Exterior Cyclic Polytopes and Convexity of Amplituhedra

Lizzie Pratt
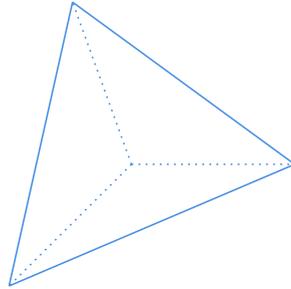
Joint with Elia Mazzucchelli
https://lizziepratt.com/notes

October 26, 2025

The positive Grassmannian

The *projective simplex* is

$$\Delta_n := \mathsf{conv}\{e_0, \ldots, e_n\} \subset \mathbb{P}^n.$$

The *Grassmannian* parameterizes k-spaces in $\mathbb{R}^n$, and is a projective variety via

$$\mathsf{Gr}(k, n) \to \mathbb{P}(\wedge^k \mathbb{R}^n)$$
$$\mathsf{span}(v_1, \ldots, v_k) \mapsto v_1 \wedge \ldots \wedge v_k.$$

The *positive Grassmannian* is

$$\mathsf{Gr}_{\geq 0}(k, n) := \Delta_{\binom{n}{k}-1} \cap \mathsf{Gr}(k, n).$$

The amplituhedron

Let Z be a $(k+m) \times n$ matrix with positive maximal minors.

$$\wedge^k Z : \ \mathbb{P}(\wedge^k \mathbb{R}^n) \ \dashrightarrow \ \mathbb{P}(\wedge^k \mathbb{R}^{k+m})$$

$$v_1 \wedge \ldots \wedge v_k \ \mapsto Z v_1 \wedge \ldots \wedge Z v_k.$$

The *amplituhedron* $\mathcal{A}_{k,m,n}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k,n)$.

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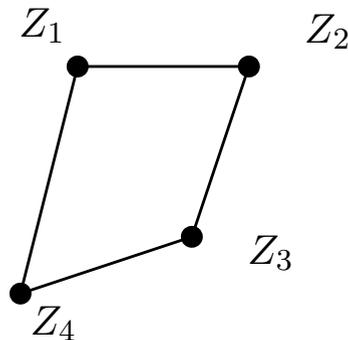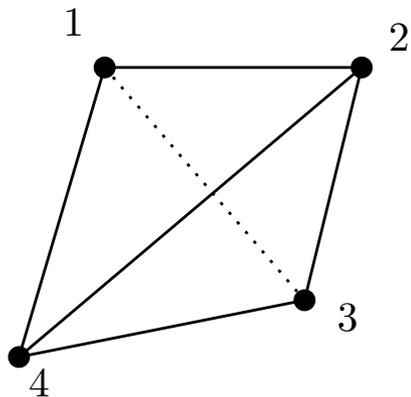$$\wedge^k Z : \ \mathbb{P}(\wedge^k \mathbb{R}^n) \ \dashrightarrow \ \mathbb{P}(\wedge^k \mathbb{R}^{k+m})$$

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Example $(k = 1)$
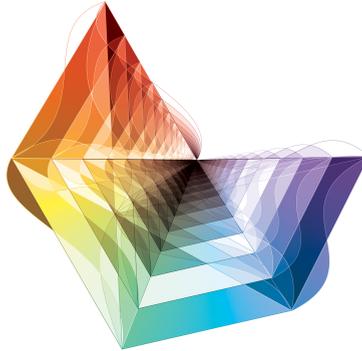
$$Z : \Delta_{n-1} \to \mathbb{P}^m$$

$$e_i \mapsto Z_i$$

The image is a *cyclic polytope*.
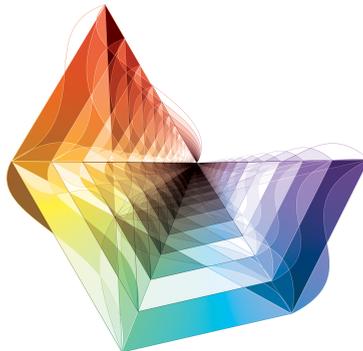
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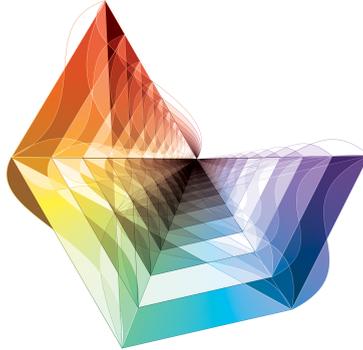
The *twistor coordinates* wrt Z on $\mathsf{Gr}(k, k+2)$ are

$$\langle ij \rangle := \det[Z_i \, Z_j \, Y^T], \qquad [Y] \in \mathsf{Gr}(k, k+2).$$

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On $\mathsf{Gr}(2, 4)$, we have

$$\langle 12 \rangle = (z_{1i}z_{2j} - z_{2i}z_{1j})p_{34} - (z_{1i}z_{3j} - z_{3i}z_{1j})p_{24} + (z_{2i}z_{3j} - z_{3i}z_{2j})p_{14} + \dots$$

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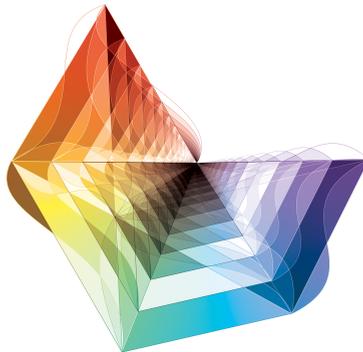The *twistor coordinates* wrt Z on $\mathsf{Gr}(k, k+2)$ are

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This vanishes on lines $[Y]$ meeting the line $Z_1 \wedge Z_2$ in $\mathbb{P}^3$.

Boundaries of the amplituhedron

Theorem (Ranestad–Sinn–Telen 24)

The algebraic boundary of the $m = 2$ amplituhedron is given by $\langle 12 \rangle, \ldots, \langle n-1\,n \rangle, \langle 1n \rangle = 0$.

Theorem (Even-Zohar–Lakrec–Tessler 25)

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Exterior cyclic polytopes

The *exterior cyclic polytope* of Z is

$$C_{k,m,n}(Z) := \wedge^k Z(\Delta_{\binom{n}{k}-1}) \ \subset \ \mathbb{P}(\wedge^k \mathbb{R}^{k+m})$$

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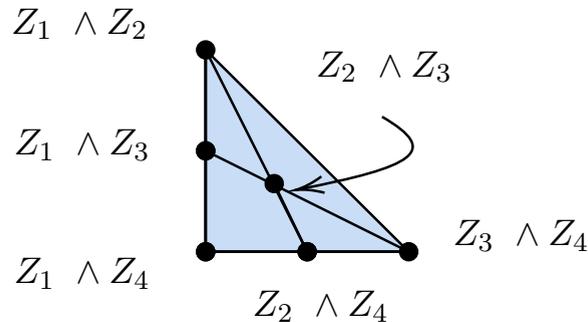$$= \mathsf{conv}(Z_{i_1} \wedge \ldots \wedge Z_{i_k} \ : \ \{i_1, \ldots, i_k\} \subset [n]).$$

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Example (The polytope $C_{2,1,4}(Z)$)

In $(\mathbb{P}^2)^*$, we have
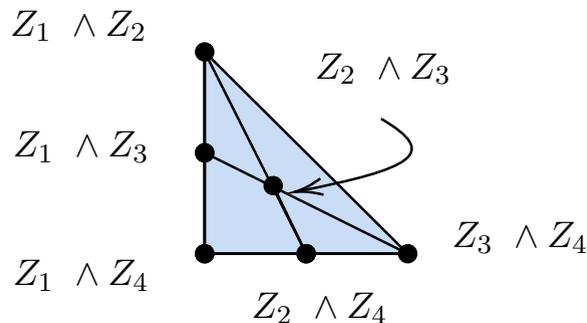
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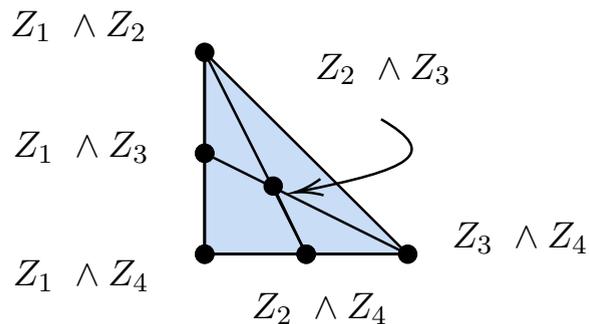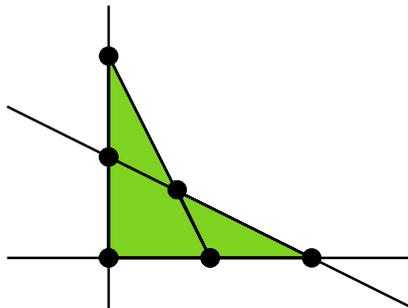In $(\mathbb{P}^2)^*$, we have

Theorem (Mazzucchelli–P)

The polytope $C_{k,m,n}(Z)$ is the convex hull of $\mathcal{A}_{k,m,n}(Z)$.

An example

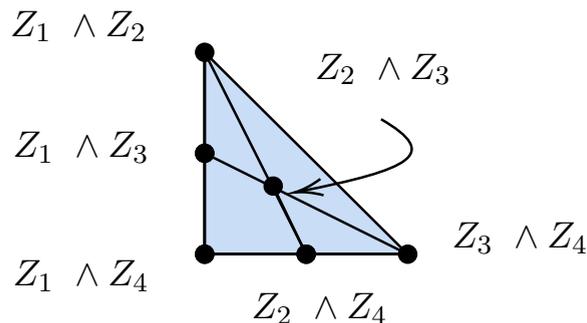The polytope $C_{2,1,4}(Z)$ looks like

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Theorem (Mazzucchelli–P)

The amplituhedron $\mathcal{A}_{2,2,n}(Z)$ equals $C_{2,2,n}(Z) \cap Gr(2,4)$.

An example with $k = m = 2$ and $n = 6$

Fix real numbers $0 < a < b < c < d < e < f$ and consider

$$
Z \ = \ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & f \\ a^2 & b^2 & c^2 & d^2 & e^2 & f^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 & f^3 \end{pmatrix} .
$$

Then $C_{2,2,6}(Z)$ is the convex hull in $\mathbb{P}^5$ of the 15 columns of $\wedge^2 Z$:

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\begin{pmatrix}
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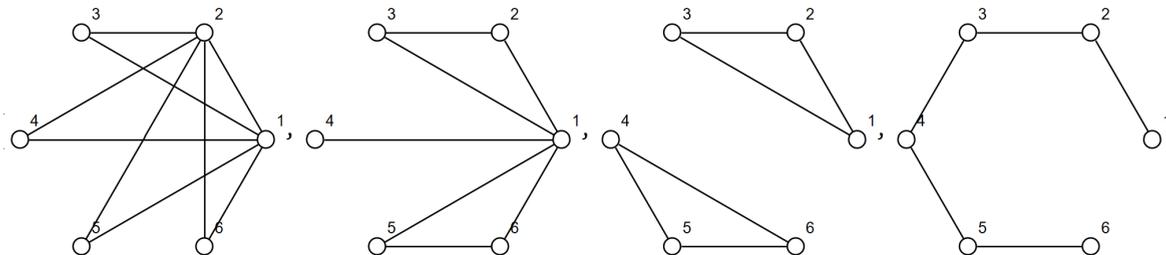
$$\begin{pmatrix} a - b & a - c & a - d & a - e & \cdots & d - f & e - f \\ a^2 - b^2 & a^2 - c^2 & a^2 - d^2 & a^2 - e^2 & \cdots & d^2 - f^2 & e^2 - f^2 \\ a^3 - b^3 & a^3 - c^3 & a^3 - d^3 & a^3 - e^3 & \cdots & d^3 - f^3 & e^3 - f^3 \\ a^2 b - ab^2 & a^2 c - ac^2 & a^2 d - ad^2 & a^2 e - ae^2 & \cdots & d^2 f - df^2 & e^2 f - ef^2 \\ a^3 b - ab^3 & a^3 c - ac^3 & a^3 d - ad^3 & a^3 e - ae^3 & \cdots & d^3 f - df^3 & e^3 f - ef^3 \\ a^3 b^2 - a^2 b^3 & a^3 c^2 - a^2 c^3 & a^3 d^2 - a^2 d^3 & a^3 e^2 - a^2 e^3 & \cdots & d^3 f^2 - d^2 f^3 & e^3 f^2 - e^2 f^3 \end{pmatrix}.$$

Substituting $(1, 3, 4, 7, 8, 9)$, it has f-vector $(15, 75, 143, 111, 30)$. Among the 30 facets, there are 15 4-simplices, six double pyramids over pentagons, three cyclic polytopes $C(4, 6)$, and three with f-vector $(9, 26, 30, 13)$.

Combinatorics changes as Z varies

Identify vectors $Z_i \wedge Z_j$ with edges ij of a complete graph. There are 30 facets, with four types of supporting hyperplanes:

For $(1, 3, 4, 7, 8, f)$, three facets for $f < 45/7$ are

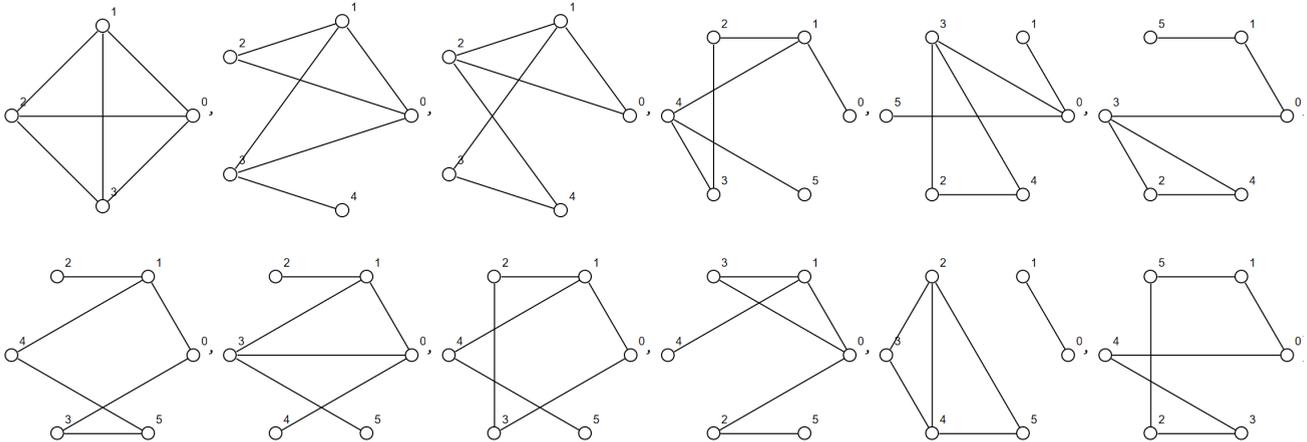$$\{12, 23, 34, 45, 56\}, \ \{12, 23, 34, 56, 16\}, \ \{12, 16, 34, 45, 56\}.$$

and for $f > 45/7$ change to

$$\{12, 16, 23, 34, 45\}, \ \{12, 16, 23, 45, 56\} \ \{16, 23, 34, 45, 56\}.$$

Example, continued

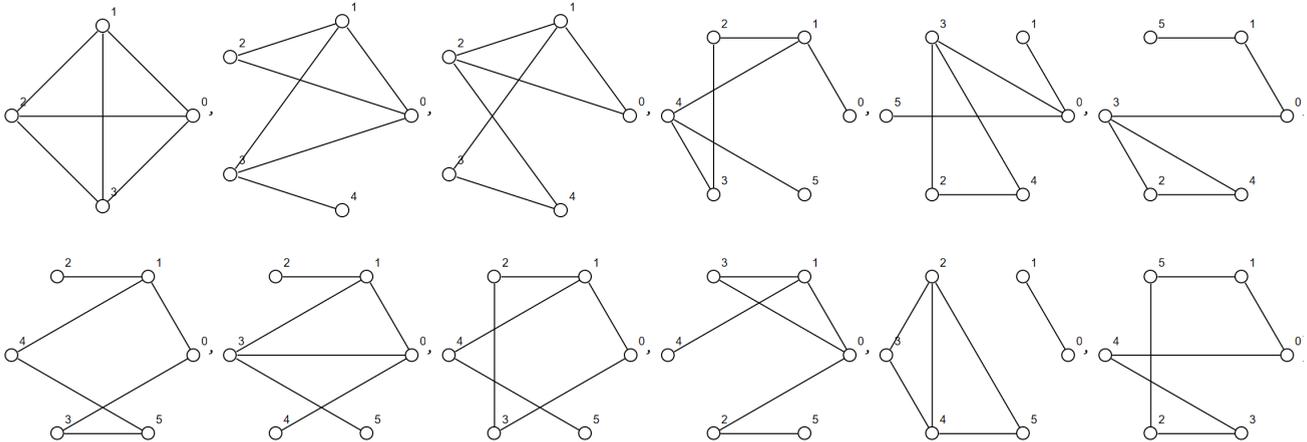Of the $\binom{15}{6}$ minors of $\wedge^2 Z$, 1660 are zero and 3345 are nonzero.

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Example, continued

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Symmetry classes of minors:

Sign of each minor is fixed by $a < \ldots < f$ except for

$$[12, 23, 34, 45, 56, 16] =$$
$$(a-c)(a-d)(a-e)(b-d)(b-e)(b-f)(d-f)(c-e)(c-f)$$
$$\cdot (abd - abe - acd + acf + ade - adf + bce - bcf - bde + bef + cdf - cef).$$

Results and computations

Theorem (Mazzucchelli–P)

The combinatorial type of $C_{2,2,n}(Z)$ is constant for positive $4 \times n$ matrices Z outside the closed locus where the polynomial $\det[Z_{12} \ Z_{23} \ Z_{34} \ Z_{45} \ Z_{56} \ Z_{16}]$ or one of its permutations is zero.

In Plücker coordinates on $Z \in \mathsf{Gr}(4, n)$:

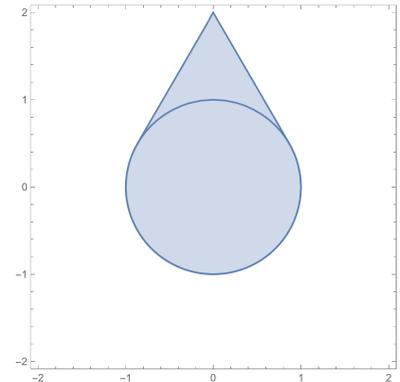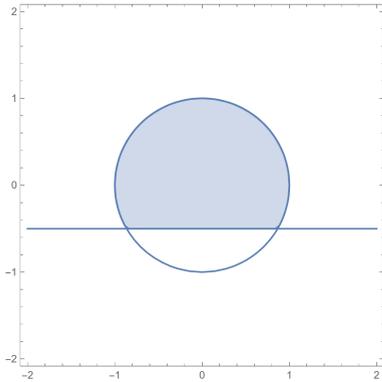$$p_{1234}p_{1356}p_{2456} - p_{1235}p_{1346}p_{2456} + p_{1235}p_{1246}p_{3456} \cdot$$

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In Plücker coordinates on $Z \in \mathsf{Gr}(4, n)$:

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For $k = m = 2$, small f-vectors include:

$n = 5$:	10	35	55	40	12	1
$n = 6$:	15	75	143	111	30	1
$n = 7$:	21	147	328	282	82	1
$n = 8$:	28	266	664	616	192	1
$n = 9$:	36	450	1217	1191	390	1

What is a *dual amplituhedron*?

The *polar dual* of a semialgebraic set $S \subset \mathbb{R}^n$ is

$$S^* := \{l \in (\mathbb{R}^n)^* \, : \, l(x) \geq -1 \; \forall \, x \in S\} \; .$$

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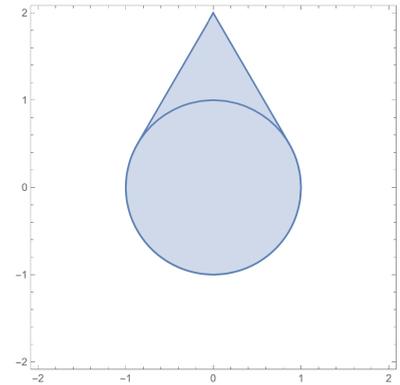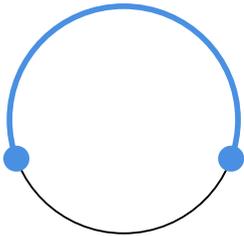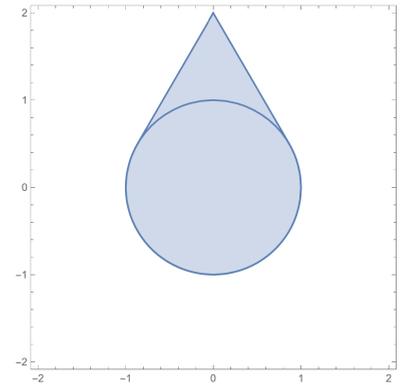The *polar dual* of a semialgebraic set $S \subset \mathbb{R}^n$ is

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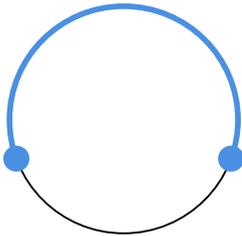The *polar dual* of a semialgebraic set $S \subset \mathbb{R}^n$ is

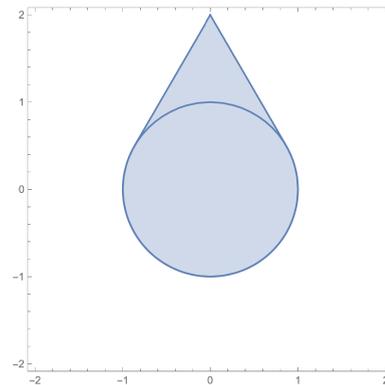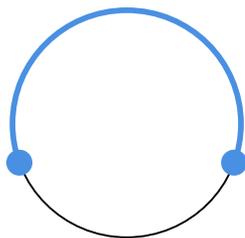$$S^* := \{l \in (\mathbb{R}^n)^* \ : \ l(x) \geq -1 \ \forall \, x \in S\} \ .$$

Observation: $S^* = \text{conv}(S)^*$. Very big!

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The *extendable dual amplituhedron* is

$$\mathcal{A}^*_{k,m,n} := Gr(m, k+m) \cap \mathsf{conv}(\mathcal{A}_{k,m,n})^* = Gr(m, k+m) \cap C^*_{k,m,n}.$$

The twist map

Define

$$W_i := Z_{i-m+1} \wedge Z_{i-m+2} \wedge \cdots \wedge Z_i \wedge \cdots \wedge Z_{i+k-1}, \qquad i \in [n].$$

The *twist map* is

$$\tau : \mathsf{Mat}_{>0}(k+m, n) \to \mathsf{Mat}_{>0}(k+m, n),$$
$$Z \mapsto W,$$

where W has columns $W_1, \ldots, W_n$. [Marsh–Scott 13]

Example

$$[Z_1 \ \ldots \ Z_6] \mapsto [Z_6 \wedge Z_1 \wedge Z_2 \quad Z_1 \wedge Z_2 \wedge Z_3 \quad \ldots \quad Z_5 \wedge Z_6 \wedge Z_1].$$

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Theorem (Mazzucchelli–P)

There is an equality

$$\mathcal{A}_{2,2,n}(Z)^* = \mathcal{A}_{2,2,n}(W).$$

$\mathcal{A}_{2,2,n}(Z)^*$ is an amplituhedron for another particle configuration!

Duality of polytopes

The *Schubert exterior cyclic polytope* $\widetilde{C}_{k,m,n}(Z)$ is obtained from $C_{k,m,n}(Z)$ by deleting all facet inequalities whose supporting hyperplanes are not Schubert divisors.

Proposition (Mazzucchelli–P)

There is an equality

$$\widetilde{C}_{2,2,n}(Z) = C_{2,2,n}(W)^*.$$
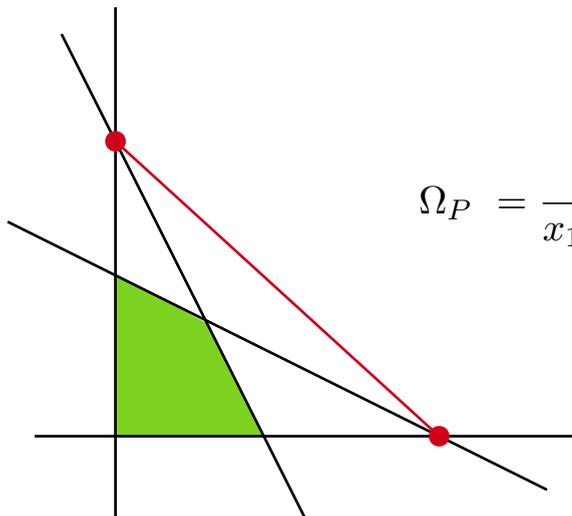
Example

The f-vector of $C_{2,2,6}$ is

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The f-vector of $\widetilde{C}_{2,2,6}$ is

$$(30, 111, 143, 75, 15).$$

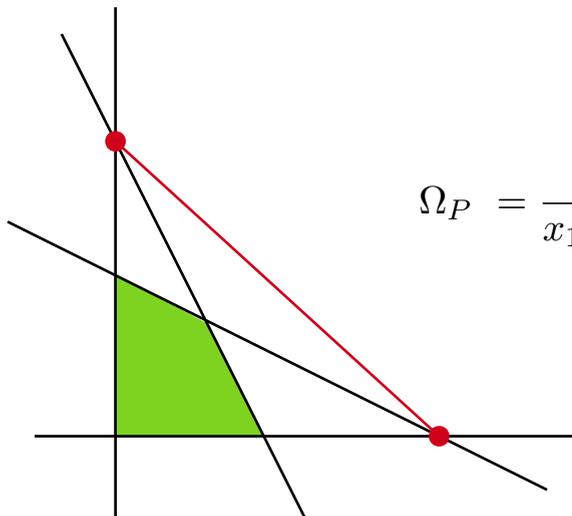# What is a *dual amplituhedron*?

A polytope P has a *canonical function* Ω_P with simple poles on ∂P and nowhere else:

$$\Omega_P = \frac{2 - x_1 - x_2}{x_1 x_2 (2 - x_1 - 2x_2)(2 - 2x_1 - x_2)}$$

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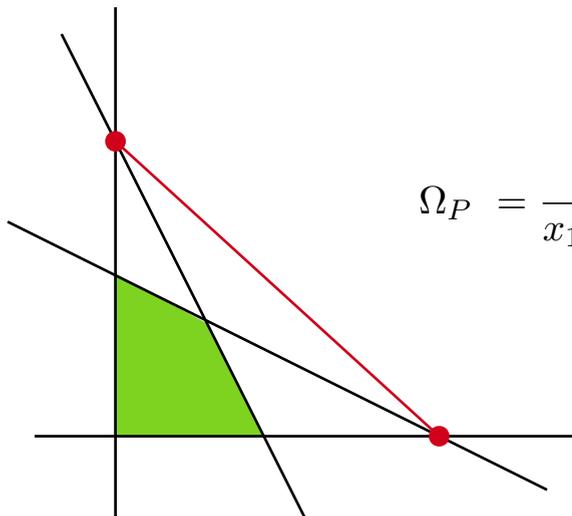A polytope P has a *canonical function* Ω_P with simple poles on ∂P and nowhere else:

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Laplace integral representation:

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What about $\mathcal{A}^*_{2,2,n}$ and the Parke-Taylor form?

Thank you for listening!