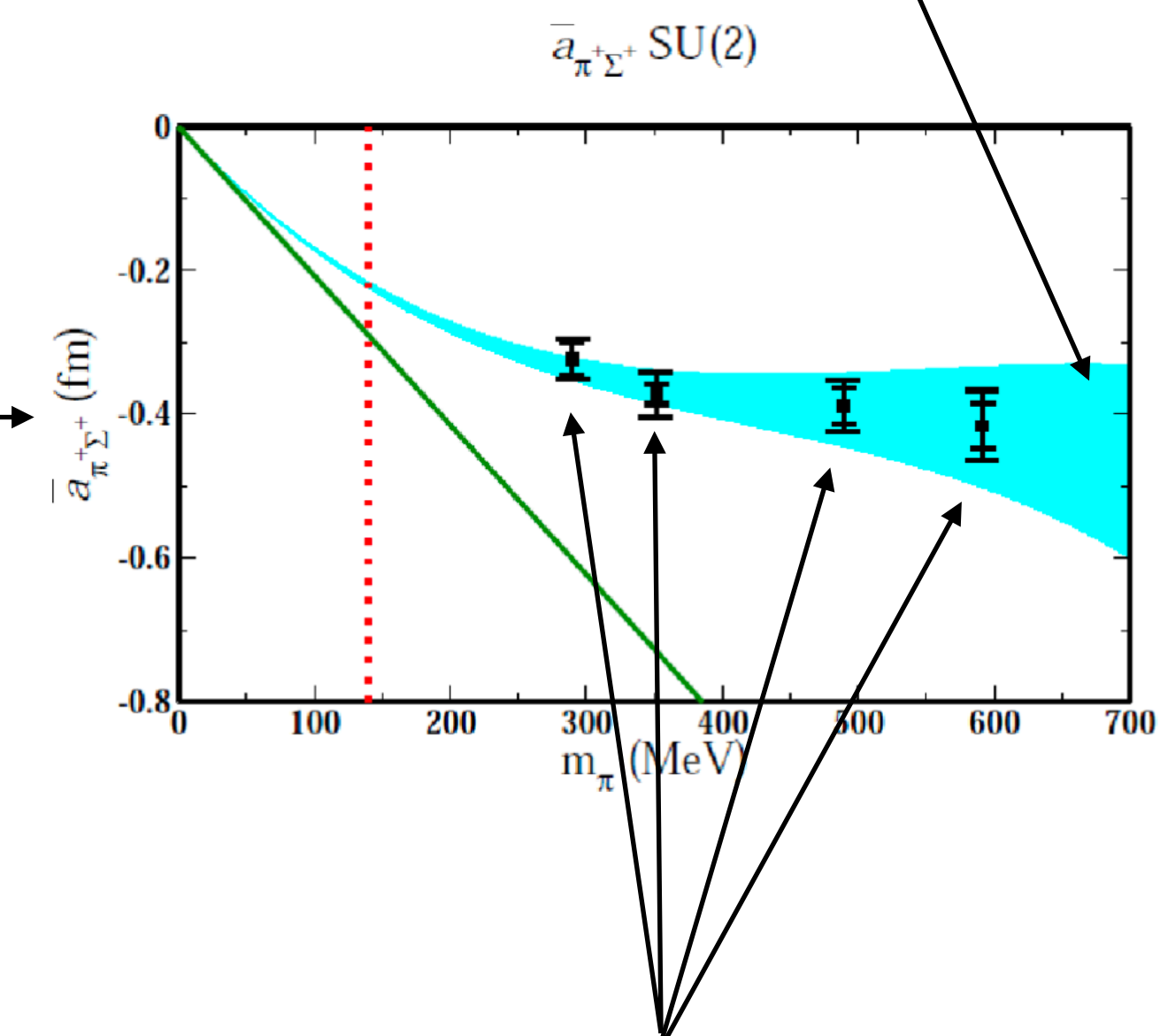
MM/P.C.Bruns/Ulf-G. Meißner/B.Kubis *Phys.Rev.D* 80 (2009) 094006

- full SU(3) dynamics near threshold
- agrees with experiment in many cases
- well controlled chiral extrapolation of Lattice QCD results ($\pi\Sigma$)

$$\begin{aligned} \mathcal{L}_{\phi B}^{(2)} = & b_{D/F} \langle \bar{B} [\chi_+, B]_{\pm} \rangle + b_0 \langle \bar{B} B \rangle \langle \chi_+ \rangle + b_{1/2} \langle \bar{B} [u_\mu, [u^\mu, B]_{\mp}] \rangle + b_3 \langle \bar{B} \{u_\mu, \{u^\mu, B\}\} \rangle + b_4 \langle \bar{B} B \rangle \langle u_\mu u^\mu \rangle \\ & + i\sigma^{\mu\nu} (b_{5/6} \langle \bar{B} [[u_\mu, u_\nu], B]_{\mp} \rangle + b_7 \langle \bar{B} u_\mu \rangle \langle u_\nu B \rangle) + \frac{ib_{8/9}}{2m_0} (\langle \bar{B} \gamma^\mu [u_\mu, [u_\nu, [D^\nu, B]]_{\mp}] \rangle + \langle \bar{B} \gamma^\mu [D_\nu, [u^\nu, [u_\mu, B]]_{\mp}] \rangle) \\ & + \frac{ib_{10}}{2m_0} (\langle \bar{B} \gamma^\mu \{u_\mu, \{u_\nu, [D^\nu, B]\}\} \rangle + \langle \bar{B} \gamma^\mu [D_\nu, \{u^\nu, \{u_\mu, B\}\}] \rangle) + \frac{ib_{11}}{2m_0} (2 \langle \bar{B} \gamma^\mu [D_\nu, B] \rangle \langle u_\mu u^\nu \rangle \\ & + \langle \bar{B} \gamma^\mu B \rangle \langle [D_\nu, u_\mu] u^\nu + u_\mu [D_\nu, u^\nu] \rangle), \end{aligned} \quad (8)$$



MM/P.C.Bruns/Ulf-G. Meißner/B.Kubis *Phys.Rev.D* 80 (2009) 094006



LQCD at unphysical pion masses
Torok/Beane/Detmold/Luu/... *Phys.Rev.D* 81 (2010) 074506

AXIAL-VECTOR MESON*

Excited axial-vector meson: $a_1(1420)$

Observed by COMPASS/Belle in $\pi^- \pi^+ \pi^-$ final state

Creation mechanisms:

COMPASS:2015kdx, Rabusov:2023tna

- Excited state of $a_1(1260)$
- “Triangle singularity” $K^*(892)\bar{K} \rightarrow \pi f_0(980)$

COMPASS:2020yhb

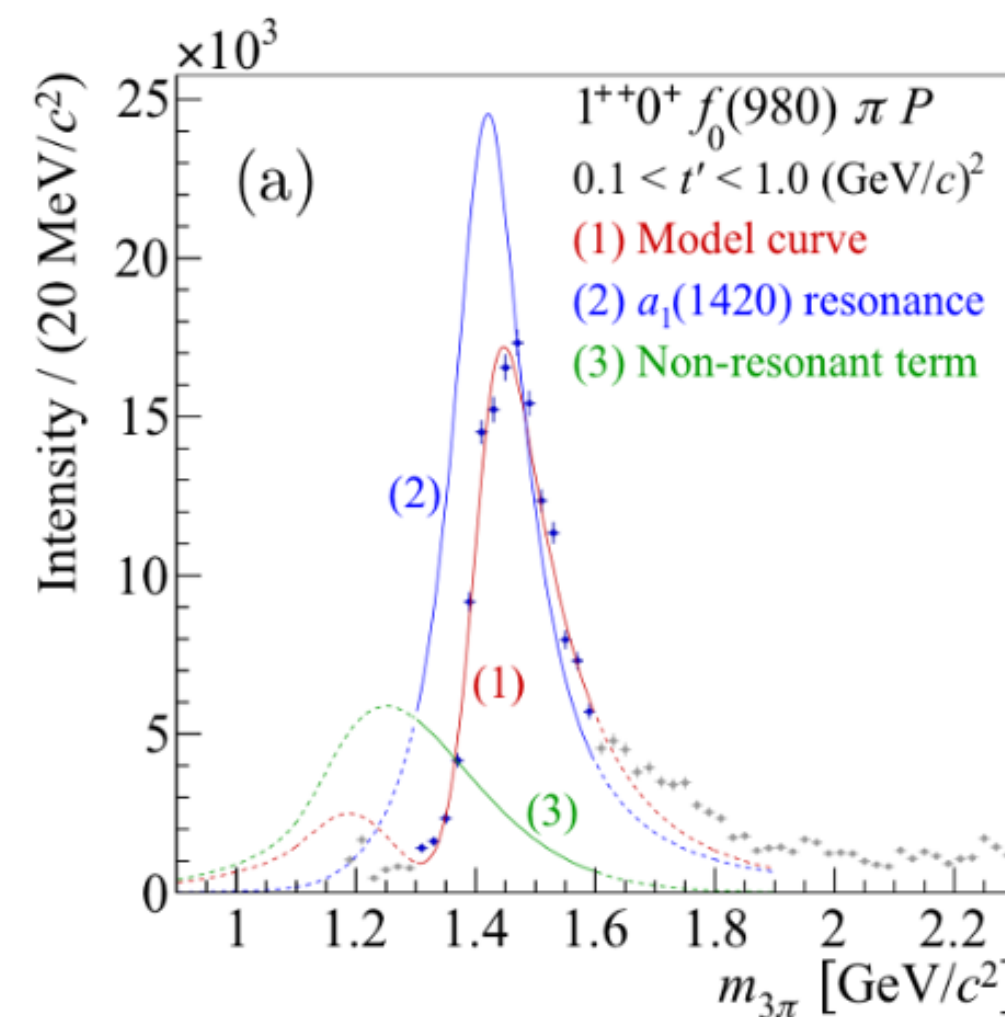
Mikhasenko:2015oxp Review: Guo:2019twa

Related: Dai:2018hqb, Dai:2018rra, Liang:2019jtr, Jing:2019cbw, Du:2021zdg,

Duan:2023dky, Wang:2016dtb, Nakamura:2023obk, Zhang:2024dth, Achasov:2022onn,

Nakamura:2023hbt, arXiv:1609.04133 [hep-ph].

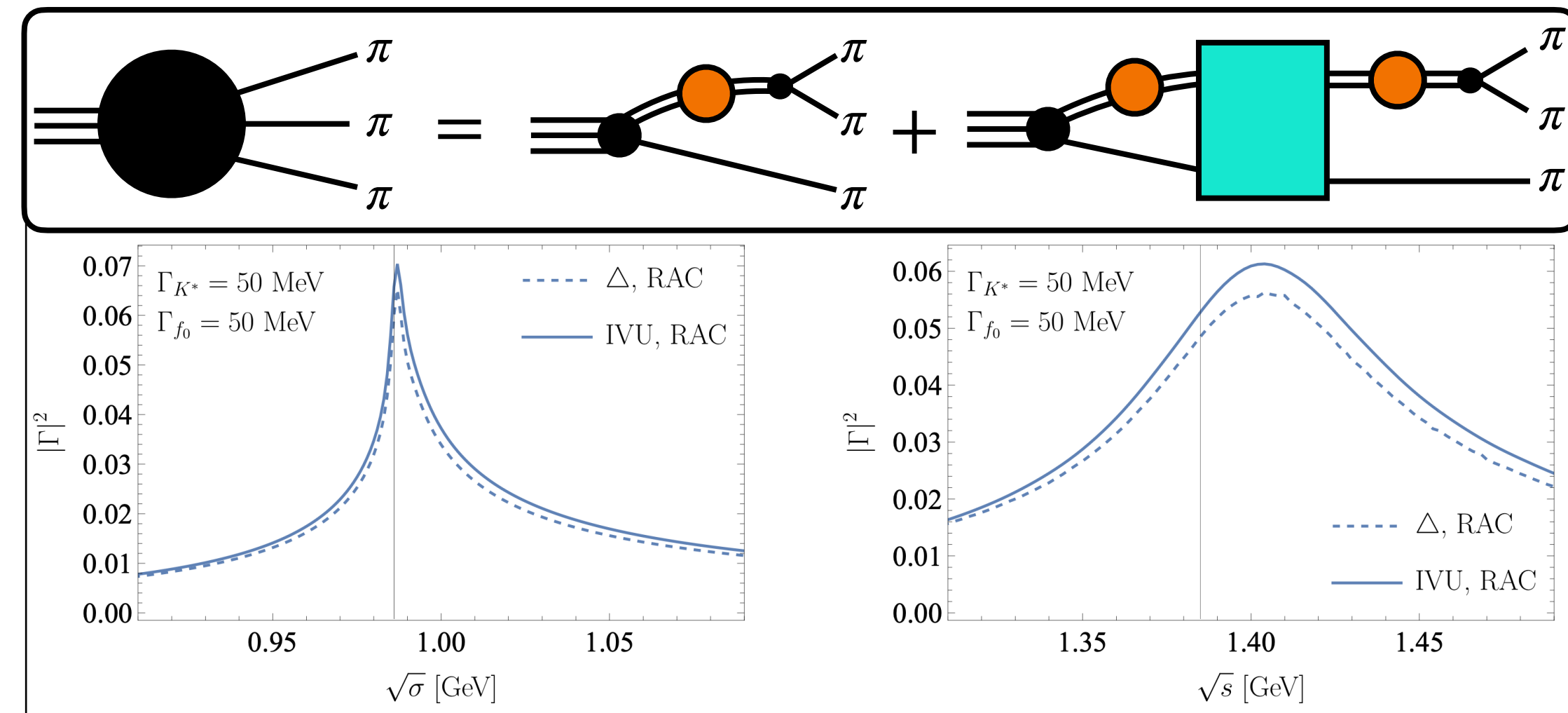
Talks: J.J.Wu — Z.Zhang



3b unitary formalism IVU

Ajay Sakthivasan/MM/Döring/Rusetsky 2407.17969 [hep-ph]

- formalism to incorporate both hypothesis
- full 2- and 3-body re-scattering
- **for now:** only kinematic/analytical properties (no spin)



➡ Effect is small but distinguishable

➡ Add spin, fit to the line-shapes ... in progress

Mapping to infinite volume

- 3-body quantization condition

$$\text{FVU}$$

$$\det \left[2L^3 E_p \left(\tilde{K}^{-1} - \Sigma^L \right) - B - C \right]^\Lambda \equiv 0$$

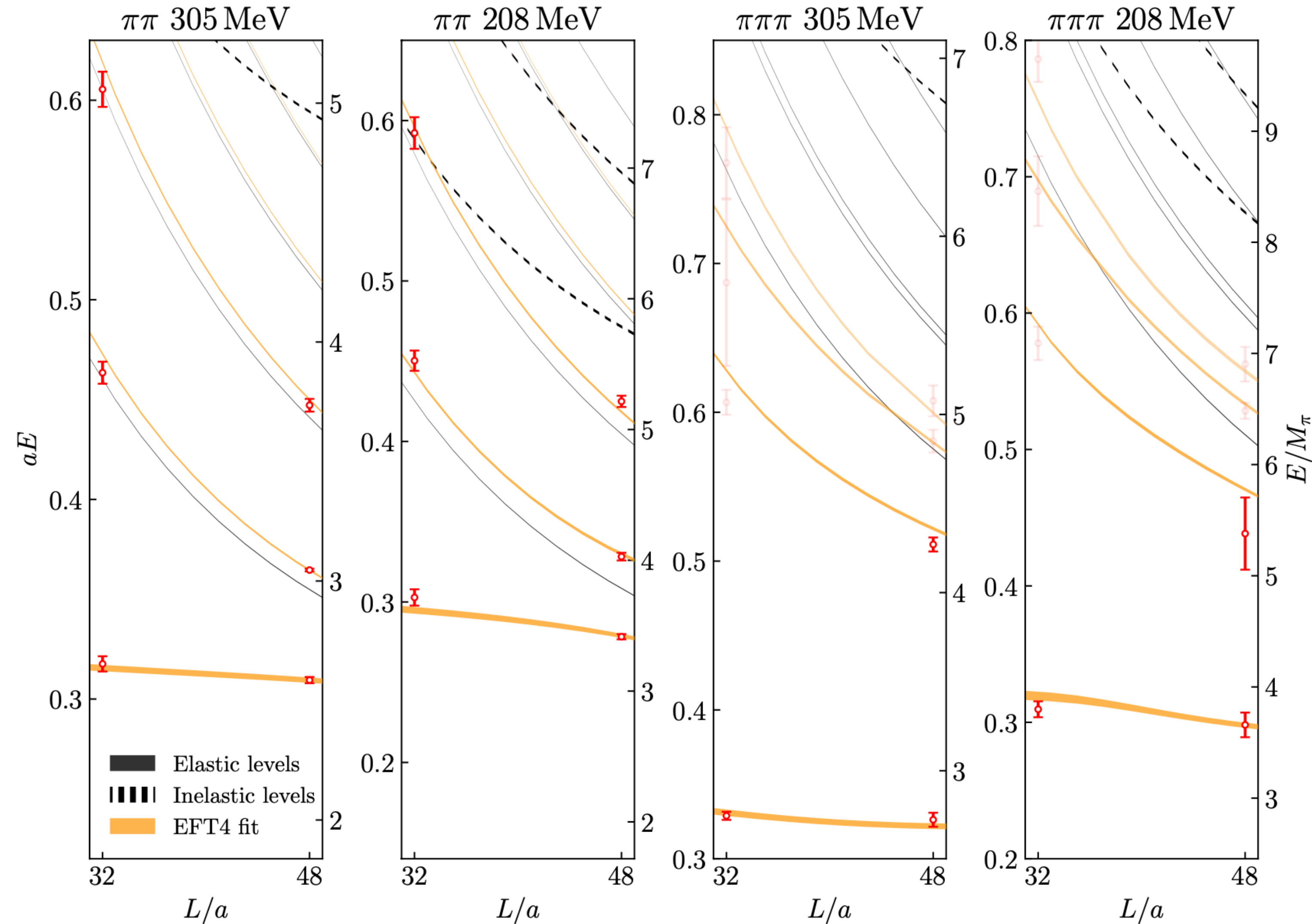
MM/Döring
Eur.Phys.J.A 53 (2017) 12,

- Volume-independent 2-,3-body force $C, \tilde{K}$
- saturated by meson s-channel interaction — **EFT form**

$$\left[\tilde{K}^{-1} \right]_{\mathbf{p}'\lambda', \mathbf{p}\lambda} = \delta_{\lambda'\lambda} \delta_{\mathbf{p}'\mathbf{p}} \frac{\sigma_p - M_\rho^2}{2g^2},$$

$$\tilde{c}_{11} = \frac{6s(M_\rho^2 - \sigma_q + 6g^2 f_\pi^2)(M_\rho^2 - \sigma_p + 6g^2 f_\pi^2)}{64g^2 \pi^3 f_\pi^6 (s - M_\omega^2)},$$

Finite-volume spectrum = Energy eigenvalues



Current frontier: 3-body dynamics from LQCD

➡ 3-body Quantization Conditions¹

➡ RFT / FVU / NREFT

➡ many perturbatively interacting systems are studied²

$$0 = \det \left(L^3 \left(\tilde{F}/3 - \tilde{F}(\tilde{K}_2^{-1} + \tilde{F} + \tilde{G})^{-1} \tilde{F} \right)^{-1} + K_{\text{df},3} \right) \quad \text{RFT}$$

$$0 = \det \left(B_0 + C_0 - E_L \left(K^{-1}/(32\pi) + \Sigma_L \right) \right) \quad \text{FVU}$$

 3-body force

 one-particle exchange

 2-body interaction

 2-body self-energy

1) Rusetsky, Bedaque, Grißhammer, Sharpe, Meißner, Döring, Hansen, Davoudi, Guo....

Reviews:

Hansen/Sharpe *Ann.Rev.Nucl.Part.Sci.* 69 (2019);

MM/Döring/Rusetsky *Eur.Phys.J.ST* 230 (2021);

2) MM/Döring *PRL* 122(2019); Blanton et al. *PRL* 124 (2020); Hansen et al. *PRL* 126 (2021);

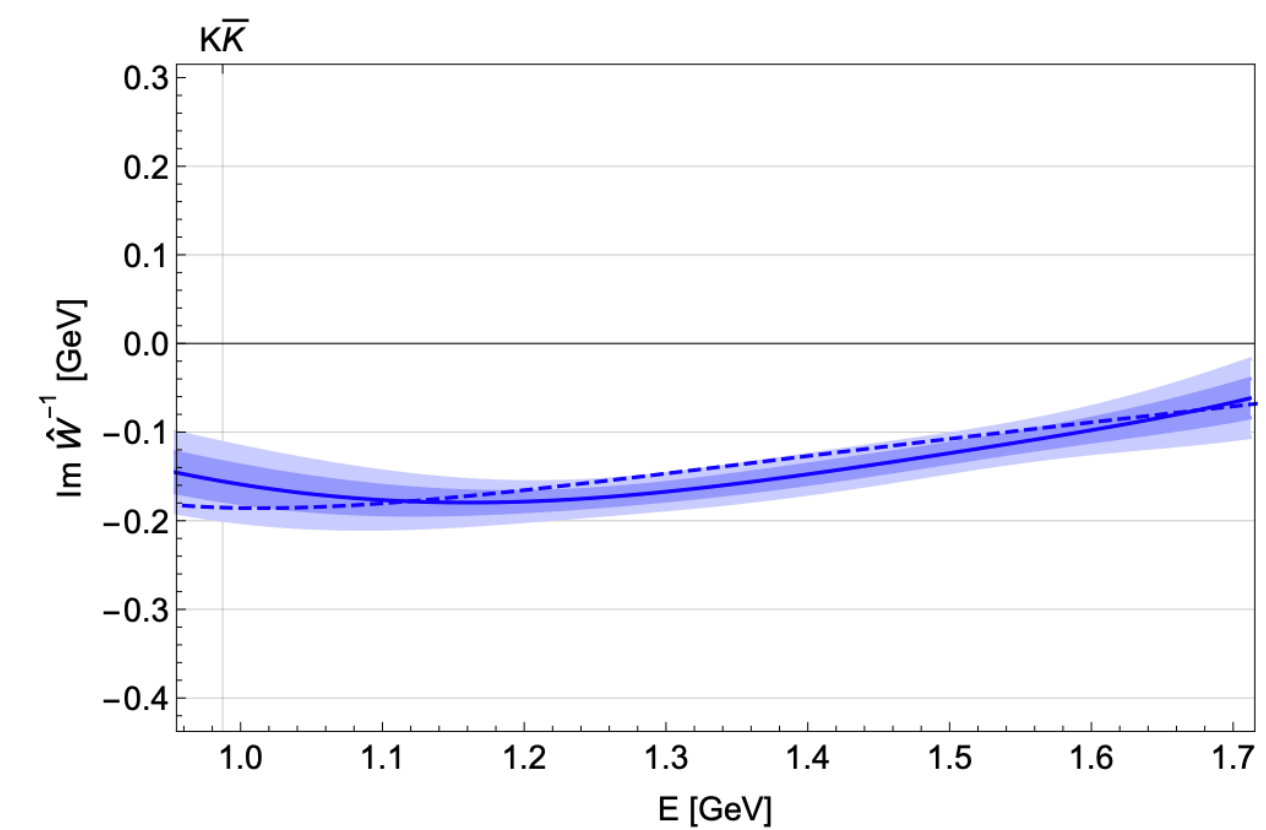
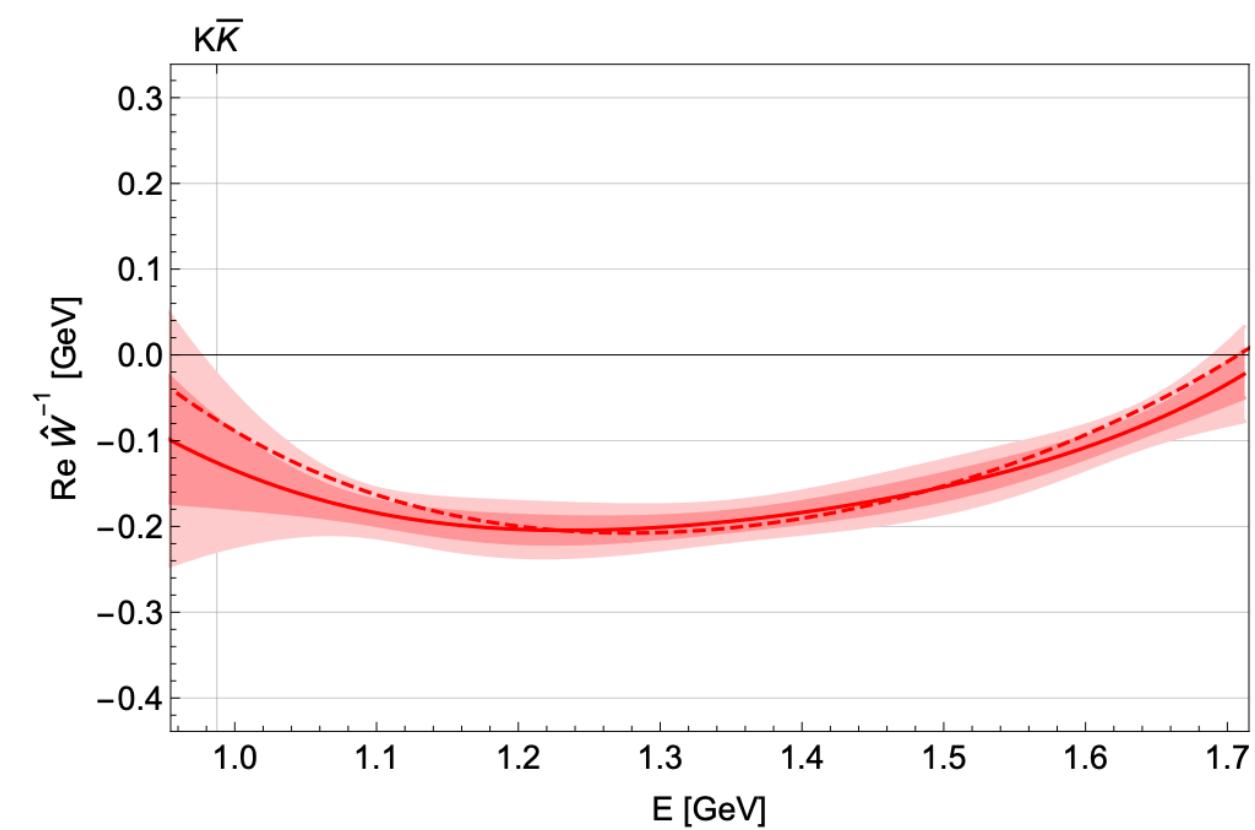
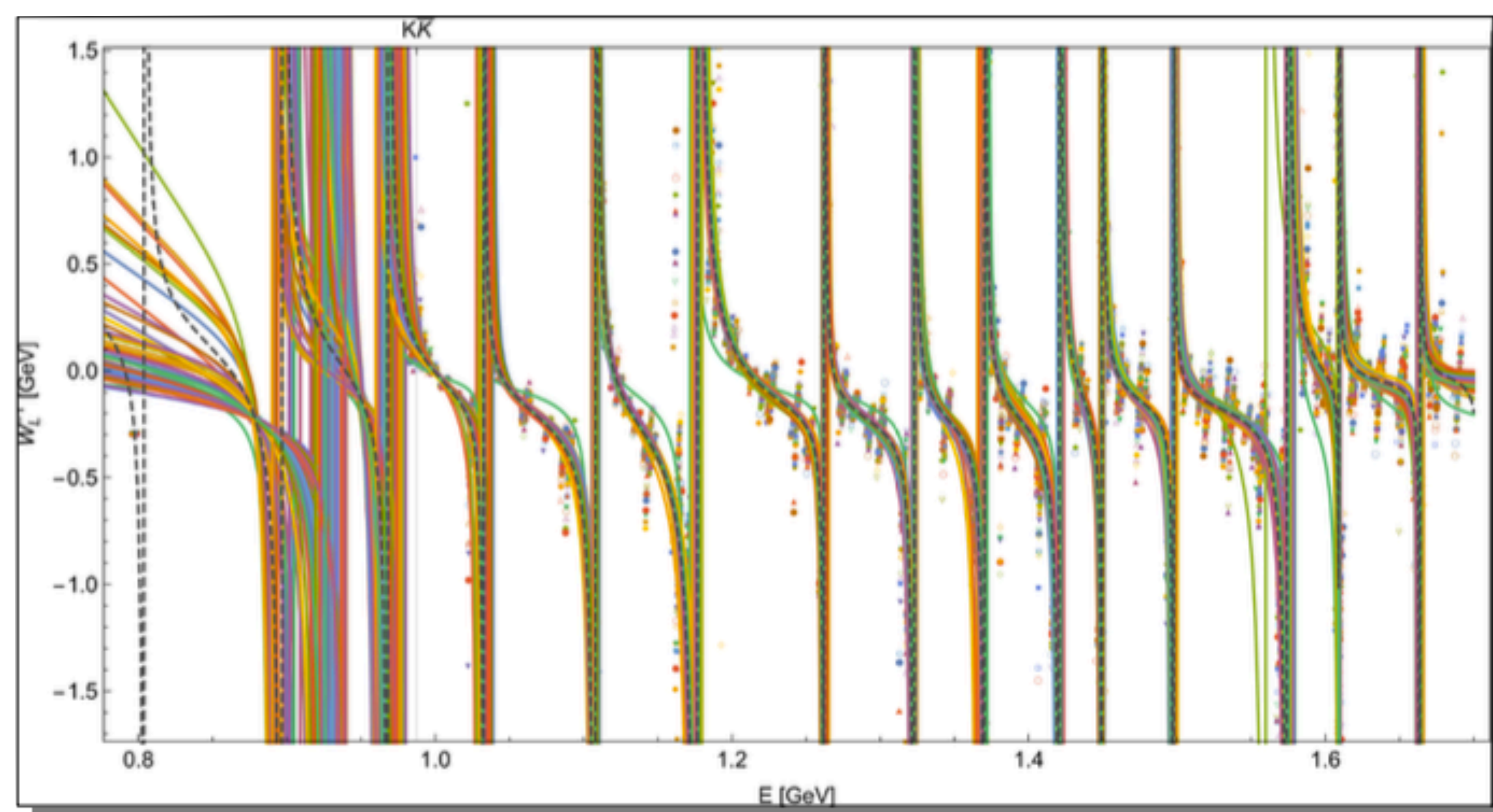
OPTICAL POTENTIAL

Alternatively: bulk properties of int. spectra

- double-limit prescription¹

$$W^{-1}(E) = \lim_{\epsilon \rightarrow 0} \lim_{L \rightarrow \infty} W_L^{-1}(E + i\epsilon)$$

- smoothing and inverse problem²
- typically many input (EEVs) required



Agadjanov/MM/.. JHEP 06 (2016)

¹) DeWitt, *Phys. Rev.* 103 (1956)

²) Agadjanov/MM/.. JHEP 06 (2016); Hansen et al. *Phys.Rev.D* 96 (2017) Bulava/Hansen *Phys.Rev.D* 100 (2019); Briceño et al. *Phys.Rev.D* 103 (2021)