The exchange term of self-consistent relativistic mean field calculations

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Dirac mean fields

The Dirac equation for a nucleon of momentum k in nuclear matter is

$$(\gamma^\mu k_\mu - M - \Sigma(k))\, u(k,s) = 0$$

The Dirac mean field can be decomposed as

$$\Sigma_t(k) = \Sigma_t^s(k) I - \Sigma_t^0(k)\gamma^0 + \Sigma_t^v(k)\vec{\gamma}\cdot\vec{k} \qquad t = n,\, p$$

In terms of these, we can define the effective mass and momentum,

$$M^*(k) = M + \Sigma^s(k), \quad \vec{k}^* = \vec{k}\,(1 + \Sigma^v(k))$$

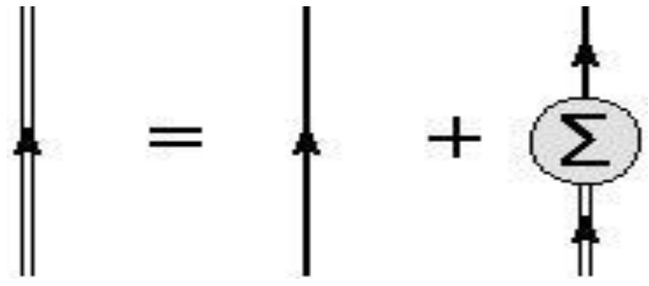
and the single-particle energy of the state, *u(k,s)*,

$$E^*(k) = E + \Sigma^0(k) = \sqrt{k^{*2} + M^{*2}}$$

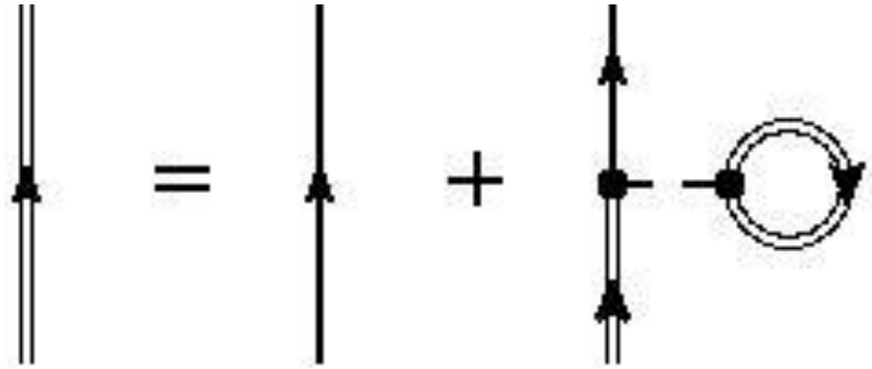
Approximations to the mean field

Dyson equation for the single-particle propagator

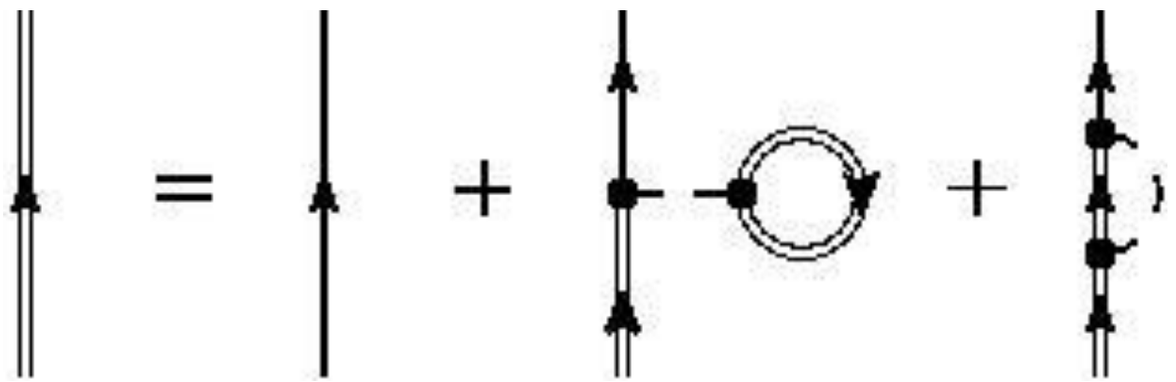
$$G_t(k) = G_0(k) + G_0(k)\Sigma_t(k)G_t(k)$$



Hartree – includes direct interaction term with all nucleons



Hartree-Fock – includes direct interaction and nucleon exchange terms



Meson exchange - Hartree vs. Hartree-Fock

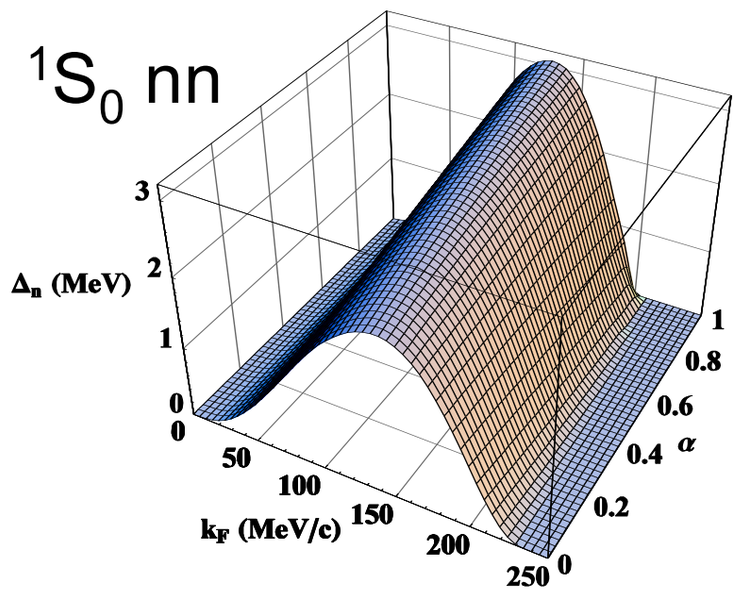
Low-mass mesons included in meson-exchange NN interactions, such as the Bonn interactions.

Only the first four enter the Hartree mean field – no pions.

meson	J^π, I	mass (MeV)
σ	0^+, 0	550.0
δ	0^+, 1	983.0
ω	1^-, 0	782.6
ρ	1^-, 1	769.0
η	0^-, 0	548.4
π	0^-, 1	138.03

Without the long-range of pion exchange, the Hartree approximation uses non-linear meson interactions or density-dependent coupling coefficients to model the nuclear surface.

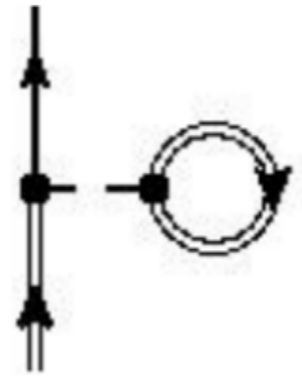
Clustering effects, such as pairing, dominate at low density – at the nuclear surface.
These are of special importance in lightly bound nuclei.



Hartree, Fock, and the Fierz identities

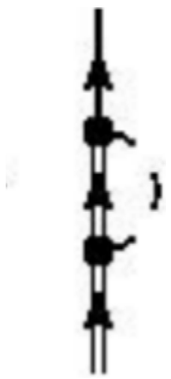
The Hartree term:

$$H = \sum_{j,\alpha\beta} \Gamma_{j\alpha}(0) D_j^{\alpha\beta}(0) \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[\Gamma_{j\beta}(0) G(q)\right]$$



The Fock term:

$$F = -\sum_{j,\alpha\beta} \int \frac{d^3q}{(2\pi)^3} \Gamma_{j\alpha}(k-q) D_j^{\alpha\beta}(k-q)\, G(q)\, \Gamma_{j\beta}(q-k)$$



Using the Fierz identities, the contribution of each Fock term to the mean fields can be identified.

$$\begin{aligned}
H_\sigma &\rightarrow H_\sigma + (-F_\sigma - 3F_\delta + 4F_\omega + 12F_\rho + 3F_\pi)/8 \\
H_\delta &\rightarrow H_\delta + (-F_\sigma + F_\delta + 4F_\omega - 4F_\rho - F_\pi)/8 \\
H_\omega &\rightarrow H_\omega + (F_\sigma + 3F_\delta + 2F_\omega + 6F_\rho - 3F_\pi)/8 \\
H_\rho &\rightarrow H_\rho + (F_\sigma - F_\delta + 2F_\omega - 2F_\rho + F_\pi)/8
\end{aligned}$$

Hartree-Fock and Brueckner

The Hartree-Fock approximation can describe the nuclear surface in a more fundamental manner than the Hartree approximation.

However, neither the Hartree nor the Hartree-Fock approximation provides a good description of the interaction of two nucleons -- they do not take into account properly the hard core of the nucleon-nucleon interaction.

Density-dependent Hartree models try to correct for this by adjusting to Brueckner mean fields, which take the hard core into account.

Our objective: to adjust Brueckner mean fields with a density-dependent Hartree-Fock interaction - one that includes the pion explicitly.

Hasn’t this been done before? Of course it has!

- Properties of nuclear and neutron matter in a relativistic Hartree-Fock theory, C. J. Horowitz and B. D. Serot, Nucl. Phys. A 399, 529 (1983).

- Relativistic description of nuclear systems in the Hartree-Fock approximation, A. Bouyssy, J.-F. Mathiot, Nguyen Van Giai, S.Marcos, Phys. Rev. C36, 380 (1987).

- Nuclear matter problem in the relativistic Green's function approach, P. Poschenrieder and M. K. Weigel, Phys. Rev. C 38, 471 (1988).

- N N correlations and relativistic Hartree-Fock in finite nuclei, R. Fritz and H. Müther, Phys. Rev. C 49, 633 (1994).

- From nuclear matter to finite nuclei. I. Parametrization of the Dirac-Brueckner G matrix, Hielke Freerk Boersma and Rudi Malfliet, Phys. Rev. C 49, 233 (1994).

- A density-dependent effective interaction for relativistic Hartree-Fock calculations, S. Marcos, J.-F. Mathiot, M. López-Quelle, R. Niembro, P. Bernardos, Nucl. Phys. A 600, 529(1996).

- Relativistic Hartree-Fock-Bogoliubov theory with Density Dependent Meson-Nucleon Couplings, WenHui Long, Peter Ring, Nguyen Van Giai, and Jie Meng, Phys. Rev. C 81, 024308 (2010).

- Relativistic Hartree-Fock-Bogoliubov model for deformed nuclei, J.-P. Ebran, E. Khan, D. Peña Arteaga, and D. Vretenar, Phys. Rev. C 83, 064323 (2011).

- Localized form of Fock terms in nuclear covariant density functional theory, Haozhao Liang, Pengwei Zhao, Peter Ring, Xavier Roca-Maza, Jia Meng, Phys. Rev. C 86, 021302 (2012).

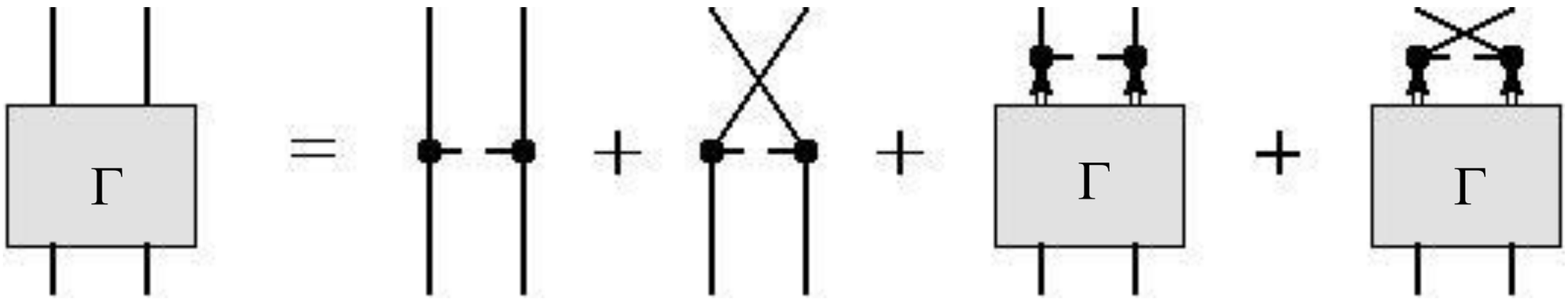
- Pairing phase transition: A Finite-Temperature Relativistic Hartree-Fock-Bogoliubov study, Jia Jie Li, Jérôme Margueron, Wen Hui Long, and Nguyen Van Giai, Phys. Rev. C 92, 014302 (2015).

- Tensor force effect on the evolution of single-particle energies in some isotopic chains in the relativistic Hartree–Fock approximation , M. López-Quelle, S. Marcos, R. Niembro , L.N. Savushkin , Nucl. Phys. A 971, 149 (2018).

- Hypernuclear stars from relativistic Hartree-Fock density functional theory, Jia Jie Li, Wen Hui Long, and Armen Sedrakian, Eur. Phys. J A 54, 113 (2018).

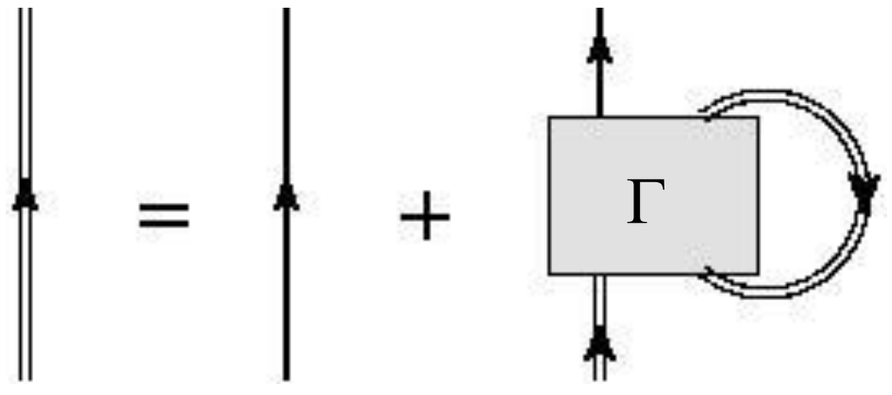
The Breuckner G matrix and mean field

The Brueckner G-matrix Γ satisfies a Bethe-Salpeter equation.



In free space, the G-matrix reduces to the two-nucleon scattering T-matrix.

Dyson equation for the single-particle propagator



$$G_t(k) = G_0(k) + G_0(k)\Sigma_t(k)G_t(k)$$

Brueckner Calculations

We show calculations as a function of the density and the asymmetry α,

$$\alpha = \frac{\rho_n - \rho_p}{\rho}$$

α = 0 nuclear matter

α = 1 neutron matter

We represent the density by an effective Fermi momentum, given by

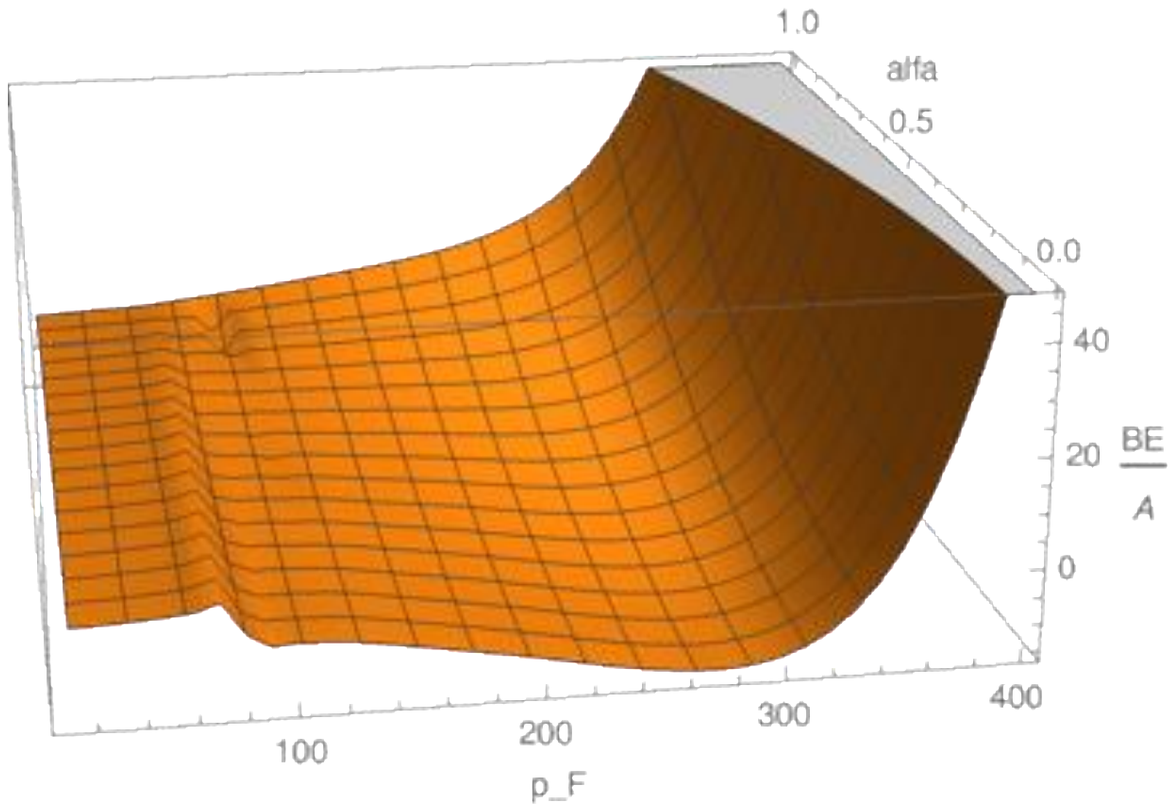
$$\frac{2}{3\pi^2} k_F^3 = \rho_p + \rho_n = \rho$$

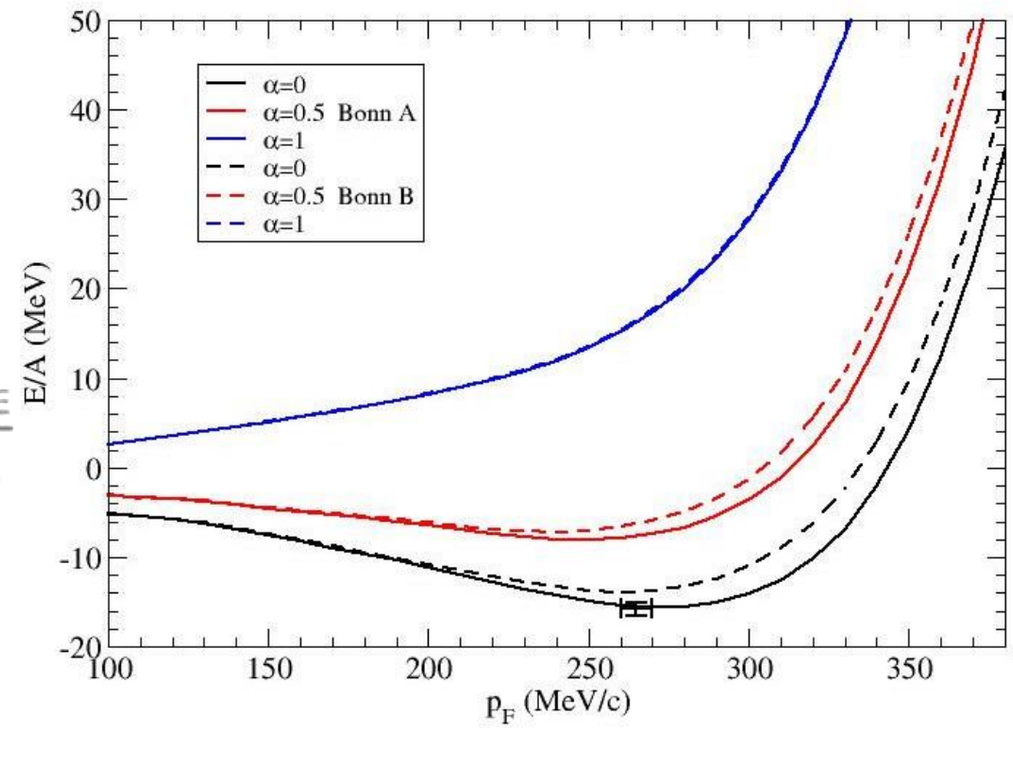
We use the Bonn A and B interactions *(B. Brockmann and R. Machleidt, Phys. Rev, C42 (1990) 1965.)* with pseudovector coupling for the η and π.

	$p_{F,sat}$ [MeV/c]	$E_{b,sat}$ [MeV]	K [MeV]	a_{sym} [MeV]
Bonn A	272.7	-15.60	218.7	32.3
Bonn B	262.0	-13.85	240.6	30.8

meson	J^π, I	mass (MeV)
σ	0^+, 0	550.0
δ	0^+, 1	983.0
ω	1^-, 0	782.6
ρ	1^-, 1	769.0
η	0^-, 0	548.4
π	0^-, 1	138.03

Brueckner calculations – Bonn A



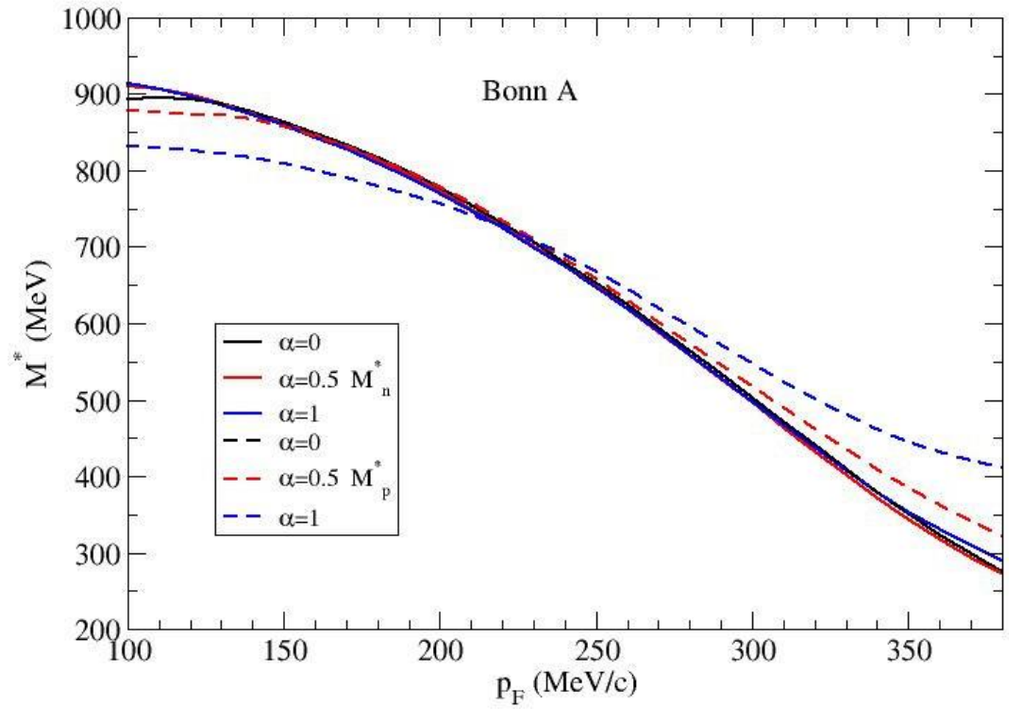


Blip in E/A below P_F = 100 MeV/c

M* - effective mass at the Fermi momentum

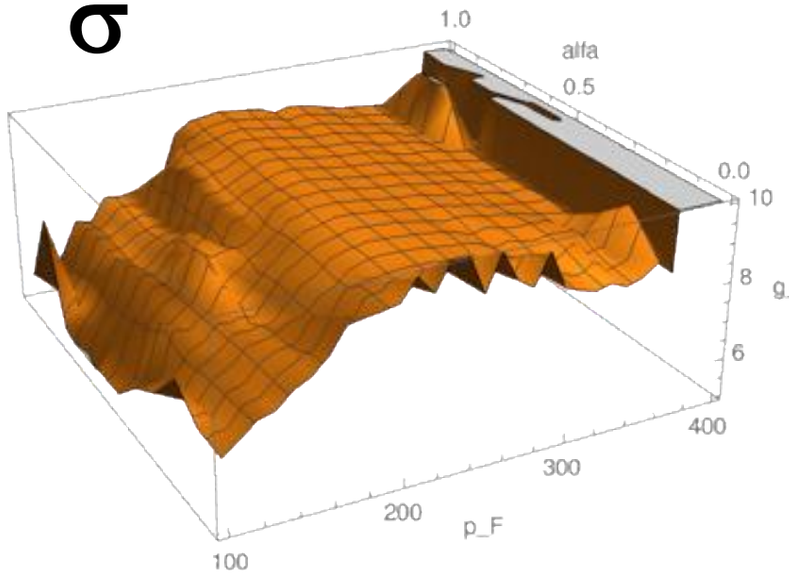
Neutrons – almost the same M* in nuclear and neutron matter

Protons – differs form neutron M* in asymmetric matter

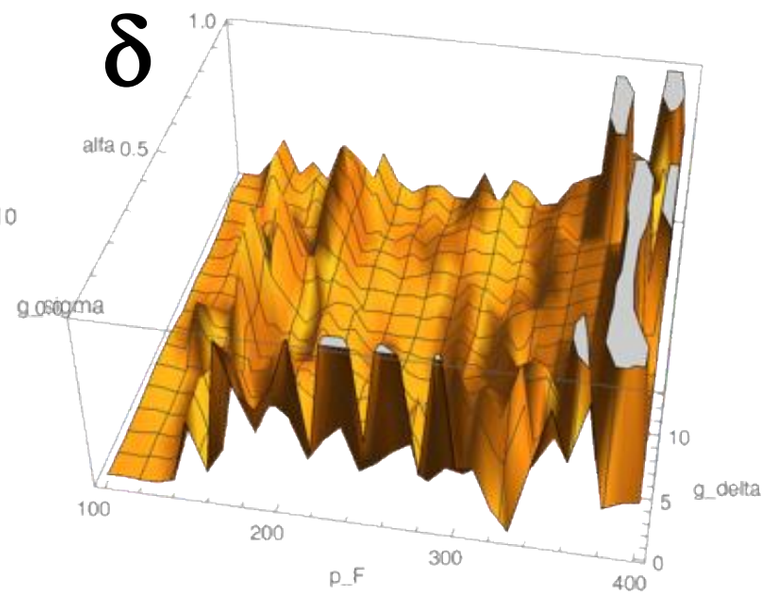


Hartree-Fock fit with 6 mesons – σ, δ, ω, ρ, η, π

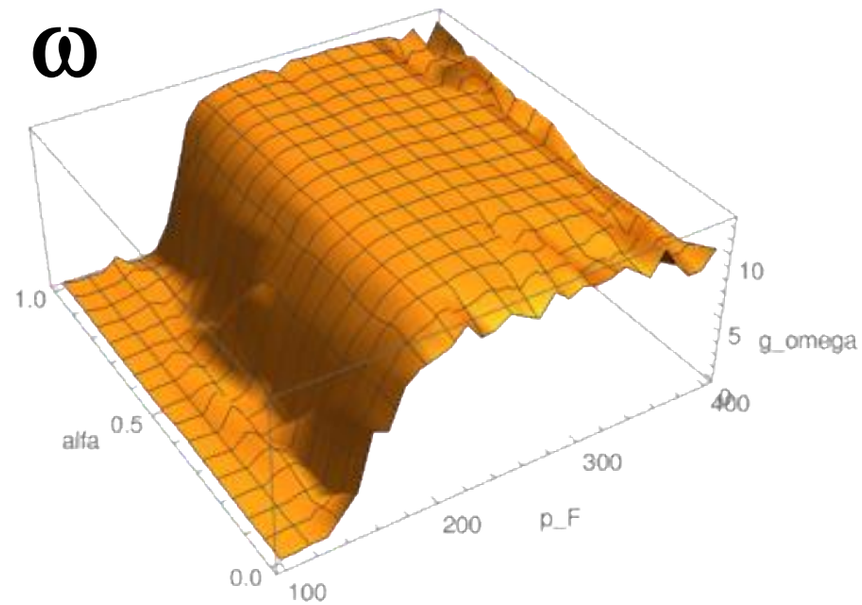
σ



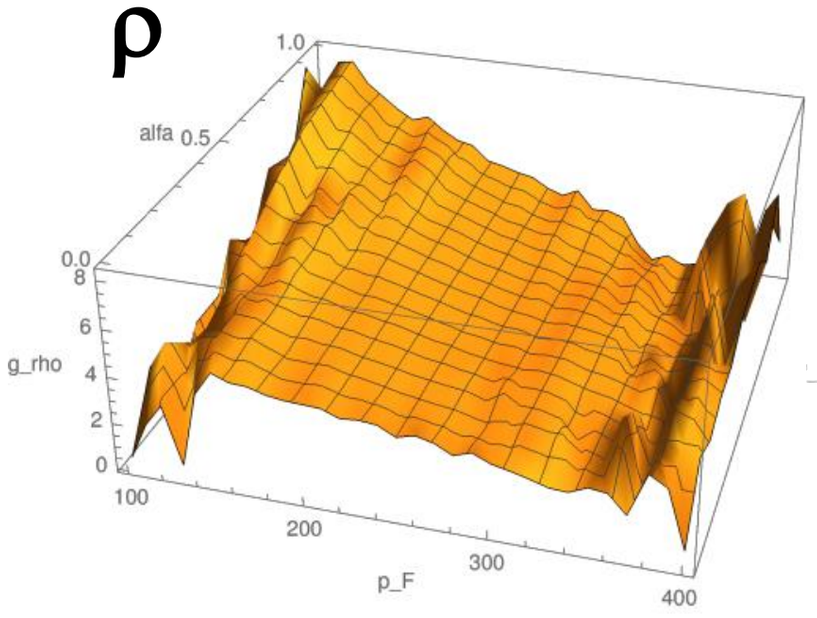
δ



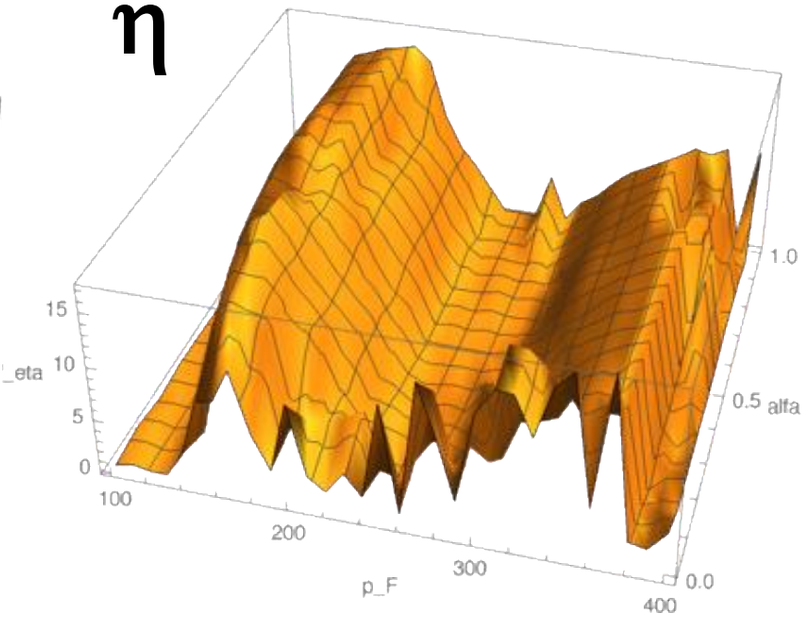
ω



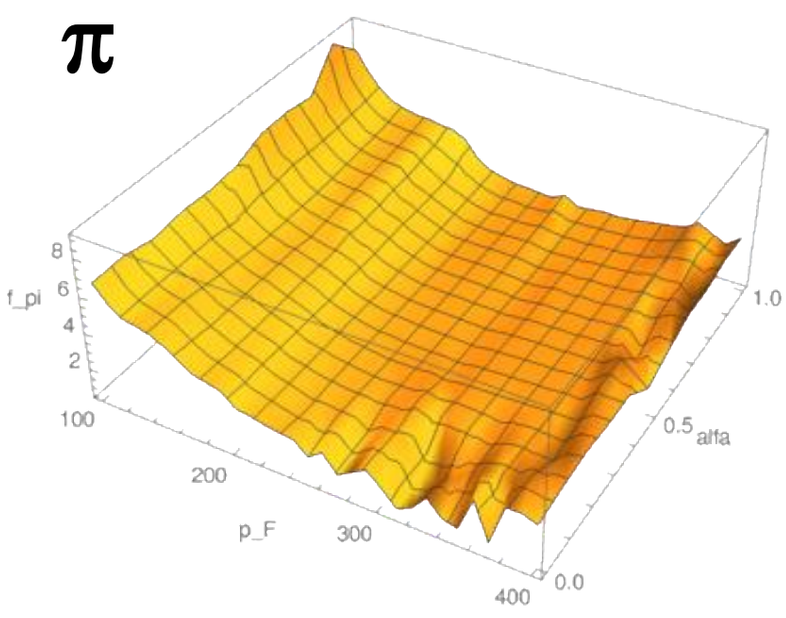
ρ



η



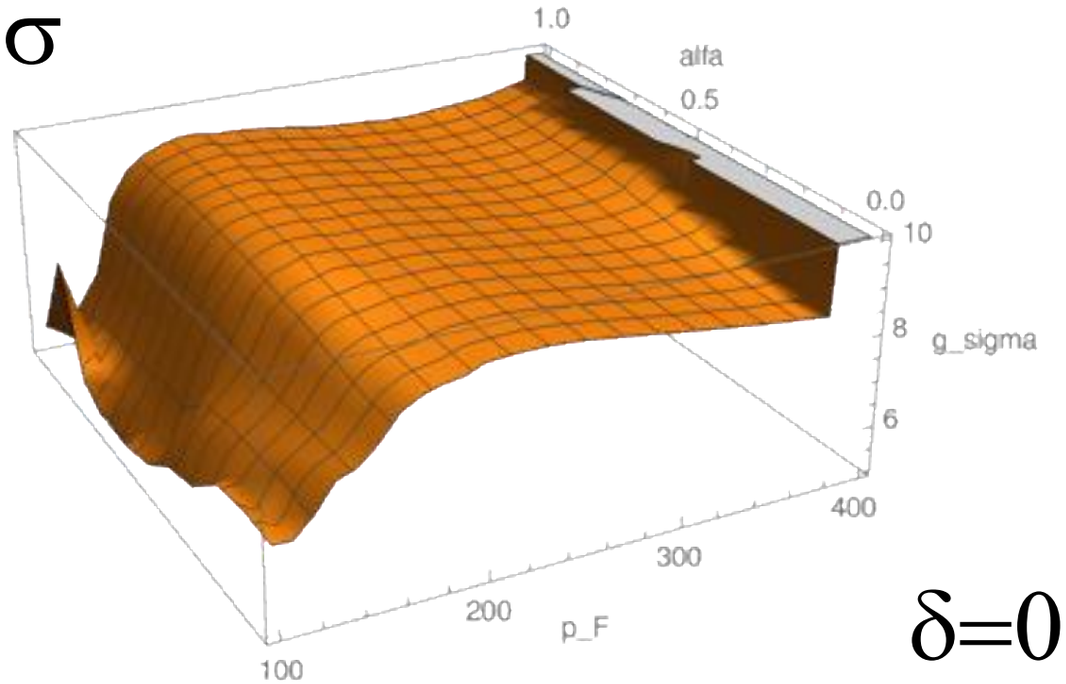
π



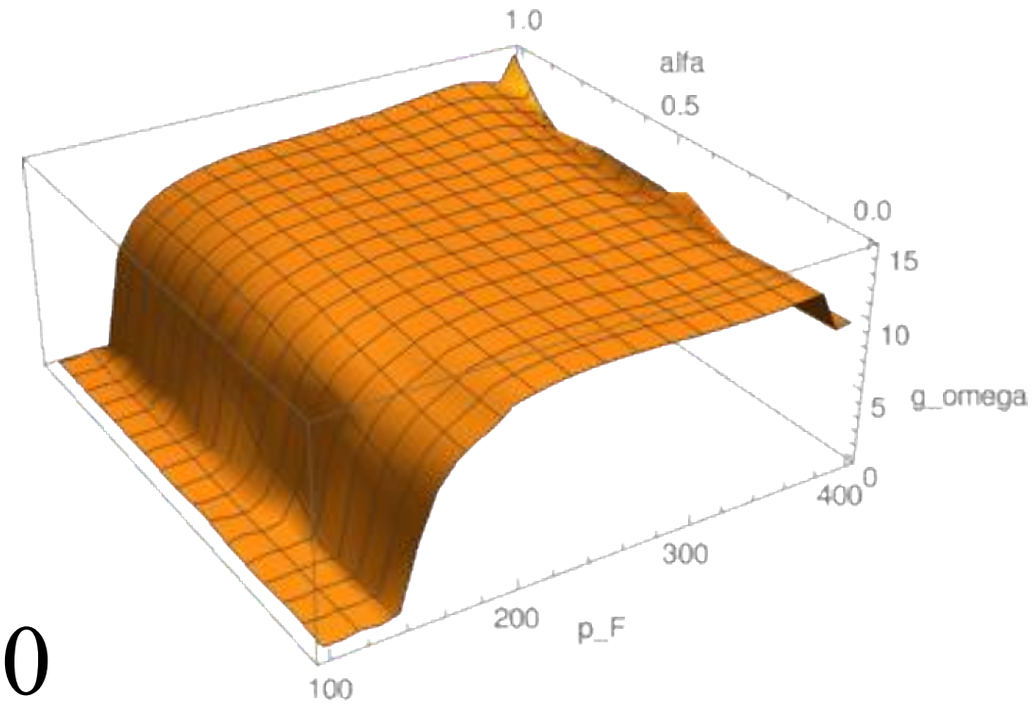
"Overfitting"

Hartree-Fock fit with 4 mesons – σ, ω, ρ, π

σ

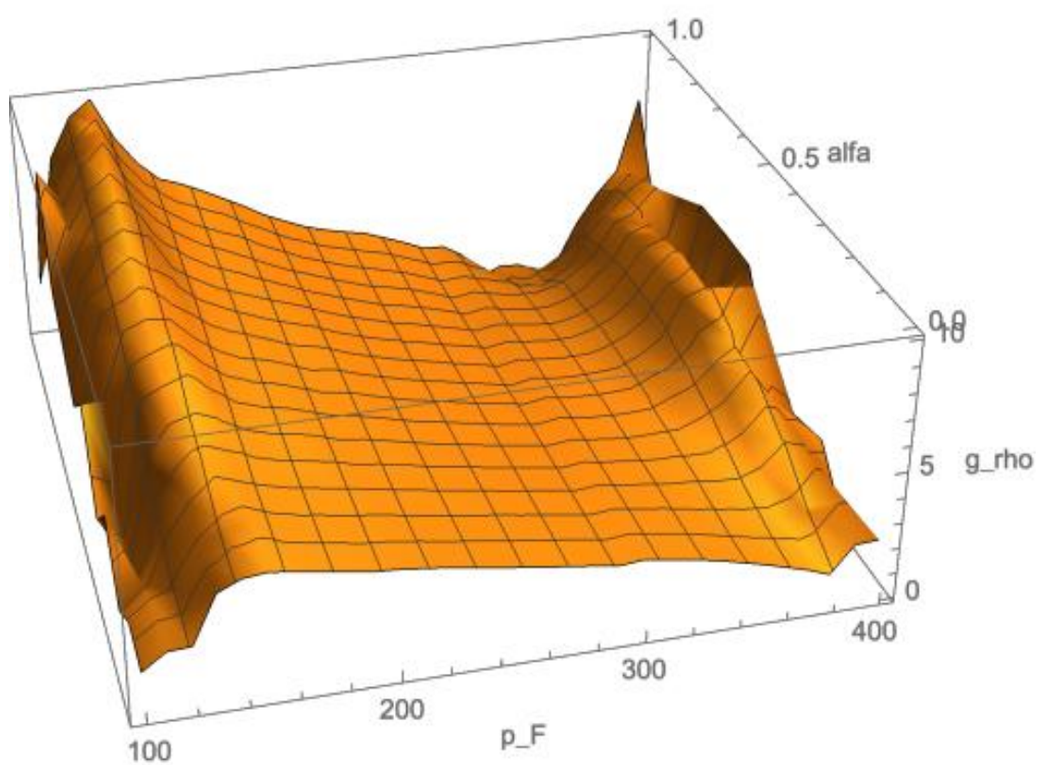


ω

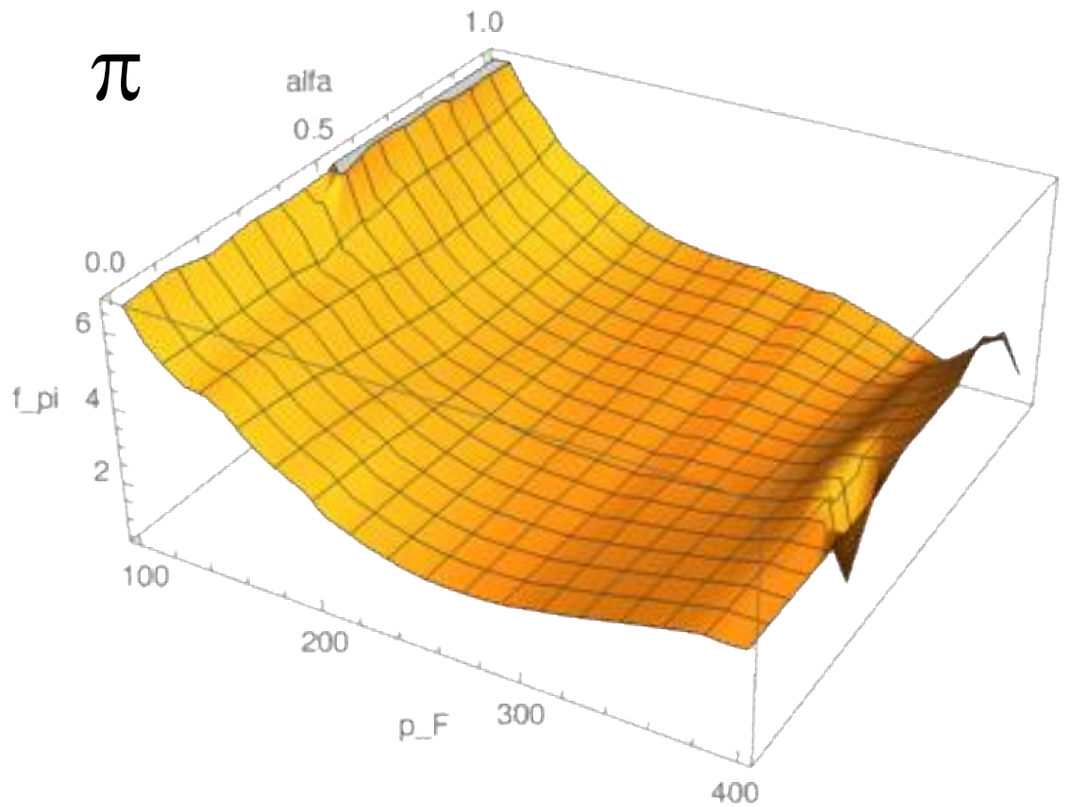


$\delta=0 \quad \eta=0$

ρ



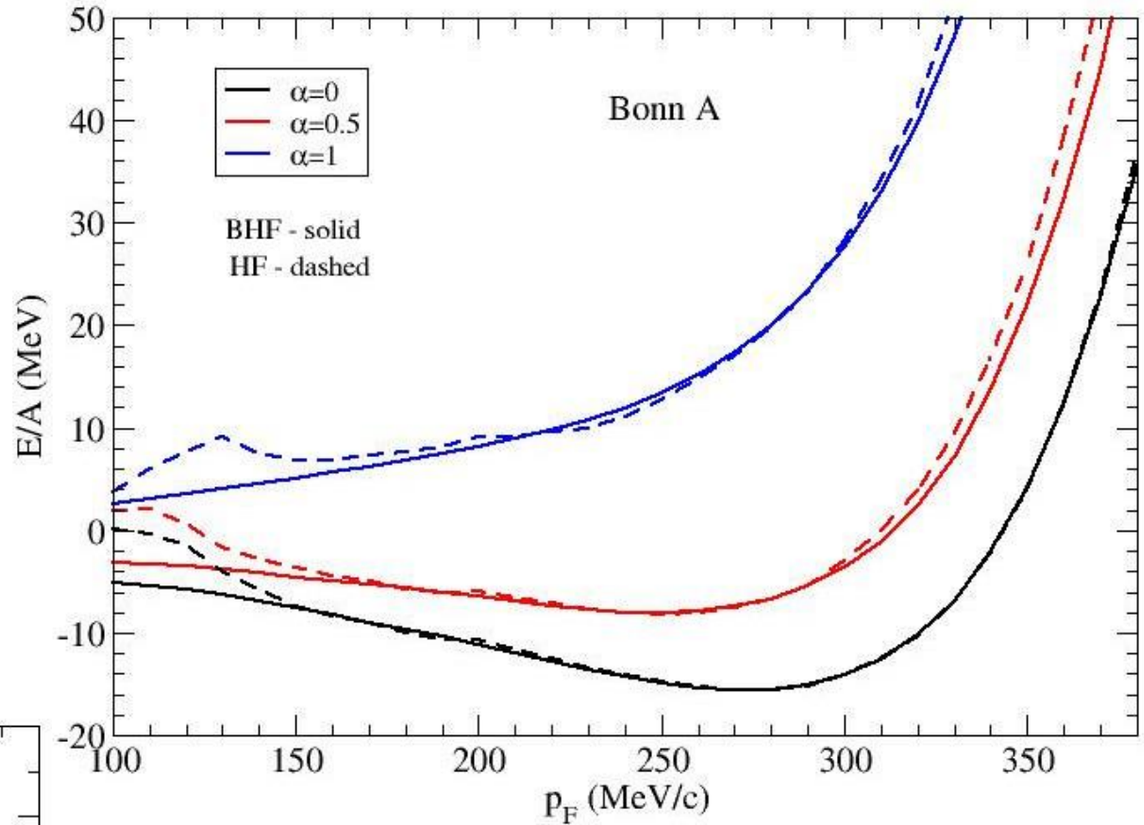
π

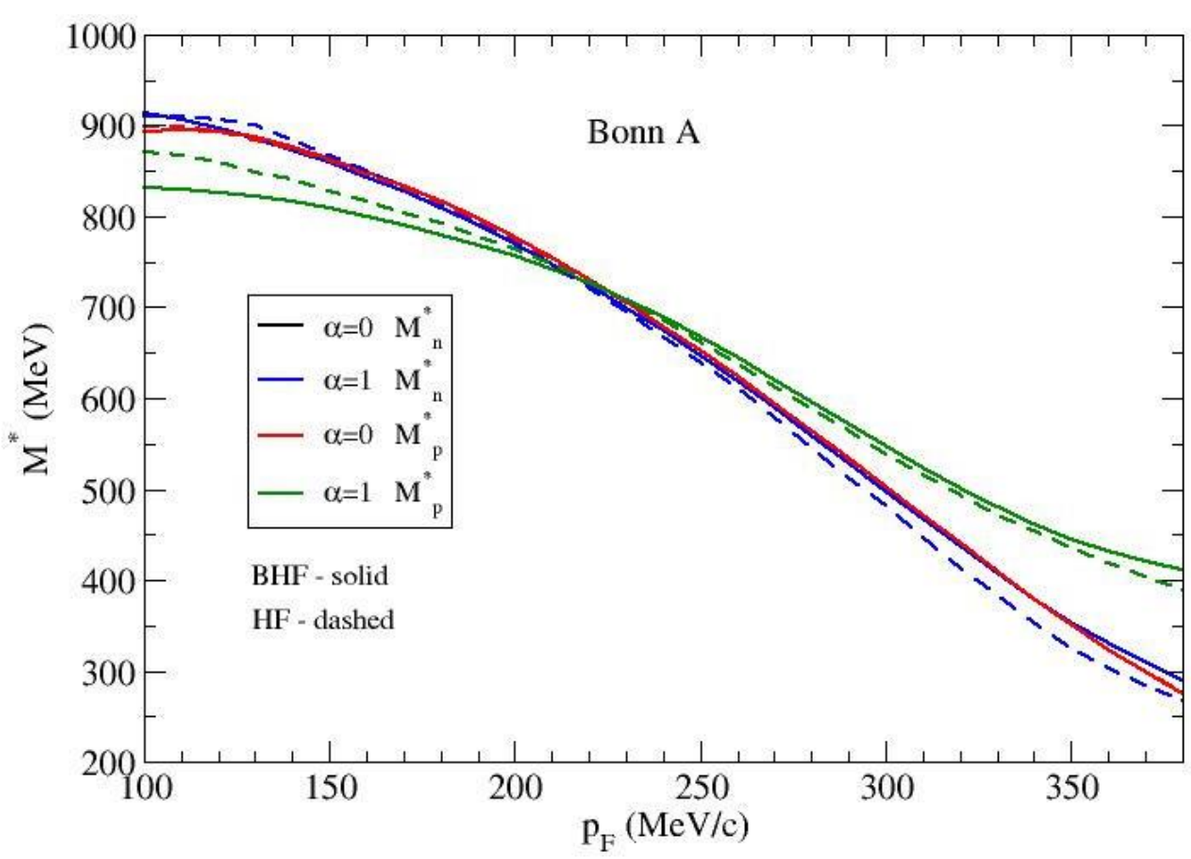


A Hartree-Fock fit to the Brueckner mean fields

The average trend of the binding energy is well described at Fermi momenta above about 150 MeV/c.

At lower Fermi momenta, the HF parametrization is underbound.





The HF fit describes the effective mass at the Fermi momentum quite well.

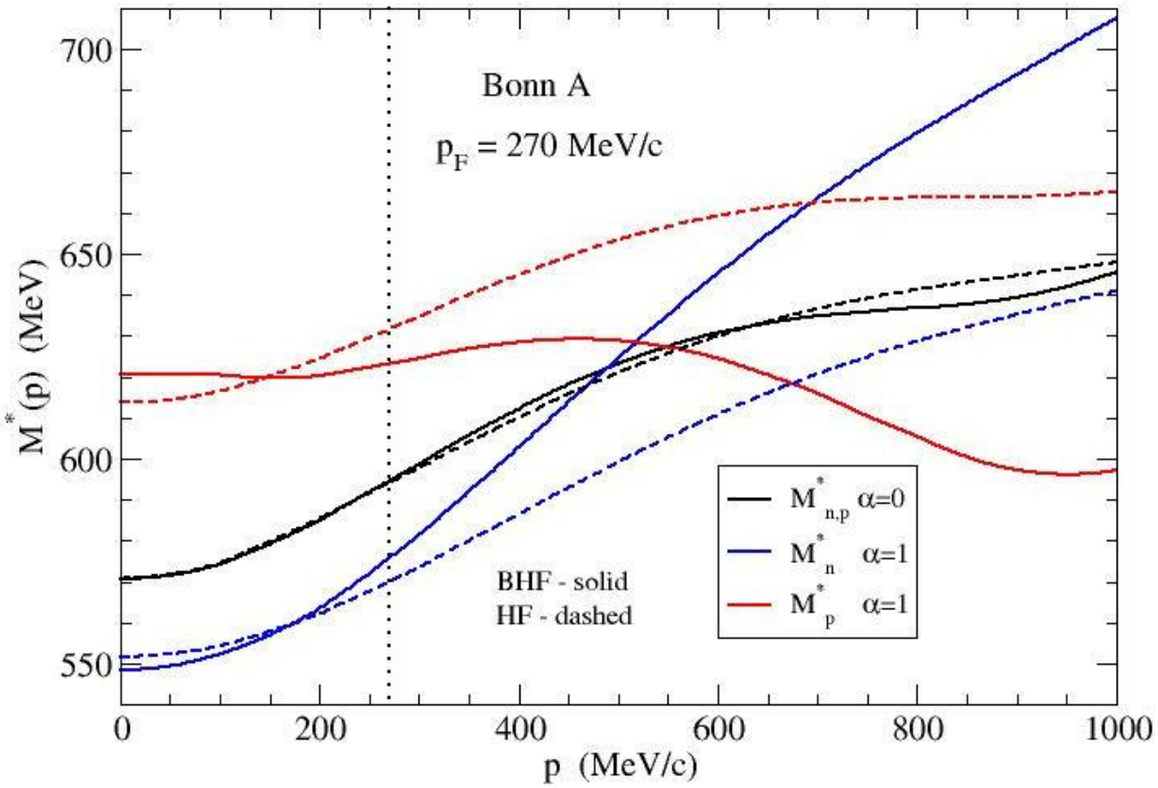
Momentum dependence of the mean fields

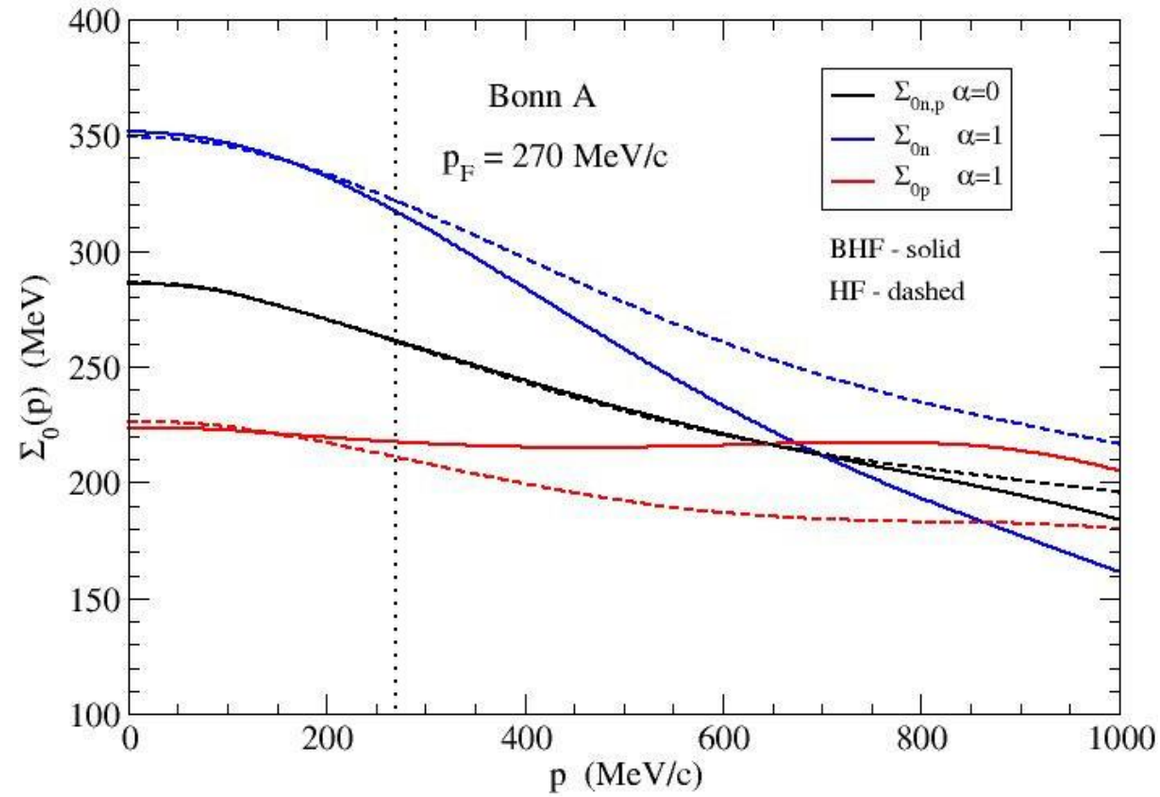
HF fields should be compared to BHF ones below the Fermi momentum.

Neutron fields - Momentum dependence becomes stronger as asymmetry grows.

Proton fields – momentum dependence becomes weaker.

Divergence between BHF and HF fields larger for asymmetric matter.





Summary

HF fits to BHF mean fields and energy show that:

- π, σ, ω and ρ exchange are sufficient to describe mean fields of nuclear matter at Fermi momenta between 150 and 380 MeV/c;
- HF fits fail to describe the BHF fields and energy at Fermi momenta above about 380 MeV/c.
- π and σ exchange alone seem to be sufficient to describe the BHF mean fields at Fermi momentum below about 150 MeV/c, but do not reproduce the binding energy;

The clustering / pairing at low nuclear density that we hope to describe in the HF approximation is also what complicates the Brueckner approach.

A simplified HF description of the nuclear surface, in terms of π, σ and, perhaps, ρ exchange, still might provide a way forward.

Extra stuff

A few things we've left out:

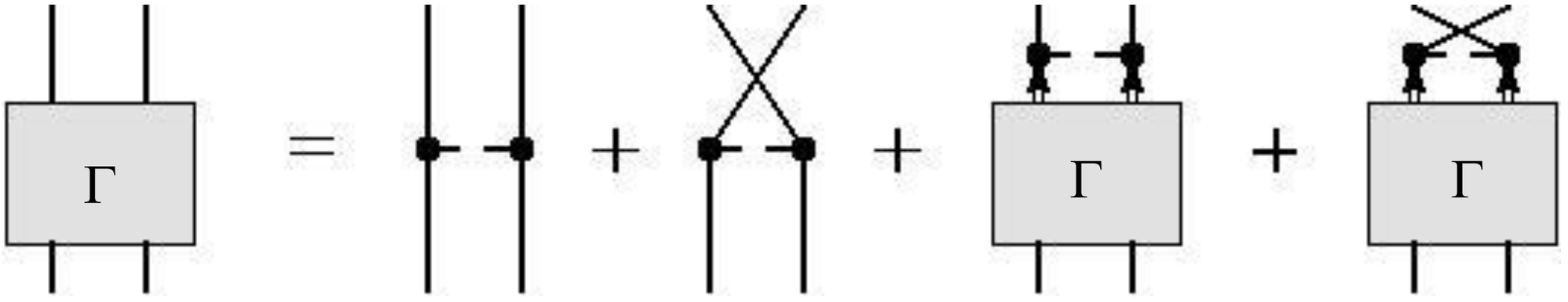
1) Retardation terms → covariance of the interaction;
2) Negative energy states and full Dirac structure;
3) Pairing → a density dependent gap;
4) Quasiparticle approximation → Re[Γ] → Γ, Re[Σ] → Σ;
→ spectral functions
A(k,ω);
5) σ meson dynamics → correlated π's and π – Δ dynamics;
6) Δ's;
7) Nucleon structure;
8) RPA/particle-hole and higher-order correlations;

The Bethe-Salpeter equation

The Brueckner G-matrix Γ satisfies a Bethe-Salpeter equation.

We calculate it in the ladder approximation, including the anti-symmetrized bare nucleon-nucleon interaction to all orders in the nn,np,pn and pp channels.

In free space, the G-matrix reduces to the two-nucleon scattering T-matrix.



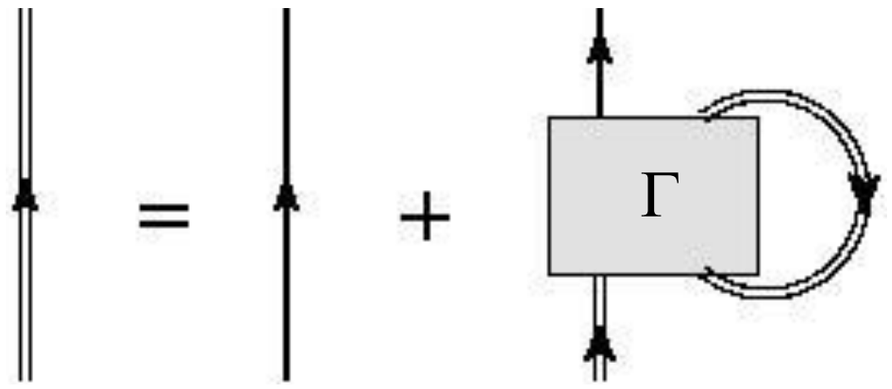
$$\Gamma(q',q|P) = V(q',q) + \frac{i}{2}\int \frac{d^4q''}{(2\pi)^4} V(q',q'') \times G_1\left(\frac{P}{2}+q''\right) G_2\left(\frac{P}{2}-q''\right) \Gamma(q'',q|P)$$

nn or pp

The Brueckner mean field

$$G_t(k) = G_0(k) + G_0(k)\Sigma_t(k)G_t(k)$$

Dyson equation for the single-particle propagator



$$\Sigma_t(k) = -i \int \frac{d^4p}{(2\pi)^4} \sum_{t'=n,p} Tr\left[\Gamma_{tt'}(k,p;k,p)G_{t'}(p)\right]$$

Quasi-particle approximation $\Gamma \rightarrow \mathrm{Re}[\Gamma]$, $\Sigma \rightarrow \mathrm{Re}[\Sigma]$:

$$\Sigma_t^s(k) = \sum_{t'=n,p} 2 \int \frac{d^3p}{(2\pi)^3} \Gamma^S_{tt'}\left(k-p, k-p|k+p\right) \frac{M^*_{t'}(p)}{E^*_{t'}(p)} \theta\left(k_{Ft'} - |\vec{p}|\right)$$

$$\Sigma_t^0(k) = - \sum_{t'=n,p} 2 \int \frac{d^3p}{(2\pi)^3} \Gamma^V_{tt'}\left(k-p, k-p|k+p\right) \theta\left(k_{Ft'} - |\vec{p}|\right) \qquad t = n,p$$

$$\Sigma_t^v(k) = - \sum_{t'=n,p} \frac{2}{k^2} \int \frac{d^3p}{(2\pi)^3} \Gamma^V_{tt'}\left(k-p, k-p|k+p\right) \frac{\vec{k}\cdot\vec{p}^*}{E^*_{t'}(p)} \theta\left(k_{Ft'} - |\vec{p}|\right)$$

The reduced Bethe-Salpeter equation

We simplify the Bethe-Salpeter equation further by
1) projecting onto a set of positive-energy single-particle states
2) and reducing the integral to a three-dimensional one.

$$\gamma\left(q',q|P\right) = v\left(q',q\right) + \frac{1}{2}\int \frac{d^3q''}{(2\pi)^3} v\left(q',q''\right) \times Q\left(q''|P\right) g\left(q''|P\right) \gamma\left(q'',q|P\right)$$

nn or pp

where γ and v are the reduced G-matrix and antisymmetrized bare interaction, Q is a Pauli blocking factor and g is a reduced two-particle propagator.

This equation is solved by reducing it to coupled equations for the principal parts of 5 (or 6) independent helicity amplitudes, expanding these in partial waves and solving them as matrix equations.

(K. Erkelenz, Phys. Reports, C13 (1974) 191.)

Reconstructing the G-matrix

To calculate the mean field, we need the complete G-matrix, not just its projection. To reconstruct it, we solve for the component functions of a covariant expansion, projected on positive-energy states with distinct masses,

$$\begin{aligned}\Gamma\left(q',q|P\right) &= \Gamma^{S}\left(q',q|P\right) I^{(1)} I^{(2)} + \Gamma^{V}\left(q',q|P\right) \gamma_{\mu}^{(1)} \gamma^{(2)\mu} \\ &+ \Gamma^{T}\left(q',q|P\right) \sigma_{\mu\nu}^{(1)} \sigma^{(2)\mu\nu} + \Gamma^{P}\left(q',q|P\right) \gamma_{5}^{(1)} \gamma_{5}^{(2)} \\ &+ \Gamma^{A}\left(q',q|P\right) \gamma_{5}^{(1)} \gamma_{\mu}^{(1)} \gamma_{5}^{(2)} \gamma^{(2)\mu}\end{aligned}$$

(C.J. Horowitz and B. Serot, Nucl. Phys. A464 (1987) 613.)

Due to the ambiguity between the pseudoscalar and pseudovector pion- and eta-nucleon coupling, we reconstruct the difference between the G-matrix and the bare interaction *V* and use the unprojected expression for the latter.

$$\Gamma\left(q',q|P\right) = V\left(q',q\right) + \Delta\Gamma\left(q',q|P\right)$$

(E. Schiller and H. Müther, Eur. Phys. J. A 11 (2001) 227.)