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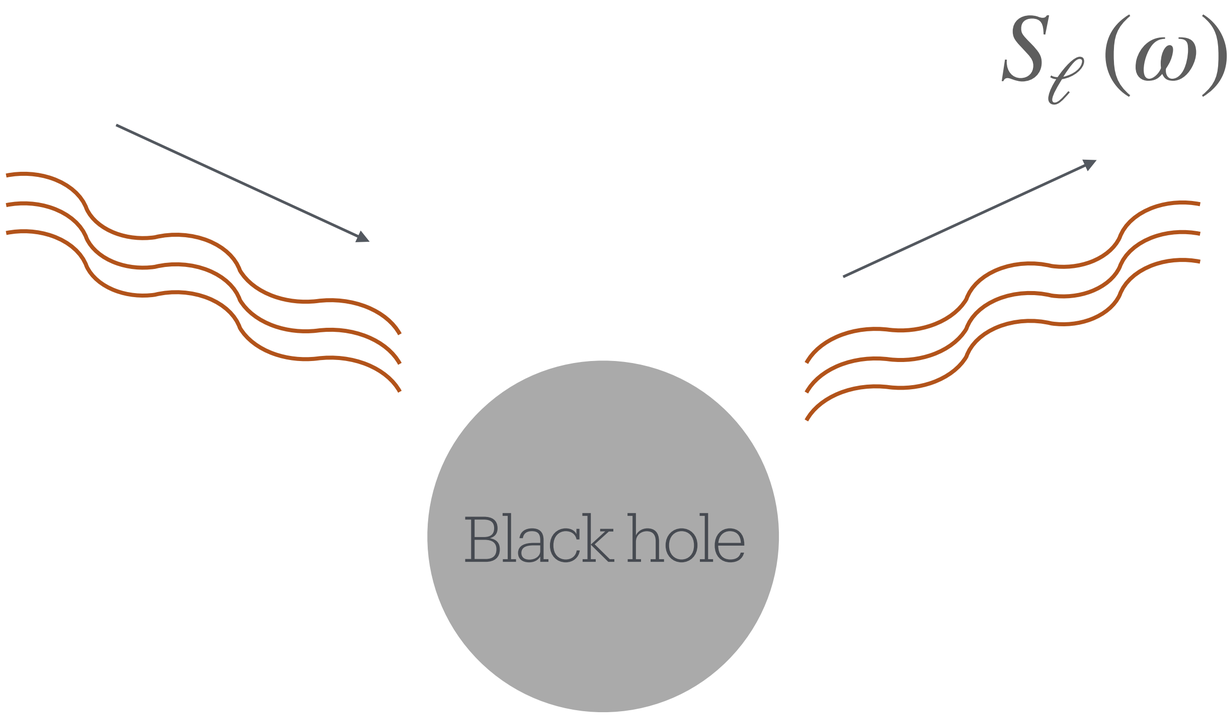
UCLA Mani L. Bhaumik Institute for Theoretical Physics

Analyticity of the Black Hole S-matrix

25xx.xxxx [M. Correia, T. Gopalka, GI, A. Wolz]

Introduction



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Theoretical

- Poles/Branch cuts capture the response of the black hole

- S-matrix in an open system/ background breaking translations

Causality/Unitarity?

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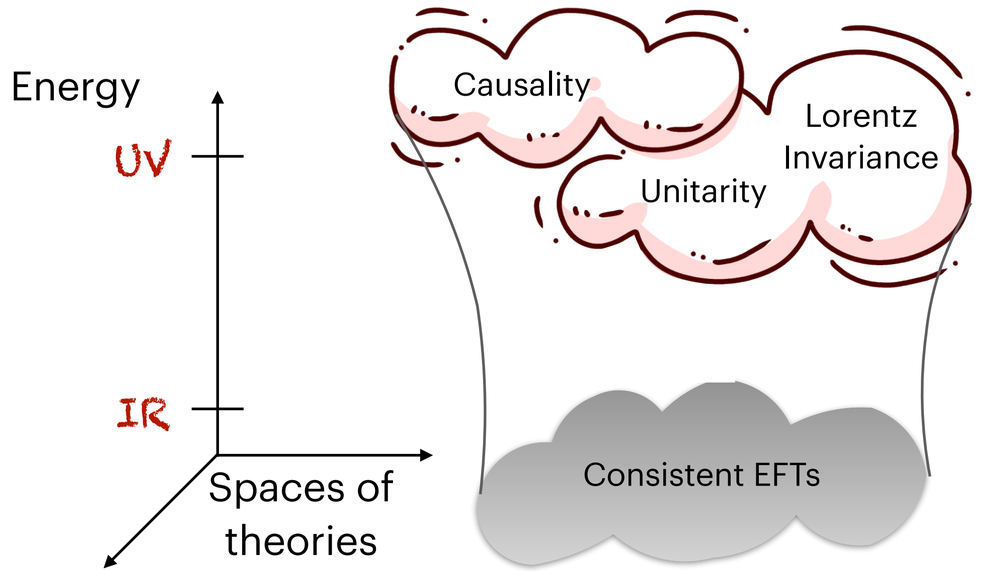

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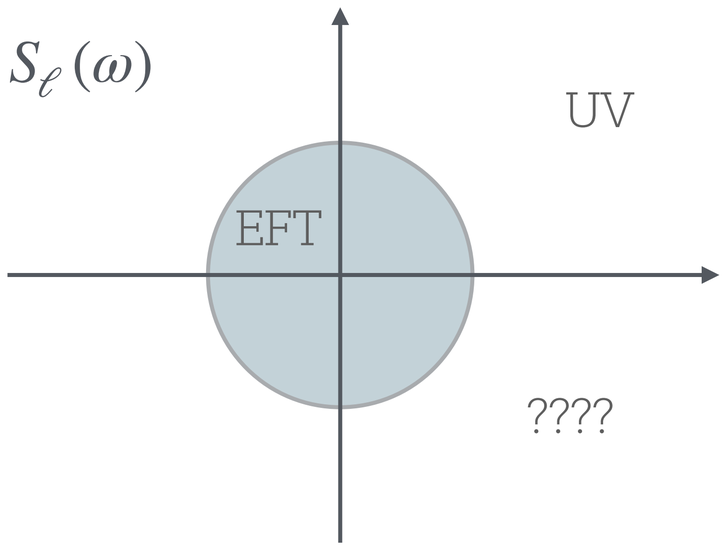

Outline

- The setup
- Analyticity for generic potentials
- Analyticity of the Black Hole S-matrix
- Examples

Setup

Setup

Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$

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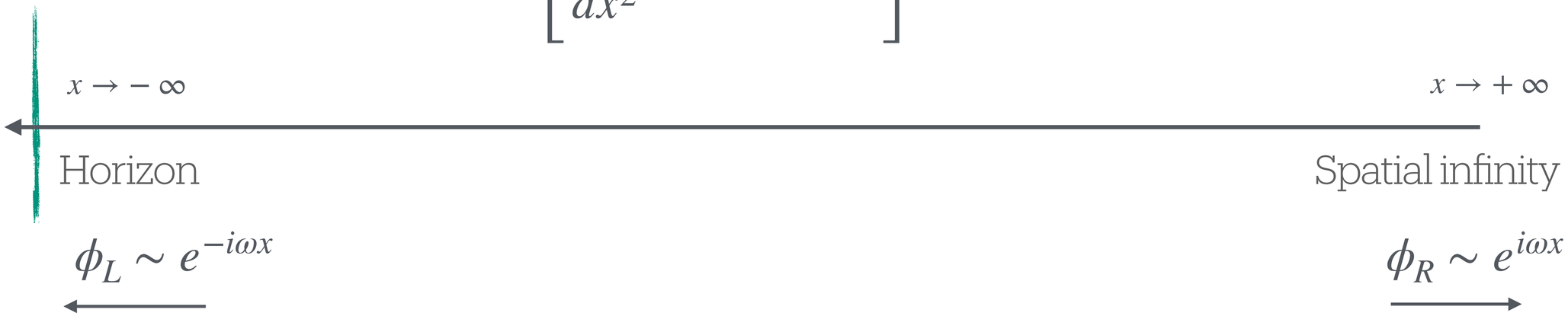

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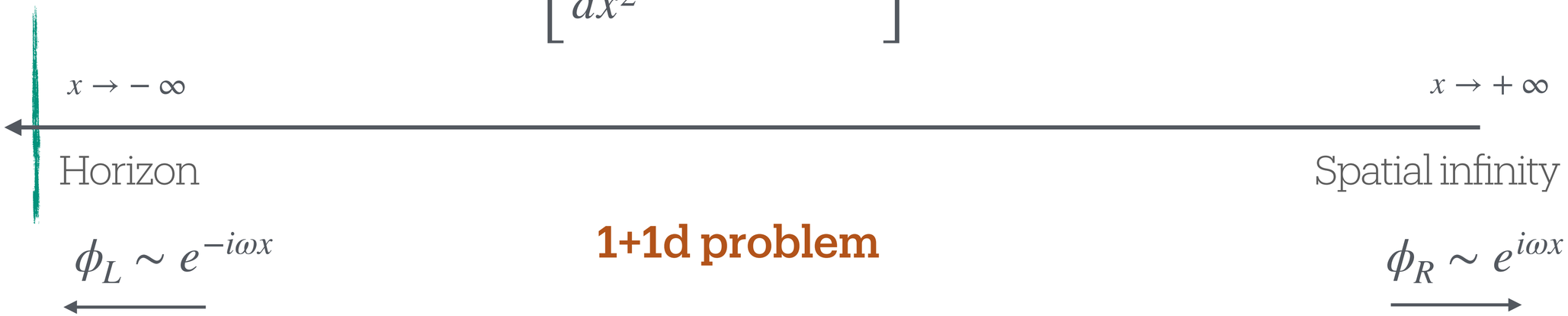

Relation S-matrix and retarded Green's function

$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$

$\phi_R(\omega, x \to -\infty) = e^{i\omega x}$

Relation S-matrix and retarded Green's function

S-matrix
(or reflection coefficient)

$$\phi_L(\omega, x \to +\infty) = A_{\text{in}}(\omega)\, e^{-i\omega x} + A_{\text{out}}(\omega)\, e^{i\omega x}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$

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Retarded Green's function

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right] G_R(\omega, x) = \delta(x - x')$$

$$G_R(\omega, x, x') = \frac{1}{2i\omega A_{\text{in}}}\left[\phi_L(x)\,\phi_R(x')\,\theta(x' - x) + \phi_L(x')\,\phi_R(x)\,\theta(x - x')\right]$$

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$$x, x' \to \infty$$

$$G_R(\omega, x, x') \longrightarrow -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$

Analyticity for generic potentials

J.R. Taylor (1983)
R.G. Newton (2013)

Warmup: Volterra equations for the wavefunction

Integral solutions

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right]\ V(x')\,\phi_L(\omega, x')\,dx'$$

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- Right BCs
- Solutions of wave equation

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Warmup: Volterra equations for the wavefunction

Integral solutions

$$\chi_L(\omega, x) = 1 + \int_{-\infty}^{x} G_0(\omega, x - x')\, V(x')\, \chi_L(\omega, x')\, dx'$$

$$\chi_R(\omega, x) = 1 + \int_{x}^{\infty} G_0(\omega, x - x')\, V(x')\, \chi_R(\omega, x')\, dx'$$

$$G_0(\omega, x - x') = \frac{e^{2i\omega(x-x')} - 1}{2i\omega}$$

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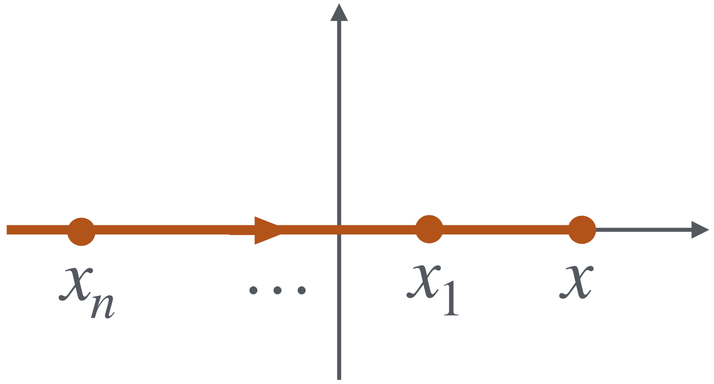
Solve the Volterra equations iteratively

$$\chi_L(\omega, x) = \sum_n \chi_L^{(n)}(\omega, x)$$

Warmup: Volterra equations for the wavefunction



$$\chi_L^{(n)}(\omega, x) = \int_{x > x_1 > \ldots > x_n > -\infty} G_0(\omega, x - x_1) \cdots G_0(\omega, x_{n-1} - x_n)\, V(x_1) \cdots V(x_n)\, dx_1 \cdots dx_n$$

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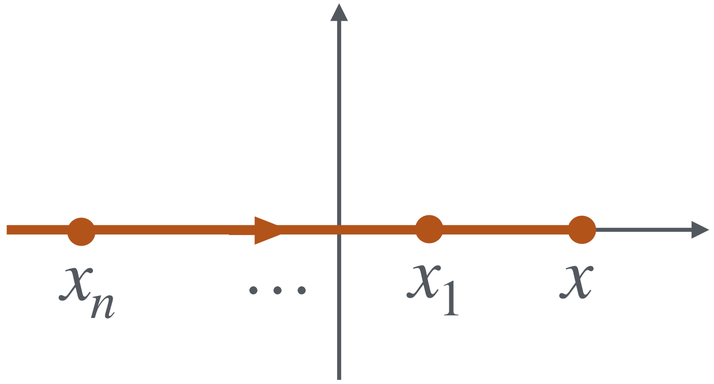

$$|\chi_L^{(n)}(\omega, x)| \leq \int |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)|\, |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n$$

$$x > x_1 > \ldots > x_n > -\infty$$

$$|G_0(\omega, x - x')| = \left| \frac{e^{2i\omega \overbrace{(x-x')}^{<0}} - 1}{2i\omega} \right|$$

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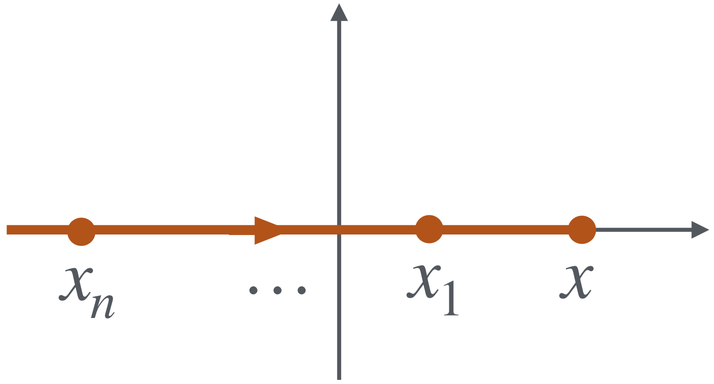

$$|\chi_L^{(n)}(\omega, x)| \leq \int |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)|\, |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n$$

$$x > x_1 > \ldots > x_n > -\infty$$

$$|G_0(\omega, x - x')| = \left| \frac{e^{2i\omega \overbrace{(x-x')}^{<0}} - 1}{2i\omega} \right| < \frac{1}{|\omega|}$$

$\mathrm{Im}\,\omega > 0$

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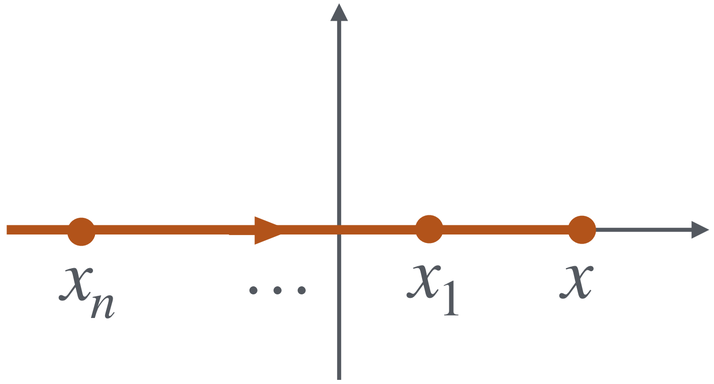

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$$x > x_1 > \ldots > x_n > -\infty$$

$$\leq \frac{1}{|\omega|^n} \int |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n$$

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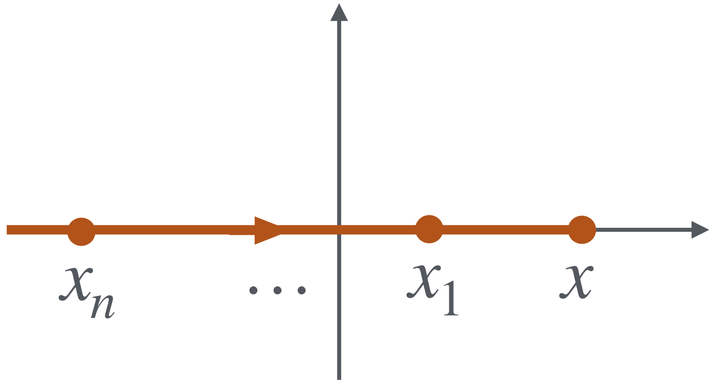

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$$x > x_1 > \ldots > x_n > -\infty$$

$$\leq \frac{1}{|\omega|^n} \int |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n = \frac{1}{n!} \left[\frac{1}{|\omega|} \int_{-\infty}^{x} |V(x_1)|\, dx_1 \right]^n$$

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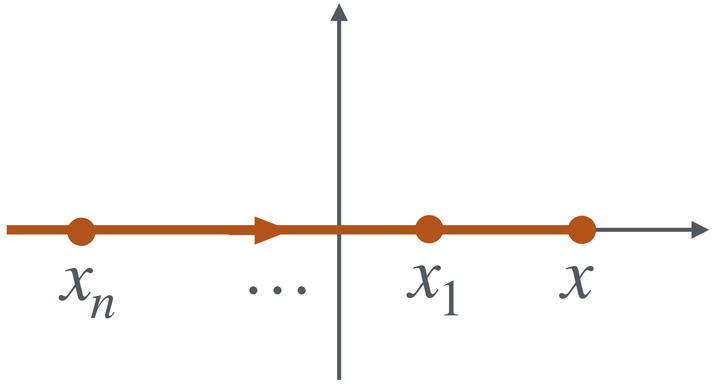
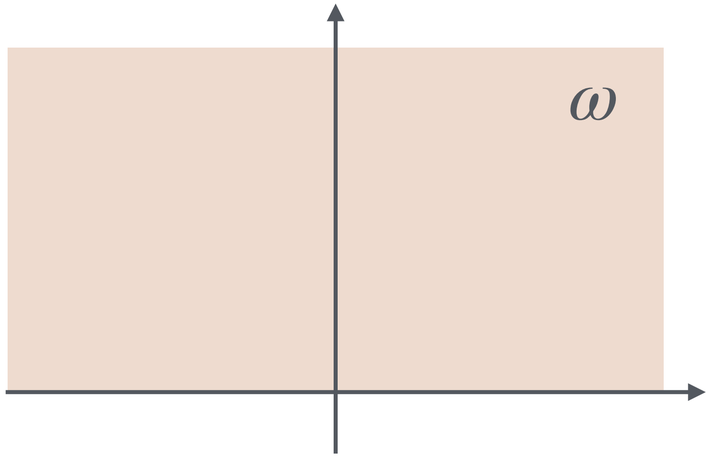


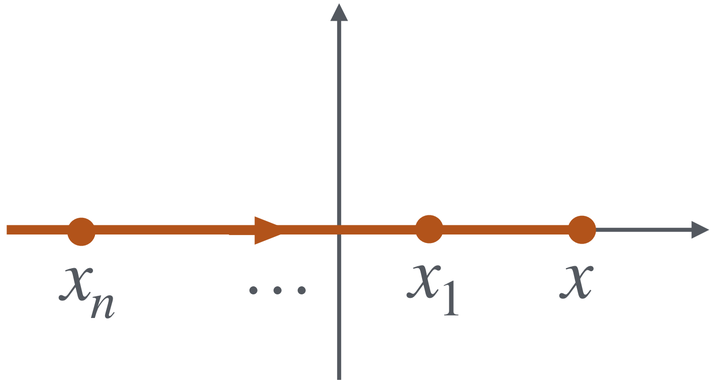


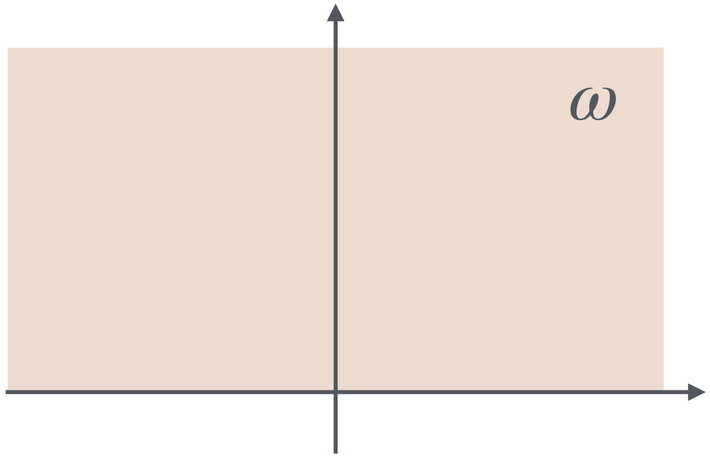
$$|\chi_L(\omega,x)| = |\sum_n \chi_L^{(n)}(\omega,x)| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{x}|V(x')|dx'}$$

uniform convergence

$\chi_L(\omega,x)$ is analytic in the UHP

Warmup: Volterra equations for the wavefunction





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uniform convergence

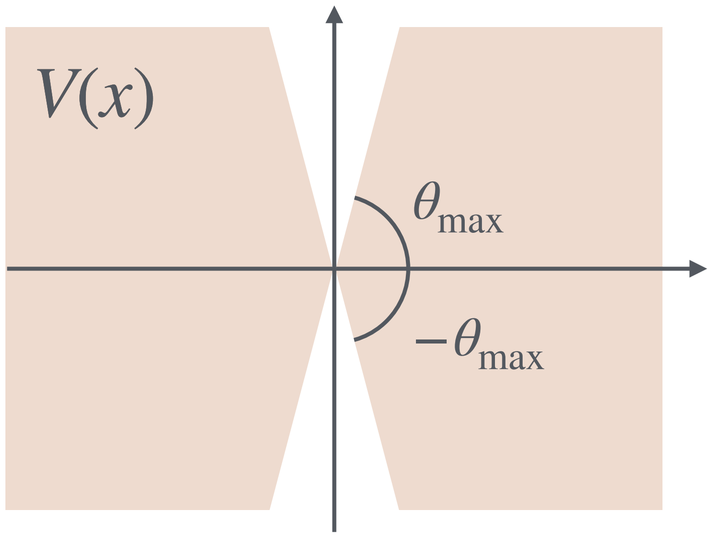
$\chi_L(\omega,x)$ is analytic in the UHP

$G_R(\omega,x)$ is analytic in the UHP

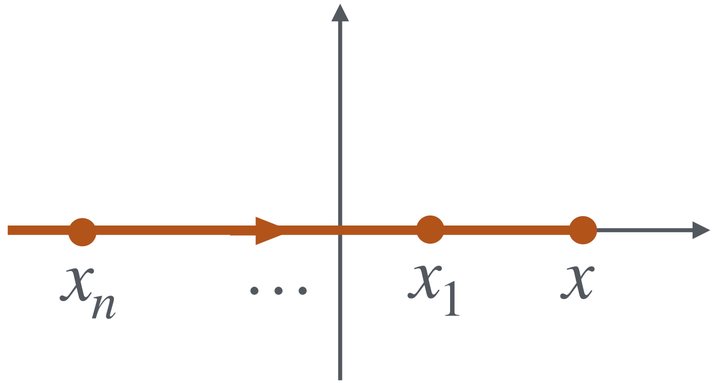
Simplest incarnation of causality

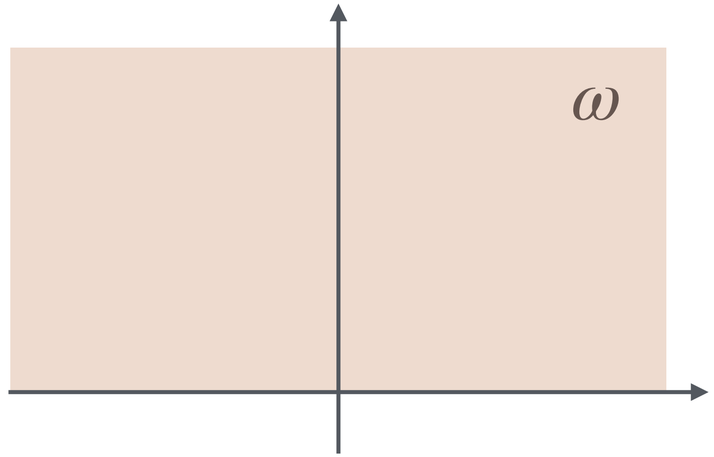
$$G_R(t-t',x) = 0\,, \quad t-t' < 0$$

More analyticity from the potential



$$V(x \to \infty) \lesssim \frac{1}{|x|}$$





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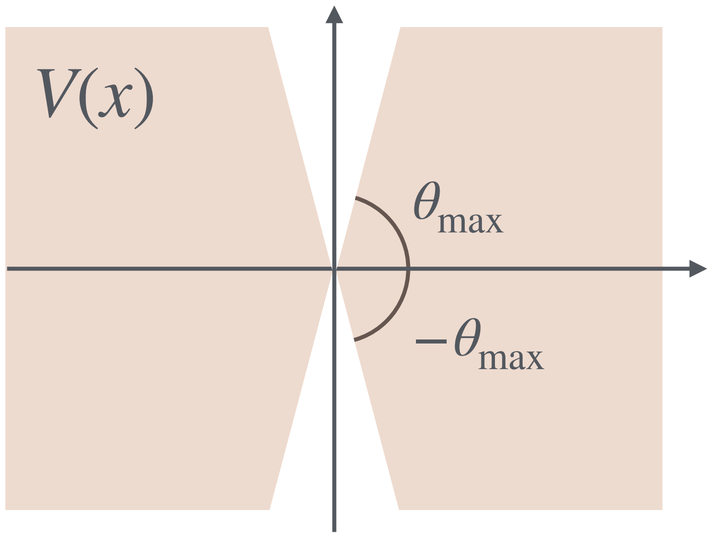

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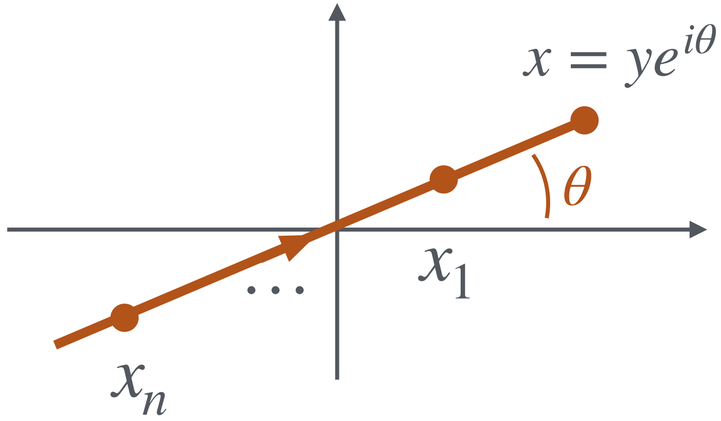
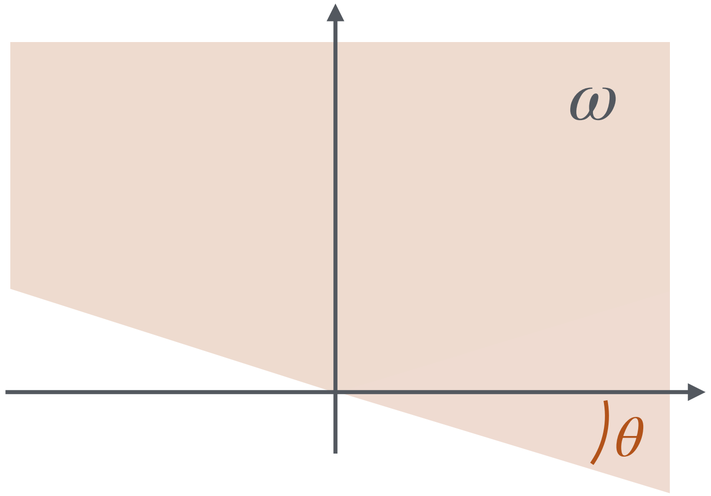
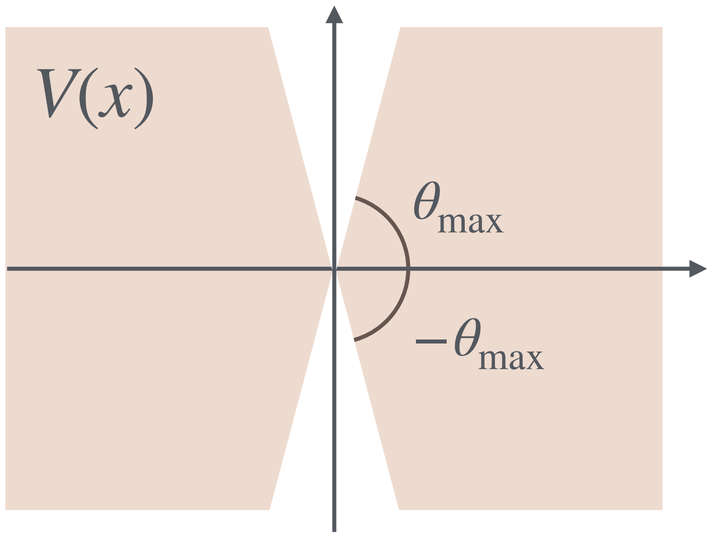




$$|\chi_L(\omega, ye^{i\theta})| = |\sum_n \chi_L^{(n)}(\omega, ye^{i\theta})| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{y} |V(e^{i\theta}y')|dy'}$$

$$\mathrm{Im}\, e^{i\theta}\omega > 0$$

More analyticity from the potential



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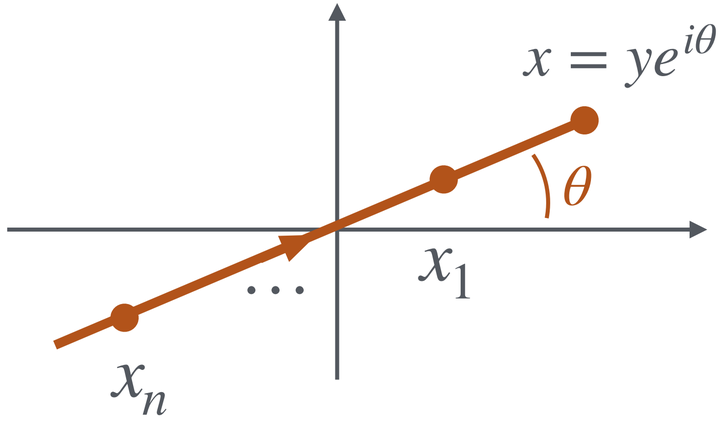

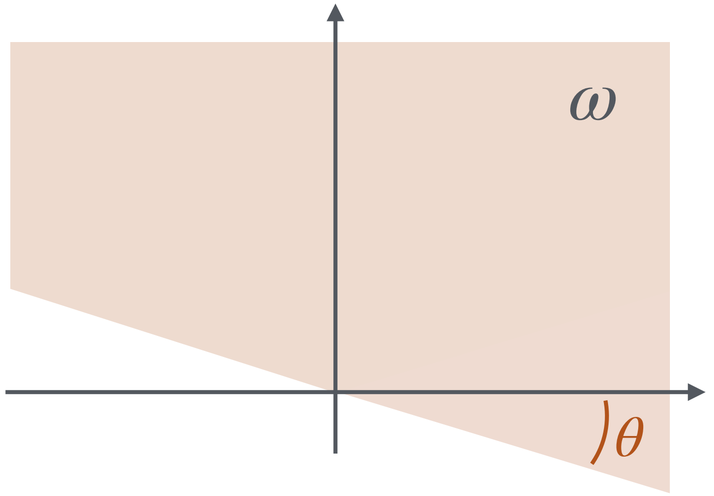

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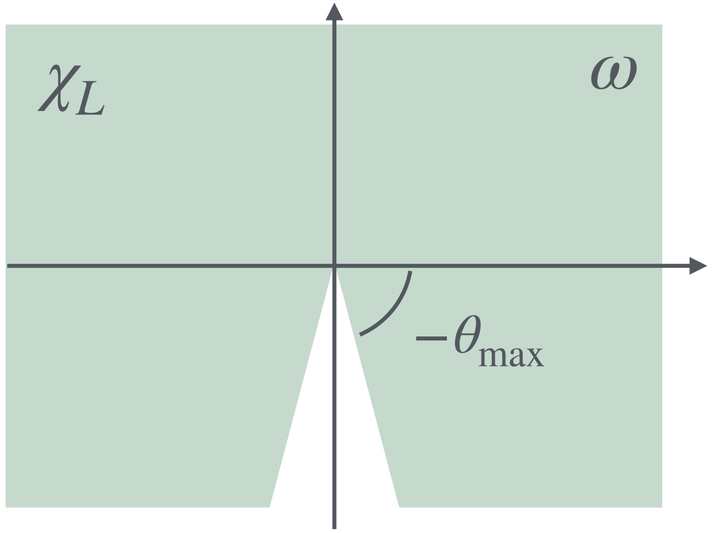

Analyticity of connection coefficients

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_L(\omega, x')\,dx'$$

$x \to \infty$

$$A_{\text{in}}(\omega) = 1 - \frac{1}{2i\omega}\int_{-\infty}^{\infty} V(x')\,\chi_L(\omega, x')\,dx'$$

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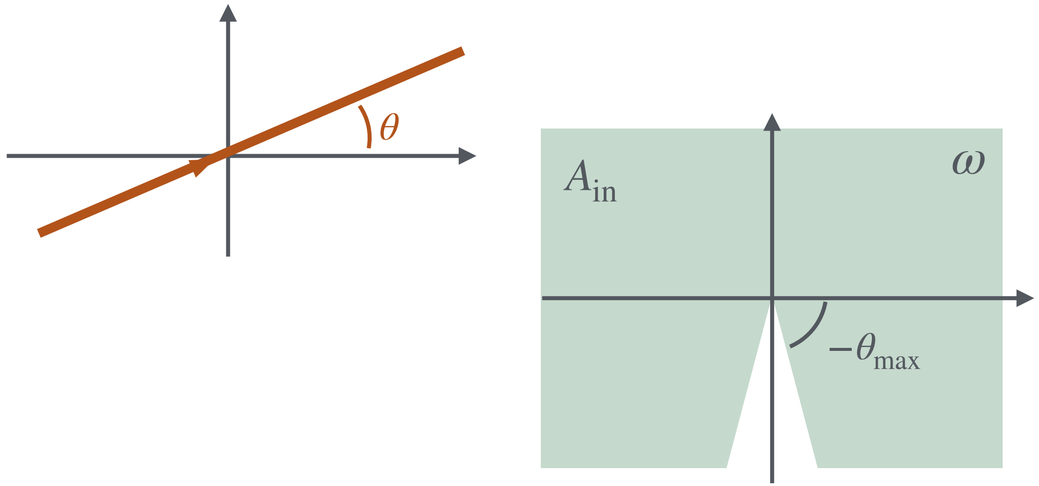

$$A_{\text{out}}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega, x')\,dx'$$

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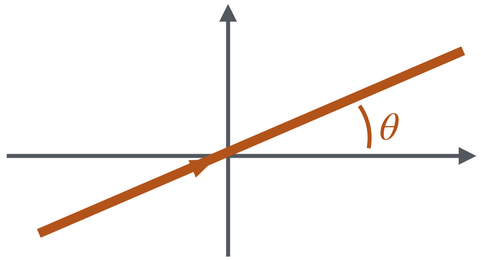
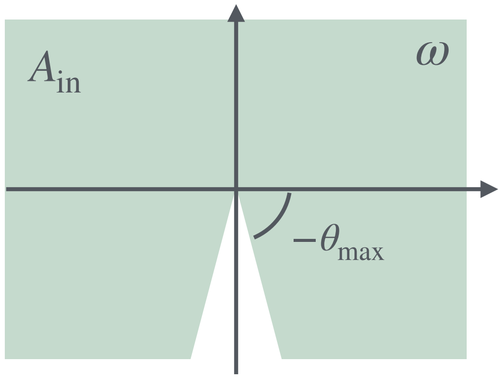
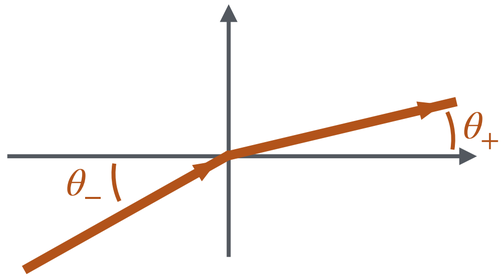
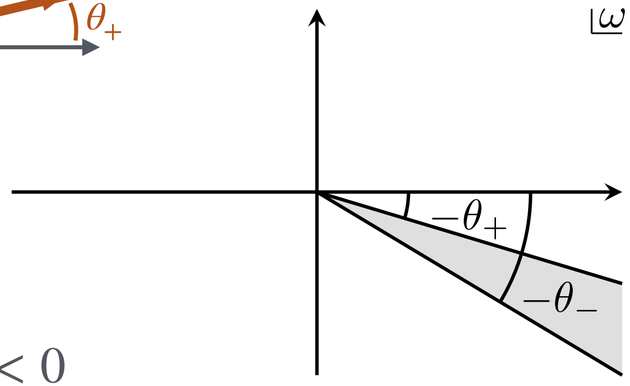




$$A_{\text{out}}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega, x')\,dx'$$





$\text{Im}(e^{i\theta_-}\omega) > 0 \;\cap$

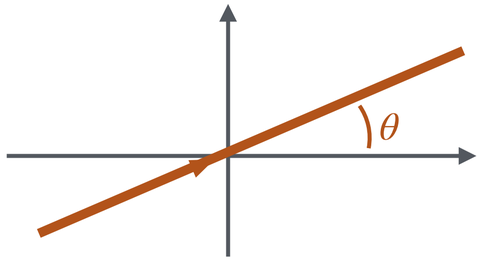
$\text{Im}(e^{i\theta_+}\omega) < 0$

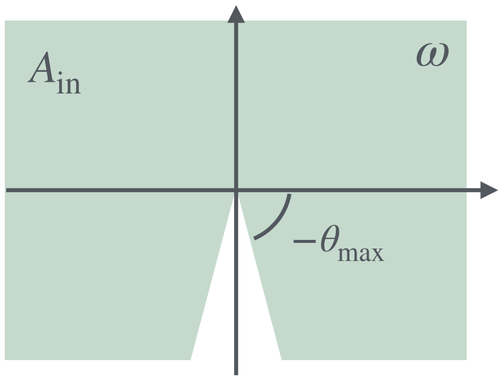
Analyticity of connection coefficients

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_L(\omega, x')\,dx'$$

$x \to \infty$

$$A_{\text{in}}(\omega) = 1 - \frac{1}{2i\omega}\int_{-\infty}^{\infty} V(x')\,\chi_L(\omega, x')\,dx'$$





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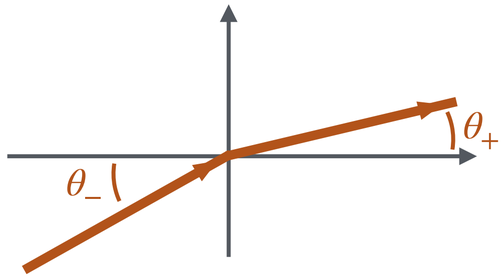
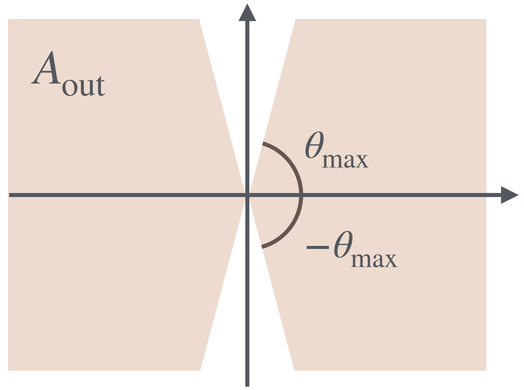
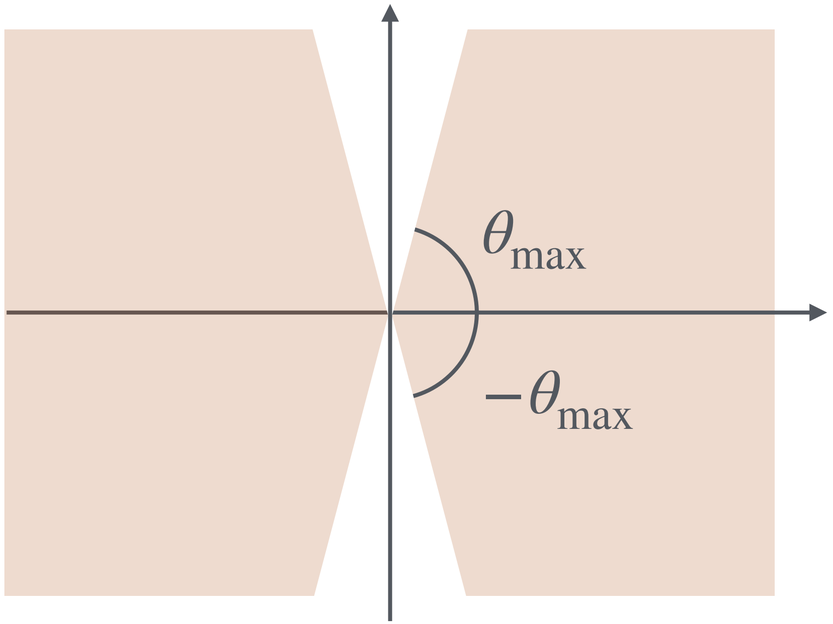


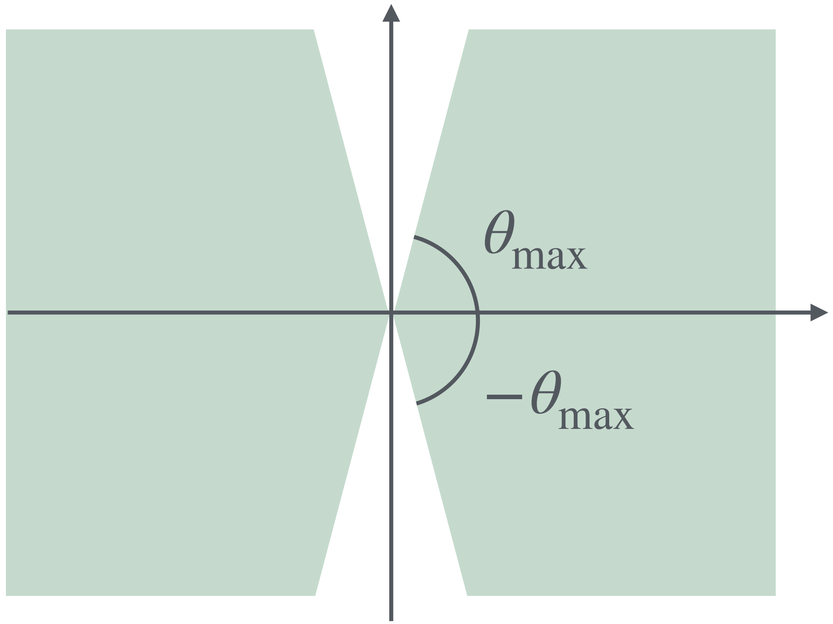


Analyticity of the S-matrix

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$





Meromorphic

Analyticity of the S-matrix

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

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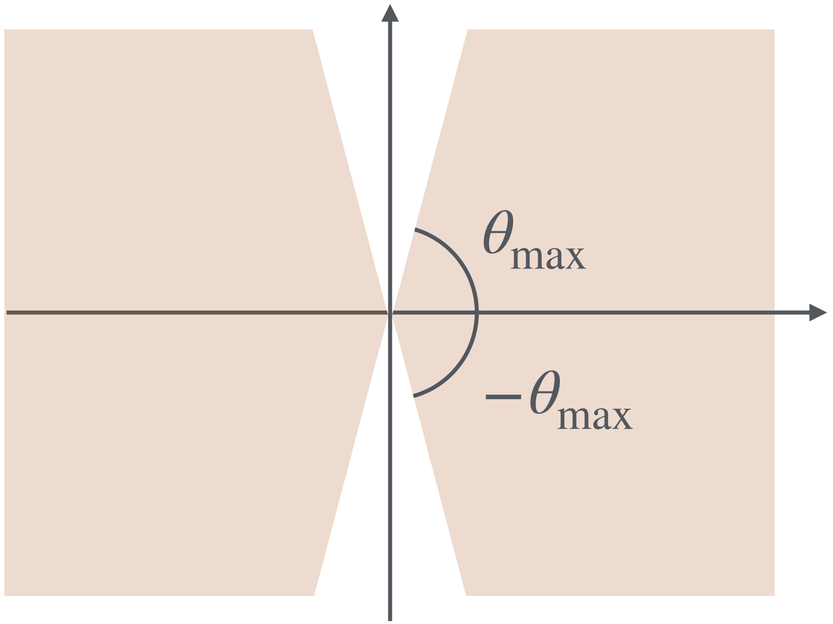
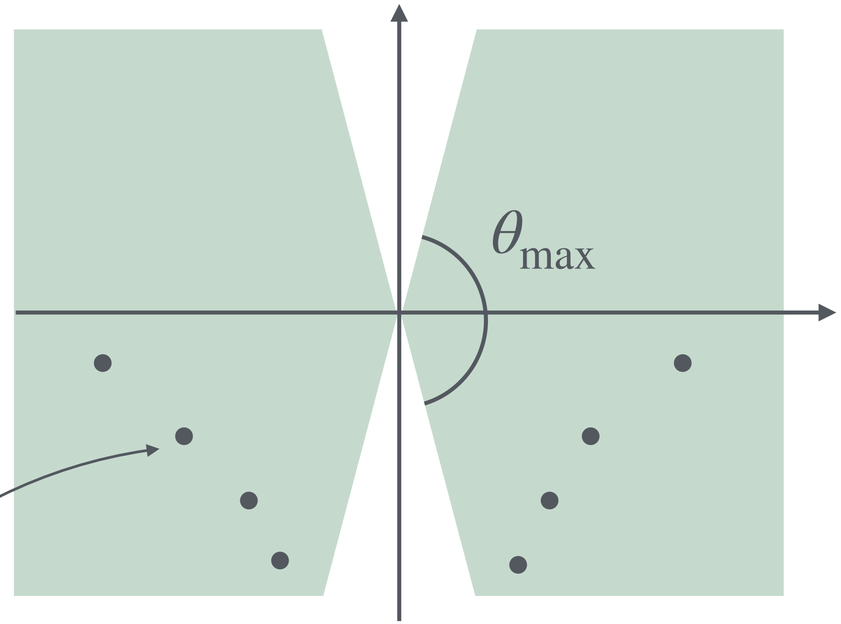




Quasi-normal modes (QNMs)

Meromorphic $A_{\text{in}} = 0$

Analyticity of the Black Hole S-matrix

The Schwarzschild potential

Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log\left(\frac{r}{R_s} - 1 \right)$

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x) \right] \phi(\omega, x) = 0$$

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1 - s^2) \right)$$

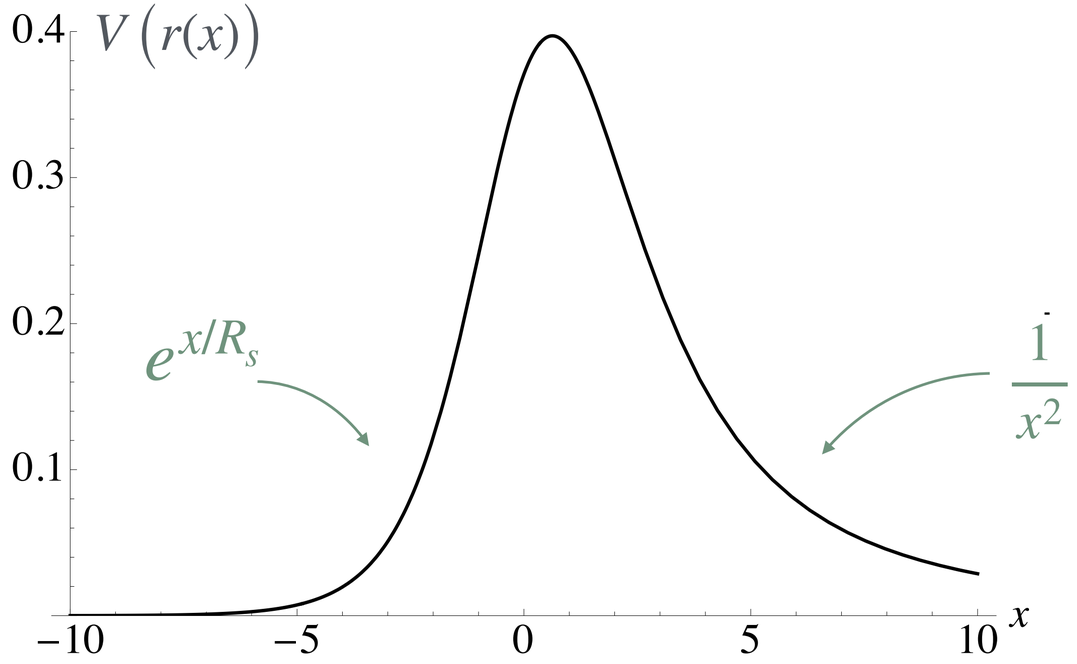
Angular momentum

Spin of the scattered wave

The Schwarzschild potential

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1 - s^2)\right)$$

$$x = r + R_s \log\left(\frac{r}{R_s} - 1\right)$$



The Schwarzschild potential

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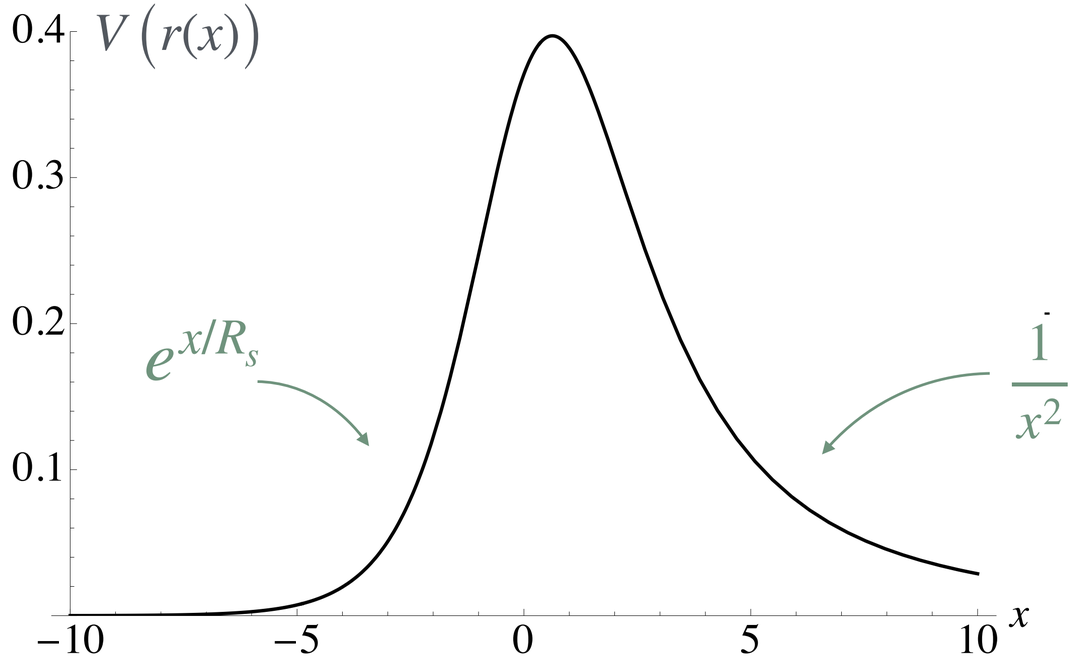

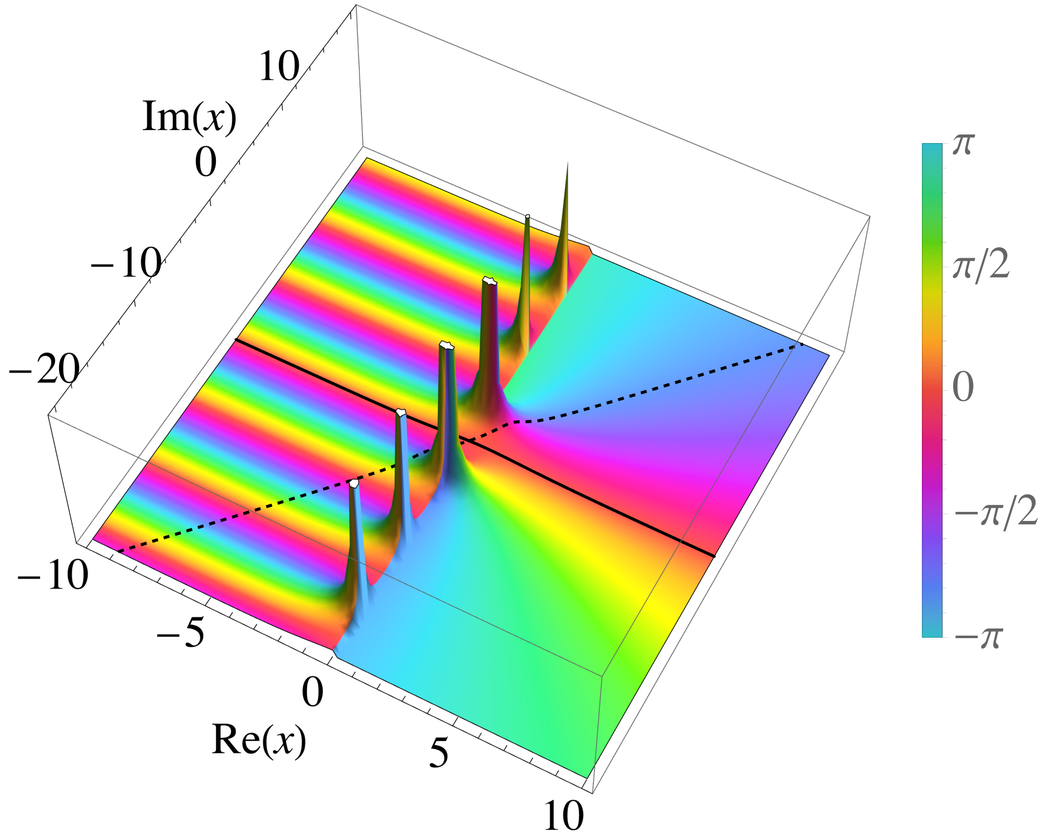

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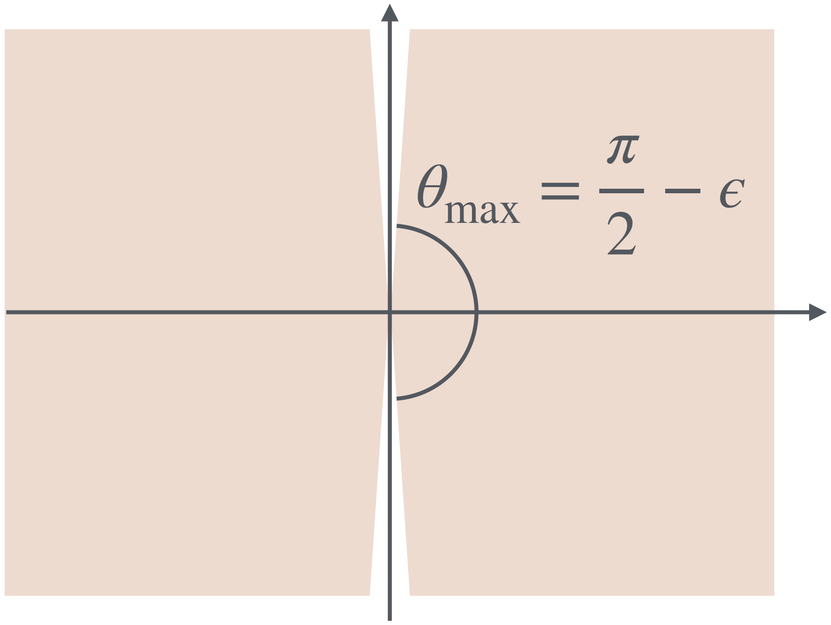
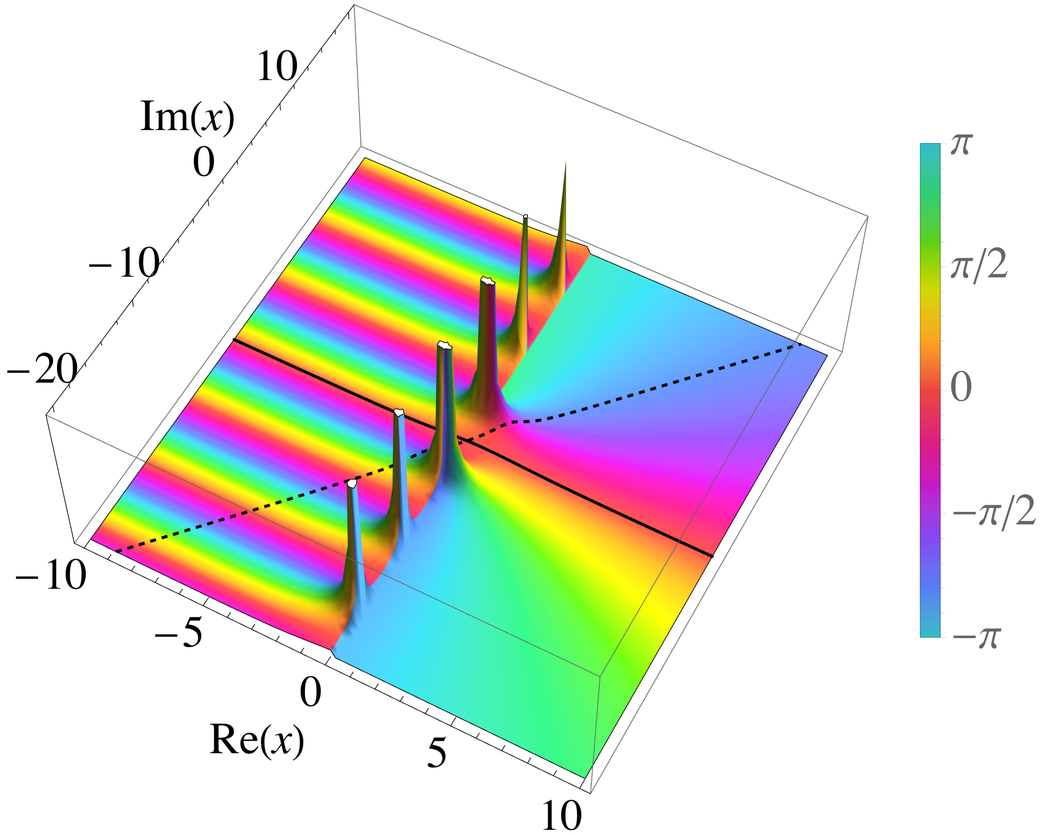
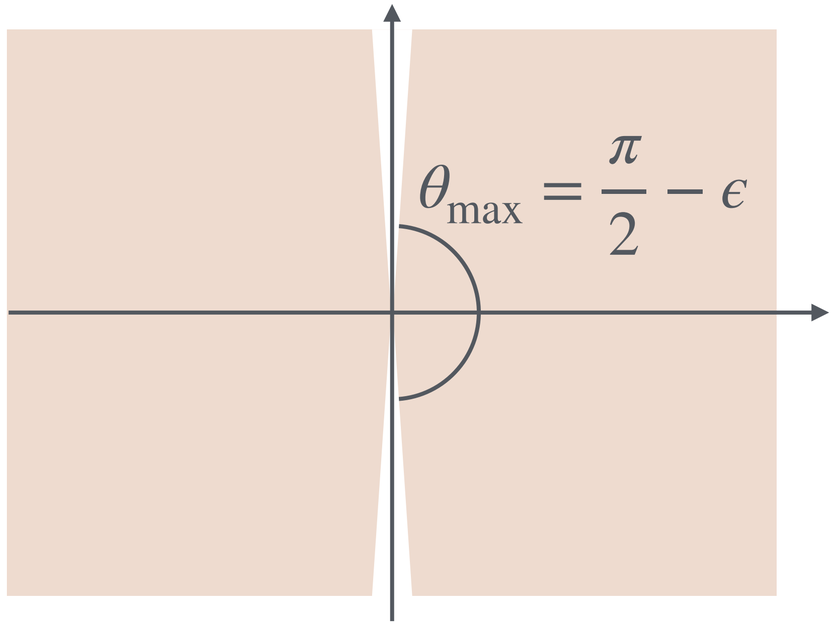
$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

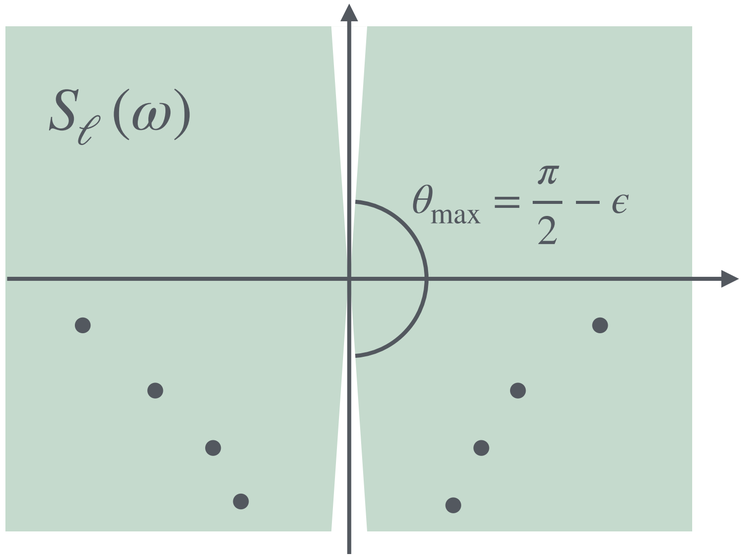


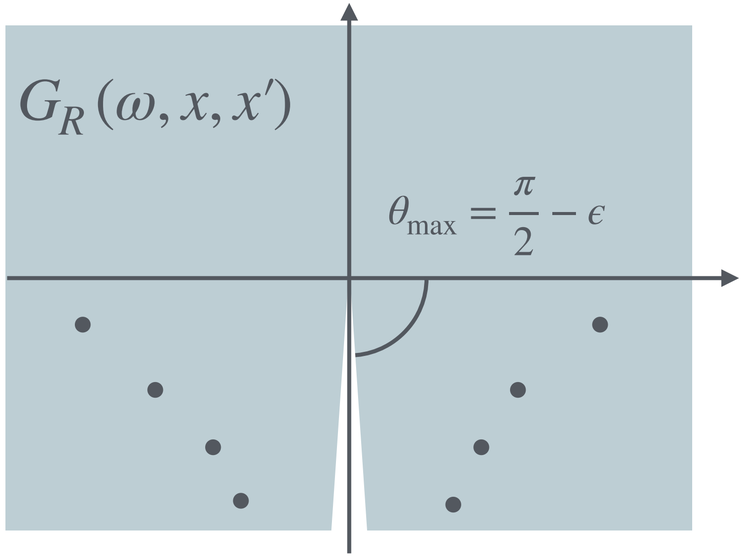


Proven analyticity of the Black Hole amplitude

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$





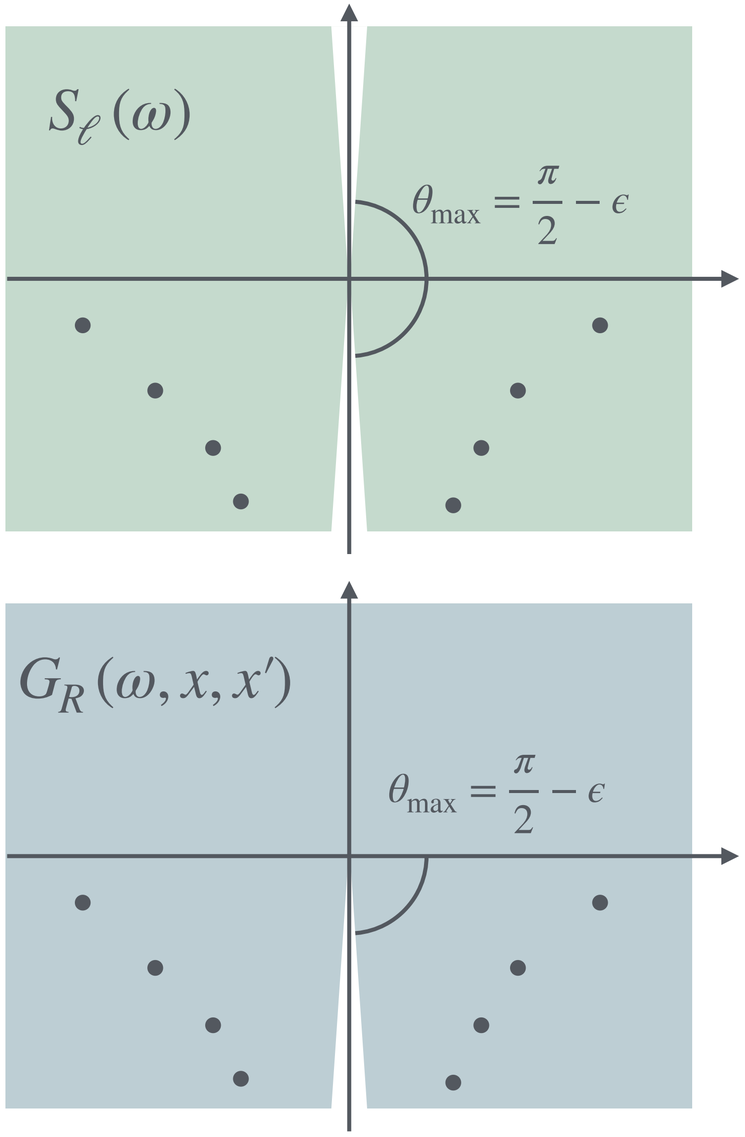


Proven analyticity of the Black Hole amplitude

- Explore solvable regimes
  - Small frequency expansion
  - Coulomb potential
  - High frequencies limit

- Consistency with

$$G_R(\omega, x, x') \overset{x, x' \to \infty}{\longrightarrow} -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$



Explicit solutions

Small frequency expansion

$S_\ell = e^{2i\delta_\ell}$

$\ell = 0$

[MST, 1976]
[Dodelson,Grassi,Iossa,Lichtig,Zhiboedov, 2022]

$$\delta_0(\omega) = R_s\,\omega\left(\frac{1}{2}\log(4R_s^2\omega^2)+\gamma_E-\frac{1}{2}\right)+(R_s\,\omega)^2\left(\frac{11\pi}{12}+i\right)$$

$$+(R_s\,\omega)^3\left[\frac{1}{2}-\frac{1}{2}\log(4R_s^2\omega^2)+\frac{11}{6}\zeta_2-\frac{1}{3}\zeta_3-\gamma_E+i\pi\right]$$

$$+(R_s\,\omega)^4\left[-\left(\frac{\pi}{2}+\frac{11i}{6}\right)\left[\log(4R_s^2\,\omega^2)+2\gamma_E\right]+4i\zeta_2+\frac{227i}{36}+\frac{2299\pi}{2160}\right]$$

$$+(R_s\,\omega)^5\left[\frac{11}{24}\left[\log(4R_s^2\,\omega^2)+2\gamma_E\right]^2-\left(\zeta_2+\frac{17}{36}+\frac{11i\pi}{6}\right)\left[\log(4R_s^2\,\omega^2)+2\gamma_E\right]\right.$$

$$\left.+\left(2i\pi-\frac{8591}{1080}\right)\zeta_2-\frac{11}{15}\zeta_2^2-\frac{59}{36}\zeta_3+\frac{1}{5}\zeta_5-\frac{1111}{540}+\frac{191i}{36}\pi\right]+\mathcal{O}(\omega^6)$$

Small frequency expansion

$S_\ell = e^{2i\delta_\ell}$

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$V \sim \frac{GM}{r}$

[MST, 1976]
[Dodelson,Grassi,Iossa,Lichtig,Zhiboedov, 2022]

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$$+(R_s\,\omega)^3\left[\frac{1}{2} - \frac{1}{2}\log(4R_s^2\omega^2) + \frac{11}{6}\zeta_2 - \frac{1}{3}\zeta_3 - \gamma_E + i\pi\right]$$

$$+(R_s\,\omega)^4\left[-\left(\frac{\pi}{2} + \frac{11i}{6}\right)\left[\log(4R_s^2\,\omega^2) + 2\gamma_E\right] + 4i\zeta_2 + \frac{227i}{36} + \frac{2299\pi}{2160}\right]$$

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- Unitarity

$$|S_\ell(\omega)| \leq 1$$

- Crossing

$$S^*(\omega) = S(-\omega^*)$$

Small frequency expansion

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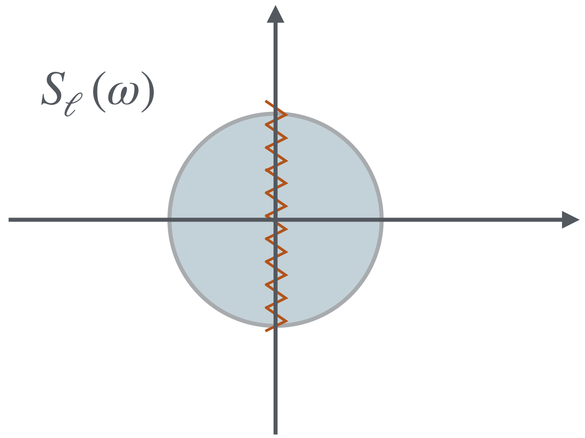


- Unitarity

$$|S_\ell(\omega)| \leq 1$$

- Crossing

$$S^*(\omega) = S(-\omega^*)$$

Large frequency limit of Regge-Wheeler $|GM\omega| \gg 1$

gr-qc/9607064 [N. Andersson]

$$\hat{u}^{\text{in}} = \left(\frac{r}{2GM}\right)^{1/2}\left(\frac{r}{2GM}-1\right)^{-2i\omega GM} M\left(1/2-4i\omega GM, 1-4i\omega GM, -2i\omega(r-2GM)\right) e^{i\omega(r-4GM)}$$

$$\hat{u}^{\text{up}} = (-4iGM\omega)^{1/2-4iGM\omega}\left(\frac{r}{2M}\right)^{1/2}\left(\frac{r}{2GM}-1\right)^{-4iGM\omega} U\left(1/2-4iGM\omega, 1-4iGM\omega, -2i\omega(r-2GM)\right) e^{i\omega x}$$

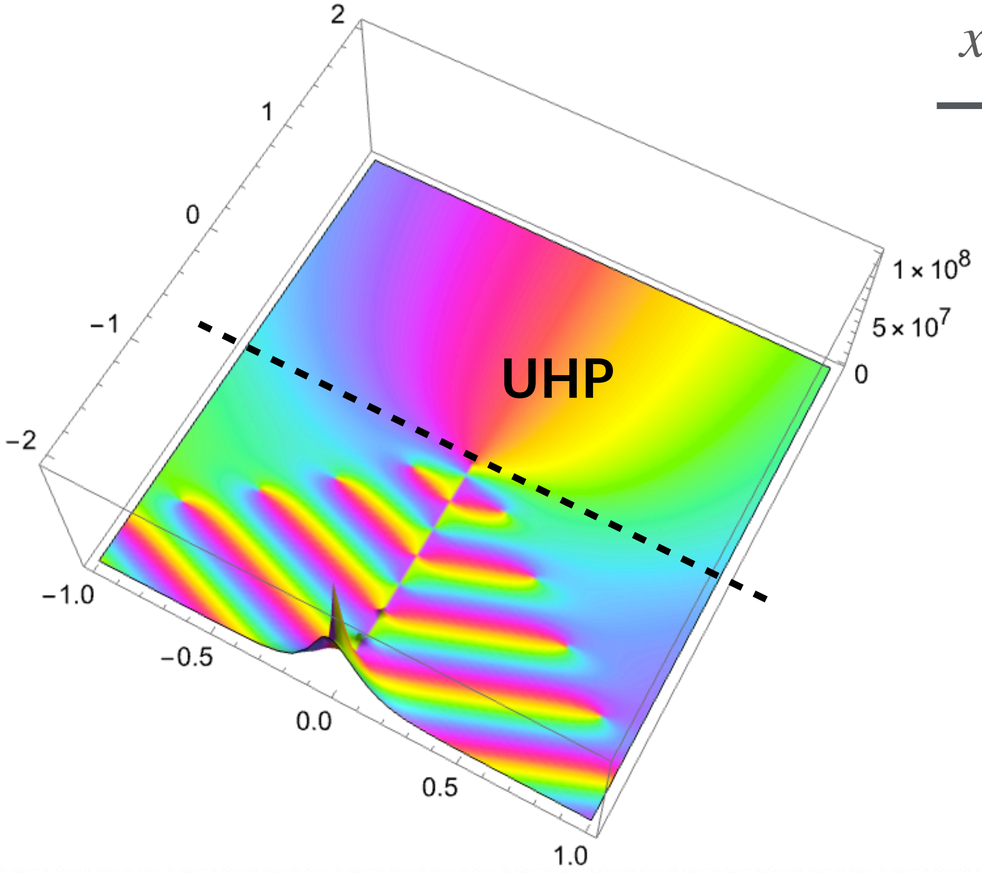
$$G_R{}^{\ell}(\omega, x, x') \xrightarrow{x, x' \to \infty} S_\ell(\omega) = -\frac{2^{8iGM\omega} e^{-4iGM\omega}\sqrt{-iGM\omega}\,(iGM\omega)^{-1/2+4iGM\omega}\,\Gamma(1/2-4iGM\omega)}{\sqrt{\pi}}$$

Large frequency limit of Regge-Wheeler $|GM\omega| \gg 1$

gr-qc/9607064 [N. Andersson]

Retarded Green's function

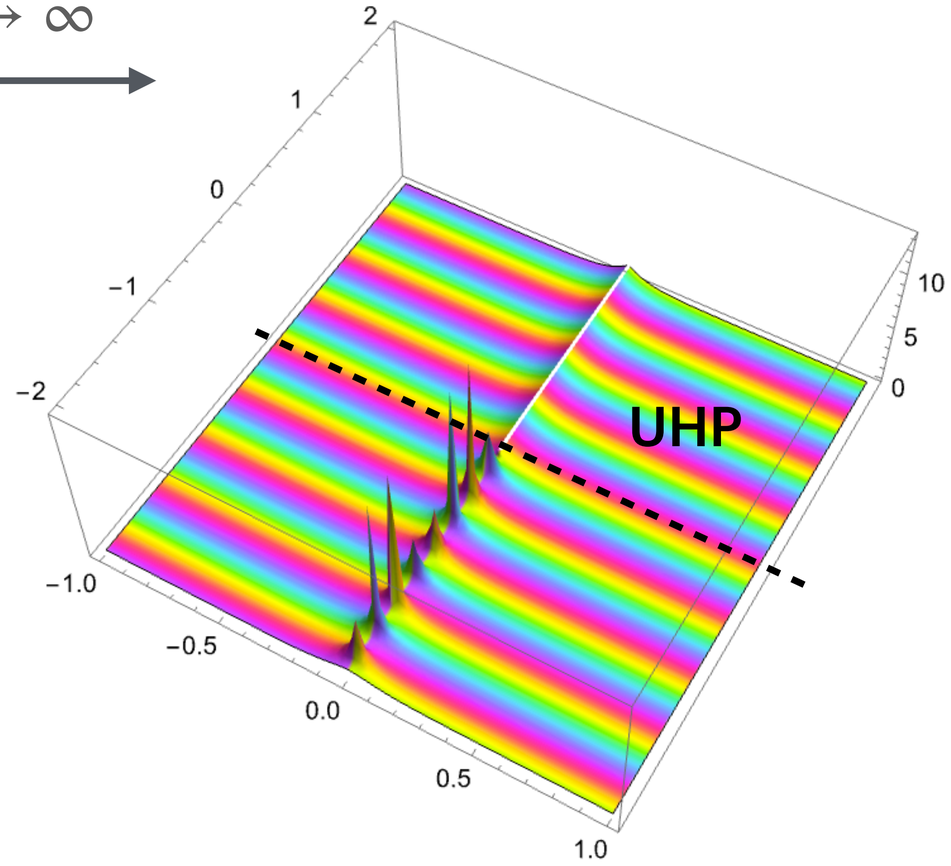
$G_R{}^{\ell}(\omega, x, x')$



$x, x' \to \infty$

S-matrix

$S_{\ell}(\omega)$



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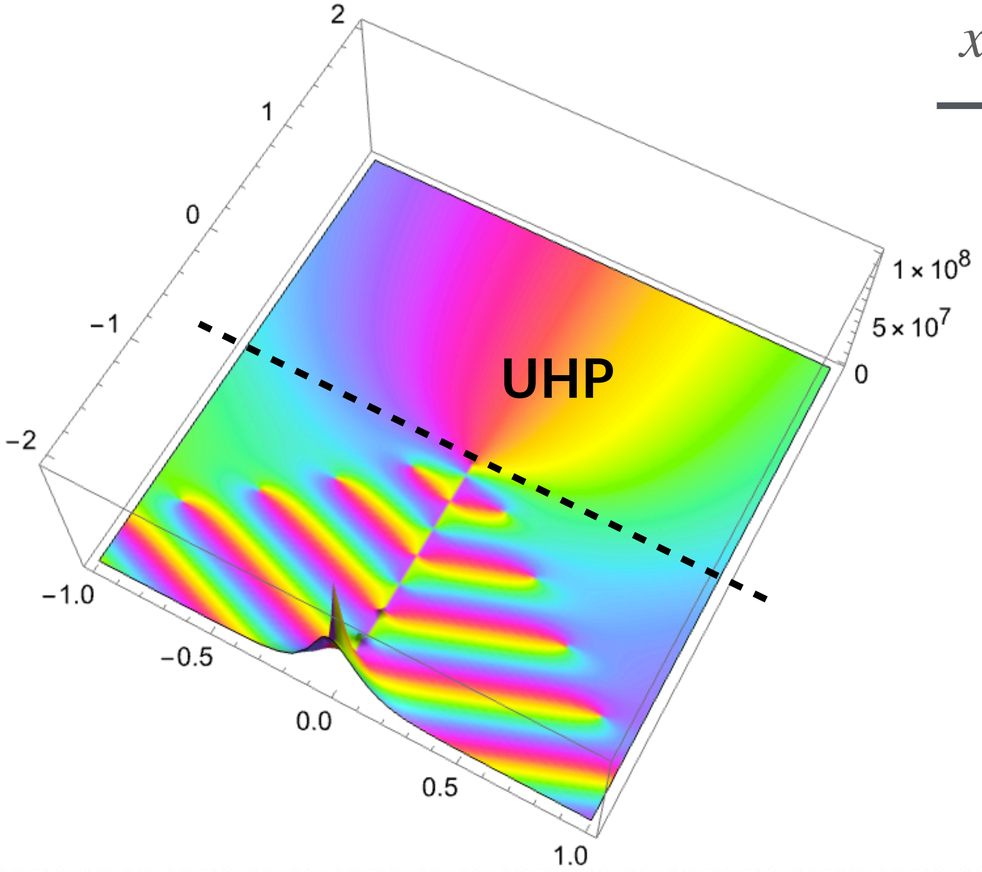

$x, x' \to \infty$

S-matrix

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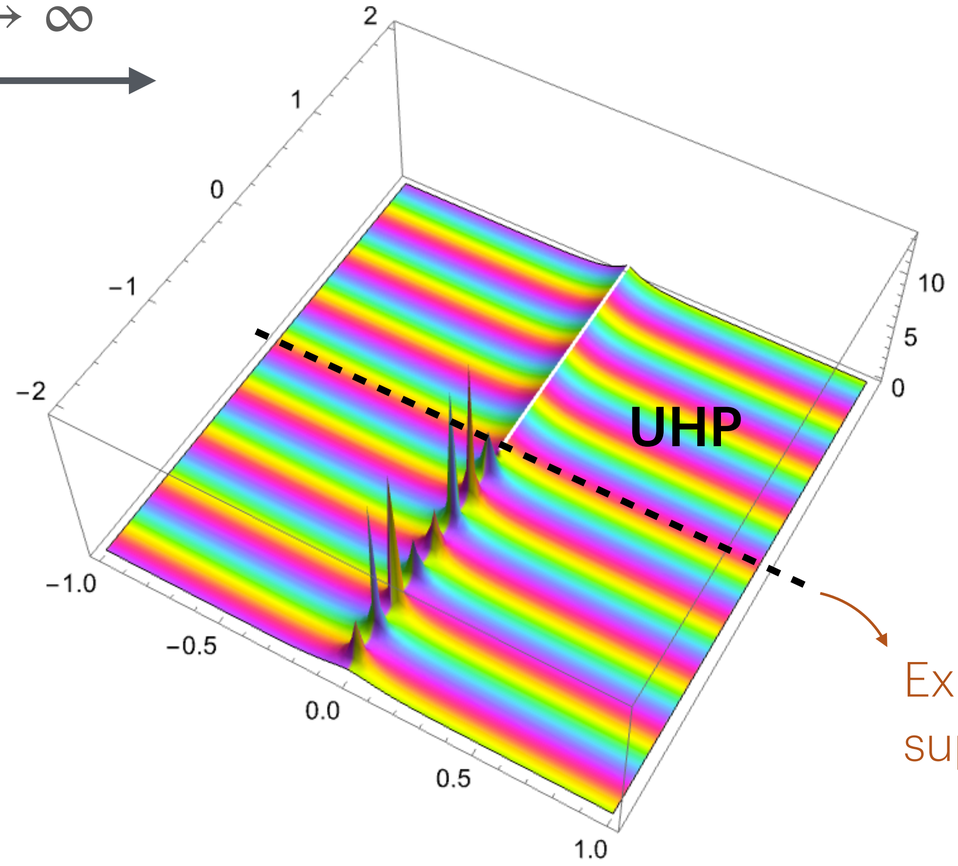


Exponentially suppressed

Newtonian potential

$$V \sim \frac{4GM\omega^2}{r}$$

$$\hat{u}_\ell^{\text{in}} \sim \left(\frac{r}{GM}\right)^{\ell+1} e^{i\omega r} M\left(\ell+1-2iGM\omega, 2\ell+2, -2i\omega r\right)$$

$$\hat{u}_\ell^{\text{up}} \sim \left(\frac{r}{GM}\right)^{\ell+1} e^{i\omega r} U\left(\ell+1-2iGM\omega, 2\ell+2, -2i\omega r\right)$$

Confluent Hypergeometric functions

$$x > x'$$

$$G_R^\ell(x, x', \omega) = -\frac{1}{2i\omega A_{\text{in}}} \hat{u}_\ell^{\text{in}}(x, \omega)\, \hat{u}_\ell^{\text{up}}(x', \omega)$$

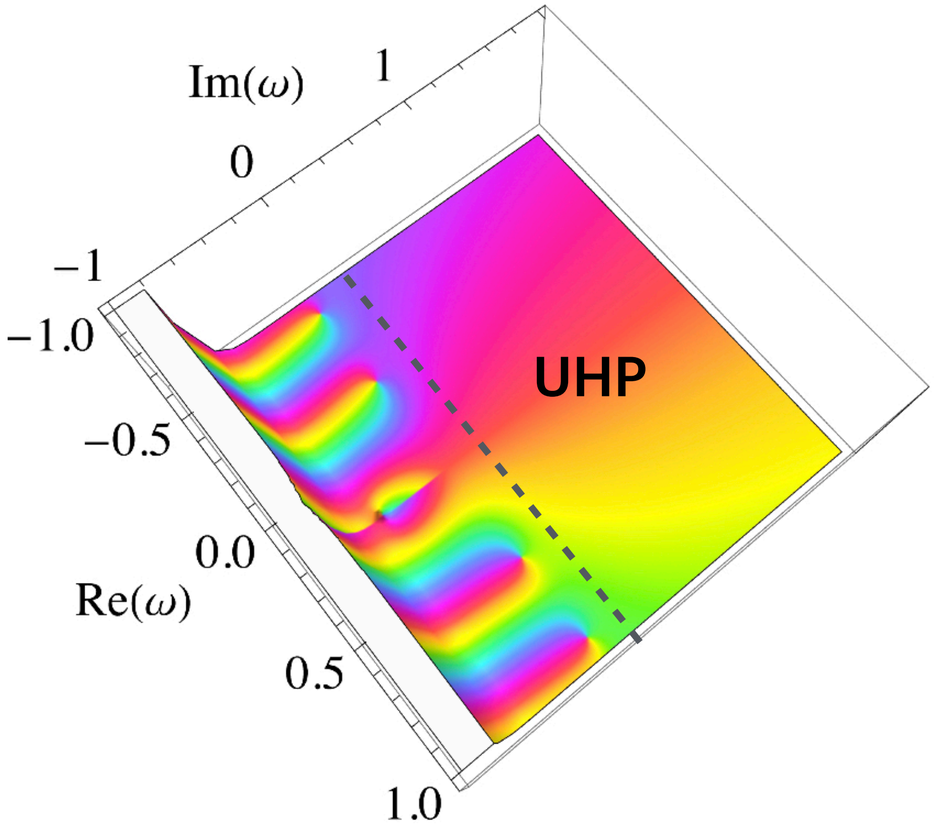
$$G_R{}^\ell(\omega, x, x') \xrightarrow{x, x' \to \infty} S_\ell(\omega) = \frac{\left[(2GM\omega)^2\right]^{2iGM\omega} \Gamma\left(1+\ell-2iGM\omega\right)}{\Gamma\left(1+\ell+2iGM\omega\right)}$$

Newtonian potential

$$V \sim \frac{4GM\omega^2}{r}$$

Retarded Green's function

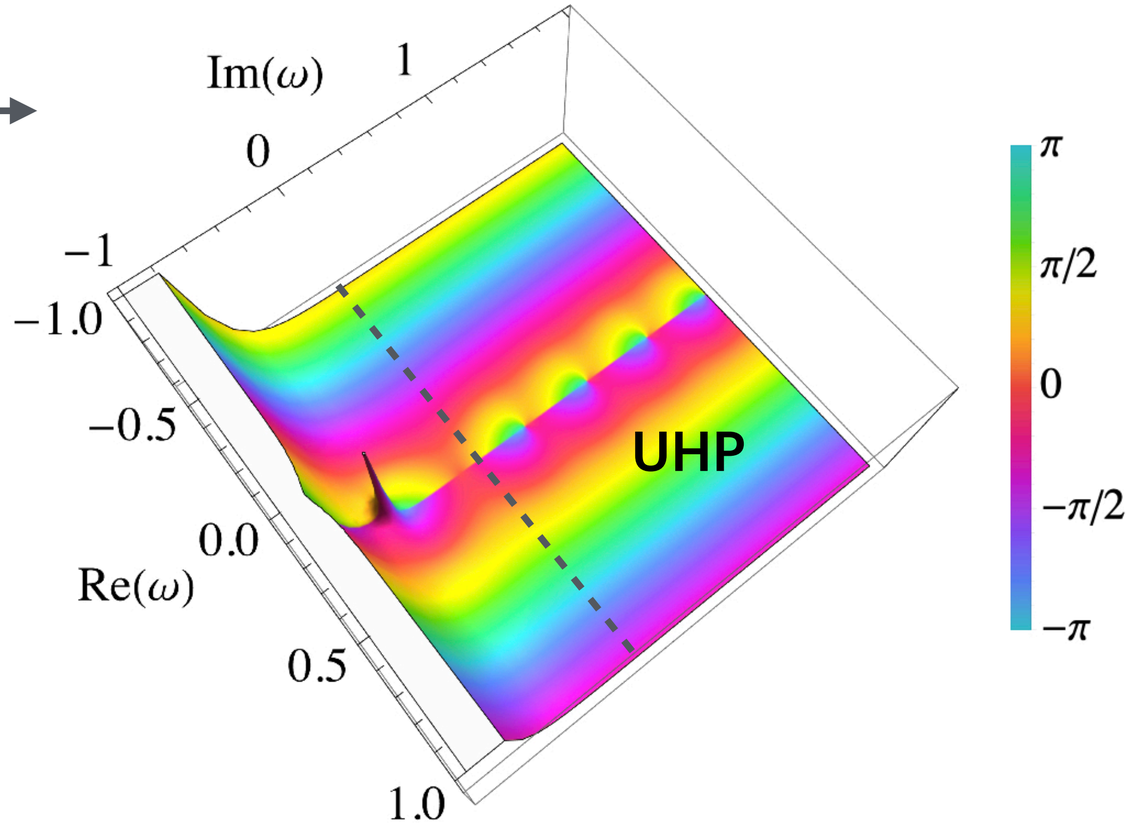
$$G_R{}^{\ell}(\omega, x, x')$$



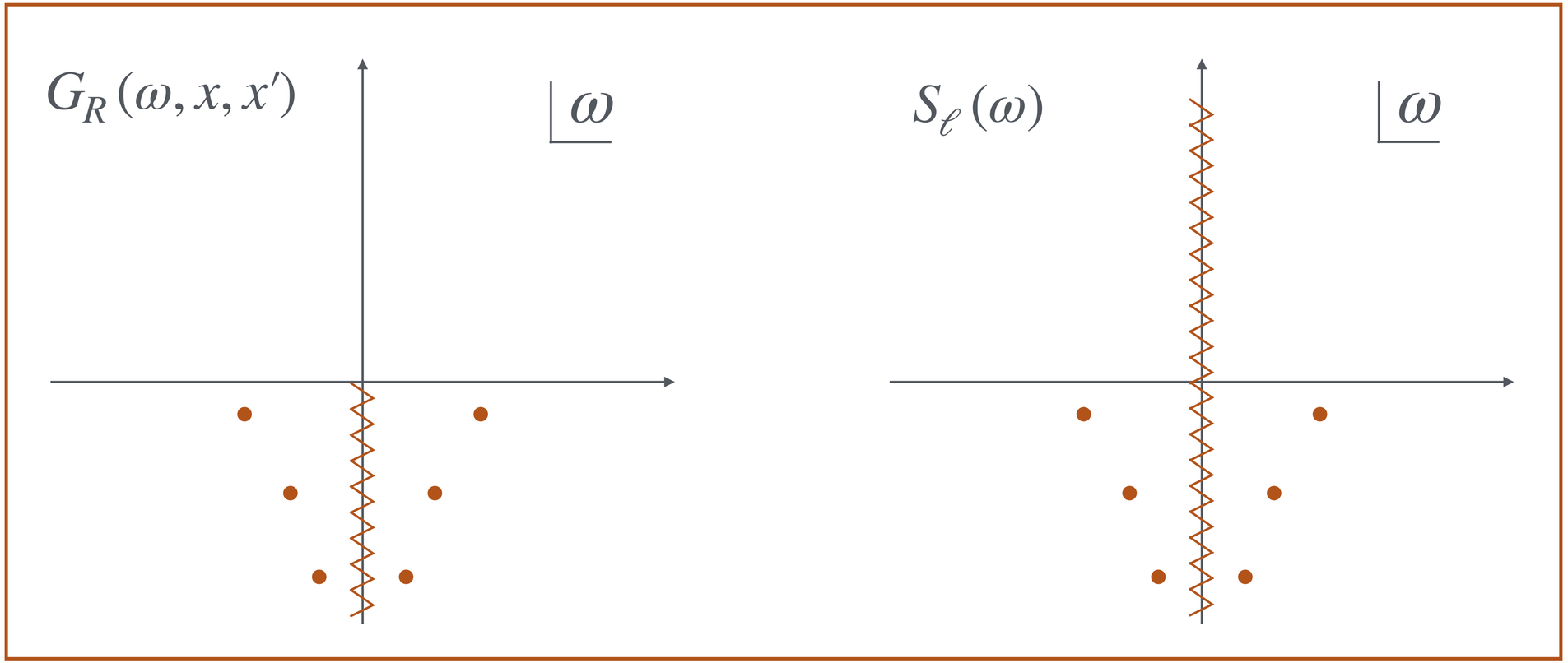
$$x, x' \to \infty$$

S-matrix

$$S_{\ell}(\omega)$$



So what is happening?



Stokes phenomena

$$G_R(\omega, x, x') \overset{x, x' \to \infty}{\longrightarrow} -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$

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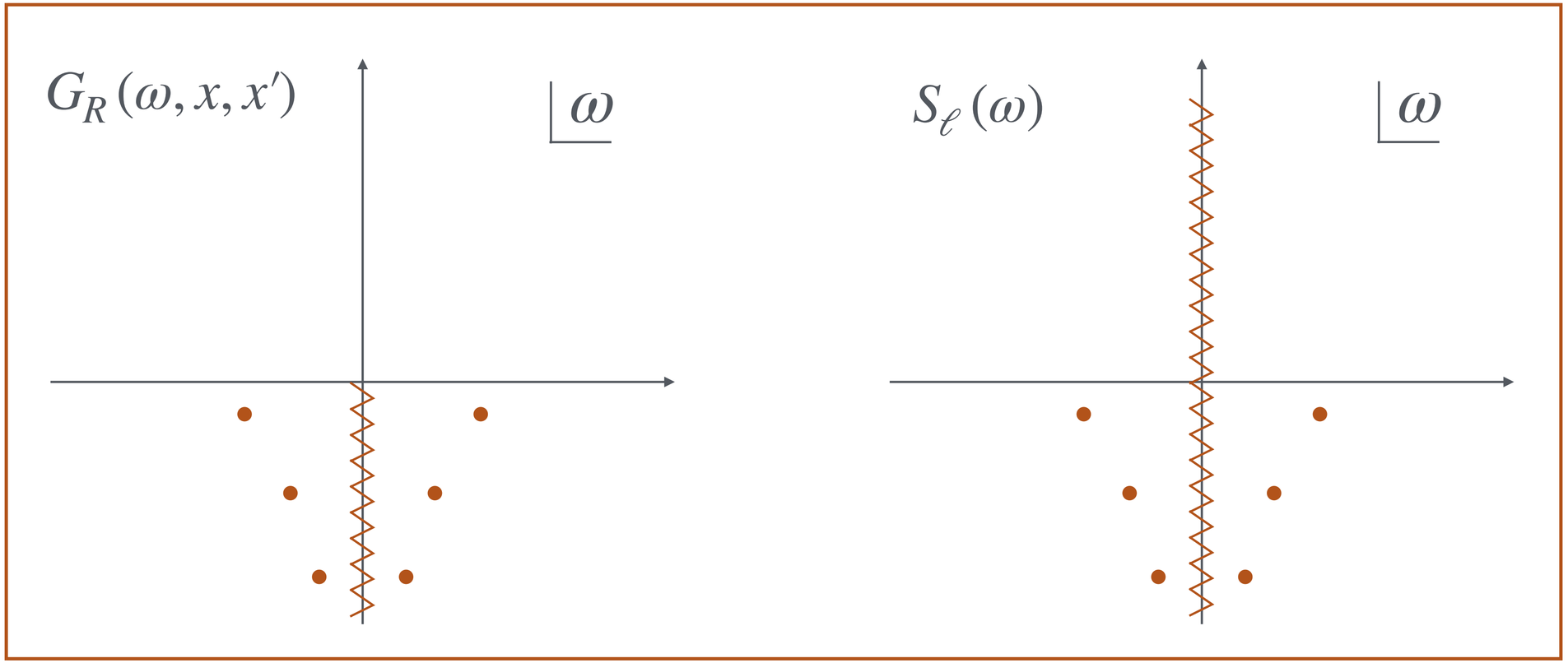

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Thank you!