Loop Quantum Cosmology

Parampreet Singh

Louisiana State University

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Yangzou University

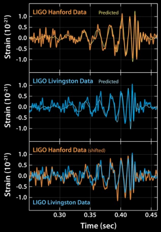
Outline:

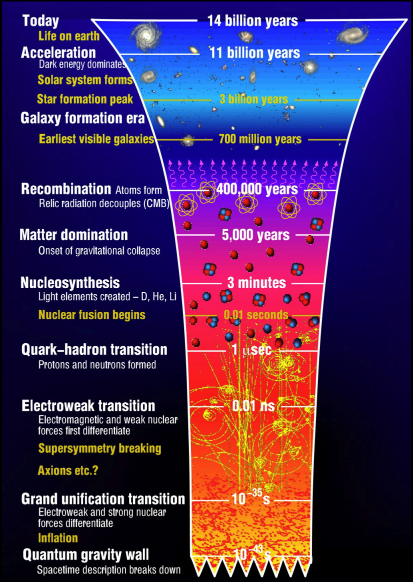
- Lecture 1: Preliminaries of Cosmological Framework, Genericness and Types of Singularities in GR
- Lecture 2: An Overview of Singularity Resolution in LQC
- Lecture 3: Hamiltonian Framework for Cosmological Perturbations: Exploring Quantum Geometry Effects in CMB

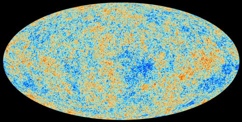
Lecture 1

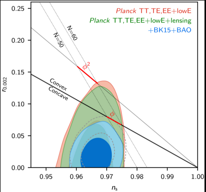
Preliminaries of Cosmological Framework, Genericness and Types of Singularities in GR

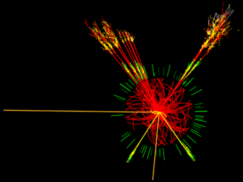
An almost complete history of our Universe











The Expanding Universe

From the spectra emitted by the galaxies, Hubble in 1920's discovered that they are moving farther from each other. Fainter the galaxy, faster it recedes.

Friedmann and Lemaitre found a solution in GR in which the universe was expanding. Primevial Atom as the seed of the Universe.

The expansion of the universe led to many questions. How did the universe begin and how did the galaxies form?

And the problem of initial singularity!

Universe without initial singularity

Eddington believed there is nothing wrong with GR and there is no initial singularity. It was believed to be an artifact of simplifying assumptions of isotropy and homogeneity.

"Philosophically, the notion of a beginning of the present order of Nature is repugnant to me. I should like to find a genuine loophole." (Eddington, 1931)

Eddington's ideas greatly influenced Hoyle who later pioneered the Steady State Theory based on Perfect Cosmological Principle – the universe looks same not only in space but also in time. No initial singularity.

Einstein's lost theory uncovered

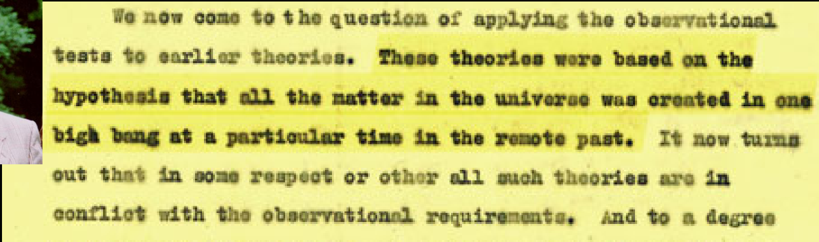
Davide Castelvecchi

Nature **506**, 418–419 (2014) | Cite this article

The "Big Bang"

Gamow developed in detail Lemaitre's preliminary ideas and explored the ultra dense state of matter from which elements would originate. Predicted the universe will have a very cold background radiation (CMB).

Big Bang was coined by (his arch rival) Hoyle in a BBC interview in April 1949.



We now come to the question of applying the observational tests to earlier theories. These theories were based on the hypothesis that all the matter in the universe was created in one big bang at a particular time in the remote past. It now turns out that in some respect or other all such theories are in conflict with the observational requirements. And to a degree

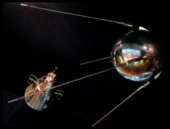
Genericness of Singularities

In 1950's Raychaudhuri proved Eddington wrong by showing existence of singularities in anisotropic and homogeneous spacetimes. Discovered Raychaudhri equation which plays an important role in understanding the attractive nature of gravity and divergence of geodesics.

In 1960's Geroch, Penrose and Hawking proved that singularities are generic in GR. Null energy condition must be violated to avoid singularities; $(\rho + P \geq 0)$

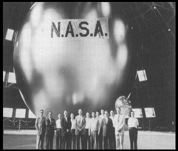
Something unexpected from geopolitics

1945: Arthur C Clarke conceptualized communication satellites.



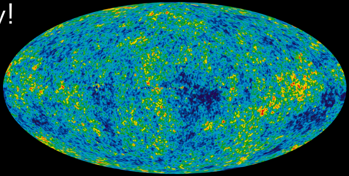
1957: Sputnik 1 launched.

1958: NASA born.



1959: Project Echo (balloon satellites)

1964: Penzias and Wilson found a mysterious noise coming from outside our galaxy!



Fundamental questions:

Is our universe described by the classical continuum spacetime at all the scales?

What is the quantum nature of spacetime?

If the spacetime has an "atomic structure", what is the fate of big bang and black hole singularities?

How does a quantum spacetime affect the physics of very early universe and in the interior of black holes?

Does quantum spacetime leave any signatures in the phenomenology of very early universe in the CMB?

Motivation: Why symmetry-reduced models?

Full quantum gravity is QFT of spacetime – an enormously complex, largely unsolved problem. Cosmological, anisotropic, and black hole interior spacetimes are symmetry-reduced: they have finitely many degrees of freedom and allow complete, rigorous quantization while retaining genuine non-trivial physics.

More than a toy model:

- BKL conjecture tells us that near any spacelike singularity the dynamics is *locally* governed by Bianchi IX dynamics. So homogeneous and Bianchi models do not merely approximate – they capture the generic singularity structure of GR.
- Wheeler-DeWitt theory applied these ideas first – but with no input from a full theory of QG. It fails to resolve singularities.
- LQC repeats the program using the genuine kinematic structure of LQG: connection-triad variables, holonomies, and the discrete quantum geometry of spin networks. The result is qualitatively different. Big Bang replaced by Big Bounce.

Motivation: What can we learn?

What can one learn in this quantum gravity playground?

- Rigorous self-consistent quantum spacetimes. Construct physical Hilbert space, inner product, Dirac observables, and physical states from first principles. No approximations forced by complexity.
- Test tools and techniques. Develop and rigorously validate methods to extract reliable physics using HPC.
- Rule out quantization ambiguities. An enormous space of mathematically possible quantizations exists. Internal consistency conditions and physical predictions drastically restrict them **(Corichi, PS (08-09))**.
- Fundamental insights. Address deep issues: consistent quantum probabilities across a bounce, emergence of effective descriptions, black hole singularities and information loss etc.

Motivation: A window into Planck scale via CMB

The pre-inflationary bounce in LQC modifies the comoving horizon. Modes can cross in and out of the horizon before inflation starts, departing from the standard Bunch-Davies vacuum. This opens a direct observational window into quantum geometry.

CMB anomalies at large angular scales are precisely where pre-inflationary physics can leave an imprint:

- Power suppression: Deficit of two-point correlations at low multipoles ($\ell \lesssim 30$) compared to ΛCDM.
- Dipolar modulation: Scale-dependent asymmetry between multipoles ℓ and $\ell + 1$.
- Parity anomaly: Excess power in odd multipoles; ΛCDM predicts parity neutrality.
- Lensing anomaly: CMB lensing amplitude incompatible with ΛCDM predictions.

LQC can potentially explain these anomalies.

Caveats:

(i) Quantization of homogeneous spacetimes is "quantum mechanics of spacetime." Where as full quantum gravity is "QFT of spacetime." Assuming homogeneity of spacetime, various hurdles of the full quantum gravity can be bypassed. Hope is that some qualitative aspects are captured.

(ii) The phenomenology of LQC is based on assuming the validity of effective description of spacetime. Tested extensively for a set of models. Provides reliable description of bounce for a class of states.

Cosmological model: spacetime and matter

Spatially flat homogeneous and isotropic FLRW spacetime with lapse $N = 1$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) \quad (1)$$

where $a(t)$ is the scale factor.

The universe is filled with a perfect fluid with stress-energy tensor

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \quad (2)$$

where u^μ is the 4-velocity of comoving observers (following the Hubble flow).

Energy density ρ and pressure $P = -\partial H_m / \partial a^3$ satisfy the conservation law from $T^\mu_{\nu;\mu} = 0$:

$$\dot{\rho} + 3H(\rho + P) = 0 \quad (3)$$

where $H = \dot{a}/a$ is the Hubble rate. This is known as the continuity equation.

Cosmological model: equations of state

For a fixed equation of state $w = P/\rho$, the conservation law integrates to $\rho \propto a^{-3(1+w)}$.

Pressurless dust: $\rho \propto a^{-3}$, $w = 0$

Relativistic matter/radiation: $\rho \propto a^{-4}$, $w = -1/3$

Dark energy: $\rho \propto a^{0\pm\epsilon}$, $w \approx -1$ Cosmological constant: $\rho = \mathrm{constt.}$, $w = -1$.

Why is the massless scalar field so important in LQC?

- With $w = 1$, stiff matter scales as a^{-6}. It dominates over all other matter near the singularity where $a \to 0$ (except anisotropies).
- The scalar field momentum is a constant of motion, and ϕ grows monotonically. It can serve as an internal clock for the quantum evolution – a relational observable that replaces coordinate time.

Cosmological model: dynamics and the big bang

From Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ one obtains the Friedmann equation:

$$H^2 := \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \tag{4}$$

and the Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{5}$$

Gravity is attractive when $\rho + 3P > 0$ (strong energy condition). The Ricci scalar is

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) = 8\pi G(\rho - 3P) \tag{6}$$

Integrating: $a \propto t^{2/3(1+w)}$ for $w \neq -1$, and $a \propto e^{\sqrt{\Lambda}t}$ for $w = -1$.

The big bang: when $a \to 0$, $\rho \to \infty$ in finite time. Spacetime curvature, tidal forces, and the Ricci scalar all diverge. This is the breakdown of classical spacetime description.

Anisotropic models and the BKL conjecture

The isotropic (FLRW) model is the simplest setting but the physical universe is not exactly isotropic. In the presence of anisotropies the Friedmann equation acquires an additional term:

$$H^2 = \frac{8\pi G}{3}\rho + \sigma^2, \qquad \sigma^2 \propto a^{-6} \tag{7}$$

where σ^2 is the anisotropic shear scalar. When present, anisotropies dominate the dynamics near the singularity since $\sigma^2 \propto a^{-6}$ grows faster than any ordinary matter.

BKL conjecture **(Belinskii, Khalatnikov, Lifshitz (1970s))**: Near a generic spacelike singularity, spatial points decouple from each other. The dynamics at each point is governed by a homogeneous but anisotropic cosmology – the Bianchi IX (Mixmaster) model.

This means: generic singularities in GR are not isotropic. The isotropic big bang is a special case. Any quantum gravity theory that claims to resolve singularities must address the anisotropic case.

Cigar singularities: the generic endpoint

The Bianchi-I model provides the simplest anisotropic setting. The metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1^2(t)\mathrm{d}x^2 + a_2^2(t)\mathrm{d}y^2 + a_3^2(t)\mathrm{d}z^2 \tag{8}$$

with three independent scale factors. Classical dynamics drives two of them to zero while one can diverge – the cigar singularity: collapse along two directions while the third stays finite or expands.

In the full Bianchi IX (Mixmaster) model, the approach to the singularity involves chaotic oscillations between different cigar orientations – "the Mixmaster" chaos. Each bounce between cigar axes is driven by curvature of the spatial sections. **(Misner (69); BKL (70))**

Key lesson: Generic cosmological singularities in GR are cigar-type, not point-like. Resolving only the isotropic big bang is insufficient. LQC must demonstrate singularity resolution in Bianchi models – which we will see it does in Lecture 2.

Big bang is not the only kind of cosmological singularity!

Depending on the equation of state there can be various types of singularities which can be classified as strong and weak.

Big Bang/Crunch: ρ, P, R diverge when $a \to 0$ in finite time. NEC: $(\rho + P) \geq 0$, is always satisfied. Strong singularity.

Big Rip/Type I singularity: NEC violated. In finite time, $a(t) \to \infty$. Accompanied with a divergence of ρ, P, R. Strong singularity.

Sudden or Type II singularity: At a finite value of the scale factor and energy density, R diverges. Needs exoitic equation of state. Weak singularity.

Type III singularity: Occurs at a finite value of scale factor. Both the energy density and pressure diverge. Strong singularity.

Type IV singularity: Only curvature derivatives blow up. Weak singularity.

Horizon Problem and Inflation

Particle horizon: Maximum comoving distance light can travel in given time. For fixed equation of state $w = P/\rho$:

$$\eta = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)} = \int_{\ln a_i}^{\ln a_f} \frac{\mathrm{d}\ln a}{aH} = H_0^{-1} \int_{\ln a_i}^{\ln a_f} a^{(1+3w)/2}\, \mathrm{d}\ln a \qquad (9)$$

If strong energy condition (SEC) is satisfied, comoving Hubble radius $(aH)^{-1}$ increases during expansion. For dust, radiation, massless scalar, the Hubble sphere grows.

Horizon Problem: How do we explain almost perfect isotropic nature of CMB in standard big bang model? There are roughly 10000 disconnected patches! Any two points which are more than a degree apart were never in causal contact.

If Hubble sphere decreases during expansion in the early universe, it can explain causal connection between different points in CMB.

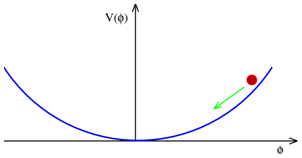
Implies violation of SEC. Results in necessity of **inflation**.

A phase of accelerated expansion in the early universe where SEC is violated. Popular paradigm to explain observations by introducing scalar field potentials.

$$\rho = \dot{\phi}^2/2 + U, \quad P = \dot{\phi}^2/2 - U \tag{10}$$

Conservation law results in Klein-Gordon eq:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + U_{,\phi} = 0 \tag{11}$$

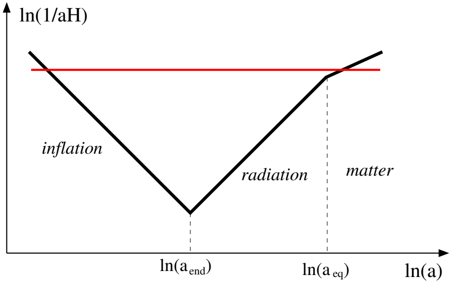


Past incomplete **(Borde, Guth, Vilenkin (03))**

Inflaton slow rolls down: $\dot{\phi}^2 \ll U$ implying $w \approx -1$ and causing an almost exponential expansion measured in number of e-foldings $N := \ln(a_e/a_i)$.

Inflation

Comoving horizon shrinks allowing causal contact between different points in the CMB. Inhomogenities arise from the quantum fluctuations of the inflaton which freeze out on exiting horizon and generate classical density perturbations on re-entry.



In the Fourier space, the power spectrum of these primordial perturbations turns out to be almost scale invariant – that is, almost independent of the wavenumber of the Fourier modes in the observational regime.

Summary for Lecture 1

- Singularities are generic in GR. Raychaudhuri (1950s) showed them in anisotropic spacetimes; Penrose-Hawking-Geroch (1960s) proved they are generic. NEC must be violated to avoid them.
- Generic singularities are cigar-type. BKL conjecture: near any spacelike singularity, dynamics at each spatial point is governed by Bianchi IX (Mixmaster) with chaotic cigar oscillations. Isotropic big bang is a special case.
- Inflation is past-incomplete. Borde-Guth-Vilenkin theorem: any expanding spacetime is geodesically past-incomplete. Inflation requires a pre-inflationary phase. In GR that includes the Big Bang.
- Looking ahead: Lecture 2 will show how QG effects convert avoid big bang and lead to quantum bounce for a wide range of models.

Lecture 2

An Overview of Singularity Resolution in LQC

Useful References for this Lecture

- A. Ashtekar, M. Bojowald and J. Lewandowski, "Mathematical structure of loop quantum cosmology," Adv. Theor. Math. Phys. **7**, no.2, 233-268 (2003)
- A. Ashtekar, T. Pawlowski and P. Singh, "Quantum Nature of the Big Bang: Improved dynamics," Phys. Rev. D **74**, 084003 (2006)
- A. Ashtekar and P. Singh, "Loop Quantum Cosmology: A Status Report," Class. Quant. Grav. **28**, 213001 (2011)
- I. Agullo and P. Singh, "Loop Quantum Cosmology," [arXiv:1612.01236 [gr-qc]]
- B. F. Li and P. Singh, "Loop Quantum Cosmology: Physics of Singularity Resolution and its Implications," [arXiv:2304.05426 [gr-qc]]

Strategy to extract quantum cosmological effects

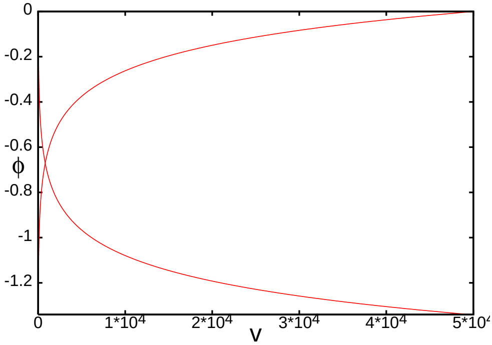
- Quantize the classical system. Find physical Hilbert space: inner product, Dirac observables, physical states.
- Consider physical initial states (such as in the GR epoch) and evolve using quantum Hamiltonian constraint. Almost on all occasions, models not exactly solvable therefore numerical simulations necessary.
- Compute expectation values of observables (and their fluctuations). Compare with the classical trajectory. Obtain departures between GR and quantum model.
- Make precise statements about how singularity resolution occurs. Behavior of energy density, shear scalar etc.
- Extract robust phenomenological predictions.

Due to the underlying symmetries, spatial diffeomorphism constraint is satisfied and the only non-trivial constraint is the Hamiltonian constraint.

Matter Hamiltonian: $\mathcal{H}_\phi = P_\phi^2/2V$

$V = V_o\, a^3$ where V_o is the volume of the fiducial cell introduced to define symplectic structure.

Hamilton's equations yield: $P_\phi = \text{constant}, \;\; \phi \sim \log V, \;\; \rho \propto a^{-6}$



ϕ acts as a "clock." Two solutions: an expanding and a contracting universe (both solutions are singular).

Wheeler-DeWitt quantization

Quantize geometry and matter for a homogeneous universe.
Only finite number of degrees of freedom, system can be treated quantum mechanically.

Earlier attempts based on treating spatial metric and its conjugate as phase space variables **(Misner, Wheeler, DeWitt 1970's)**:

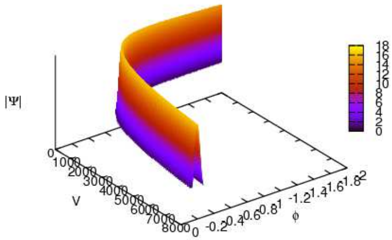
- Basic variables: $v, p_v \propto \dot{v}$ (geometry), ϕ, p_ϕ (matter).
- Operators: $\hat{v}\,\Psi(v,\phi) = v\,\Psi(v,\phi)$, $\hat{p}_v\,\Psi(v,\phi) = -i\hbar\frac{\partial}{\partial v}\Psi(v,\phi)$
- Hamiltonian: $(\hat{v}\,\hat{p}_v)^2\,\Psi(v,\phi) = \hat{\mathcal{H}}_\phi\,\Psi(v,\phi)$
- For a massless scalar, quantum Hamiltonian is:

$$\frac{\partial^2}{\partial\alpha^2}\,\Psi(\alpha,\phi) = \frac{\partial^2}{\partial\phi^2}\,\Psi(\alpha,\phi), \qquad \alpha = \log v$$

- Observables, inner product available.
- To extract departures from General Relativity, consider a semi-classical state at late times (present epoch) and evolve backwards towards big bang.

Is singularity resolved in the backward evolution?

Wheeler-DeWitt states just follow the classical trajectory, all the way to the big bang.



Singularity is not resolved! What went wrong?

- A straight forward union of quantum theory and gravity does not work. No guidance from a full theory of quantum gravity.
- Properties of spacetime same as in the classical theory.

Towards loop quantization

Due to the symmetries of the isotropic and homogeneous spacetime, the connection A_a^i and triad E_i^a can be written as

$$A_a^i = c\, V_o^{-1/3}\, \mathring{\omega}_a^i, \quad E_i^a = p\, V_o^{-2/3}\, \sqrt{\mathring{q}}\, \mathring{e}_i^a\ , \tag{12}$$

where c and p denote the isotropic connection and triad, and $\mathring{e}_i^a$ and $\mathring{\omega}_a^i$ are the fiducial triads and co-triads compatible with the fiducial metric $\mathring{q}_{ab}$. The pair (c, p) satisfies

$$\{c, p\} = \frac{8\pi G\gamma}{3}; \qquad \gamma \approx 0.2375 (\text{from BH thermo}) \tag{13}$$

Related to the metric variables as

$$|p| = V_o^{2/3} a^2 \tag{14}$$

and (only in GR as)

$$c = \gamma V_o^{1/3} \frac{\dot{a}}{N}. \tag{15}$$

Promote the classical phase variables and the classical Hamiltonian constraint to their quantum operator analogs. Holonomies of the connection A_a^i along edges, and the fluxes of the triads along 2-surfaces. Due to homogeneity the latter is proportioinal to triad.

The holonomy of the symmetry reduced connection A_a^i along a straight edge $\mathring{e}_k^a$ with fiducial length μ is,

$$h_k^{(\mu)} = \cos\left(\frac{\mu c}{2}\right)\mathbb{I} + 2\sin\left(\frac{\mu c}{2}\right)\tau_k \tag{16}$$

$\mathbb{I}$ is a unit 2×2 matrix and $\tau_k = -i\sigma_k/2$, where σ_k are the Pauli spin matrices.

Captured by functions $N_\mu(c) := e^{i\mu c/2}$. Since μ can take arbitrary values, N_μ are almost periodic functions of the connection c.

There exists a unique kinematical representation of algebra generated by these functions (Ashtekar, Campiglia (12); Engle, Hanusch, Thiemann (16)). Parallels existence of a unique irreducible representation of the holonomy-flux algebra in full LQG (Lewandowski, Okolow, Sahlmann, Thiemann (06); Fleishchack (09))

The gravitational sector of $\mathcal{H}_{\rm kin}$ is $L^2(\mathbb{R}_{\rm Bohr}, {\rm d}\mu_{\rm Bohr})$. The kinematical Hilbert space in LQC is fundamentally different from one in the Wheeler-DeWitt theory. von-Neumann theorem is bypassed.

Action of operators:

$$\hat{N}_\zeta \, \Psi(\mu) = \Psi(\mu + \zeta), \tag{17}$$

where ζ is a constant

$$\hat{p} \, \Psi(\mu) = \frac{8\pi\gamma l_{\rm Pl}^2}{6} \, \mu\Psi(\mu) \ . \tag{18}$$

Change in the orientation of the triads in absence of fermions is a large gauge transformation by a parity operator: $\hat{\Pi}\Psi(\mu) = \Psi(-\mu)$. We choose symmetric states satisfying $\Psi(\mu) = \Psi(-\mu)$.

The field strength $F_{ab}^{\ k}$ in the Hamiltonian constraint is expressed in terms of the holonomies over a square plaquette $\square_{ij}$ with length $\bar{\mu}V_o^{1/3}$ in the $i-j$ plane spanned by fiducial triads:

$$F_{ab}^{\ k} = -2 \lim_{Ar\square \to 0} \mathrm{Tr}\left(\frac{h_{\square_{ij}} - \mathbb{I}}{Ar\square}\tau^k\right) \mathring{\omega}_a^i \, \mathring{\omega}_b^j \ . \qquad (19)$$

Ar$\square$ denotes the area of the square plaquette, and $h_{\square_{ij}} = h_i^{(\bar{\mu})} h_j^{(\bar{\mu})} (h_i^{\bar{\mu}})^{-1} (h_j^{\bar{\mu}})^{-1}$, with $\bar{\mu}$ denoting the edge length of the plaquette.

Note that due to the underlying quantum geometry, the limit Ar$\square \to 0$ does not exist. Shrink the area of the loop to the minimum non-zero eigenvalue of the area operator in LQG.

Denote this minimum area as Δl_{Pl}^2 where $\Delta = 4\sqrt{3}\pi\gamma$.

Using physical area of the loop equalling $\bar{\mu}^2|p|$ results in **(Ashtekar, PS, Pawlowski (06))**

$$\bar{\mu}^2 = \frac{\Delta l_{\mathrm{Pl}}^2}{|p|} \qquad (20)$$

Action of $N_{\bar{\mu}}$ on the triad eigenstates is not by a simple translation.

More convenient to work with variables b and v which are defined in terms of c and p as:

$$\mathrm{b} := \frac{c}{|p|^{\frac{1}{2}}}, \quad \mathrm{v} := \mathrm{sgn}(p)\frac{|p|^{\frac{3}{2}}}{2\pi G}, \tag{21}$$

which when promoted to operators have action:

$$\widehat{\exp(i\lambda \mathrm{b})}\,|\nu\rangle \;=\; |\nu - 2\lambda\rangle, \qquad \hat{V}\,|\nu\rangle \;=\; 2\pi\gamma l_{\mathrm{Pl}}^2\,|\nu|\,|\nu\rangle \tag{22}$$

where $\nu = \mathrm{v}/\gamma\hbar$. Quantum Hamiltonian constraint equation:

$$\partial_\phi^2\,\Psi(\nu,\phi) \;=\; 3\pi G\,\nu\,\frac{\sin\lambda \mathrm{b}}{\lambda}\nu\,\frac{\sin\lambda \mathrm{b}}{\lambda}\,\Psi(\nu,\phi) \;=:\; -\,\Theta\Psi(\nu,\phi) \tag{23}$$

where Θ is a positive definite, second order difference operator:

$$\Theta\Psi(\nu,\phi) \;:=\; -\frac{3\pi G}{4\lambda^2}\,\nu\,((\nu+2\lambda)\Psi(\nu+4\lambda) - 2\nu\Psi(\nu,\phi) + (\nu-2\lambda)\Psi(\nu-4\lambda)) \tag{24}$$

(Ashtekar, PS, Pawlowski (06); Ashtekar, Corichi, PS (08))

Quantum difference equation resulting from quantum geometry results in Wheeler-DeWitt differential equation at large volumes.

Quantum constraint similar to the Klein-Gordon theory, ϕ plays the role of time and Θ acts like a spatial Laplacian operator. Physical states can be chosen as solutions of the positive frequency square root of Θ:

$$-i\,\partial_\phi \Psi(\nu,\phi) = \sqrt{\Theta}\,\Psi(\nu,\phi)\ . \tag{25}$$

Inner product:

$$\langle \Psi_1 | \Psi_2 \rangle \;=\; \sum_{\nu} \,\bar{\Psi}_1(\nu,\phi_o)|\nu|^{-1}\Psi(\nu_2,\phi_o)\ . \tag{26}$$

Dirac observables:

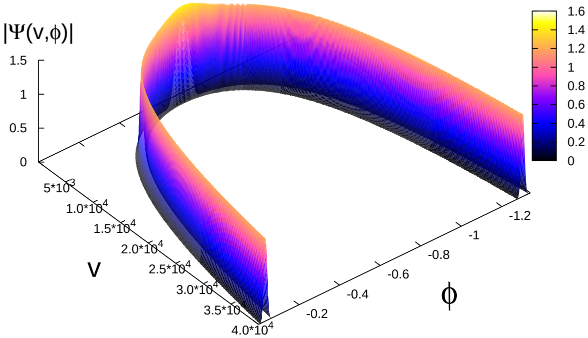
$$\hat{V}|_{\phi_o}\Psi(\nu,\phi) \;=\; 2\pi\gamma l_{\rm Pl}^2\, e^{i\sqrt{\Theta}(\phi-\phi_o)}|\nu|\Psi(\nu,\phi_o) \tag{27}$$

and

$$\hat{p}_\phi \Psi(\nu,\phi) \;=\; -i\hbar\,\partial_\phi\,\Psi(\nu,\phi) = \hbar\sqrt{\Theta}\Psi(\nu,\phi)\ . \tag{28}$$

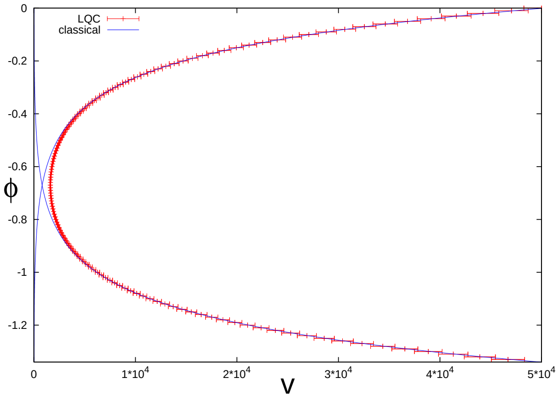
Quantum Bounce

Due to quantum geometry effects in loop quantum gravity, big bang is replaced by a quantum bounce! **(Ashtekar, Pawlowski, PS, (06))**



For sharply peaked states universe bounces at a maximum of energy density $\rho = \rho_{\rm max} = 3/8\pi G \Delta^2 \approx 0.41 \rho_{\rm Planck}$

Comparison of Classical and Quantum Evolution



Universe follows classical trajectory till curvature is approximately a percent of the Planck curvature. GR an excellent approximation when gravity is weak. Singularity recovered when $\Delta \to 0$.

Probability of bounce to occur is unity! **(Craig, PS (2013))**

Robustness of singularity resolution: examples

- Exactly solvable model (flat, isotropic with a massless scalar) **(Ashtekar, Corichi, PS (2008))**
- In presence of spatial curvature $k = \pm 1$ **(Ashtekar, Pawlowski, PS, Vandersloot (2007); Kaminski, Lewandowski, Szulc (2007); Vandersloot (2007); Szulc (2009))**
- Bianchi models **(Ashtekar, Wilson-Ewing (2009-2010); Martin-Benito, Mena-Marugan, Pawlowski (2009); Diener, Joe, Megevand, PS (2018))**
- Negative cosmological constant **(Bentivegna, Pawlowski (2007))**
- Positive cosmological constant **(Pawlowski, Ashtekar (2012))**
- ϕ^2 inflationary potential **(Ashtekar, Pawlowski, PS (unpublished); Giesel, Li, PS (2021))**
- Extremely wide states not corresponding to a classical universe at late times **(Diener, Gupt, PS (2014))**
- Non-gaussian and highly squeezed states corresponding to highly quantum universes **(Diener, Gupt, PS (2014))**
- Radiation **(Pawlowski, Pierini, Wilson-Ewing (2014))**

For suitably chosen coherent states, following geometric formulation of QM, one can obtain an effective description

(Taveras (2008))

$$C_H^{(\mathrm{eff})} = -\frac{3\hbar}{4\gamma\lambda^2}\nu \sin^2(\lambda \mathrm{b}) + \frac{1}{4\pi\gamma l_{\mathrm{Pl}}^2}\frac{P_\phi^2}{\nu} \, . \tag{29}$$

Vanishing of Hamiltonian constraint yields

$$\frac{3}{8\pi G\gamma^2\lambda^2} V \sin^2(\lambda \mathrm{b}) = \frac{P_\phi^2}{2V} \, . \tag{30}$$

The modified Friedmann and Raychaudhuri equations can be found using Hamilton's equation for V and b

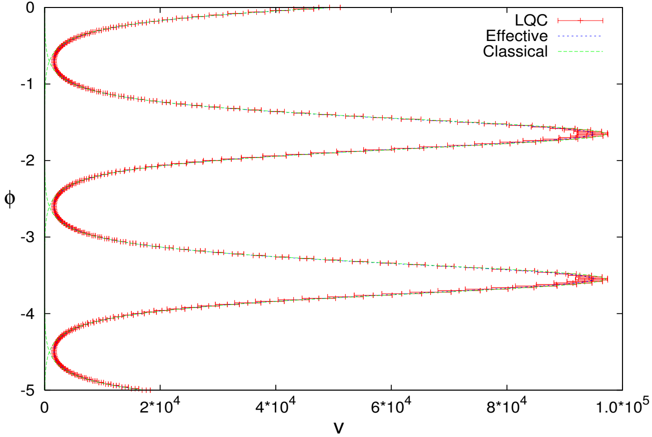
$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \quad \text{with} \quad \rho_{\max} = \frac{3}{8\pi G\gamma^2\lambda^2} \, . \tag{31}$$

The quantum gravitational correction thus appears as a ρ^2 modification to the classical Friedmann equation. Bounce occurs when $\rho = \rho_{\max}$. Gravity becomes very repulsive for $\rho > \rho_{\max}/2$.

An accurate test of recovering GR: $k = 1$ FLRW model

(Ashtekar, Pawlowski, PS, Vandersloot; Szulc, Kaminski, Lewandowski (2007))

The closed model has past and future singularities.

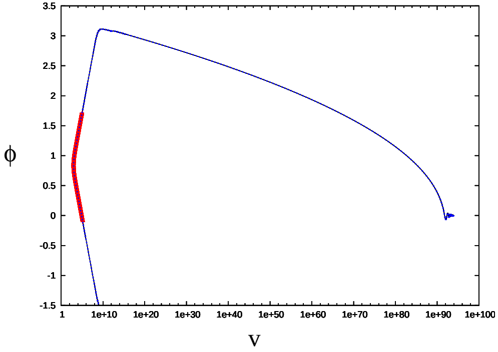


LQC predicts recollapse accurately and avoids both big bang and big crunch in various cycles. Effective trajectory completely captures quantum evolution in all cycles.

- Is quantum bounce a generic feature of states in the theory?
  Bounce happens for all the states in the physical Hilbert space for the spatially flat, isotropic model (Ashtekar, Corichi, PS (2008)).
- What is the state of the universe on the other side? Is it a quantum foam or a classical spacetime?
  A macroscopic universe, such as ours, bounces from a macroscopic universe similar to ours. Spacetime fluctuations severely constrained on both sides of the bounce (Corichi, PS; Kaminski, Pawlowski; Corichi, Montoya (2008-2010))
- What about quantization ambiguities? Do they affect results?
  Surprisingly, an enormous number of 'possible' quantizations in LQC can be ruled out. Consistency conditions have been proposed, which restrict many mathematically possible choices (Corichi, PS (2008-2009)). Recently new regularizations studied where bounce is asymmetric for massless scalar field (mLQC-I (Li, PS, Wang (2020)))

Does quantum gravity resolve problems of inflation?

Quantum gravity resolves the past singularity in inflation (Ashtekar, Pawlowski, PS (unpublished)).

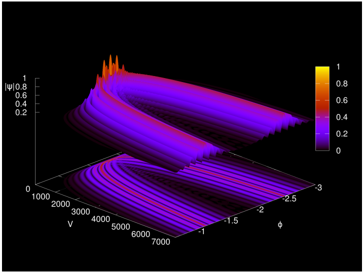


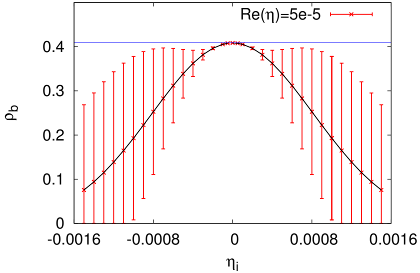
Non-singular difference equation for inflation with additional test field as a clock (Giesel, Li, PS (2020)) Loop quantum effects also help in setting right initial conditions for inflation (PS, Vandersloot, Vereschagin (2007); Ranken, PS (2012); Gupt, PS (2013)) and give valuable insights on the probability for inflation to occur (Ashtekar, Sloan; Corichi, Karami (2009)).

Bounce for highly quantum states

Bounce not restricted to any special states. Even occurs for states which are highly non-Gaussian or squeezed.

(Diener, Gupt, Megevand, PS (2014))



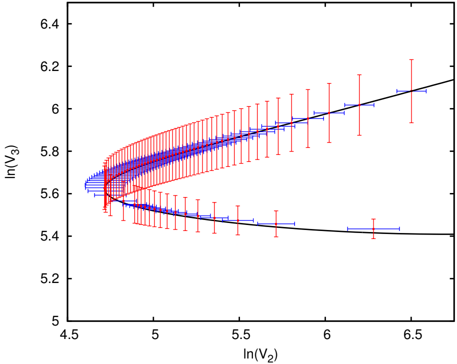


In the isotropic model, quantum fluctuations are found to always lower the curvature scale at which the bounce occurs. Quantum fluctuations in the state enhance the "repulsive nature of gravity" in the quantum regime.

Anisotropic quantum bounce

Computationally challenging and expensive. Limited early results on bounce in Bianchi-I vacuum model **(Martin-Benito, Mena Marugan, Pawlowski (2008))**. Using HPC framework, physics of quantum bounce in Bianchi-I vacuum spacetime rigorously understood

(Diener, Joe, Megevand, PS (2018))



Anisotropic shear remains bounded throughout the evolution.
Effective description turns out to be a good approximation.

Modifications of standard LQC

(Yang, Ding, Ma (09); Li, PS, Wang (18); Assanioussi, Dapor, Liegener, Pawlowski (18))

Hamiltonian constraint composed of Euclidean and Lorentzian terms:

$$\mathcal{C}_{\rm grav} = \mathcal{C}_{\rm grav}^{(E)} - (1+\gamma^2)\mathcal{C}_{\rm grav}^{(L)}$$

where

$$\mathcal{C}_{\rm grav}^{(E)} = \frac{1}{2}\int {\rm d}^3x \, \epsilon_{ijk} F^i_{ab} \frac{E^{aj}E^{bk}}{\sqrt{\det(q)}}$$

and

$$\mathcal{C}_{\rm grav}^{(L)} = \int {\rm d}^3x \, K^j_{[a} K^k_{b]} \frac{E^{aj}E^{bk}}{\sqrt{\det(q)}}$$

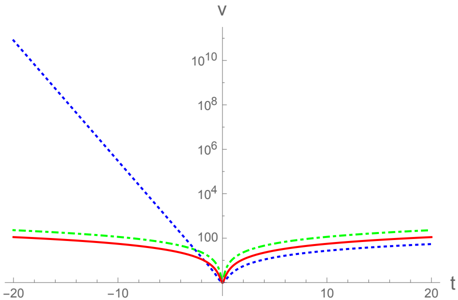
In LQC, quantization of spatially flat models obtained after combining $\mathcal{C}_{\rm grav}^{(E)}$ and $\mathcal{C}_{\rm grav}^{(L)}$. If terms are treated distinct, then form of quantum Hamiltonian constraint significantly different.

Two ambiguities at this level:

- Quantize $\mathcal{C}_{\rm grav}^{(L)}$ as above after using identities on classical phase space and expressing in terms of holonomies. Leads to mLQC-I.
- Use $K^i_a = \gamma^{-1} A^i_a$ in $\mathcal{C}_{\rm grav}^{(L)}$, and then quantize. Results in mLQC-II.

Comparison of mLQC-I and mLQC-II with LQC

Non-trivial modifications to Friedmann dynamics for mLQC-I and mLQC-II in comparison to LQC in Planck regime **(Li, PS, Wang (18))**



In mLQC-I, spacetime curvature remains Planckian before the bounce yet satisfies Einstein field equations but with a quantum gravitational origin matter.

In mLQC-II, spacetime curvature decreases quickly on both sides of the bounce as in LQC. No emergent matter or a rescaled G.

No cyclic models possible in mLQC-I **(Li, PS (22))**

Does LQC resolve all the singularities?

Spacetime curvature invariants can in principle diverge for for various spacetimes in loop quantum gravity **(PS (09,11); Saini, PS (16-17))**

Example: In the spatially flat isotropic model in loop quantum cosmology, spacetime curvature captured by

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) = 8\pi G\rho\left(1 - 3w + 2\frac{\rho}{\rho_{\rm max}}(1+3w)\right), \quad w = p/\rho$$

Energy density and Hubble rate have upper bound in loop quantum cosmology, but pressure is not bounded.

For highly exotic equations of state, pressure can diverge at a finite value of energy density causing some special singularities **(Barrow, Tsagas (04))**

Resolution of all strong singularities in LQC

When is a singularity physically relevant? The singularity at $\tau = \tau_o$ is strong and physically relevant if $\int_0^\tau d\tau' |R_{i\ 4j4}|$ diverges as $\tau \to \tau_o$. Else the singularity is weak.

For all the events where curvature invariants diverge in loop quantum gravity, singularities are weak and geodesics can be extended beyond such events. Interestingly, quantum geometry effects ignore weak singularities.

Strong curvature singularities are forbidden in loop quantum gravity at least for isotropic and anisotropic spacetimes.

(PS (09,11); PS, Vidotto (10); Saini, PS (16-17))

As in LQC, in mLQC-I and mLQC-II scale factor remains finite and non-zero for all finite time evolution. All strong singularities resolved for mLQC-I and mLQC-II **(Saini, PS (18))**

Summary for Lecture 2

- Loop quantum cosmology provides a glimpse on the origin of the Universe in non-perturbative quantum gravity for homogeneous universes. Emerging picture from simple models: Big bang not the beginning, big crunch not the end.
- Singularity resolution achieved in various isotropic and anisotropic models. No need to introduce any exotic matter/ad-hoc assumptions/fine tuning. Existence of bounce tested for extreme conditions using high performance computers.
- Bounce occurs for states in a dense subspace of the physical Hilbert space (not only for those which are semi-classical at late times).
- Discreteness of quantum geometry bounds the energy density, anisotropic shear and curvature scalars.
- Main open question: Is singularity resolution an artifact of symmetry reduced models? Or do these results point towards a generic resolution of singularities in LQG?

Lecture 3

Hamiltonian Framework for Cosmological Perturbations:
Exploring Quantum Geometry Effects in CMB

Useful References for Lecture 3

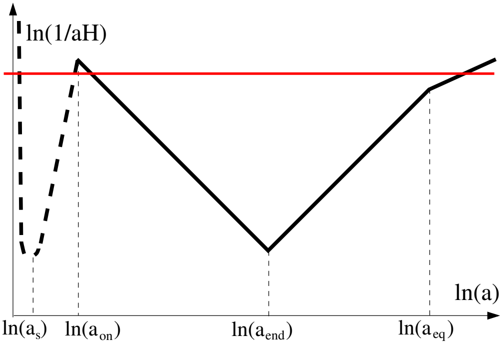
- D. Langlois, Hamiltonian formalism and gauge invariance for linear perturbations in inflation, Class. Quant. Grav. **11** 389 (1994)
- I. Agullo, A. Ashtekar and W. Nelson, "Extension of the quantum theory of cosmological perturbations to the Planck era," Phys. Rev. D **87**, 043507 (2013)
- L. C. Gomar, M. Martín-Benito and G. A. M. Marugán, "Gauge-Invariant Perturbations in Hybrid Quantum Cosmology," JCAP **06**, 045 (2015)
- A. Ashtekar, B. Gupt, D. Jeong and V. Sreenath, "Alleviating the Tension in the Cosmic Microwave Background using Planck-Scale Physics," Phys. Rev. Lett. **125**, 051302 (2020)
- I. Agullo, D. Kranas and V. Sreenath, "Anomalies in the Cosmic Microwave Background and Their Non-Gaussian Origin in Loop Quantum Cosmology," Front. Astron. Space Sci. **8**, 703845 (2021)
- B. F. Li and P. Singh, "Close relationship between the dressed metric and the hybrid approach to perturbations in effective loop quantum cosmology," Phys. Rev. D **106**, 086015 (2022)

Motivation

- The homogeneity assumption made rigorous quantization possible, but the physical universe is not exactly homogeneous. Small inhomogeneities in the early universe are the seeds of all large-scale structure.
- These inhomogeneities arise from quantum fluctuations of the inflaton. They freeze out on exiting the Hubble horizon and generate classical density perturbations on re-entry.
- The CMB is the most precise snapshot of these primordial perturbations. It is our main observational window into the physics of the very early universe.
- While the inflationary paradigm fits observations extremely well at small angular scales, there are anomalies at the largest angular scales that are difficult to explain within standard ΛCDM. These are precisely the scales where a pre-inflationary quantum gravity epoch could leave an imprint.

Motivation: the bounce and the comoving horizon

In LQC, the pre-inflationary bounce changes the behavior of the comoving Hubble horizon.



- Modes can cross in and out of the horizon even before inflation starts. Their quantum state at the onset of inflation can differ from the Bunch-Davies vacuum.
- Nature of the bounce – and hence the imprint – depends on the regularization (LQC, mLQC-I, mLQC-II) and quantum ambiguities.

The CMB: what do we expect?

The Cosmic Microwave Background is a snapshot of the universe at $\sim 380{,}000$ years after the big bang – light that has been travelling to us ever since.

What the standard ΛCDM model predicts:

- Temperature fluctuations $\delta T/T \sim 10^{-5}$ arising from quantum fluctuations of the inflaton
- Gaussian: fluct. at different scales are statistically ind.
- Statistically isotropic: no preferred direction in the sky
- Nearly scale-invariant: roughly equal power at all angular scales – the primordial power spectrum $P_S \propto k^{n_s - 1}$ with $n_s \approx 1$

Anomalies are places where the observed CMB sky persistently disagrees with these predictions – all at the largest angular scales (low multipoles $\ell = 2, 3, 4 \ldots$)

CMB anomalies: a potential window for quantum gravity

The CMB power spectrum from Planck 2018 reveals several statistically significant anomalies at large angular scales ($\ell \lesssim 30$):

- **Power suppression anomaly:** Lack of two-point correlations at large angular scales compared to the ΛCDM prediction. Power deficit at $\ell = 2$–30 at ~ 2–3σ.
- **Dipolar modulation anomaly:** Scale-dependent dipolar asymmetry between multipoles ℓ and $\ell + 1$. Preferred direction in the sky.
- **Parity anomaly:** ΛCDM predicts parity neutrality (equal power in even and odd multipoles). Observed CMB has excess power in odd multipoles at large scales.
- **Lensing anomaly:** CMB lensing amplitude is larger than predicted by ΛCDM, incompatible at $\sim 2\sigma$.

These anomalies occur at the angular scales most sensitive to the pre-inflationary epoch – exactly where LQC effects are expected.

Main approaches: dressed metric and hybrid

Strategy: Fock-quantized perturbations on a loop-quantized background.

- **Dressed metric approach** (Agullo, Ashtekar, Nelson (12)): classical framework follows Langlois (1994). The quantum corrected background "dresses" the modes.
- **Hybrid approach** (Fernandez-Mendez, Mena Marug'an, Olmedo (12)): classical framework follows Halliwell-Hawking (1985). Background and perturbations quantized simultaneously.
- In the classical theory, both approaches lead to the same second-order Hamiltonian and therefore the same Mukhanov-Sasaki equation. Differences arise only in the Planck regime when quantum geometric effects are imported.
- Differences between the two at the effective level can be traced to quantum ambiguities in polymerization (Li, PS (22)).

Road map for Cosmological Perturbations

- **Step 1: LQC effective background.** Modified Friedmann equation with bounce at $\rho = \rho_{\max}$. Establishes the pre-inflationary spacetime.
- **Step 2: Hamiltonian framework.** ADM action for GR + scalar field. Background equations. Expand to second order in perturbations. SVT decomposition and gauge fixing.
- **Step 3: Mukhanov-Sasaki equation.** Eliminate constraints using linearized Hamiltonian and diffeomorphism constraints. Second-order Hamiltonian $\to$ MS equation for Q and rescaled variable $\nu = aQ$.
- **Step 4: Import quantum geometry.** Consistently polymerize background quantities. Modified MS equation carries LQC imprint.
- **Step 5: Observational output.** Initial conditions at bounce $\to$ propagate modes $\to$ primordial power spectrum $P_S(k) \to$ angular power spectrum $C_\ell^{TT} \to$ CMB anomalies.

Brief overview of results

Significant advances in connecting quantum geometry effects with precision CMB observations:

- Agreement with observations at UV (small) scales. Quantum gravity effects encoded at intermediate and infrared (large) scales as power amplification.
- Special vacuum state (Ashtekar, Gupt (17)) can potentially alleviate power suppression and lensing anomalies (Ashtekar, Gupt, Jeong, Sreenath (20)).
- Non-Gaussianities can alleviate anomalies including the parity asymmetry (Agullo, Kranas, Sreenath (20)).
- Non-oscillatory vacuum and states of low energy can also alleviate anomalies (Mart'ın-Benito, Neves, Olmedo (21)).
- Modified LQC (mLQC-I, mLQC-II) leave distinct imprints in CMB (Li, PS, Wang (22)).

(Langlois (94)) Consider a massive scalar field ϕ minimally coupled in GR:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) \qquad (32)$$

In the ADM formalism, foliate spacetime into spatial hypersurfaces Σ_t using lapse $N(t, x^i)$ and shift $N^i(t, x^i)$.

Phase space variables:

- Gravity: induced metric γ_{ij} and conjugate momentum π^{ij}
- Matter: scalar field ϕ and momentum π_ϕ

Poisson brackets:

$$\{\gamma_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \delta^k_{(i}\delta^l_{j)}\delta^3(\mathbf{x}, \mathbf{y}), \quad \{\phi, \pi_\phi\} = \delta^3(\mathbf{x}, \mathbf{y}) \qquad (33)$$

Hamiltonian and diffeomorphism constraints

The ADM action in canonical variables:

$$S = \int \mathrm{d}^4x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\phi \dot{\phi} - N\mathcal{H} - N^i \mathcal{H}i \right) \tag{34}$$

Lapse and shift are Lagrange multipliers. Variation gives two constraints:

Hamiltonian constraint:

$$\mathcal{H} = \frac{2\kappa}{\sqrt{\gamma}} \left(\pi^{ij}\pi ij - \frac{\pi^2}{2} \right) - \frac{\sqrt{\gamma}}{2\kappa}{}^{(3)}R + \sqrt{\gamma} \left(\frac{\pi_\phi^2}{2\gamma} + \frac{1}{2}\partial_i\phi\partial^i\phi + U(\phi) \right) \approx 0 \tag{35}$$

Spatial diffeomorphism constraint:

$$\mathcal{H}i = -2\partial_k(\gamma_{ij}\pi^{jk}) + \pi^{jk}\partial_i\gamma jk + \pi_\phi\partial_i\phi \approx 0 \tag{36}$$

Physical solutions lie on the constrained phase space. First-class constraints: Hamiltonian evolution preserves them.

The cosmological FLRW background

Spatially flat FLRW metric:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + e^{2\alpha(t)}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{37}$$

With $a = e^\alpha$, $\pi_a = e^{-\alpha}\pi_\alpha$:

$$\{a, \pi_a\} = \frac{1}{V_o}, \qquad \{\bar{\phi}, \bar{\pi}_\phi\} = \frac{1}{V_o} \tag{38}$$

Background Hamiltonian constraint (spatially flat, ${}^{(3)}R = 0$):

$$\mathcal{H}^{(0)} = -\frac{\kappa}{12}\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + U(\phi)a^3 \approx 0 \tag{39}$$

Hamilton's equations with $N = 1$ recover classical dynamics:

$$H^2 = \frac{\kappa}{3}\rho, \qquad \frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3P), \qquad \ddot{\phi} + 3H\dot{\phi} + U_{,\phi} = 0 \tag{40}$$

In LQC, the modified Friedmann equation $H^2 = \frac{8\pi G}{3}\rho\,(1 - \rho/\rho_{\max})$ replaces this. Recall that the bounce occurs at $\rho = \rho_{\max}$.

Expand around the FLRW background:

$$\gamma_{ij} = \bar{\gamma}_{ij} + \delta\gamma_{ij}, \quad \pi^{ij} = \bar{\pi}^{ij} + \delta\pi^{ij}, \quad \phi = \bar{\phi} + \delta\phi, \quad \pi_\phi = \bar{\pi}_\phi + \delta\pi_\phi \quad (41)$$

In Fourier space ($\delta\phi(\mathbf{k}) = \delta\phi^*(-\mathbf{k})$ for real fields):

$$\delta\phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k, \delta\phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (42)$$

Poisson brackets:

$$\{\delta\phi(\mathbf{k}), \delta\pi_\phi(\mathbf{k}')\} = \delta^{(3)}(\mathbf{k} + \mathbf{k}') \quad (43)$$

The symmetric matrix $\delta\gamma_{ij}(\mathbf{k})$ generates a $6 \times \infty^3$ dimensional space. For each $\mathbf{k}$ this decomposes into scalar (S), vector (V), and tensor (T) subspaces.

Introduce orthonormal basis $A^{(n)}_{ij}(\mathbf{k})$, $n = 1 \ldots 6$: $n = 1, 2$ scalar, $n = 3, 4$ vector, $n = 5, 6$ tensor.

SVT decomposition and gauge fixing

Linearized constraints eliminate degrees of freedom:

- Linearized spatial diffeomorphism constraint $\mathcal{H}_i^{(1)} \approx 0$: 3 constraints, each first-class
- First two fix vector perturbations entirely
- Third + linearized Hamiltonian constraint $\mathcal{H}^{(1)} \approx 0$ eliminates 4 d.o.f.
- Starting from 10 d.o.f. in scalar+vector subspace $\rightarrow$ 2 physical d.o.f. remain

Spatially flat gauge: set perturbation of intrinsic curvature to zero,

$$\gamma_{(1)}(\mathbf{k}) = \gamma_{(2)}(\mathbf{k}) = 0 \tag{44}$$

In this gauge the Mukhanov-Sasaki variable Q coincides with the scalar field perturbation:

$$Q = \delta\phi \tag{45}$$

The two physical degrees of freedom are then (Q, P_Q) for scalar modes, and the two tensor polarizations $(\tilde{\gamma}_{(n)}, \tilde{\pi}_{(n)})$ for $n = 5, 6$.

Second order Hamiltonian for scalar perturbations

After imposing the spatially-flat gauge and eliminating the linearized constraints, the second-order Hamiltonian for scalar perturbations in (Q, P_Q) is:

$$\mathbf{H}^{(2)}\mathrm{s} = \int \mathrm{d}^3k \left[\frac{1}{a^3} P_Q(\mathbf{k}) P_Q(-\mathbf{k}) + a\left(k^2 + \Omega_Q^2\right) Q(\mathbf{k}) Q(-\mathbf{k})\right] \tag{46}$$

where $k = ak_{\mathrm{phy}}$ and the effective potential is

$$\Omega_Q^2 = \frac{3\kappa\bar{\pi}_\phi^2}{a^4} - \frac{18\bar{\pi}_\phi^4}{a^6\pi_a^2} - \frac{12a\bar{\pi}_\phi}{\pi_a} U_{,\phi} + a^2 U_{,\phi\phi} \tag{47}$$

Hamilton's equations:

$$\dot{Q}(\mathbf{k}) = \frac{1}{a^3} P_Q(\mathbf{k}), \qquad \dot{P}_Q(\mathbf{k}) = a\left(k^2 + \Omega_Q^2\right) Q(\mathbf{k}) \tag{48}$$

Tensor modes: same structure without Ω_Q^2, gauge-invariant.

The Mukhanov-Sasaki equation

Taking the time derivative of $\dot{Q}$ and using $\dot{P}_Q$:

$$\ddot{Q}(\mathbf{k}) + 3H\dot{Q}(\mathbf{k}) + \frac{1}{a^2}\left(k^2 + \Omega_Q^2\right)Q(\mathbf{k}) = 0 \qquad (49)$$

This is the Mukhanov-Sasaki equation. (Mukhanov (88))

Rescale $\nu = aQ$, use conformal time $\eta = \int \mathrm{d}t/a$:

$$\nu''(\mathbf{k}) + \left(k^2 + m^2\right)\nu(\mathbf{k}) = 0, \qquad m^2 = \Omega_Q^2 - \frac{a''}{a} \qquad (50)$$

This is a harmonic oscillator with time-dependent mass.

- $k\eta \gg 1$ (subhorizon): modes oscillate freely like flat-space plane waves
- $k\eta \ll 1$ (superhorizon): modes freeze; $\mathcal{R} = \nu/z$ conserved with $z = a\dot{\phi}/H$

The primordial power spectrum:

$$P_S(k) = \frac{k^3}{2\pi^2}\frac{|\nu_k|^2}{z^2} \longrightarrow A_s\left(\frac{k}{k_*}\right)^{n_s-1} \qquad \text{(Bunch-Davies + slow roll)} \qquad (51)$$

Importing quantum geometry: polymerization

Strategy: replace classical background quantities in the MS equation with effective LQC expressions.

The key LQC substitution in the background:

$$b^2 \longrightarrow \frac{\sin^2(\lambda b)}{\lambda^2} \quad \Longrightarrow \quad H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{52}$$

The background quantities $1/\pi_a^2$ and $1/\pi_a$ appearing in Ω_Q^2 must be consistently polymerized:

$$\frac{1}{\pi_a^2} \to \frac{\kappa}{12v^{4/3}\rho}, \qquad \frac{1}{\pi_a} \to -\frac{H}{2v^{2/3}\rho} \tag{53}$$

(Gomar et al (15); Li, PS, Wang (20))

Key requirements: (i) Use the same polymerization in the perturbation equation as in the background. (ii) Ensure Ω_Q is continuous at the bounce so initial conditions can be set and propagated across.

Modified MS equation and the bounce curvature scale

At the bounce, for kinetic-dominated evolution ($\Omega_Q^2 \ll a''/a$), the MS equation becomes:

$$\nu_k'' + a^2 \left(\frac{k^2}{a^2} - \frac{R}{6} \right) \nu_k = 0 \tag{54}$$

The Ricci scalar defines a bounce curvature scale:

$$\lambda_{R_B} := 2\pi \sqrt{\frac{6}{|R_B|}} \approx 1.96 l_{\text{Pl}} \tag{55}$$

- $\lambda_{\text{phy}} \ll \lambda_{R_B}$: curvature term negligible $\rightarrow$ propagate as in Minkowski $\rightarrow$ unaffected by quantum geometry
- $\lambda_{\text{phy}} \gg \lambda_{R_B}$: modes feel bounce curvature $\rightarrow$ excited $\rightarrow$ depart from Bunch-Davies

The departure from the Bunch-Davies vacuum at the bounce is a prominent source of all LQC imprints in the CMB.

The observational window

Not all modes are affected. There is a window of wavenumbers whose physical size at the bounce exceeds λ_{R_B}.

Find the mode with minimum physical wavenumber at the bounce observable today. If

$$k_{\rm min}^{\rm phy} < k_{R_B}^{\rm phy} \tag{56}$$

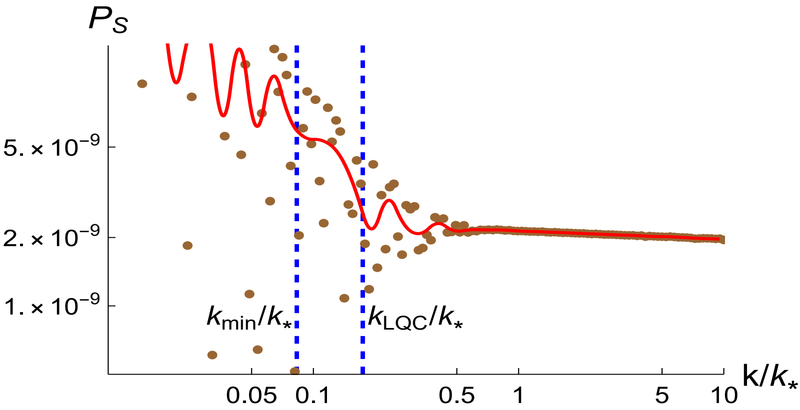
a window of wavenumbers carries a signature of the pre-inflationary branch.

Example: ϕ^2 potential with $\phi_B = 1.0, m_{\rm Pl}$

- Pivot mode at Hubble crossing: $k_*^{\rm phy} = 4.72 l_{\rm Pl}^{-1}$
- Observable minimum: $k_{\rm min}^{\rm phy} = 0.02 l_{\rm Pl}^{-1} < k_{\rm LQC}^{\rm phy}$ (window exists)

Sensitivity to ϕ_B:

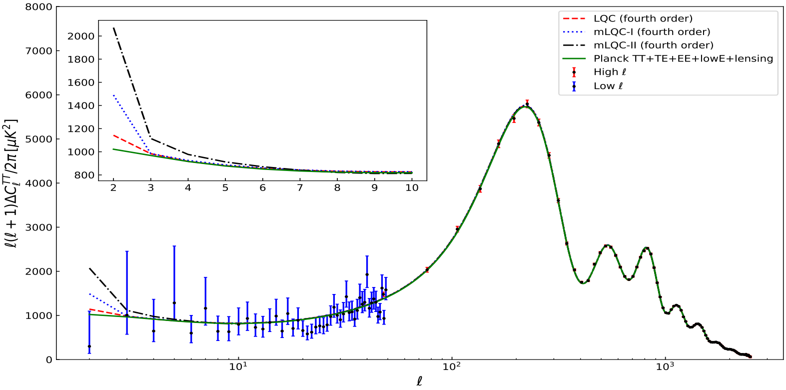
- $\phi_B \gtrsim 1.13 m_{\rm Pl}$: no window – inflation long enough to wash out pre-inflationary effects
- $\phi_B \lesssim 0.95 m_{\rm Pl}$: large window – but must remain consistent with near-scale-invariant spectrum



$$P_S = \frac{k^3}{2\pi^2}\frac{|\nu_k|^2}{z_s^2}$$

Modes inside the observational window (low k) are excited at the bounce.

Angular power spectrum



Departures from the ΛCDM prediction appear at low multipoles ($\ell \lesssim 30$) – consistent with the observed power suppression anomaly. The exercise can be repeated for different potentials (Starobinsky, ϕ^2) and different regularizations (LQC, mLQC-I, mLQC-II).

Role of the vacuum state

A key ingredient is the choice of initial state at the bounce:

- **Bunch-Davies vacuum:** natural for de Sitter inflation; not well-motivated at the bounce where spacetime is not de Sitter.
- **4th order adiabatic vacuum:** minimizes particle production at the bounce. Leads to power amplification at low k.
- **Ashtekar-Gupt (AG) state** (Ashtekar, Gupt (17)): selected by requiring the state to be as classical as possible at the bounce consistent with the uncertainty principle. Can alleviate power suppression and lensing anomalies (Ashtekar, Gupt, Jeong, Sreenath (20)).
- **Non-oscillatory vacuum** (Mart'ın-Benito, Neves, Olmedo (21)): another well-motivated choice that also alleviates anomalies.

The power spectrum depends sensitively on the initial state.

Non-Gaussianities and the parity anomaly

Beyond the power spectrum, non-Gaussianities offer an independent probe:

- The parity anomaly (excess power in odd multipoles) cannot be explained by a modification of the two-point function alone. It requires non-Gaussian correlations between modes.
- Agullo, Kranas, Sreenath (2020) showed that non-Gaussianities generated during the LQC bounce can alleviate the parity anomaly through mode coupling between ℓ and $\ell + 1$.
- The bispectrum in LQC is consistent with Planck 2018 non-Gaussianity constraints at UV scales while departing at IR scales.
- Modified LQC (mLQC-I) predicts a qualitatively different bispectrum due to the asymmetric bounce – another potential observational discriminant.

Open questions

- Uniqueness of signatures: Can we identify observable features in the CMB that uniquely distinguish LQC from other pre-inflationary models (string gas cosmology, bouncing cosmologies in modified gravity)?
- Quantum treatment of perturbations: Current results are at the effective level. Is singularity resolution maintained and are the anomaly-alleviating effects robust when perturbations are treated fully quantum mechanically?
- Consistency of quantum ambiguities: How sensitive are the observational predictions to the choice of vacuum state, regularization, and polymerization scheme? Is there a preferred choice from first principles?
- Full quantum gravity: How do these results change when the full LQG framework (beyond symmetry reduction) is used for the background?

Summary for Lecture 3

- The Hamiltonian framework (ADM + SVT + spatially-flat gauge) gives the Mukhanov-Sasaki equation – the central equation for cosmological perturbations. Both dressed metric and hybrid approaches give the same MS equation classically.
- Quantum geometry enters by consistently polymerizing background quantities in the MS equation. Certain modes are excited at the bounce and depart from the Bunch-Davies vacuum.
- This generates observable effects potentially explaining CMB anomalies
- Vacuum choice, regularization (LQC vs mLQC-I vs mLQC-II), and non-Gaussianities all play a role in connecting theory to observations.
- Open question: Can unique, unambiguous quantum gravity signatures be identified in the CMB? Future missions will sharpen this test.