

Relative Twisted Cohomology for Amplitudes and Cosmology

Andrzej Pokraka

mainly based on work with S. Caron-Huot, S. De and R. Glew

(1910.11852, 2104.06898, 2112.00055, 2308.03753, 2411.09695, 2508.11568)

Michigan Interdisciplinary Meeting on Amplitudes:
Bridges between Physics & Mathematics

Overview

Algebraic Geometry

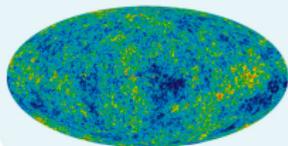
(Relative) Twisted Cohomology
Intersection Theory

String Integrals

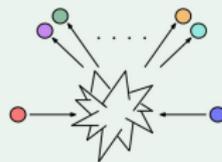
Dimensionally regulated
Feynman Integrals

(FRW)
Cosmological Integrals

Cosmological
Correlators



Scattering
Amplitudes



Anatomy of twisted integrals

universal multi-valued function called **twist** $u = \prod_{i=1}^m T_i$ ^{polynomial} $\alpha_i \in \mathbb{C} \setminus \mathbb{Z}$ (dim-reg/background cosmology)

Twisted integrals: $\int_{\gamma} \boxed{u} \boxed{\varphi}$

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Twisted integrals: $\int_{\gamma} u \varphi$ \rightarrow meromorphic differential form

Twisted variety $\mathcal{T} := \cup_{i=1}^m \{T_i = 0\}$

On-shell variety $\mathcal{S} := \cup_{i=1}^n \{S_i = 0\}$

$$\varphi \sim \frac{\text{dvol}}{T_1^{\mathbb{Z}} \cdots T_m^{\mathbb{Z}} S_1^{\mathbb{Z}} \cdots S_n^{\mathbb{Z}}}$$

↓
propagators

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As long as the twist u is generic, framework can be generalized to accommodate integer exponents [Caron-Huot, AP '21, '22]

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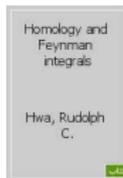
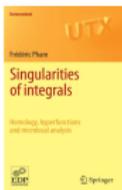
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Twisted cohomology forms finite dim vector space with an **inner product** (non-degenerate pairing) called the **intersection number**

$$\check{\varphi} \in \text{Dual (relative) twisted cohom} \longleftrightarrow \text{Twisted cohomology w/ } S_i^{\mathbb{Z}} \ni \varphi$$

(dual forms/integrands) $\langle \check{\varphi} | \varphi \rangle$ (forms/integrands)

Philosophy: why relative?



Tools/ideas from algebraic geometry (i.e., twisted (co)homology) often not directly applicable unless the physical integrals unless they are sufficiently **generic** or **regularized** in some way

$$u \rightarrow u \prod_i S_i^{\delta \in \mathbb{C}^{\mathbb{Z}}} \text{ with } \delta \rightarrow 0 \text{ at end} \quad \text{obscures physics : (}$$

Relative twisted cohomology is a rigorous mathematical framework for studying the **non-generic** class of integrals encountered in amplitudes and cosmology

- avoids needless regularization
- preserves important physical principles:
grades geometry/forms via GU cuts/ S_i -residues

$$\text{Res}_{S_i=0}[u\varphi] \quad \checkmark$$

factorization

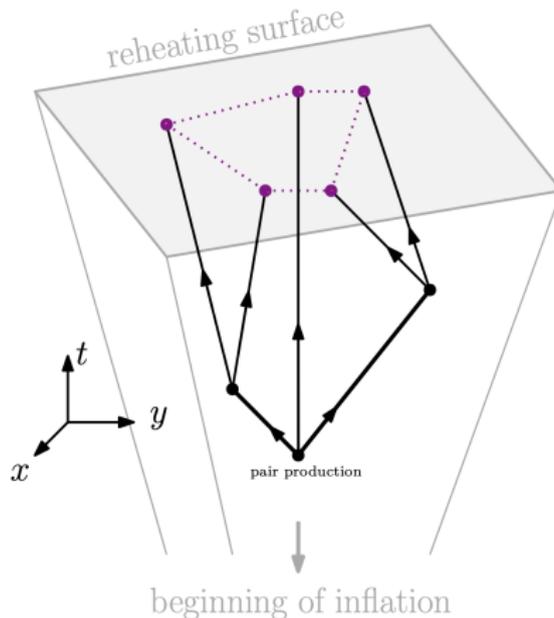
$$\text{Res}_{T_i=0}[u\varphi] \quad \times$$

DNE

Goal: Explain relative twisted cohomology and intersection theory in the context of cosmology

- 1) Introduce FRW wave function coefficients as twisted integrals
 - Too hard for direct integration \implies canonical DEQs
 - Amazingly: cosmological DEQs **purely combinatorial** (kinematic flow) whose geometric origin is obscure
- 2) Use the dual (relative) twisted cohomology to
 - Build a canonical basis from only the geometry (no bulk physics)
 - Combinatorial structure of DEQs straightforward consequence of the structure of the dual cohomology
- 3) Conclusion/future directions

Cosmological correlators



Cosmological correlators: measure spatial correlations of LSS or temperature (CMB)

Consequence of quantum fluctuations imprinted into reheating surface after inflation

~ initial conditions for time evolution of universe

Wavefunction coefficients

Mathematically, **cosmological correlators** can be decomposed into more fundamental objects called **wavefunction coefficients** ψ_\bullet , which are the cosmological analogues of **scattering amplitudes**

The toy model: Conformally-coupled scalar field in a power-law FRW spacetime [Arkani-Hamed, Maldacena '15, Arkani-Hamed, Benincasa, Postnikov '17 + many more]

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x} \cdot d\mathbf{x}) \quad a(\eta) := \frac{1}{\eta^{1+\varepsilon}} \begin{cases} \varepsilon = 0 & \text{dS} \\ \varepsilon = -1 & \text{flat} \\ \varepsilon = -2 & \text{RD} \\ \varepsilon = -3 & \text{MD} \end{cases}$$

Wavefunction coefficients at any ε

Wavefunction coeff for any ε :*

$$\psi_G^{(\varepsilon)}(\underbrace{\mathbf{X}, \mathbf{Y}}_{\text{kin. data}}) = \int_0^\infty \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{\substack{\text{twist: } u \\ \mathcal{T} = \cup_{i=1}^n \{x_i=0\}}} \underbrace{\psi_G^{(\text{flat})}(\mathbf{x} + \mathbf{X}, \mathbf{Y}) dx_1 \cdots dx_n}_{\substack{\varphi_{\text{phys}} \\ \frac{\Omega[\mathcal{P}_G](\mathbf{X}, \mathbf{Y})}{\text{dvol}} \Big|_{\mathbf{x} \rightarrow \mathbf{x} + \mathbf{X}}}}$$

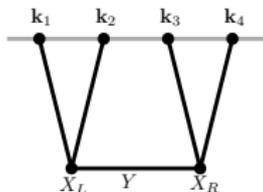
$\alpha_i := \alpha_i(\varepsilon, d, \text{valence of interaction vertices}) \in \mathbb{C} \setminus \mathbb{Z}$ assumed generic

φ_{phys} has poles outside of \mathcal{T} : $\varphi_{\text{phys}} = \frac{\text{num}}{S_1 \cdots S_m}$

On-shell variety: $\mathcal{S} = \cup_{i=1}^m \{S_i = 0\}$

[*Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel, De, **AP** Fan, Xianyu, Benincasa, Brunello, Mandal, Mastrolia, Vazão Fevola, Sattelberger, Westerdijk, Grimm + many more]

Example: the 2-site chain graph



$$Y = |\mathbf{k}_1 + \mathbf{k}_2|$$

$$X_L = |\mathbf{k}_1| + |\mathbf{k}_2|$$

$$X_R = |\mathbf{k}_3| + |\mathbf{k}_4|$$

$$\psi(\varepsilon) = \int_0^\infty \underbrace{x_1^{\alpha_1} x_2^{\alpha_2}}_u$$

$$\frac{d^3 \mathbf{x}}{S_1 S_2 S_3} \varphi_{\text{phys}}$$

Twisted variety:

$$T_1 = x_1$$

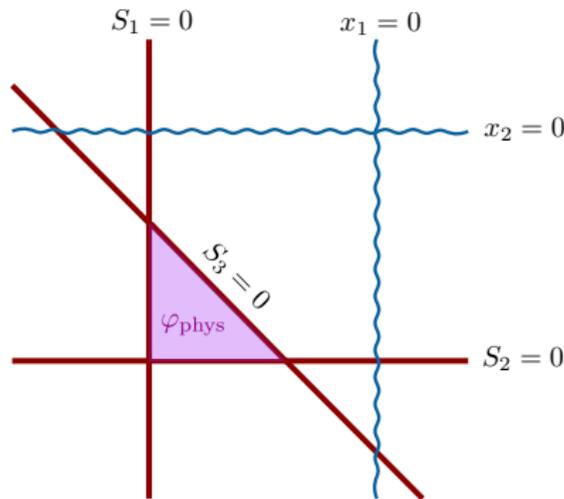
$$T_2 = x_2$$

On-shell variety:

$$S_1 = x_1 + (X_L + Y),$$

$$S_2 = x_2 + (X_R + Y),$$

$$S_3 = x_1 + x_2 + \underbrace{X_L + X_R}_{X_{\text{tot}}}$$



Too hard to integrate directly

Find **canonical DEQs** (Pfaffian system/Gauss-Manin connection) instead

Need basis of forms $\{\varphi_a\}_{a=1,\dots,\chi}$

Then, compute **DEQs** wrt kinematic parameters X_a and Y_a

$$d_{\text{kin}}\varphi_a = A_{ab} \wedge \varphi_b \quad d_{\text{kin}} = \sum_a dX_a \partial_{X_a} + \sum_a dY_a \partial_{Y_a}$$

Canonical if: $\underline{\mathbf{A}}|_{\alpha_i \rightarrow \alpha_i \alpha'} \rightarrow \alpha' \underline{\mathbf{A}}$

\implies simple solution as expansion in α' where coefficients are iterated integrals (i.e., multiple polylogarithms)

$$\int_{\gamma} u \varphi_a = I_{0,a} + \alpha' \int A_{ab} I_{0,b} + (\alpha')^2 \int A_{ab} A_{bc} I_{0,c} + \mathcal{O}((\alpha')^3)$$

How to compute $A_{\bullet\bullet}$?

Usually done via integration by parts/Stokes in presence of a **twist**

As long as $\partial\gamma = 0$ OR $\partial\gamma \in \mathcal{T}$

$$0 = \int_{\gamma} d(u \psi) = \int_{\gamma} u \nabla_{\omega} \psi$$

$$\nabla_{\omega} = d + \omega \wedge \quad \omega = d \log u = \sum_i \alpha_i \frac{dx_i}{x_i} \quad (\text{covariant derivative})$$

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Promising alternative: intersection numbers

:) clear why integrals are related and computed via localization/residues

:) map to dual cohomology and expose combinatorics of cosmo DEQs

FRW-cohomology: inequivalent integrands

FRW integrals fixed by choice of **twist** u and the **topological space**:

$$\mathbb{C}^n \setminus \underbrace{\mathcal{T} := \bigcup_{i=1}^n \{x_i = 0\}}_{\text{branch/twisted sing.}} \cup \underbrace{\mathcal{S} := \bigcup_{i=1}^m \{S_i = 0\}}_{\text{poles/untwisted sing.}}$$

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FRW twisted (co)homology:

$$H^\bullet(\mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}); \nabla_\omega) = \frac{\nabla_\omega\text{-closed forms on } \mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}) \ (\nabla_\omega \varphi = 0)}{\nabla_\omega\text{-exact forms (IBP identities } \varphi \neq \nabla_\omega \psi)}$$

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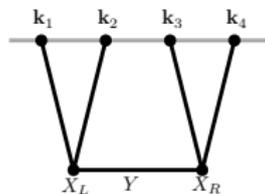
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(co)homology is finite dimensional v -space **localized to middle degree***

*(mild assumptions on u)

$$\dim H^p(\mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}); \nabla_\omega) = \begin{cases} |\chi(\mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}))| & p = n \\ 0 & \text{other} \end{cases}$$

2-site chain: topological space



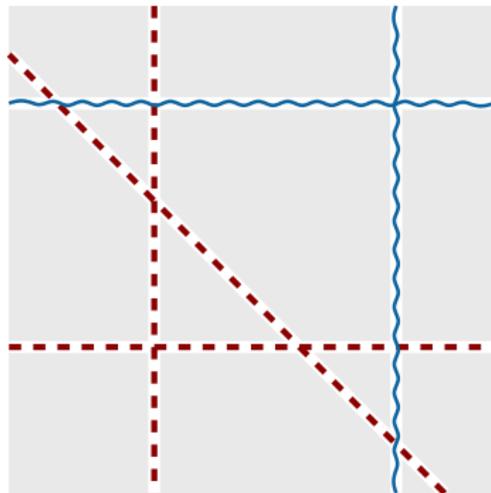
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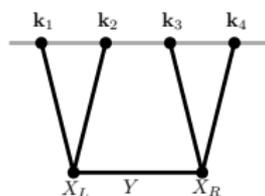
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Topological space : $\mathbb{C}^2 \setminus (\mathcal{T} \cup \mathcal{S}) =$



2-site chain: topological space



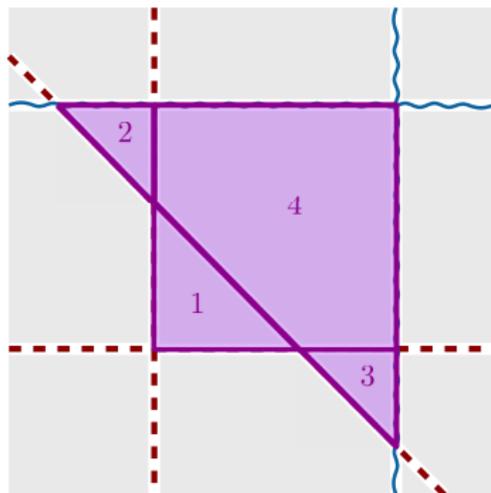
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Topological space : $\mathbb{C}^2 \setminus (\mathcal{T} \cup \mathcal{S}) =$



Dimension: $|\chi(\mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}))| = 4$ (count bounded chambers)

2-site chain: FRW forms via +geometry/Poincaré duality

Complements of hyperplane arrangements are **positive geometries** for which we can define the **canonical form** Ω :

$$\Omega : \underbrace{(\check{\gamma} : \text{region bounded by } \mathcal{T} \cup \mathcal{S})}_{\in H_n(\mathbb{C}^n \setminus \mathcal{T}, \mathcal{S}; \mathcal{L}_{-\omega})} \mapsto \text{FRW-form with simple poles on } \partial\check{\gamma}$$

such that

$$\text{Res}_{D=0} \Omega[\check{\gamma}] = \pm \Omega[\partial_D \check{\gamma}] \quad D \in \{S_i, T_i\}$$

Examples:

$$\Omega[\bullet] = 1$$

$$\Omega(a \text{---} b) = d \log \frac{x-a}{x-b}$$

$$\text{Res}_{x=a,b} \Omega[a \text{---} b] = \pm \Omega[\bullet] = \pm 1$$

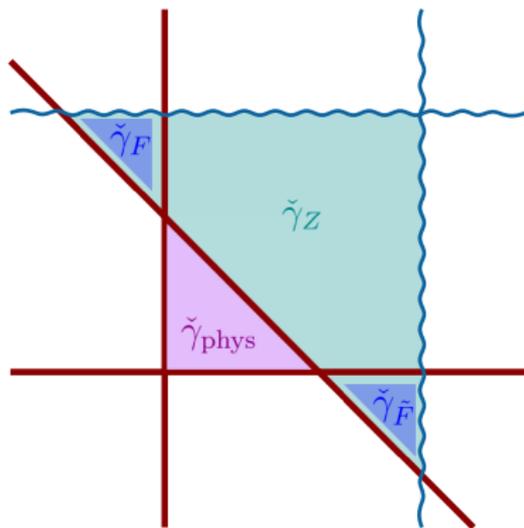
$$\Omega \left[\begin{array}{c} s_3 \\ \triangle \\ s_2 \text{---} s_1 \end{array} \right] = d \log \frac{S_1}{S_2} \wedge d \log \frac{S_2}{S_3}$$

$$\text{Res}_{S_i=0} \Omega \left[\begin{array}{c} s_3 \\ \triangle \\ s_2 \text{---} s_1 \end{array} \right] = \Omega \left[\begin{array}{c} s_i \cap s_j \text{---} s_i \cap s_k \\ \bullet \text{---} \bullet \end{array} \right]$$

2-site chain: Forms and kinematic flow

Kinematic flow: prescription for choosing basis elements along with **graphical/combinatorial** rules for computing the kinematic connection

$\underline{\mathbf{A}}_{\text{kin flow}}$ [Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel '23]



$$\Omega[\check{\gamma}_{\text{phys}}] = \frac{d^3 \mathbf{x}}{S_1 S_2 S_3} = d \log \frac{S_1}{S_2} \wedge d \log \frac{S_2}{S_3}$$

$$\Omega[\check{\gamma}_F] = d \log \frac{S_3}{S_1} \wedge d \log \frac{S_1}{x_2}$$

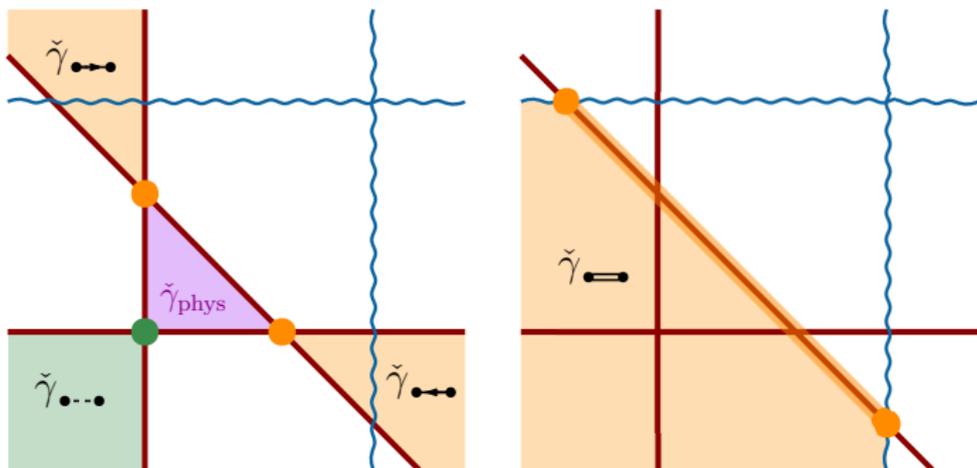
$$\Omega[\check{\gamma}_{\bar{F}}] = d \log \frac{S_2}{S_3} \wedge d \log \frac{S_3}{x_1}$$

$$\Omega[\check{\gamma}_Z] = d \log \frac{S_3}{x_1} \wedge d \log \frac{x_1}{x_2}$$

$$\underline{\mathbf{A}}_{\text{kin flow}} = \begin{pmatrix} \bullet & \bullet & \bullet & \\ & \bullet & & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{pmatrix}$$

Triangular but all elements mix

2-site chain: an improved basis



$$\Omega[\check{\gamma}_{\dots}] = d \log S_1 \wedge d \log S_2$$

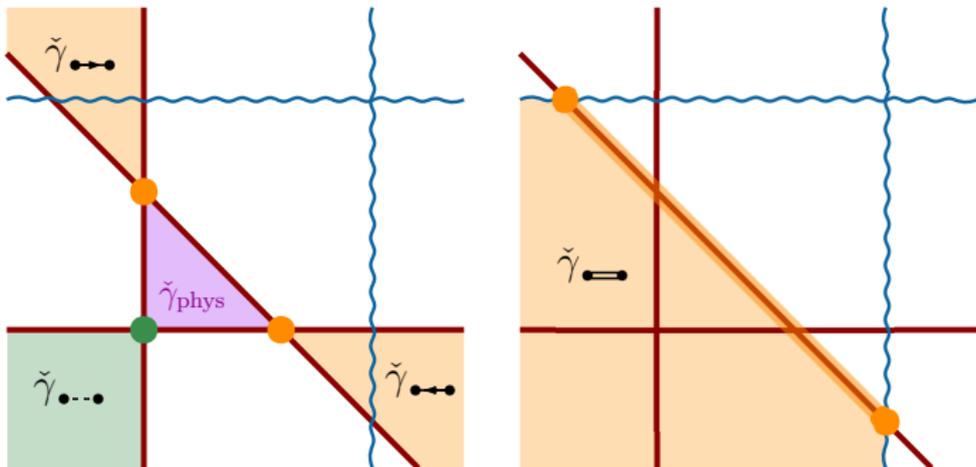
$$\Omega[\check{\gamma}_{\rightarrow\dots}] = d \log S_3 \wedge d \log S_1$$

$$\Omega[\check{\gamma}_{\leftarrow\dots}] = d \log S_3 \wedge d \log S_2$$

$$\Omega[\check{\gamma}_{\rightleftharpoons}] = d \log S_3 \wedge d \log \frac{x_1}{x_2}$$

$$\begin{aligned} \Omega[\check{\gamma}_{\text{phys}}] &= \frac{d^3 \mathbf{x}}{S_1 S_2 S_3} = d \log \frac{S_1}{S_2} \wedge d \log \frac{S_2}{S_3} \\ &= \Omega[\check{\gamma}_{\dots}] + \Omega[\check{\gamma}_{\rightarrow\dots}] + \Omega[\check{\gamma}_{\leftarrow\dots}] \end{aligned}$$

2-site chain: an improved basis



$$\underline{\mathbf{A}} = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix}$$

General feature that only gets more dramatic

Why does this basis simplify $\underline{\mathbf{A}}$?

Make clear by studying the dual cohomology + intersection #'s!

The intersection number

Intersection number $\langle \check{\bullet} | \bullet \rangle$ is an **inner product** on FRW-forms

$$\langle \check{\varphi} | \varphi \rangle : \overbrace{H^n(\mathbb{C}^n \setminus \mathcal{T}, \mathcal{S}; \check{\nabla}_{-\omega})}^{\text{dual (vector) space}} \times \overbrace{H^n(\mathbb{C}^n \setminus (\mathcal{T} \cup \mathcal{S}); \nabla_{\omega})}^{\text{FRW-forms}} \rightarrow \mathbb{C}[[\alpha_{\bullet}, X_{\bullet}, Y_{\bullet}]]$$

The intersection number **localizes**: $\langle \check{\bullet} | \bullet \rangle \sim \text{Res}_{\check{\varphi}}[\varphi]^*$

* u dependence removed $\implies \text{Res}_{T_i=0}$ is ok in this context

Efficient formula for computing the DEQs

$$A_{ac} = \sum_{b=1}^{\chi} C_{cb}^{-1} \langle \check{\varphi}_b | d_{\text{kin}} \varphi_a \rangle \quad C_{ab} = \langle \check{\varphi}_a | \varphi_b \rangle$$

Constructing the relative twisted cohomology

Theorem/Claim (Caron-Huot, AP '21,'22)

Under mild assumptions, $H^n(\mathbb{C}^n \setminus \mathcal{T}, \mathcal{S}; \check{\nabla}_{-\omega})$ is a graded vector space equal to the direct sum of twisted cohomology on each component of the **on-shell variety**/cuts

$$\mathcal{C}_J := (\mathbb{C}^n \setminus \mathcal{T}) \bigcap_{j \in J} \{S_j = 0\}$$

and **localizes to middle degree**

Proof: follows from computing the spectral sequence of double complex that degenerates at the second page

[AP WIP (parametric Feynman integrals)]

2-site chain: What are dual forms?

$$\begin{aligned}
 & H^2 \left(\left[\begin{array}{c} \text{gray square} \\ \text{blue dashed lines} \end{array} \right], \left[\begin{array}{c} \text{red lines} \\ \text{red triangle} \end{array} \right]; \check{\nabla}_{-\omega} \right) \\
 = & H^2 \left(\left[\begin{array}{c} \text{gray square} \\ \text{blue dashed lines} \end{array} \right]; \nabla_{-\omega} \right) \oplus H^1 \left(\left[\begin{array}{c} \text{red lines} \\ \text{blue dot} \end{array} \right]; \nabla_{-\omega} |_{S_1=0} \right) \oplus H^1 \left(\left[\begin{array}{c} \text{red lines} \\ \text{blue dot} \end{array} \right]; \nabla_{-\omega} |_{S_2=0} \right) \\
 & \oplus H^1 \left(\left[\begin{array}{c} \text{red lines} \\ \text{orange diagonal} \\ \text{blue dots} \end{array} \right]; \nabla_{-\omega} |_{S_3=0} \right) \oplus H^0 \left(\left[\begin{array}{c} \text{red lines} \\ \text{orange dot} \end{array} \right] \right) \oplus H^0 \left(\left[\begin{array}{c} \text{red lines} \\ \text{orange dot} \end{array} \right] \right) \\
 & \oplus H^0 \left(\left[\begin{array}{c} \text{red lines} \\ \text{green dot} \end{array} \right] \right)
 \end{aligned}$$

2-site chain: What are dual forms?

$$\begin{aligned}
 & H^2 \left(\left(\begin{array}{c} \text{[Square with blue dashed lines]} \\ \text{[Red triangle]} \end{array} ; \check{\nabla}_{-\omega} \right) \right) \\
 = & \cancel{H^2 \left(\left(\begin{array}{c} \text{[Square with blue dashed lines]} \\ \text{[Red triangle]} \end{array} ; \nabla_{-\omega} \right) \right)} \oplus \cancel{H^1 \left(\left(\begin{array}{c} \text{[Red vertical line]} \\ \text{[Red triangle]} \end{array} ; \nabla_{-\omega} |_{S_1=0} \right) \right)} \oplus H^1 \left(\left(\begin{array}{c} \text{[Red horizontal line]} \\ \text{[Red triangle]} \end{array} ; \nabla_{-\omega} |_{S_2=0} \right) \right) \\
 & \oplus H^1 \left(\left(\begin{array}{c} \text{[Red diagonal line]} \\ \text{[Red triangle]} \end{array} ; \nabla_{-\omega} |_{S_3=0} \right) \right) \oplus H^0 \left(\begin{array}{c} \text{[Red vertical line]} \\ \text{[Red triangle]} \end{array} \right) \oplus H^0 \left(\begin{array}{c} \text{[Red horizontal line]} \\ \text{[Red triangle]} \end{array} \right) \\
 & \oplus H^0 \left(\begin{array}{c} \text{[Red cross]} \\ \text{[Green dot]} \end{array} \right)
 \end{aligned}$$

2-site chain: Dual forms

Unique bounded chamber on every boundary (cut)! Already localized!

$$\begin{aligned}
 & H^2 \left(\left[\begin{array}{c} \text{shaded square with dashed lines} \\ \text{triangle with red lines} \end{array} \right]; \check{\nabla}_{-\omega} \right) \\
 &= H^1 \left(\left[\begin{array}{c} \text{diagonal line with blue dots} \\ \text{orange line with red dots} \end{array} \right]; \nabla_{\omega}|_{S_3=0} \right) \quad \Omega \left[\begin{array}{c} \text{diagonal line with blue dots} \end{array} \right] \\
 &\oplus H^0 \left(\left[\begin{array}{c} \text{vertical line with orange dot} \end{array} \right] \right) \quad \Omega \left[\begin{array}{c} \text{vertical line with orange dot} \end{array} \right] \\
 &\oplus H^0 \left(\left[\begin{array}{c} \text{diagonal line with orange dot} \end{array} \right] \right) \quad \Omega \left[\begin{array}{c} \text{diagonal line with orange dot} \end{array} \right] \\
 &\oplus H^0 \left(\left[\begin{array}{c} \text{vertical line with green dot} \end{array} \right] \right) \quad \Omega \left[\begin{array}{c} \text{vertical line with green dot} \end{array} \right]
 \end{aligned}$$

2-site chain: Dual forms

Unique bounded chamber on every boundary (cut)! Already localized!

$$H^2 \left(\left[\text{Diagram: Gray square with blue dashed lines} \right], \left[\text{Diagram: Red triangle with red lines} \right]; \check{\nabla}_{-\omega} \right)$$

$$= H^1 \left(\left[\text{Diagram: Orange diagonal line with blue dots} \right]; \nabla_{\omega}|_{S_3=0} \right)$$

$$\oplus H^0 \left(\left[\text{Diagram: Red vertical line with orange dot} \right] \right)$$

$$\oplus H^0 \left(\left[\text{Diagram: Red horizontal line with orange dot} \right] \right)$$

$$\oplus H^0 \left(\left[\text{Diagram: Red cross with green dot} \right] \right)$$

$$\Omega \left[\left[\text{Diagram: Orange diagonal line with blue dots} \right] \right] = \Omega \left[\left[\text{Diagram: Orange square with blue wavy lines and red diagonal line} \right] \right]$$

$$\Omega \left[\left[\text{Diagram: Red vertical line with orange dot} \right] \right] = \Omega \left[\left[\text{Diagram: Orange triangle with red vertical line and blue dots} \right] \right]$$

$$\Omega \left[\left[\text{Diagram: Red horizontal line with orange dot} \right] \right] = \Omega \left[\left[\text{Diagram: Orange triangle with red horizontal line and blue dots} \right] \right]$$

$$\Omega \left[\left[\text{Diagram: Red cross with green dot} \right] \right] = \Omega \left[\left[\text{Diagram: Green square with red cross and blue dots} \right] \right]$$

2-site chain: Dual forms

Unique bounded chamber on every boundary (cut)! Already localized!

$$H^2 \left(\begin{array}{c} \text{[Diagram: 2x2 grid with dashed blue lines]} \\ \text{[Diagram: red lines forming a triangle]} \end{array} ; \check{\nabla}_{-\omega} \right)$$

$$= H^1 \left(\begin{array}{c} \text{[Diagram: red line with blue dots]} \\ \text{[Diagram: orange line with blue dots]} \end{array} ; \nabla_{\omega} |_{S_3=0} \right)$$

$$\oplus H^0 \left(\begin{array}{c} \text{[Diagram: red lines forming a triangle]} \\ \text{[Diagram: orange line with blue dot]} \end{array} \right)$$

$$\oplus H^0 \left(\begin{array}{c} \text{[Diagram: red lines forming a triangle]} \\ \text{[Diagram: orange line with blue dot]} \end{array} \right)$$

$$\oplus H^0 \left(\begin{array}{c} \text{[Diagram: red lines forming a triangle]} \\ \text{[Diagram: green line with green dot]} \end{array} \right)$$

$$\Omega \left[\begin{array}{c} \text{[Diagram: blue dots]} \\ \text{[Diagram: orange line with blue dots]} \end{array} \right] = \text{Res}_{S_3=0} \Omega \left[\begin{array}{c} \text{[Diagram: orange square with red diagonal line]} \\ \text{[Diagram: blue wavy lines]} \end{array} \right]$$

$$\Omega \left[\begin{array}{c} \text{[Diagram: orange line with blue dot]} \\ \text{[Diagram: red lines forming a triangle]} \end{array} \right] = \text{Res}_{S_1=0} \circ \text{Res}_{S_3=0} \Omega \left[\begin{array}{c} \text{[Diagram: orange square with red diagonal line]} \\ \text{[Diagram: red vertical line]} \end{array} \right]$$

$$\Omega \left[\begin{array}{c} \text{[Diagram: orange line with blue dot]} \\ \text{[Diagram: red lines forming a triangle]} \end{array} \right] = \text{Res}_{S_2=0} \circ \text{Res}_{S_3=0} \Omega \left[\begin{array}{c} \text{[Diagram: orange square with red diagonal line]} \\ \text{[Diagram: red horizontal line]} \end{array} \right]$$

$$\Omega \left[\begin{array}{c} \text{[Diagram: green line with green dot]} \\ \text{[Diagram: red lines forming a triangle]} \end{array} \right] = \text{Res}_{S_2=0} \circ \text{Res}_{S_1=0} \Omega \left[\begin{array}{c} \text{[Diagram: orange square with red diagonal line]} \\ \text{[Diagram: green square with red cross]} \end{array} \right]$$

Good FRW-basis: MaxCut(FRW-basis element) = canonical form of cut

2-site chain: Intersection matrix

$$\underline{\mathbf{C}} = \begin{pmatrix} \langle \check{\bullet} | \bullet \rangle & \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] \\ \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & 1 & & & \\ \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & & 1 & & \\ \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & & & 1 & \\ \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right] & & & & \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \end{pmatrix}$$

$$\Rightarrow \underline{\mathbf{A}} = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \propto \check{\underline{\mathbf{A}}}^T$$

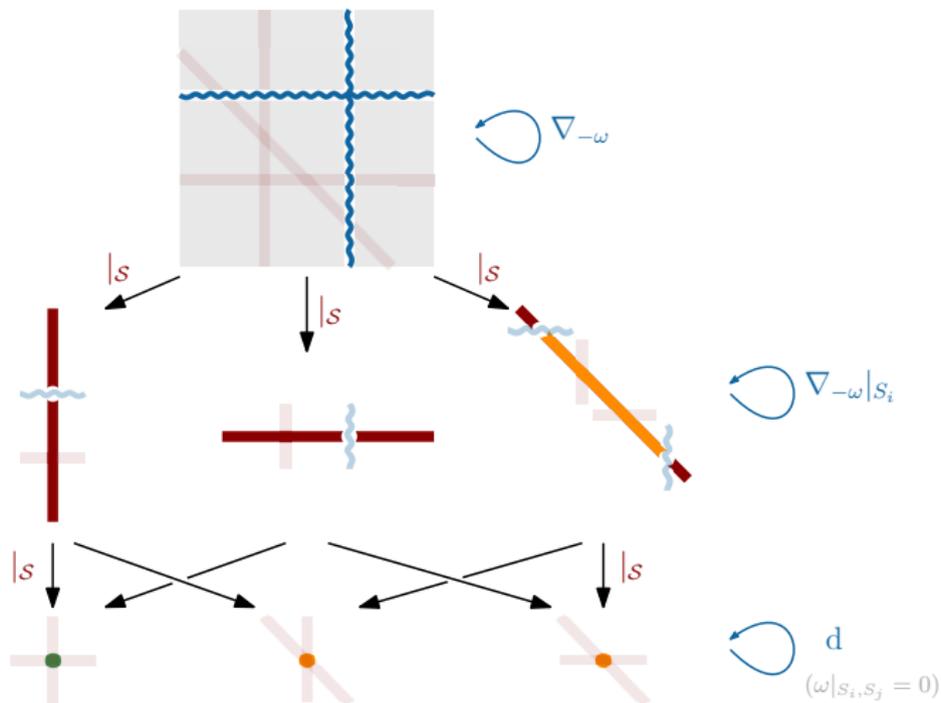
In general $\check{\underline{\mathbf{A}}}$ is block triangular
(wrt GU cuts)!

Not triangular if S_i twisted instead
($S_i^{\mathbb{Z}} \rightarrow S_i^{\beta_i \in \mathbb{C} \setminus \mathbb{Z}}$)

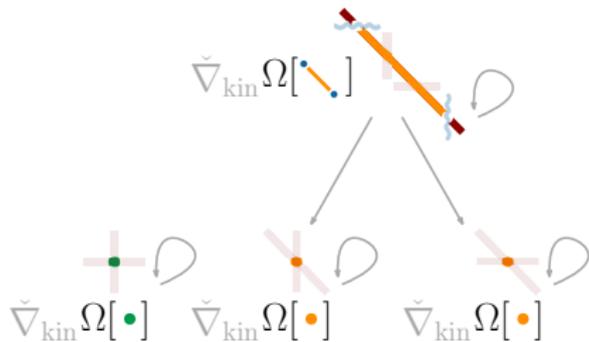
How do we see geometrically?

2-site chain: the kinematic connection $A_{\bullet\bullet}$

$$\check{\nabla}_{-\omega} := \nabla_{-\omega} + |s \quad [\text{Bott, Tu, } \textit{Differential forms in algebraic topology}, '82 \text{ (d} \rightarrow \nabla_{-\omega})]$$



2-site chain: the kinematic connection $A_{\bullet\bullet}$



$$\underline{\check{A}} = \begin{pmatrix} & \Omega[\text{---}\bullet\text{---}] & \Omega[\text{---}\bullet\text{---}] & \Omega[\text{---}\bullet\text{---}] & \Omega[\bullet\text{---}] \\ \check{\nabla}_{-\omega}\Omega[\text{---}\bullet\text{---}] & \bullet & & & \\ \check{\nabla}_{-\omega}\Omega[\text{---}\bullet\text{---}] & & \bullet & & \\ \check{\nabla}_{-\omega}\Omega[\text{---}\bullet\text{---}] & & & \bullet & \\ \check{\nabla}_{-\omega}\Omega[\bullet\text{---}] & & \bullet & \bullet & \bullet \end{pmatrix}$$

Kinematic flow and the flow of cuts

Decorated graphs specify a collection $S_{j \in J}$ such that the cut/intersection

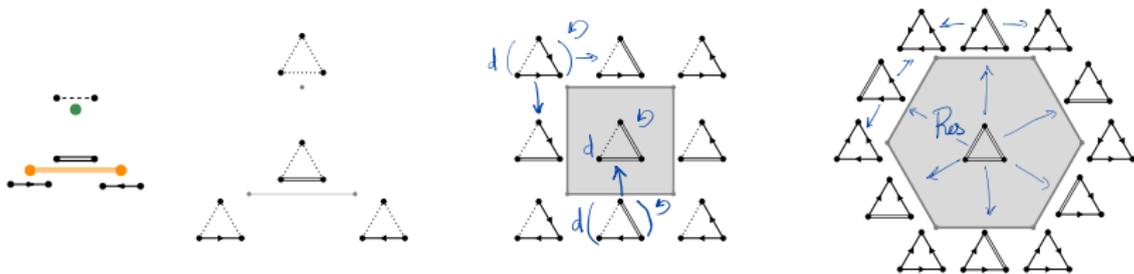
$$\mathcal{C}_J = (\mathbb{C}^n \setminus \mathcal{T}) \bigcap_{j \in J} \{S_j = 0\}$$

has a unique chamber bounded chamber Δ_J (i.e., )

Unique $\{\Omega[\Delta_J]\} \implies$ dual basis w/ $\check{\mathbf{A}}$ block triangular guaranteed by the boundary calculus of relative twisted cohomology

FRW basis $\{\Omega[\check{\gamma}_J]\}$ of unbounded chambers $\check{\gamma}_J$ (i.e., )

$$\text{Res}_{S_{j_1|j_1}} \circ \cdots \circ \text{Res}_{S_{j_1}} \Omega[\check{\gamma}_J] = \Omega[\Delta_J] \implies \underline{\mathbf{C}} \propto \mathbf{1} \implies \underline{\mathbf{A}} \propto \check{\mathbf{A}}^T$$



Conclusions

Relative twisted cohomology can be directly applied to large classes of integrals that appear in amplitudes and cosmology

Keeps the on-shell physics manifest (GU cuts)

Build a basis of FRW-forms from the **unique/canonical** dual basis with block triangular DEQs

Provided a **purely geometric origin** for the combinatorics of kinematic flow for the **improved basis** [De, AP '24; Glew, AP '25]

- “Same” basis constructed from “bulk physics” POV but antithetical to the positive geometry ethos

[Baumann, Goodhew, Joyce, Lee, Pimentel, Westerdijk '25]

Future directions

Graphical coaction for cosmological integrals (to appear soon with A. McLeod and L. Ren)

$$\Delta[\gamma|\varphi] = \sum_{a,b=1}^{\chi} C_{ab}^{-1} [\gamma|\Omega[\gamma_a]\rangle \otimes ([\gamma_b|\varphi] \bmod i\pi)$$

Encodes information about both differentials and discontinuities

Connect relative framework and cosmological story to “*Matroids and amplitudes*” [Lam '24]

More realistic cosmological models (massive/massless)

Better methods for computing intersection numbers in presence of higher degree hypersurfaces in multi-loop computations (tropicalization?)

Extra slides

Localization of the intersection number

Localization of the intersection number (for d log-forms)

$$\left\langle \underbrace{\Omega[\Delta_J]}_{\text{on cut } (\mathbb{C}^n \setminus \mathcal{T}) \cap_{j \in J} \{S_j=0\}} \mid \varphi = \Omega[\check{\gamma}] \right\rangle := \text{Res}_{\Delta_J} \circ \text{Res}_{S_{j_1|j_1}} \circ \cdots \circ \text{Res}_{S_{j_1}} [\Omega[\check{\gamma}]]$$

weighted residue that localizes
to vertices of $\Delta_J \in \mathcal{T}$

takes a GU cut

Purely combinatorial for canonical forms of positive geometries

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega \left[\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right]}_{\text{site 1}} \left| \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \right] \right\rangle = \text{Res}_{S_2=0} \circ \text{Res}_{S_1=0} \Omega \left[\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \right]$$

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega[\bullet]}_{\text{red cross}} \left| \Omega \left[\begin{array}{c} \text{red cross} \\ \text{green box} \end{array} \right] \right. \right\rangle = \Omega \left[\begin{array}{c} \text{red cross} \\ \text{green box} \end{array} \right]$$

The diagram illustrates the intersection matrix element for a 2-site chain. On the left, a bra state is shown as a red cross with a green dot at its center, labeled $\Omega[\bullet]$. A bracket underneath this state is labeled with a red cross, indicating it is the bra state of a red cross. On the right, a ket state is shown as a green box with a red cross on top and three dots inside, labeled Ω followed by the box. The inner product of these two states is equal to the ket state with the partial derivatives $\partial_{s_2} \partial_{s_1}$ applied to it.

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}}_{\text{orange arrow}} \middle| \Omega \begin{bmatrix} \text{orange box} \\ \text{red diagonal} \\ \text{blue wavy} \end{bmatrix} \right\rangle = \text{Res}_{\bullet} \circ \text{Res}_{S_3=0} \Omega \begin{bmatrix} \text{orange box} \\ \text{red diagonal} \\ \text{blue wavy} \end{bmatrix}$$

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right]}_{\substack{\bullet \\ \diagdown \\ \bullet}} \left| \Omega \left[\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \text{---} \end{array} \right] \right\rangle = \left(\frac{1}{\alpha_2} \text{Res} \bullet - \frac{1}{\alpha_1} \text{Res} \bullet \right) \circ \Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right]$$

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right]}_{\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array}} \left| \Omega \left[\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \end{array} \right] \right\rangle = \frac{1}{\alpha_2} \Omega \left[\begin{array}{c} \partial \bullet \\ \diagdown \\ \bullet \end{array} \right] - \frac{1}{\alpha_1} \Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \partial \bullet \end{array} \right]$$

2-site chain: Intersection matrix

$$\left\langle \underbrace{\Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right]}_{\substack{\text{red} \\ \text{orange} \\ \text{blue}}} \left| \Omega \left[\begin{array}{c} \text{orange box} \\ \text{red diagonal} \\ \text{blue wavy line} \\ \text{black arrow} \end{array} \right] \right\rangle = \frac{1}{\alpha_2} \Omega \left[\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right] + \frac{1}{\alpha_1} \Omega \left[\begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} \right]$$

