

Towards Quantum Gravity

as the theory of

General Quantum Relativity

or Q-number Geometrodynamics

— talk at IAS Program on Fundamental Physics, Jan 2026

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Prologue : Quantum Gravity

(Geometro-)

Dynamics of Quantum Spacetime

rather than

Quantum Dynamics of (Classical)

Spacetime

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. \Leftrightarrow Quantum Gravity

Non-Euclidean G. \Leftrightarrow Classical Gravity

The Symplectic Geometry — NC Vs C :-

- Heisenberg — $\frac{d}{ds} \alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar} [\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- Schrödinger — $\frac{d}{ds} f_\alpha(z_n, \bar{z}_n) = \{f_\alpha(z_n, \bar{z}_n), f_{H_s(z_n, \bar{z}_n)}\}$

- $f_\alpha(z_n, \bar{z}_n) \equiv \frac{g\langle \phi | \alpha(\hat{P}_\mu, \hat{X}_\mu) | \phi \rangle}{g\langle \phi | \phi \rangle} \quad \left(| \phi \rangle = \sum_n z_n | n \rangle \right)$

— as the pull-back of $\hat{\alpha}$ under $(z_n, \bar{z}_n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- \rightarrow bijective homomorphism between NC Poisson algebras

— NC Kähler product $f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$

Cirelli et.al 90

Heisenberg & Dirac 1925/26 :-

— classical to quantum

only needs a new kinematic

- **H** : physical quantities *not* real number variables
- Quine : real number as ‘convenient fiction’
- **D** : q-number as the new convenient fiction

Dirac 1925/26 :-

— Hamiltonian formulation ; q-number

- \hat{x}^i, \hat{p}_i as canonical variables; P.B. = $\frac{1}{i\hbar}[\cdot, \cdot]$

“To distinguish the *two kinds of numbers*, we shall call the quantum variables q-numbers and the numbers of classical mathematics which satisfy the commutative law c-numbers”

“Owing to the fact that we count the time as a c-number, we are allowed to use the notion of *the value of the dynamical variable* at any instance of time. This value is a q-number, capable of being represented by a generalized ‘matrix’, ...”

“At present one can form no picture of what a q-number is like.”

Concept of Numbers (in history) :

- $x + 2 = 0$ → negative numbers
- $2x - 1 = 0$ → rational numbers
- $x^2 - 2 = 0$ → real numbers
- $x^2 + 1 = 0$ → complex numbers
- $xy - x - i = 0$ → $(i, 2), (\frac{1}{i-1}, -i), \dots$
- $xy - yx - 1 = 0$ → noncommutative numbers

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

Noncommutative Number Systems :-

- **observables** are dynamical *variables*
— a state is an evaluative **homomorphism**
- **matrices as Dirac's q-numbers**
- **convention** : **representation** of observables as
— **DH-matrix value** for **a reference state**, *e.g.*

$$[\hat{s}^i]_\phi = U_\phi^\dagger \sigma^i U_\phi, \quad |\phi\rangle = U_\phi |0\rangle, \quad U_\phi = \begin{pmatrix} c & -\bar{s} \\ s & \bar{c} \end{pmatrix}$$

- need to fix a 'reference frame' for the states

Quantum Physics is simply
a q-number version of classical physics
about the q-number reality

Quantum geometry is then
q-number geometry
— *a true geometric picture of NCG*

Quantum Spacetime Geometry :-

— *NC (number) geometry*

● simple free particle motion $\hat{p}(t) = \hat{p}(0)$

$$\hat{x}(t) = \hat{p}(0) \frac{t}{m} + \hat{x}(0) \quad \Rightarrow \quad [\hat{x}(0), \hat{x}(t)] = \frac{i\hbar t}{m}$$

— NCNG : phase space \rightarrow configuration space

● ‘cotangent bundle’ (metric independent)

$$\text{—} \quad \{\hat{x}^a, \hat{p}_b\} = \delta_b^a, \quad \{\hat{x}^a, \hat{x}^b\} = 0 = \{\hat{p}_a, \hat{p}_b\}$$

$$\frac{\partial}{\partial \hat{x}^a} \equiv \{\cdot, \hat{p}_a\}, \quad \frac{\partial}{\partial \hat{p}_a} \equiv -\{\cdot, \hat{x}^a\}$$

Classical Geodesic from Hamiltonian: -

- $S_o = \int ds L_o(s) = \int ds \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}$

— manifold of any dimension and metric (\pm)

- **Lagrangian** $L(s) = -\frac{m}{2} L_o^2$ **from free particle motion**

- **Hamiltonian** : phase space as cotangent bundle

— **symplectic structure independent of $g_{\mu\nu}(x)$**

- Rindler frame: **particle at $\rho(\tau) = 0$, acceleration a**

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -\frac{a^2 \rho^2}{c^2} d\tau^2 + d\rho^2 + \dots$$

Goedesic from Free Particle Motion:-

- all positions coordinates Hermitian

- $\hat{H}_{\text{free}} = \frac{1}{2m} \hat{p}_A g^{Ab}(\hat{x}) \hat{p}_b$, $\hat{p}^b = \hat{p}_A g^{Ab}(\hat{x})$

— four vectors : $\hat{V}'^a = \hat{V}^i \frac{\partial \hat{x}'^a}{\partial \hat{x}^i}$, $\hat{W}'_a = \frac{\partial \hat{x}^i}{\partial \hat{x}'^a} \hat{W}_i$,

$$\hat{V}'^A \equiv \hat{V}'^{a\dagger} = \left(\frac{\partial \hat{x}'^a}{\partial \hat{x}^i} \right)^\dagger \hat{V}^{i\dagger} \equiv \left(\frac{\partial \hat{x}^A}{\partial \hat{x}'^I} \right) \hat{V}^I , \quad \hat{W}'_A = \hat{W}_I \left(\frac{\partial \hat{x}^I}{\partial \hat{x}'^A} \right) .$$

— Schrödinger representation fails

- \hat{x}^a and \hat{p}^a as \hat{g} -Hermitian within the ref. frame
- Hamilton's Eqs. \rightarrow mass-indep. E.O.M.

...

● **Quantum Geodesic Equation :** $\frac{d^2 \hat{x}^\mu}{ds^2} =$

$$\frac{d\hat{x}^\nu}{ds} \frac{\partial_\Lambda \hat{g}_{\nu\Omega}}{2} \frac{d\hat{x}^\Omega}{ds} \hat{g}^{\Lambda\mu} - \frac{d\hat{x}^\nu}{ds} \frac{\partial_\Omega \hat{g}_{\nu\Lambda}}{2} \hat{g}^{\Lambda\mu} \frac{d\hat{x}^\Omega}{ds} - \frac{d\hat{x}^\xi}{ds} \hat{g}_{\xi\Omega} \frac{d\hat{x}^\nu}{ds} \hat{g}^{\Omega\zeta} \frac{\partial_\zeta \hat{g}_{\nu\Lambda}}{2} \hat{g}^{\Lambda\mu}$$

- s a real number Hamiltonian evolution parameter
- proper time a quantum observable
- $\hat{p}^\mu = m \frac{d\hat{x}^\mu}{ds}$, $g(\hat{x}^\mu)$ -Hermitian (frame-dependent)

● $\hat{g}_{\mu\nu} = \frac{\partial \hat{x}^a}{\partial \hat{x}^\mu} \eta_{ab} \frac{\partial \hat{x}^b}{\partial \hat{x}^\nu}$ — all \hat{x} commute and Hermitian

$$\frac{\partial}{\partial \hat{x}^\mu} \equiv \{\cdot, \hat{p}_\mu\}, \quad \frac{\partial}{\partial \hat{p}_\mu} \equiv -\{\cdot, \hat{x}^\mu\}$$

— $\{\hat{x}^\mu, \hat{p}^\nu\} = \{\hat{x}^\mu, \hat{g}^{\nu\lambda} \hat{p}_\lambda\} = \hat{g}^{\mu\nu}, \quad \{\hat{x}^\mu, \hat{p}_\nu\} = \delta_\nu^\mu$

$\{\hat{x}^\mu, \hat{x}^\nu\} = 0 = \{\hat{p}_\mu, \hat{p}_\nu\}$ invariant (with $\hat{p}_\mu = \frac{\partial \hat{x}^a}{\partial \hat{x}^\mu} \hat{p}_a$)

Metric Operator \hat{g} on Krein Space :-

- proper inner product with Minkowski Signature

— effectively, bra as ${}_g\langle \cdot | = \langle \cdot | \hat{g}$

- observables (pseudo-)Hermitian ${}_\eta\langle \cdot | \hat{A}^{\dagger \eta} \cdot \rangle = {}_\eta\langle \hat{A} \cdot | \cdot \rangle$

$$\text{— } \hat{X}_a = \hat{\eta} \hat{X}^a \hat{\eta}^{-1} \quad \text{and} \quad \hat{P}_a = \hat{\eta} \hat{P}^a \hat{\eta}^{-1}$$

metric operator \leftrightarrow metric tensor $\hat{\eta} \leftrightarrow \hat{\eta}_{ab}$

- noncommutative geometric picture : g -Hermitian

— \hat{X}^μ and \hat{P}^μ as coordinates, $\hat{A}^{\dagger g} = \hat{A}$

nontrivial $\hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{x}^\zeta) \longrightarrow \hat{g} \quad (?)$

Lorentz Covariant Quantum Physics :-

- Schrödinger wavefunction $\phi(x^a)$
 - basic operators x^a and $-i\hbar\partial_a$
- abstract operators as Minkowski four-vectors
 - $\hat{X}_i \longrightarrow \hat{X}_a$ and $\hat{P}_i \longrightarrow \hat{P}_a$
 - $[\hat{X}_a, \hat{P}_b] = i\hbar\eta_{ab}$
- Heisenberg-Weyl symmetry — $[Y_a, E_b] = i\hbar c \eta_{ab} M$
 - M is an effective Casimir element \rightarrow Newtonian mass m
 - $m\hat{X}_a \longleftarrow Y_a$, different m for different irr. representations
 - $\hat{P}_a \longleftarrow \frac{1}{c}E_a$, constant c (... $c \rightarrow \infty$ limit)

Quantum Relativity Principle :-

- Penrose : Relativity Principle $\rightarrow \otimes \leftarrow$ Quantum
- Heisenberg picture – \hat{x} and \hat{p} as coordinates
— Noncommutative Geometry for Spacetime
- Rel. Sym. \leftarrow Quantum Ref. Frame Transformations
— *e.g.* translation by the NC value of $\hat{x}_A - \hat{x}_B$ (ans. Penrose)
- Quantum Gravity as General Quantum Relativity

Quantum Rindler Frame :-

$$\hat{x} = \hat{\rho} \cosh \frac{\hat{a}_N \hat{\tau}}{c}, \quad c\hat{t} = \hat{\rho} \sinh \frac{\hat{a}_N \hat{\tau}}{c}$$

- eigenstates : \hat{x} & \hat{t} \rightarrow $\hat{\rho}$ & $\hat{a}_N \hat{\tau}$
entanglement between $\hat{\tau}$ and \hat{a}_N

- metric : $\hat{g}_{c\hat{\tau},c\hat{\tau}} = \frac{\hat{a}_N^2 \hat{\rho}^2}{c^4}$

- quantum geodesic equations :

$$\frac{d^2 c\hat{\tau}}{ds^2} + \frac{dc\hat{\tau}}{ds} \frac{1}{\hat{\rho}} \frac{d\hat{\rho}}{ds} + \frac{d\hat{\rho}}{ds} \frac{1}{\hat{\rho}} \frac{dc\hat{\tau}}{ds} = 0 ,$$
$$\frac{d^2 \hat{\rho}}{ds^2} + \frac{dc\hat{\tau}}{ds} \frac{\hat{a}_N^2 \hat{\rho}}{c^4} \frac{dc\hat{\tau}}{ds} = 0 .$$

\rightarrow maintaining (Weak) **Equivalence Principle**

(Naive) Quantum Einstein Equation :-

$$\hat{G}_{aB} + \Lambda \hat{g}_{aB} = \kappa \hat{T}_{aB}$$

- Hermitian components of the geometric tensors

$$\hat{G}_{ab} = \hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{g}_{ab}$$

- more intriguing : $\hat{\Lambda}$
- Schwarzschild solution, with plausibly \hat{M}

Future (H & D \rightarrow ...) :-

- q-number physics – Q gravity as GQR
more NC physics : $[\hat{x}^i, \hat{x}^j] \neq 0$?
- q-number geometry – NC Geo. as symplectic coordinate picture : ‘Euclidean NC Geo.’
- q-number theory – algebra beyond algebra
- q-number technology – q-information

Quine: To be is to be the (q-number) value of a (physical) variable.

- C^* -/ $*$ -algebra as observable algebra
 - (Hilbert) space as NC symplectic geometry
 - local NC canonical coordinates
 - : Gel'fand-Kirillov dimension

- Lie symmetry (*cf.* Heisenberg-Weyl)
 - group C^* -algebra
 - / universal enveloping algebra (*cf.* Weyl)

- extended to q -number bimodule
 - Hilbert's Nullstellensatz
 - Lie skew field : Gel'fand-Kirillov conjecture

THANK YOU !