BCFW triangulations of loop amplituhedra

Pavel Galashin (Cornell University)
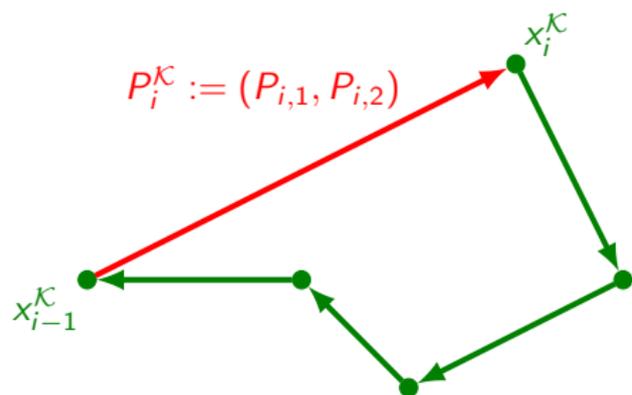
Michigan Interdisciplinary Meeting on Amplitudes:
Bridges between Physics & Mathematics
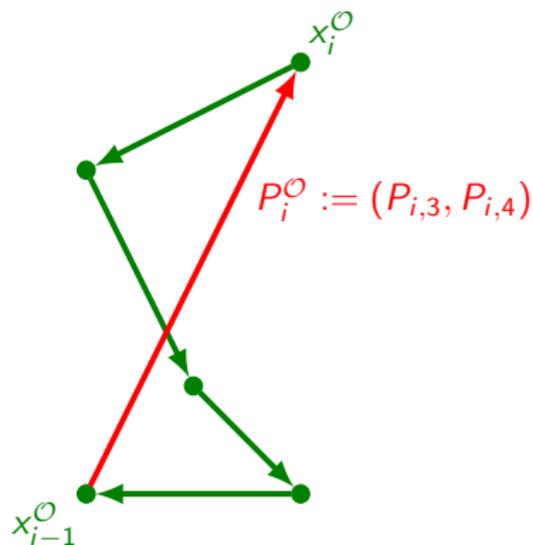
October 24, 2025

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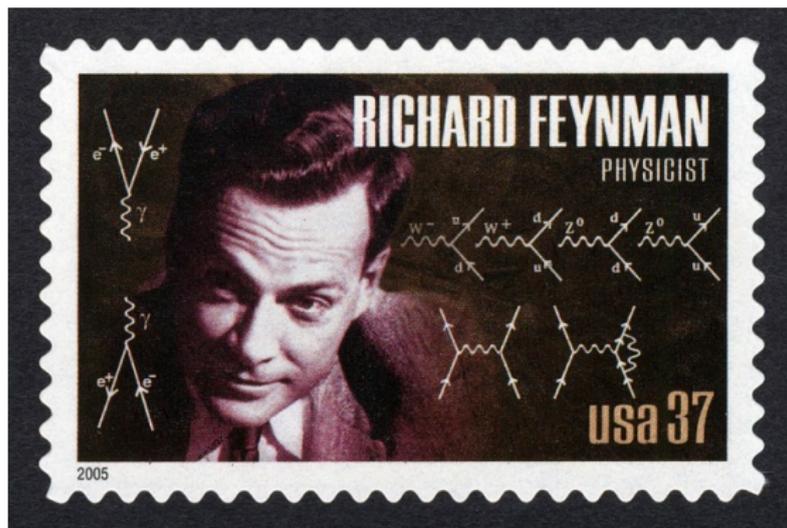$x_{i-1}^{\mathcal{O}}$

Space ($\mathbb{R}^2$)

Time ($\mathbb{R}^2$)

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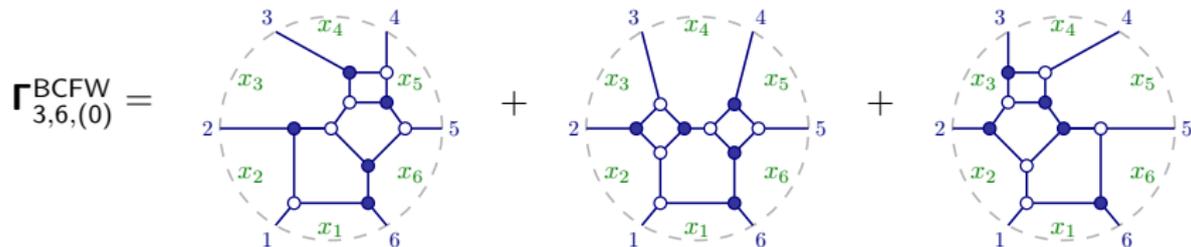
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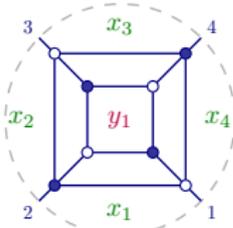$$\Gamma_{3,6,(0)}^{\mathrm{BCFW}} = \quad$$

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- **Proof** relies on the Origami–Amplituhedron Correspondence [G.'24].

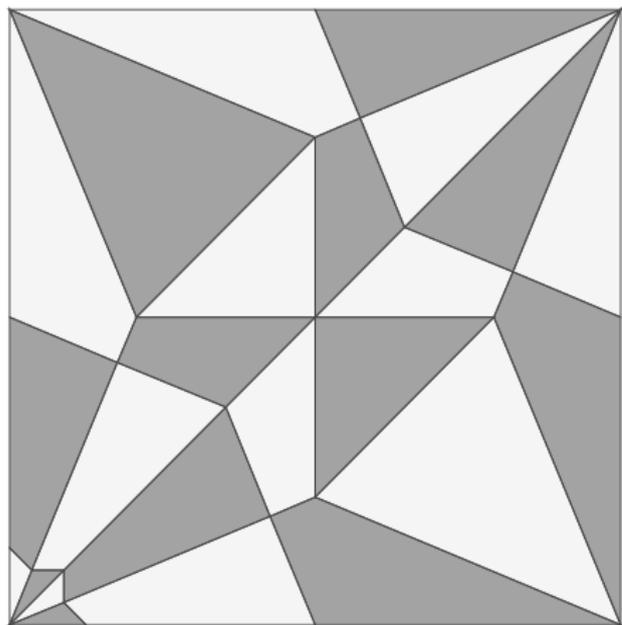**Kawasaki's theorem** or **Kawasaki–Justin theorem** is a theorem in the mathematics of paper folding that describes the crease patterns with a single vertex that may be folded to form a flat figure. It states that the pattern is flat-foldable if and only if alternatingly adding and subtracting the angles of consecutive folds around the vertex gives an alternating sum of zero. Crease patterns with more than one vertex do not obey such a simple criterion, and are NP-hard to fold.
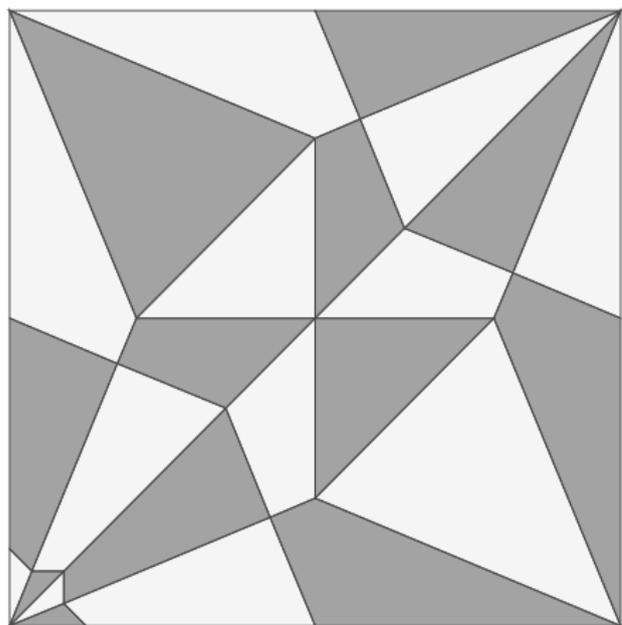

In this example, the alternating sum of angles (clockwise from the bottom) is 90° − 45° + 22.5° − 22.5° + 45° − 90° + 22.5° − 22.5° = 0°. Since it adds to zero, the crease pattern may be flat-folded.

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- Origami map $\mathcal{O}$: isometry on each face preserving/reversing the orientations of white/black faces.

$\mathcal{K}$ami plane $\mathcal{O}$rigami plane

- OCP + its origami folding = 2-dimensional discrete PL surface in $\mathbb{R}^{2,2}$.

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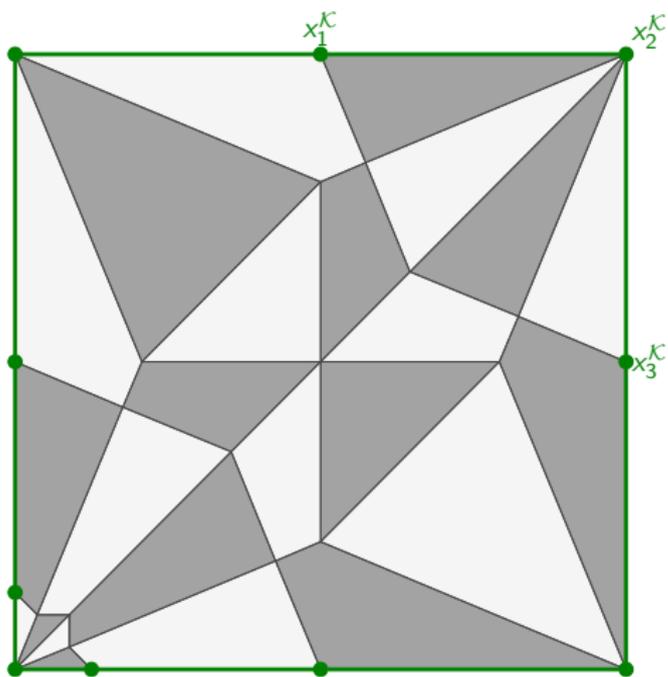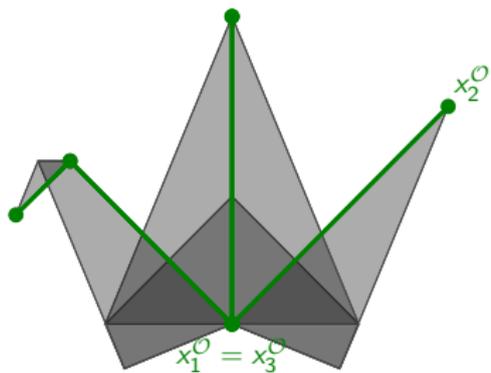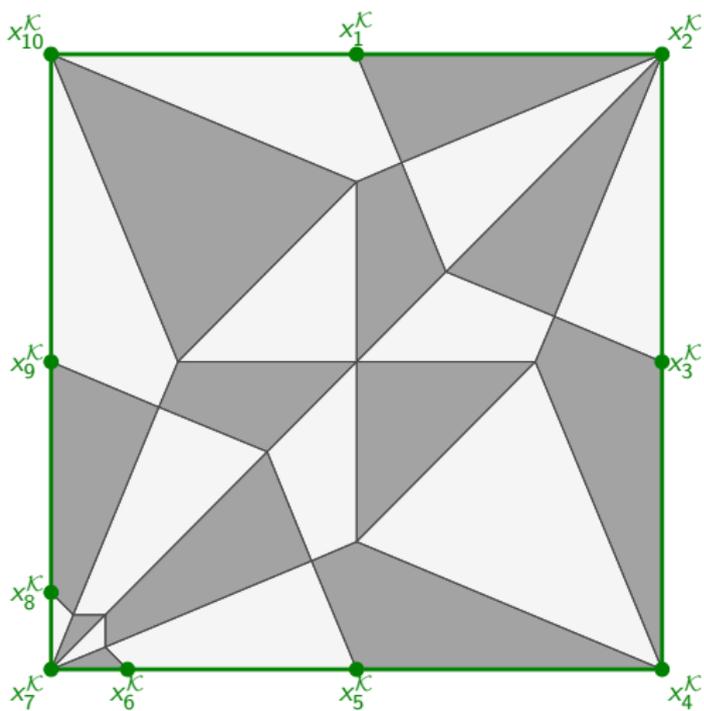- OCP + its origami folding = 2-dimensional discrete PL surface in $\mathbb{R}^{2,2}$.

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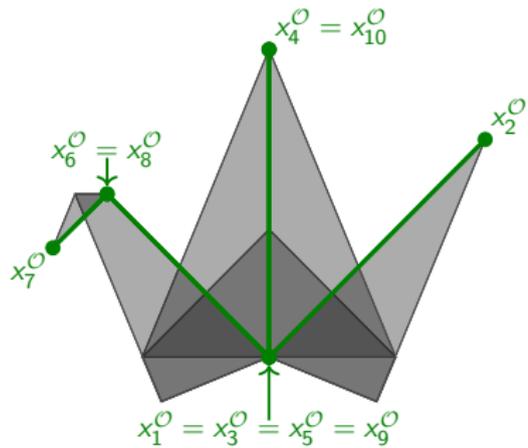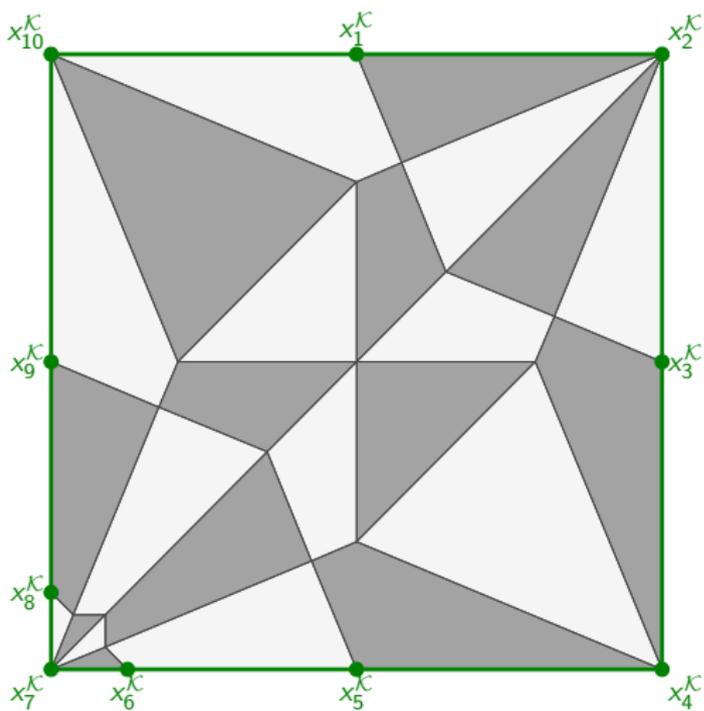$\mathcal{K}$ami plane

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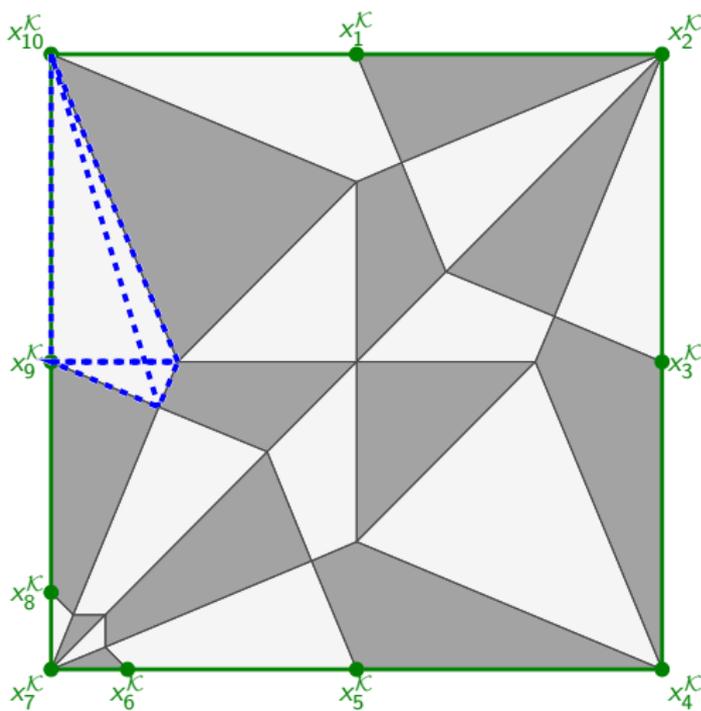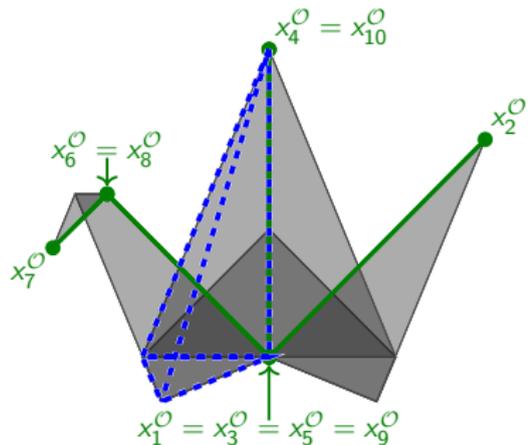$\mathcal{O}$rigami plane

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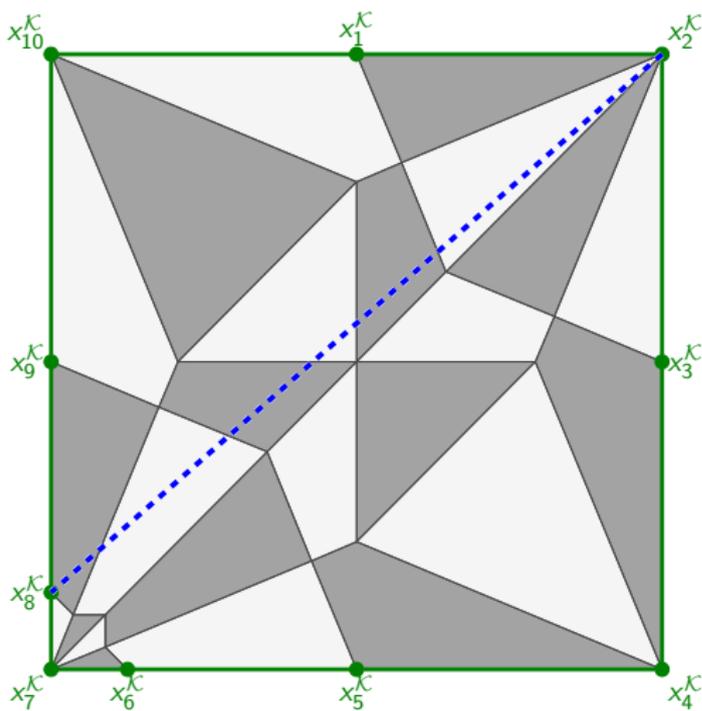$\mathcal{O}$rigami plane

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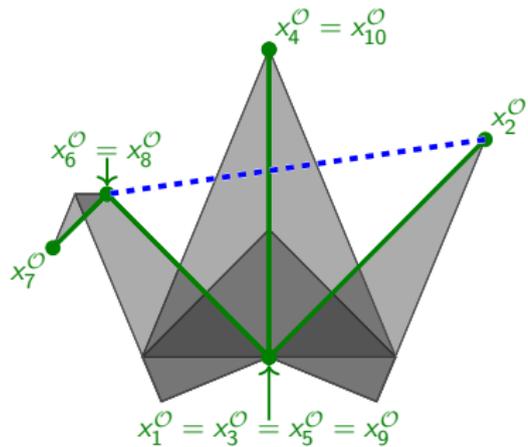
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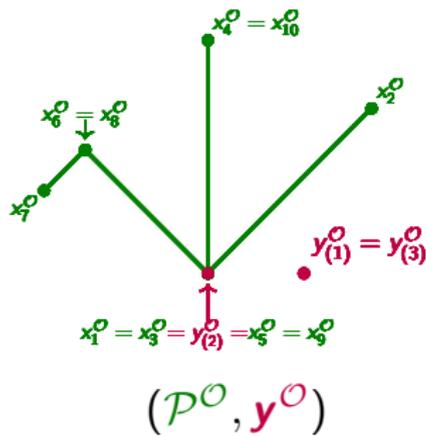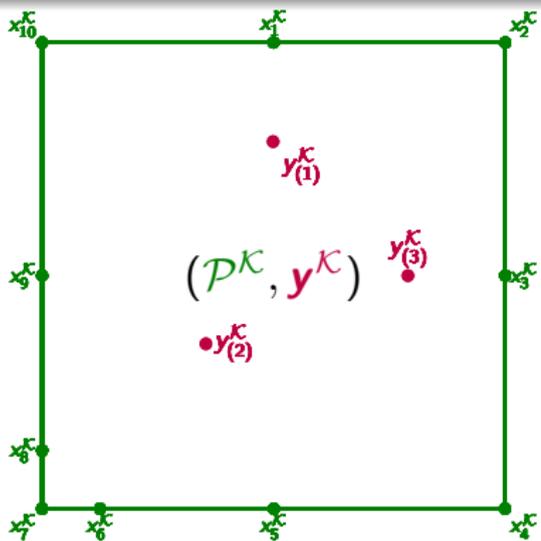
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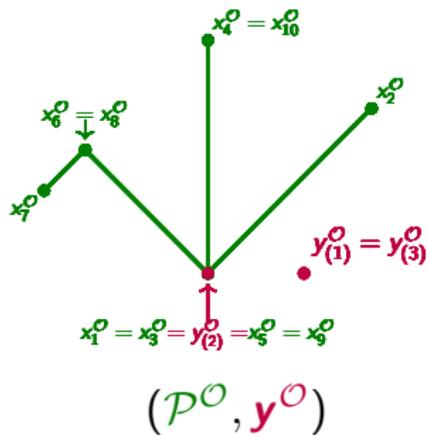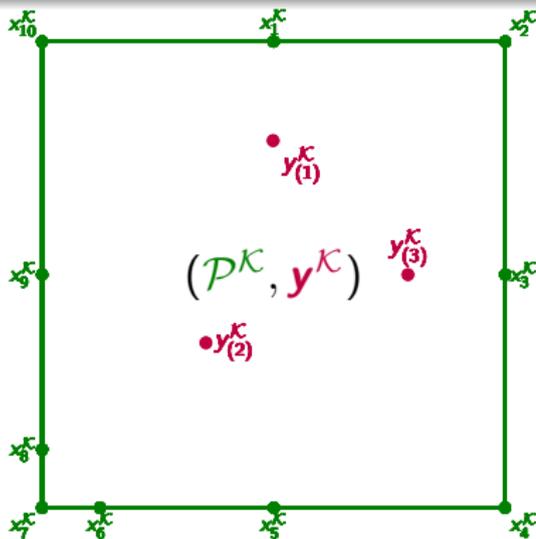
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$$(x_i - x_{i-1})^2 = 0, \quad (x_i - x_j)^2 > 0, \quad (x_i - y_{(\rho)})^2 > 0, \quad (y_{(\rho)} - y_{(\gamma)})^2 > 0$$
for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$,

- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{K}} - x_j^{\mathcal{K}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.
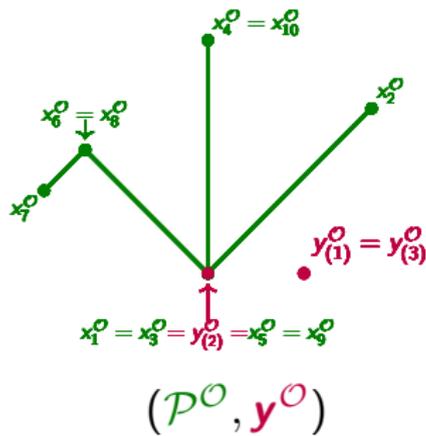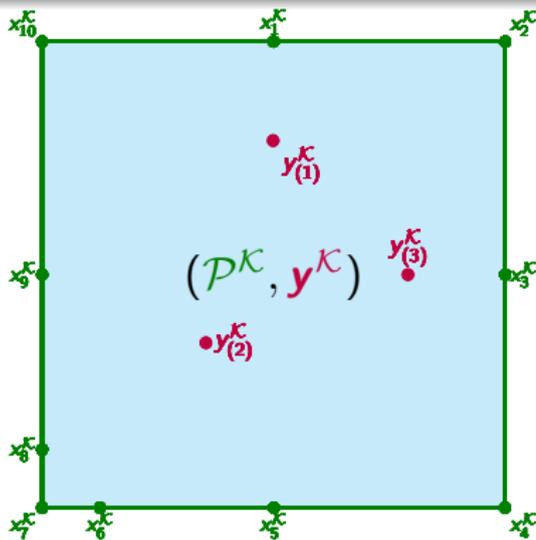
Definition

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for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$, and such that
each $y_{(\rho)}^{\mathcal{K}}$ is located inside the polygon $\mathcal{P}^{\mathcal{K}} = (x_1^{\mathcal{K}}, x_2^{\mathcal{K}}, \ldots, x_n^{\mathcal{K}})$.

- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{K}} - x_j^{\mathcal{K}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

Definition

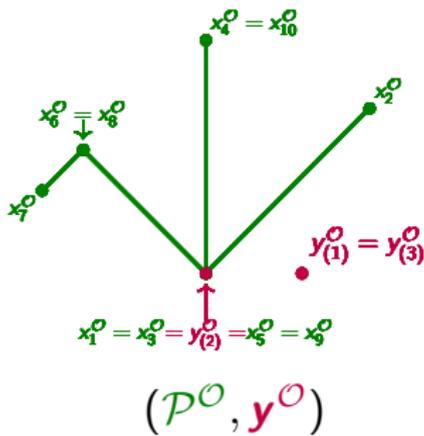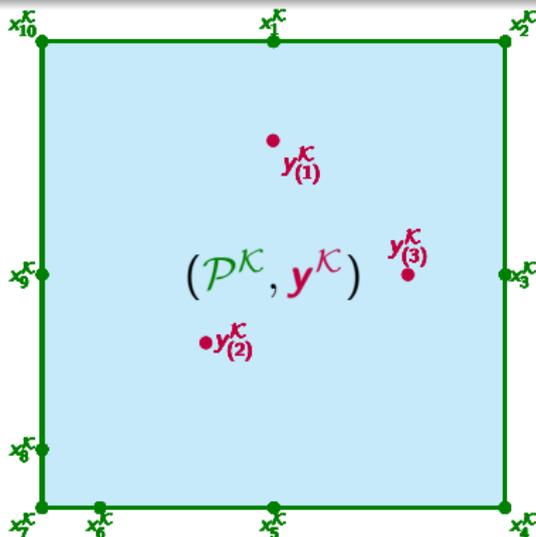Let $L \geqslant 0$. An L-punctured polygon is a pair $(\mathcal{P}, \mathbf{y})$, where
$\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\mathbf{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that
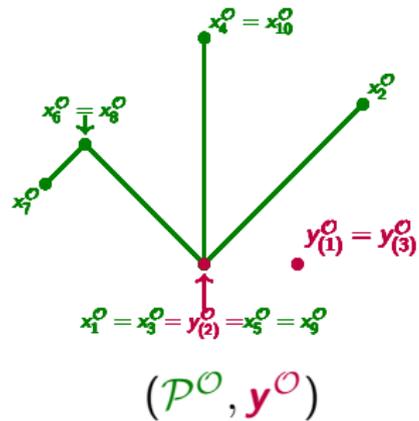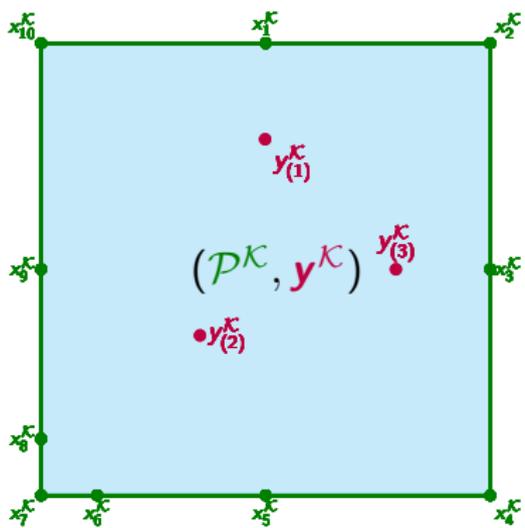$$(x_i - x_{i-1})^2 = 0, \quad (x_i - x_j)^2 > 0, \quad (x_i - y_{(\rho)})^2 > 0, \quad (y_{(\rho)} - y_{(\gamma)})^2 > 0$$
for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$, and such that
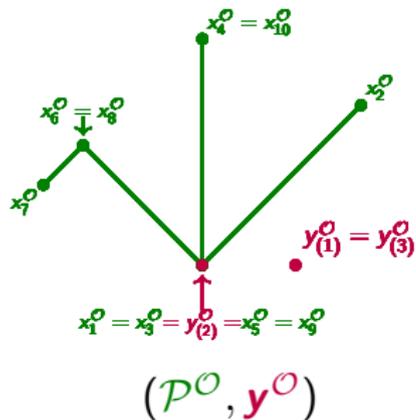each $y_{(\rho)}^{\mathcal{K}}$ is located inside the polygon $\mathcal{P}^{\mathcal{K}} = (x_1^{\mathcal{K}}, x_2^{\mathcal{K}}, \ldots, x_n^{\mathcal{K}})$.

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends
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- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:
$$\alpha_i^{\circ} + \alpha_i^{\bullet} = \alpha_i^{\mathcal{K}} \quad \text{and} \quad \alpha_i^{\circ} - \alpha_i^{\bullet} \equiv \alpha_i^{\mathcal{O}} \mod 2\pi.$$

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

- Recover white/black angle sums $(\alpha_i^\circ, \alpha_i^\bullet)$ from the geometry of $\mathcal{P}$:
$$\alpha_i^\circ + \alpha_i^\bullet = \alpha_i^{\mathcal{K}} \quad \text{and} \quad \alpha_i^\circ - \alpha_i^\bullet \equiv \alpha_i^{\mathcal{O}} \bmod 2\pi.$$

- Red folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^\bullet, \alpha_i^\circ)$.
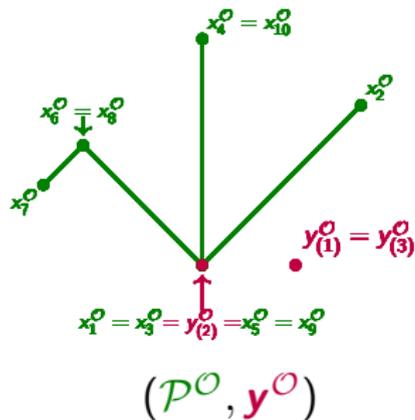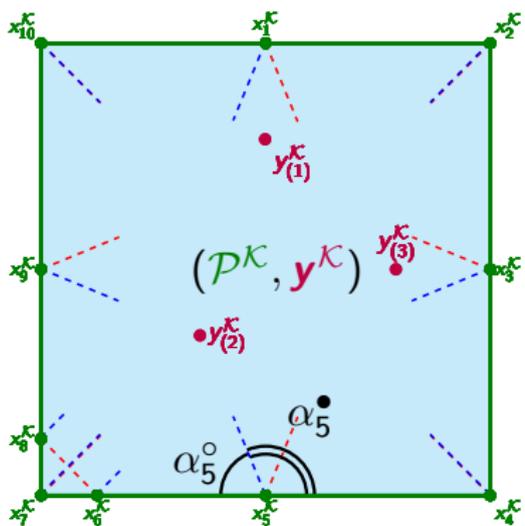
Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:
$$\alpha_i^{\circ} + \alpha_i^{\bullet} = \alpha_i^{\mathcal{K}} \quad \text{and} \quad \alpha_i^{\circ} - \alpha_i^{\bullet} \equiv \alpha_i^{\mathcal{O}} \bmod 2\pi.$$

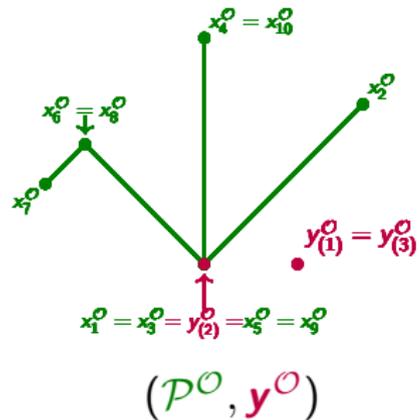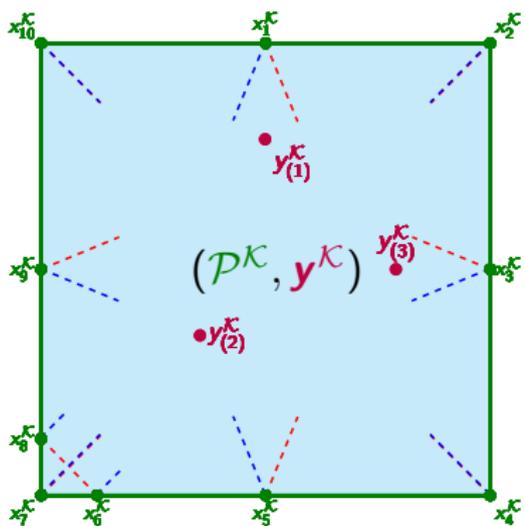- Red folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.
- Blue folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^{\circ}, \alpha_i^{\bullet})$.

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

Origami Reconstruction Algorithm

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

Origami Reconstruction Algorithm

1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

Origami Reconstruction Algorithm

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$(\mathcal{P}^{\mathcal{K}}, \mathbf{y}^{\mathcal{K}})$

$(\mathcal{P}^{\mathcal{O}}, \mathbf{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

Origami Reconstruction Algorithm

1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

2. *Let $x_i(t) = x_i + tR_i$, and let $t > 0$ be minimal such that either $\left(x_i(t) - x_j\right)^2 = 0$ or $\left(x_i(t) - y_{(\rho)}\right)^2 = 0$ for some $1 \leqslant j \leqslant n$ or $1 \leqslant \rho \leqslant L$.*

$(\mathcal{P}^{\mathcal{K}}, \mathbf{y}^{\mathcal{K}})$

$(\mathcal{P}^{\mathcal{O}}, \mathbf{y}^{\mathcal{O}})$

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Origami Reconstruction Algorithm
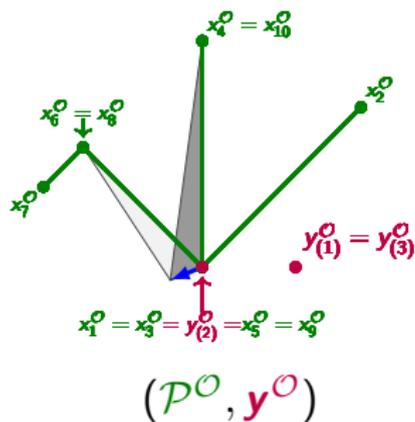
1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

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Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

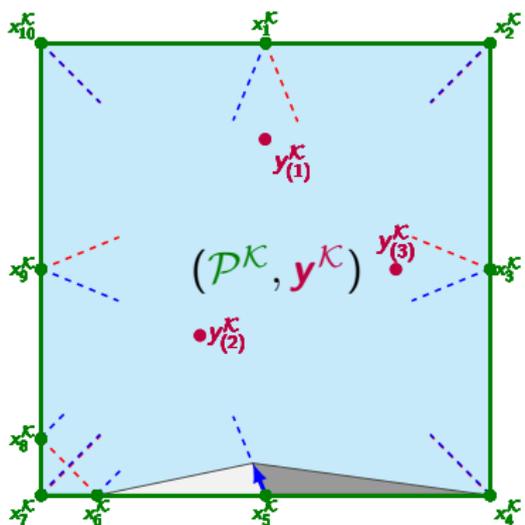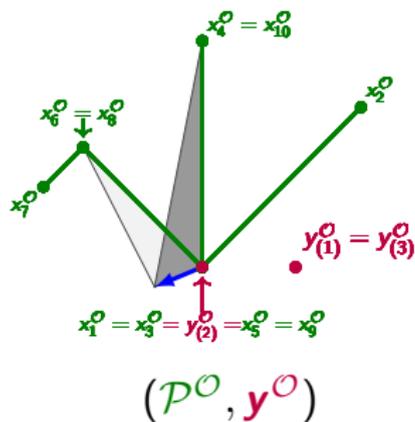## Origami Reconstruction Algorithm

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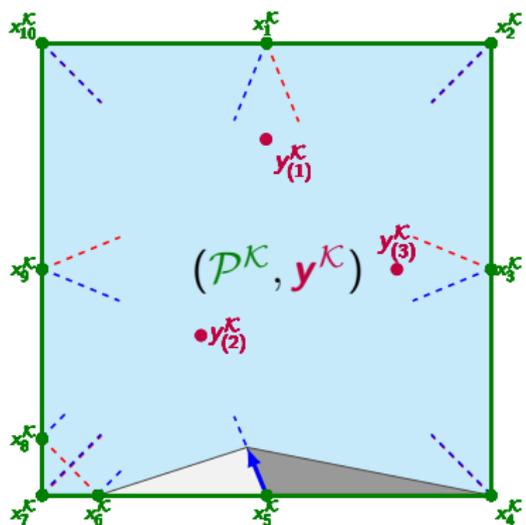
Origami Reconstruction Algorithm

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Problem: Find an OCP with boundary $\mathcal{P}^\mathcal{K}$ such that the origami map sends $x_i^\mathcal{K} \mapsto x_i^\mathcal{O}$ and $y_{(\rho)}^\mathcal{K} \mapsto y_{(\rho)}^\mathcal{O}$ for all i, ρ.
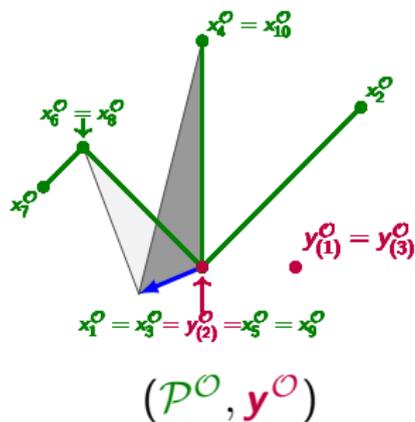
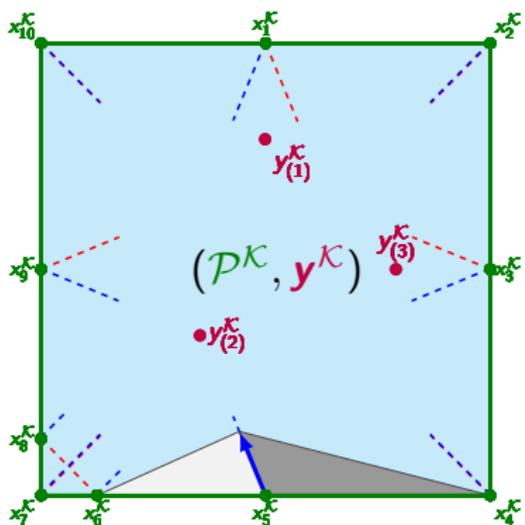## Origami Reconstruction Algorithm

1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

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Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

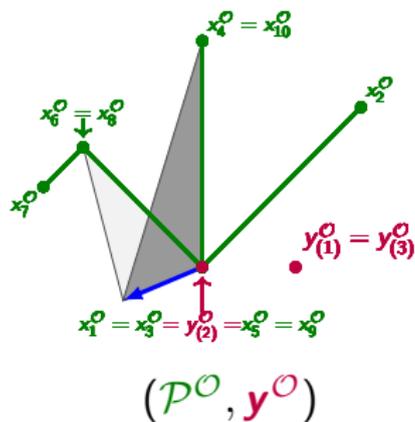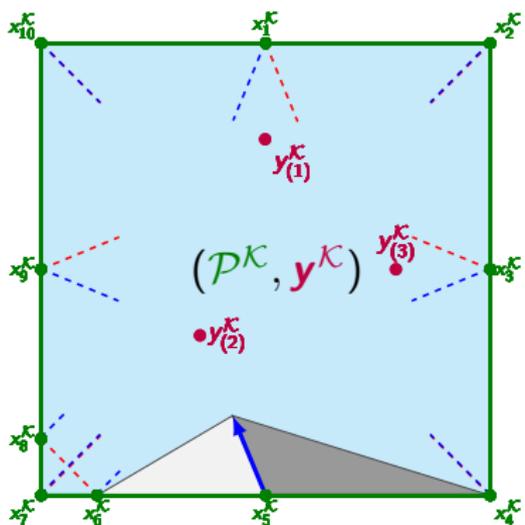## Origami Reconstruction Algorithm

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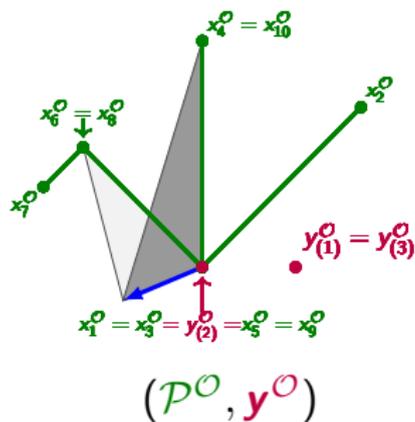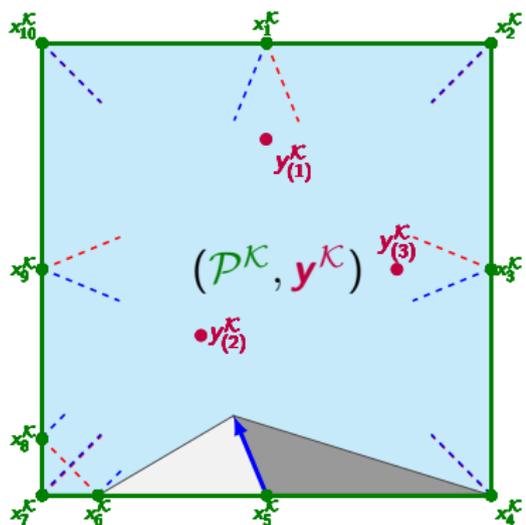## Origami Reconstruction Algorithm

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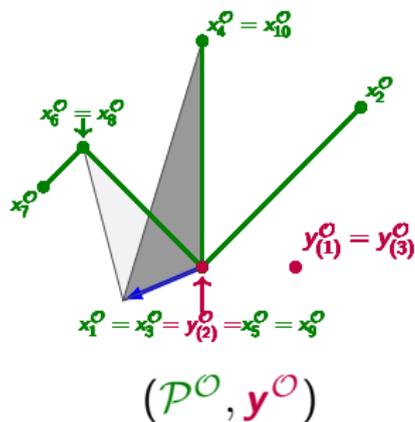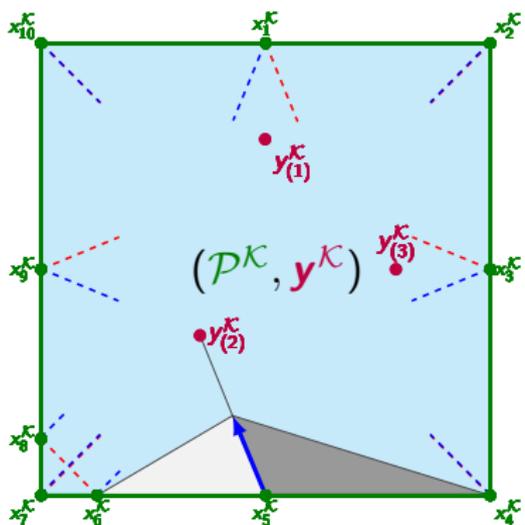## Origami Reconstruction Algorithm

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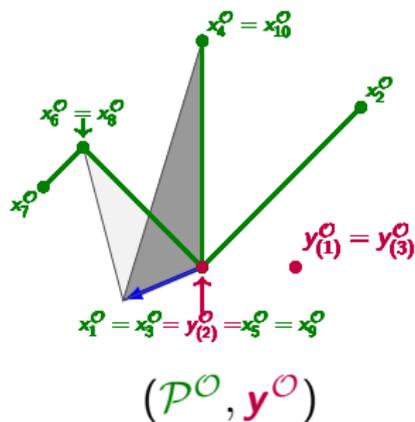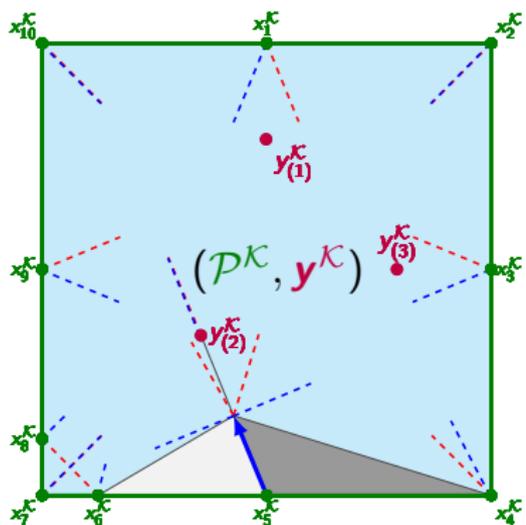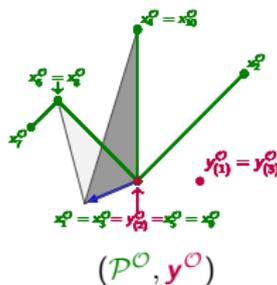## Origami Reconstruction Algorithm

1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

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3. *Draw edges from $x_i(t)$ to all vertices u such that $\left(x_i(t) - u\right)^2 = 0$.*

4. *Repeat.*

$(\mathcal{P}^{\mathcal{K}}, \mathbf{y}^{\mathcal{K}})$

$(\mathcal{P}^{\mathcal{O}}, \mathbf{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

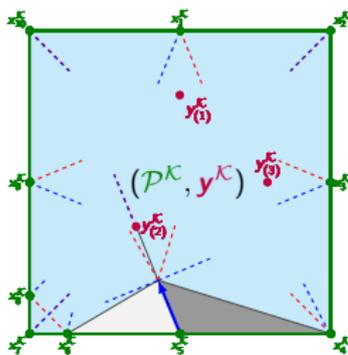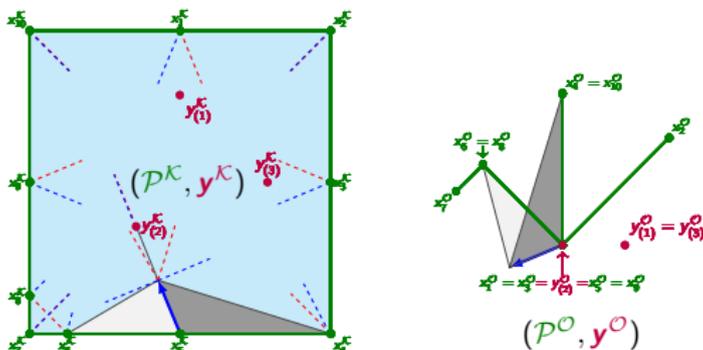### Origami Reconstruction Algorithm

1. Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.

2. Let $x_i(t) = x_i + tR_i$, and let $t > 0$ be minimal such that either $\left(x_i(t) - x_j\right)^2 = 0$ or $\left(x_i(t) - y_{(\rho)}\right)^2 = 0$ for some $1 \leqslant j \leqslant n$ or $1 \leqslant \rho \leqslant L$.

3. Draw edges from $x_i(t)$ to all vertices u such that $(x_i(t) - u)^2 = 0$.

4. Repeat.

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.

Origami Reconstruction Algorithm

1. *Choose $1 \leqslant i \leqslant n$ and a red or a blue folding ray $R_i \in \mathbb{R}^{2,2}$.*

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4. *Repeat.*

Theorem (G.'25+)

The Origami Reconstruction Algorithm works.

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends $x_i^{\mathcal{K}} \mapsto x_i^{\mathcal{O}}$ and $y_{(\rho)}^{\mathcal{K}} \mapsto y_{(\rho)}^{\mathcal{O}}$ for all i, ρ.
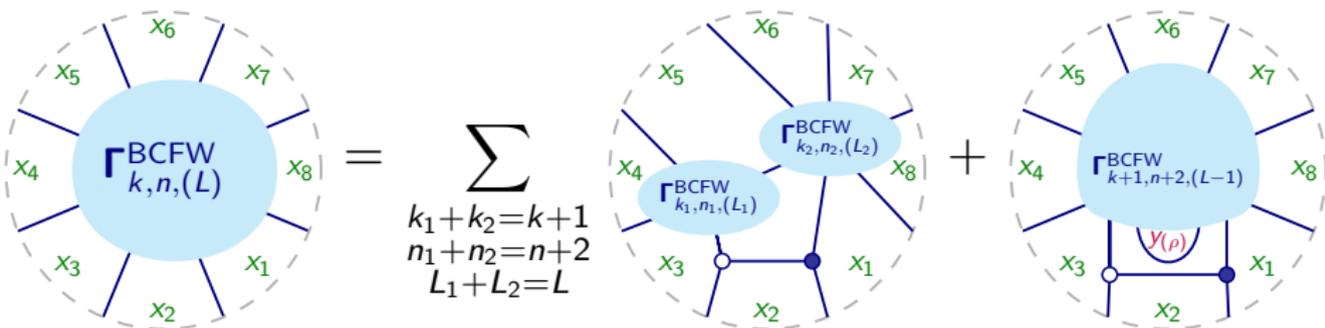
Origami Reconstruction Algorithm

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2. *Let $x_i(t) = x_i + tR_i$, and let $t > 0$ be minimal such that either $\left(x_i(t) - x_j\right)^2 = 0$ or $\left(x_i(t) - y_{(\rho)}\right)^2 = 0$ for some $1 \leqslant j \leqslant n$ or $1 \leqslant \rho \leqslant L$.*

3. *Draw edges from $x_i(t)$ to all vertices u such that $(x_i(t) - u)^2 = 0$.*

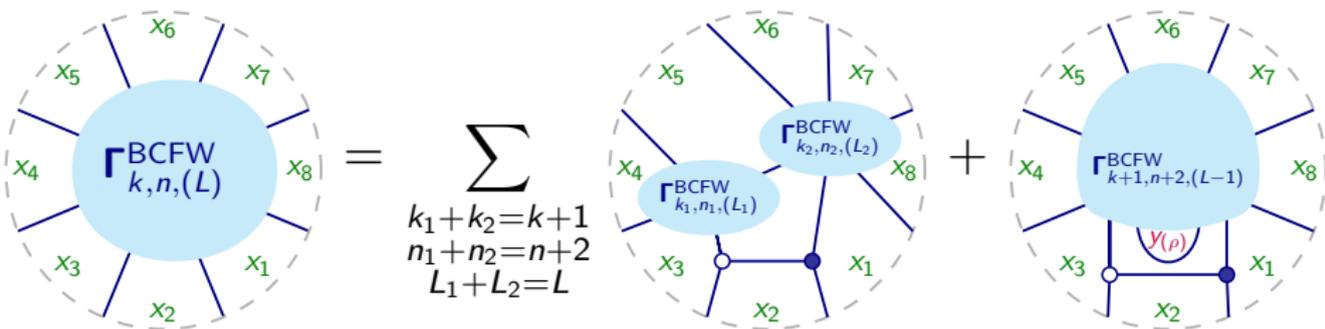4. *Repeat.*

Theorem (G.'25+)

The Origami Reconstruction Algorithm works.
For all L-punctured polygons $(\mathcal{P}, \boldsymbol{y})$, it outputs a valid, embedded OCP.

- BCFW recursion $\rightarrow$ collection $\mathbf{\Gamma}^{\mathrm{BCFW}}_{k,n,(L)}$ of L-punctured planar bicolored graphs.

- BCFW recursion $\rightarrow$ collection $\Gamma^{BCFW}_{k,n,(L)}$ of L-punctured planar bicolored graphs.

Example: $\Gamma^{BCFW}_{2,4,(1)}$:

- BCFW recursion $\rightarrow$ collection $\mathbf{\Gamma}^{\text{BCFW}}_{k,n,(L)}$ of L-punctured planar bicolored graphs.

$$\mathbf{\Gamma}^{\text{BCFW}}_{k,n,(L)} = \sum_{\substack{k_1+k_2=k+1 \\ n_1+n_2=n+2 \\ L_1+L_2=L}} \left(\ \mathbf{\Gamma}^{\text{BCFW}}_{k_2,n_2,(L_2)} \, \mathbf{\Gamma}^{\text{BCFW}}_{k_1,n_1,(L_1)} \ \right) + \ \mathbf{\Gamma}^{\text{BCFW}}_{k+1,n+2,(L-1)}$$

Example: $\mathbf{\Gamma}^{\text{BCFW}}_{2,4,(1)}$:

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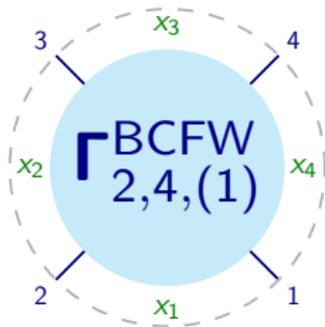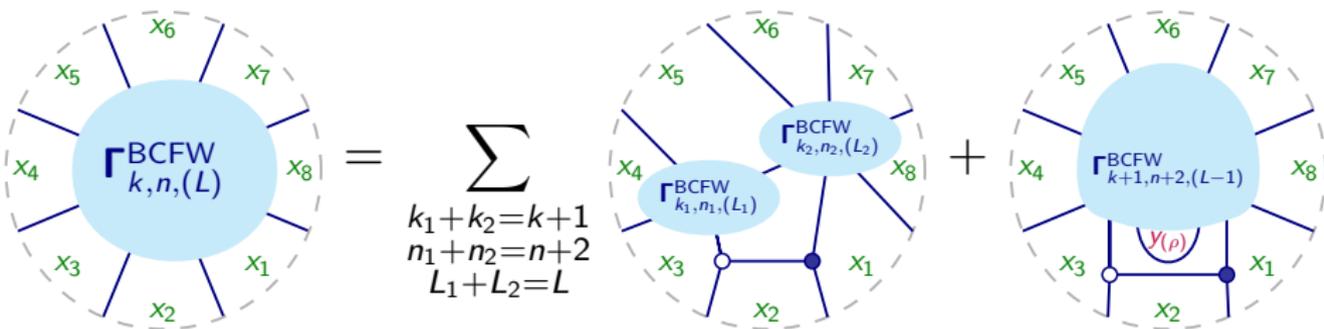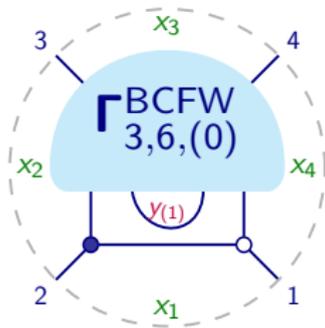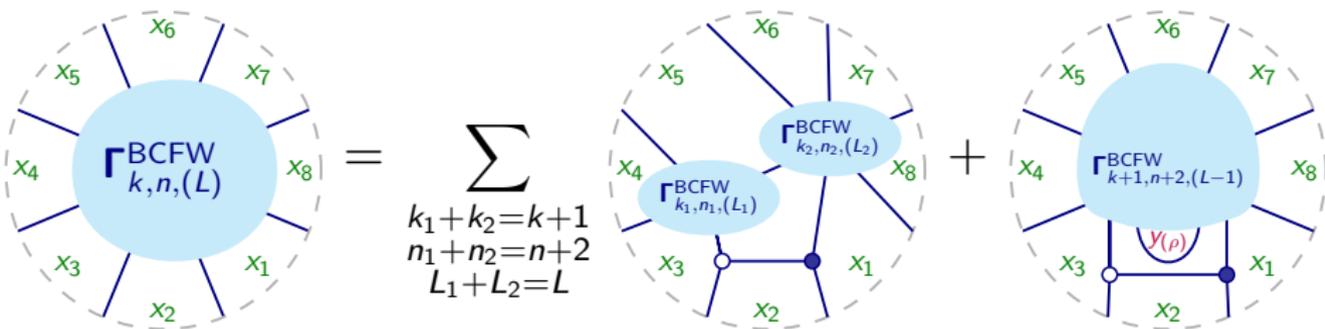

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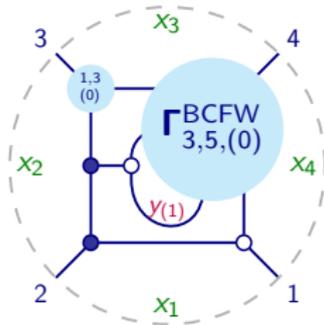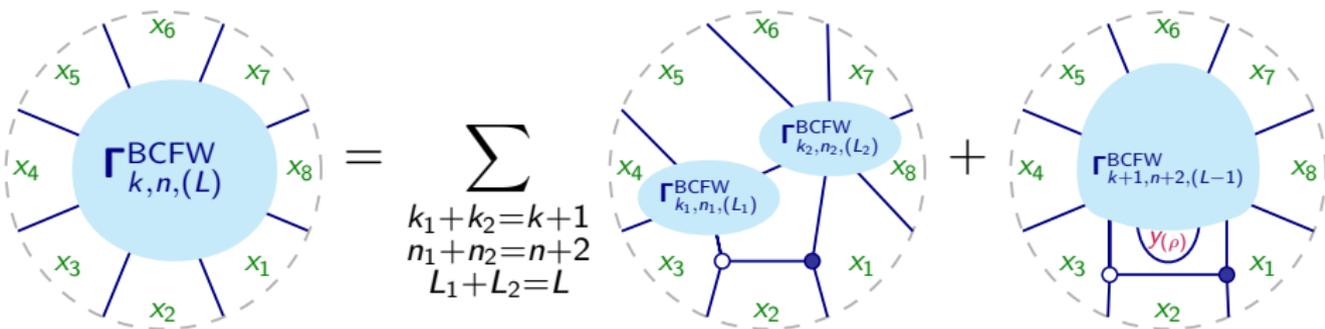
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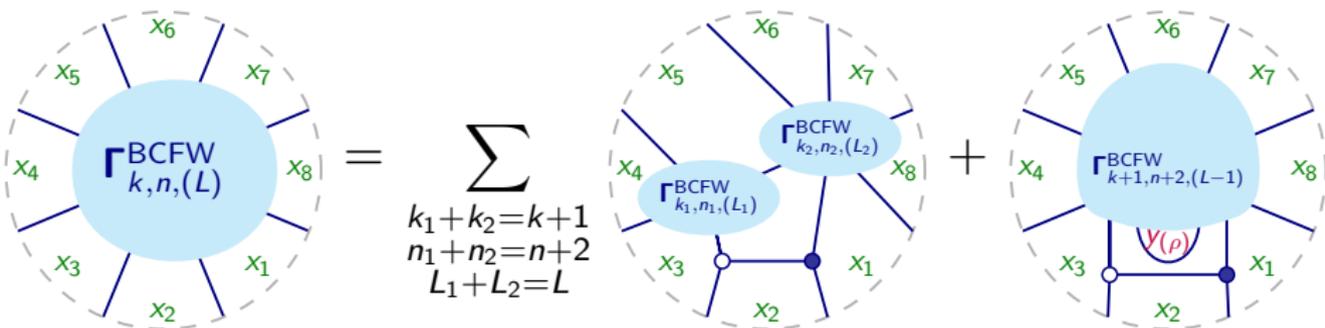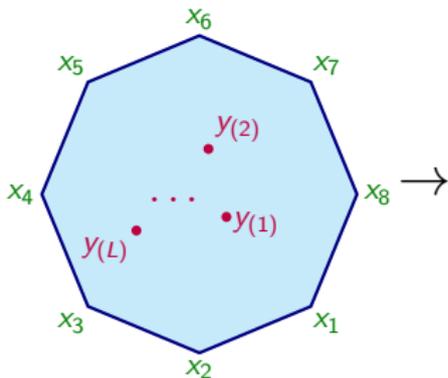- Planar dual pictures = possibilities in the Origami Reconstruction Algorithm:

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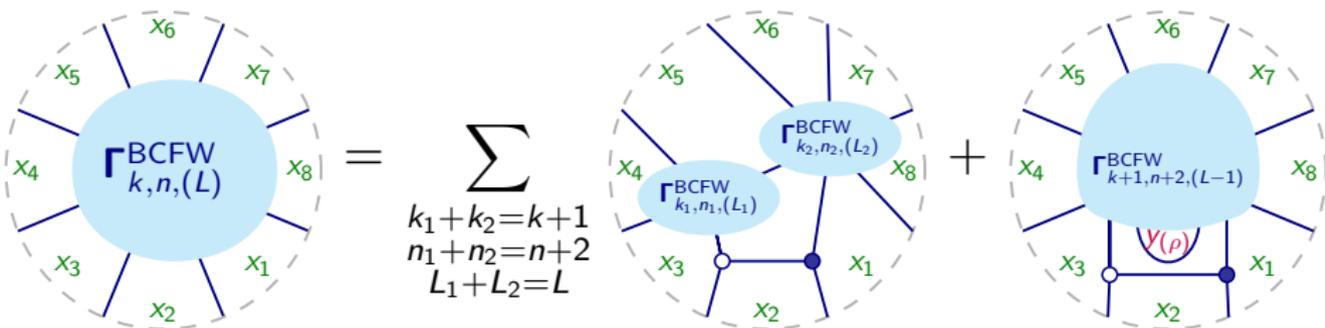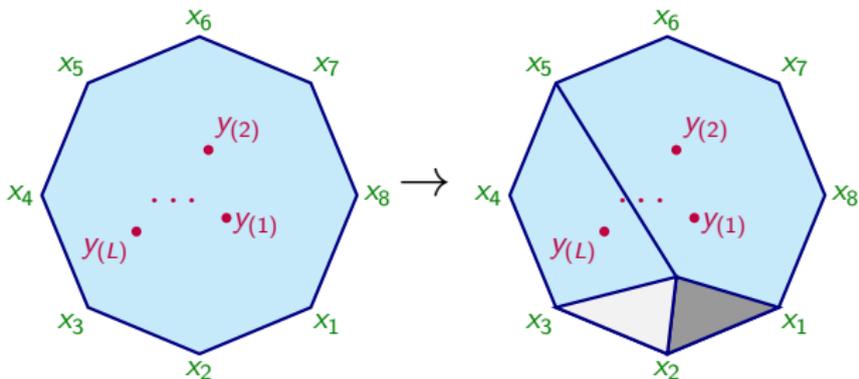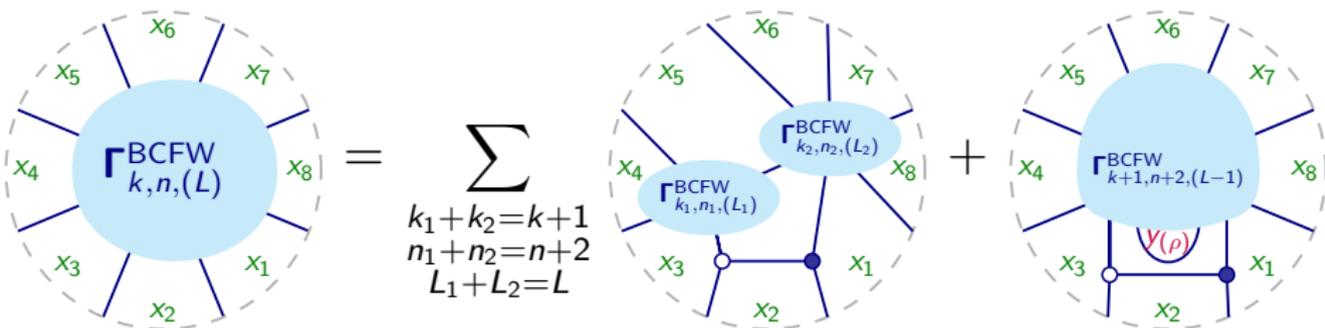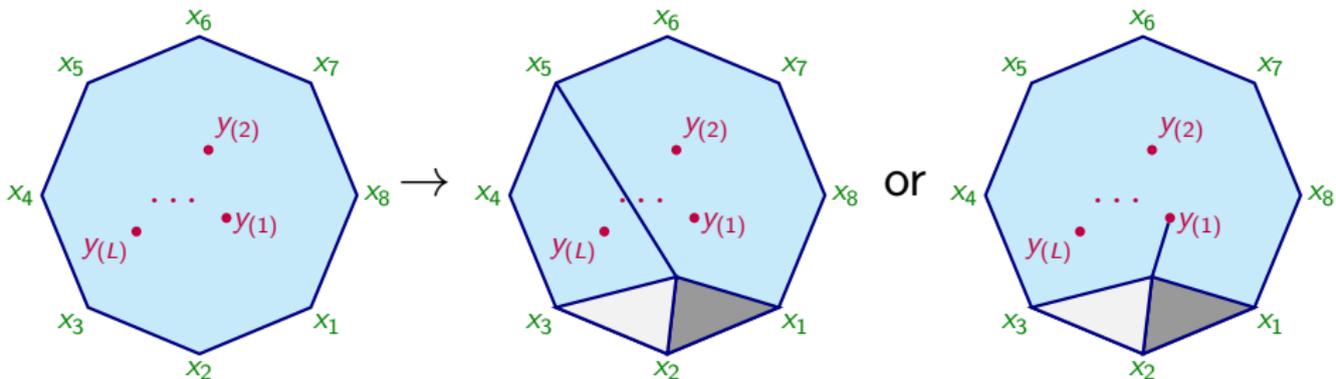- Planar dual pictures = possibilities in the Origami Reconstruction Algorithm:

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The BCFW tiles triangulate $\mathcal{M}_{k,n}^{(L)}$ and $\mathcal{A}_{k-2,n}^{(L)}$.

Proof.

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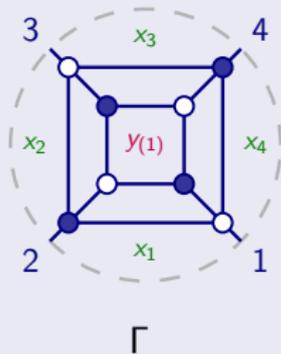

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Γ

OCP: Faces(Γ) $\to \mathbb{R}^{2,2}$
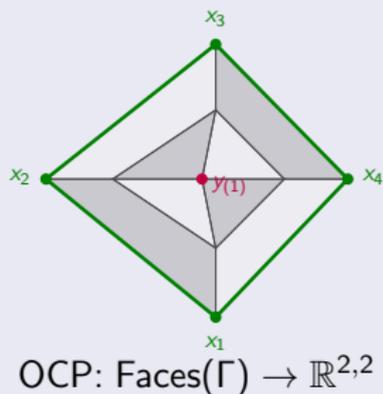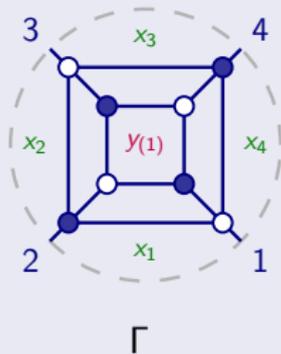
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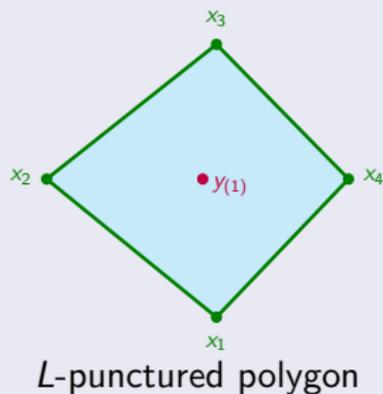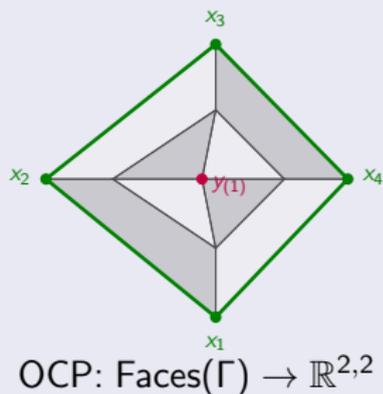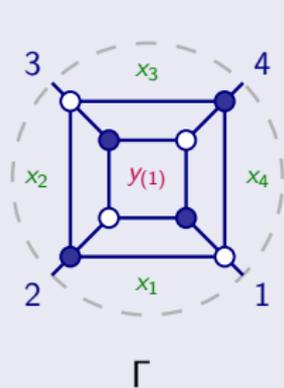
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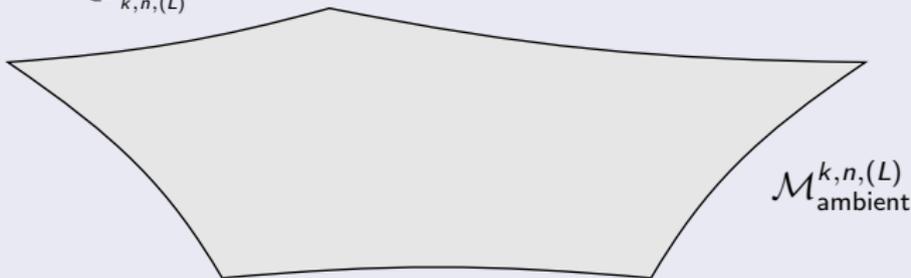$\mathcal{M}_{\text{ambient}}^{k,n,(L)}$

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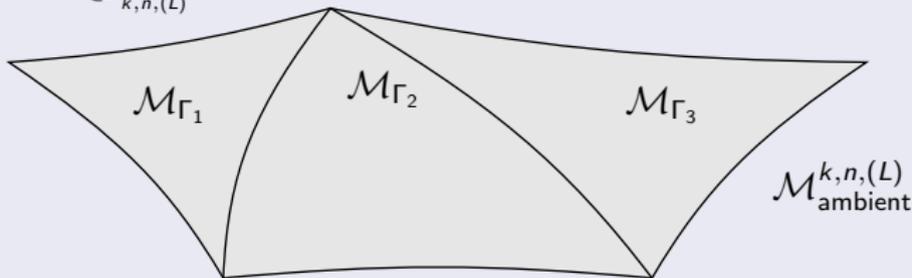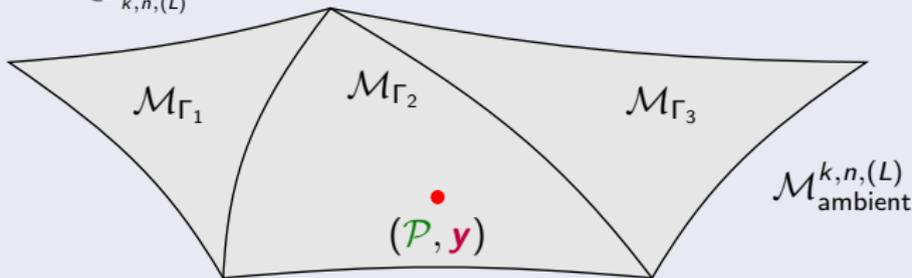

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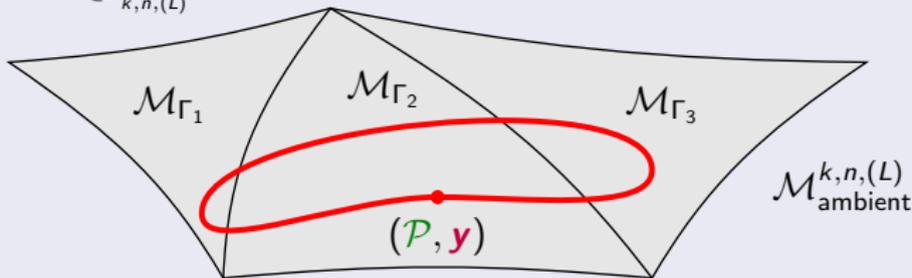

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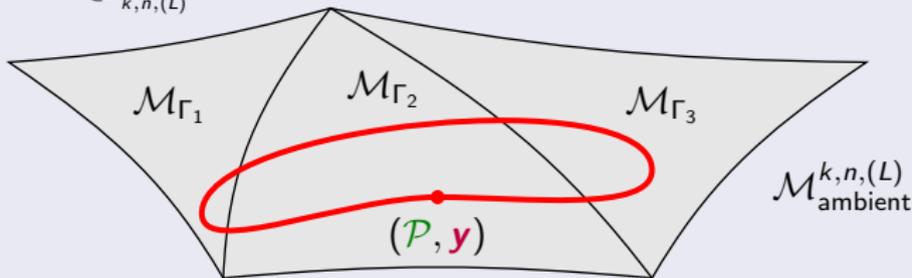

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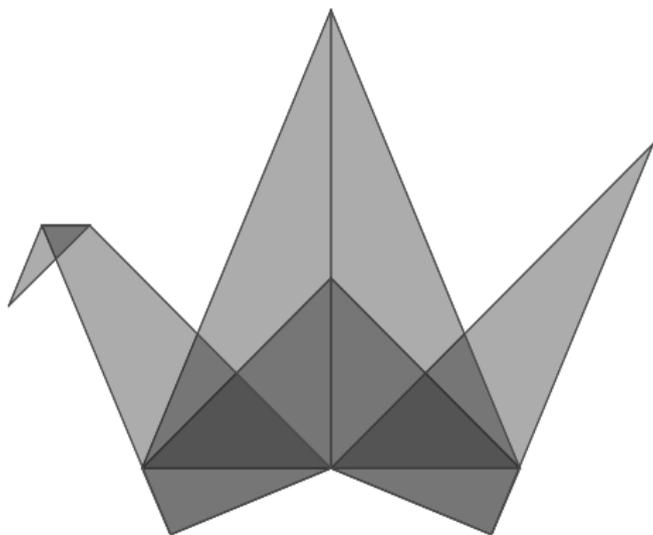## Conjecture (BCFW triangulation conjecture [AHT'13])

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- T-duality gives a triangulation-preserving map $\mathcal{M}_{\text{ambient}}^{k,n,(L)} \to \mathcal{A}_{\text{ambient}}^{k-2,n,(L)}$. $\qquad \square$

Thanks!