Loop Quantum Gravity Summer School 26

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Canonical Quantisation of Loop Quantum Gravity

LQG Summer School, Yangzhou

Aim: Brief introduction to canonical formulation of Loop Quantum Gravity

Plan of the lectures

1.) Motivation for quantum gravity

2.) The formulation of general relativity in terms of Ashtekar-Barbero variables

3.) Reduced phase space quantisation of LQG: Step 1: Reduced phase space in terms of Brown-Kuchař dust

4.) Reduced phase space quantisation of LQG: Step 2: Kinematics

5.) Reduced phase space quantisation of LQG: Step 3: Dynamics

Part 1: Motivation

1. Motivation


Über quantentheoretische Umdeutung
kinematischer und mechanischer Beziehungen.
Von W. Heisenberg in Göttingen.
(Eingegangen am 29. Juli 1925.)

- let us start with non-relativistic quantum mechanics
- Canonical quantisation:
- Classical starting point:
  NR Hamiltonian System $(q^i, p_j) \in T^*Q$

- **Kinematics**

$$(q^i, p_j) \quad \{q^i, q^k\} = 0, \{p_j, p_k\} = 0, \{q^i, p_k\} = \delta^i_k$$

- **Dynamics**

$$H(p,q) \quad \frac{dq^i}{dt} = \{q^i, H\}, \quad \frac{dp_i}{dt} = \{p_i, H\}$$

Canonical quantisation:

Representation on some Hilbert space + dynamics

(Lecture Madhavan Varadarajan)

- **Kinematics**

$$\mathcal{H} = L_2(\mathbb{R}^3, d^3x), \quad [\hat{q}^i, \hat{q}^k] = 0, [\hat{p}_j, \hat{p}_k] = 0, [\hat{q}^i, \hat{p}_k] = i\hbar \mathbb{1}_{\mathcal{H}}$$

- **Dynamics**

$$\hat{A} = H(\hat{q}, \hat{p}) \qquad i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi$$

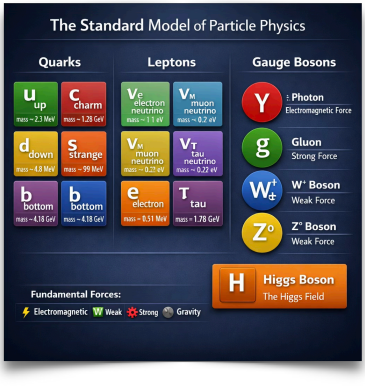
(Schrödinger)

(Heisenberg)

$$\frac{d\hat{q}^i}{dt} = \frac{1}{i\hbar}[\hat{q}^i, \hat{H}]$$

$$\frac{d\hat{p}_i}{dt} = \frac{1}{i\hbar}[\hat{p}_i, \hat{H}]$$

- Situation:
- We only consider non-relativistic models
- Global time parameter t, external not part of T*Q
- We only consider point-like particles
- We only consider finitely many degrees of freedom
- Unique step to go from classical model to corresponding quantum model (Stone-von Neumann Theorem)
  ↳ rather mild assumptions on representation
- Although foundational aspects of NR QM still part of modern research, how to apply quantisation step well understood
- Generalising models at the classical & quantum level:





- **Situation in theoretical physics**
- Two sucessfull theoretical models

Standard Model	QFT	General
• QFT, η • background dependent • only matter quantised	QFT, g non-dyn.	$(\mathcal{M}, g)$ • dynamics involve g • classical theory
Particle Physics Yu Tian	on curved spacetime (Ivan Agullo)	Relativity (Xongge Ma & Xiaolong Zhang)

- In general relativity dynamics couples geometry and matter

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{geometry}} = k \underbrace{T_{\mu\nu}}_{\text{matter}}$$

Einstein's equations

- Aim of loop quantum gravity

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} = k \hat{T}_{\mu\nu}$$

Quantum Einstein's equation

- Applications in cosmology (LQC) Parampreet Singh & Badfei Li
- Application in black holes (Cong Zhang)
- Covariant formulation of dynamics Spin foams (Muxin Han)

- Quantising systems involving gravity is often more challenging than for those where gravity is absent or treated classically

Differences & challenges compared to Quantum mechanics

I. GR is a (fully) constrained system; involves gauge dof
   - quantisation of constrained systems more ambiguities (chapter 2)

II. Even at the classical level physical observables (gauge-inv. quantities) hard to construct
   - (Quantum) Dirac observables wrt dynamical physical (quantum) reference frames (chapter 3)

III. Metric becomes operator requires background independent quantisation
   - LQG uses no Fock quantisation (chapter 4)

IV. Formulation of the dynamics: Quantum Einstein Equations (chapter 5)

Part 2:
Formulation of general relativity in terms of Ashtekar-Barbero variables

2.1. Brief recap on Hamiltonian formulation of constrained systems (Lecture Yongge Ma)

2.1.1. Classification of constraints

2.2. Quantisation of constraint systems (general remarks)

2.3 Summary main results GR in terms of ADM variables (Lecture Yongge Ma)

2.4 Hamiltonian formulation of GR in terms of Ashtekar-Barbero variables

2.1 Hamiltonian formulation of constrained systems

- Before we apply this to GR consider the situation in class. mechanics

Starting point: Action

$$S[q] = \int dt\, L(q, \dot{q}, t)$$

L: Lagrangran $\quad q = (q^1, \dots, q^f) \in Q$ configuration variables

$f = \dim(Q)$ $\quad Q$: configuration space

$\dot{q} = (\dot{q}^1, \dots, \dot{q}^f)$ (generalised) velocities

- In order to obtain the Hamiltonian formalism we need to apply a Legendre transformation

$$\begin{array}{ccc} TQ & \longrightarrow & T^*Q \quad \text{(phase space)} \\ \cap & & \cap \\ (q, \dot{q}) & & (q, p) \end{array}$$

. Canonical momenta $\quad p_j = \dfrac{\partial L}{\partial \dot{q}^j} \quad j = 1, \dots, f$

- to perform the Legendre transformation we need to solve the velocities for the momenta
- Condition that this can be done:

$$\det\left(\underbrace{\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}}_{\text{Hessian matrix}}\right) = \det\left(\frac{\partial p_i}{\partial \dot{q}^k}\right) \neq 0$$

if given: regular Lagrangian
if not given: singular Lagrangian

- If the Lagrangian is singular solving velocities for momenta does not work

↳ Legendre transformation needs to be adapted

Reason: system has constraints often occur in case of symmetries (particularly gauge-symmetries)

- Apply Dirac algorithm to

(i) perform generalised Legendre transf
(ii) determine the final set of constraints

lecture by Yongge Ma

- One simple example to pick everybody up:

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2$$

Lagrangian is singular, no $\dot{y}$ involved

- 1 primary constraint: $\phi_1 := p_y \approx 0$
- primary Hamiltonian:

$$H_{pr} = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x-y)^2 + v^1\phi_1$$

v^1 Lagrange multiplier

- Stability of ϕ_1: $\dot{\phi}_1 = \{\phi_1, H_{pr}\} \overset{!}{\approx} 0$

Case 2: $\phi_2 = p_x + x - 2y = 0$

- Stability of ϕ_2: $\dot{\phi}_2 = \{\phi_2, H_{pr}\} \overset{!}{\approx} 0$

Case 3: fixes Lagrange multiplier

$$v^1 = \frac{1}{2}(p_x - x)$$

- final set of constraints: ϕ_1, ϕ_2

2.1.1. Classification of constraints

Aim: Introduce the notion of first and second class constraints as well as Dirac bracket

- Dirac algorithm determines all constraints of a given system
- Once the set of constraints is known
  ↳ classification into two types

Def.

A constraint is called first class if its Poisson bracket with all constraints vanishes weakly.

- Hence: given a set of constraints $\{\Phi_I\}_{I\in\mathcal{I}}$ then for 1st class constraint Φ_I we have $\{\Phi_I, \Phi_J\} = f_{IJ}{}^K \Phi_K$ for all $J \in \mathcal{I}$ this is equivalent to $\{\Phi_I, \Phi_J\} \approx 0$ for all $J \in \mathcal{I}$

the remaining constraints are called second class.

- for weak equalities, first compute Poisson brackets, then apply $\Phi_I \approx 0$.

Example: from earlier example

$\Phi_1 = p_y \approx 0$, $\Phi_2 = p_x + x - 2y \approx 0$

$\{\Phi_1, \Phi_2\} = \{p_y, -2y\} = 2 \not\approx 0$

↳ Φ_1 & Φ_2 are both second class constraints

- Consider the general case again:
  Assume given set of constraints:
  $\{\Phi_I\}_{I=1}^{N}$ $A_1 = 1, \dots, N_1$ are 1st class
  $B_2 = 1, \dots, N_2$ are 2nd class
  with $N = A_1 + B_2$
  Notation: $\Phi^{(1)}_{A_1}$ 1st class constraints
  $\Phi^{(2)}_{B_2}$ 2nd class constraints
  we assume that the second class constraints are irreducible
  ↳ no linear combination of 2nd class constraints is 1st class

Question: How can we test whether a set of 2nd class constraints is irreducible?

We define the matrix $M_{A_2B_2}$

$$M_{A_2B_2} := \left(\{\Phi^{(2)}_{A_2}, \Phi^{(2)}_{B_2}\}\right)$$

- A set of 2nd class constraints is irreducible if $\det(M) \neq 0$
- If $\det(M) = 0$ then

$$V^{A_2} \{ \Phi^{(2)}_{A_2}, \Phi^{(2)}_{B_2} \} \approx 0$$

has a non-trivial solution for V^{A_2}

↳ more linear combinations of 2nd class constraints which are 1st class exist.

Def.

A set $\{\Phi_I\}_{I \in \mathcal{I}}$ of constraints is 2nd class if and only if the matrix

$$M_{IJ} := (\{\Phi_I, \Phi_J\})$$

is invertible on $T^*Q|_{\Phi=0}$

- The classification into 1st & 2nd class constraint will be important when defining Dirac observables
- With M_{IJ} and $\det(M_{IJ}) \neq 0$ given we can introduce the notion of the Dirac bracket:

$$\{f,g\}_* := \{f,g\} - \{f, \Phi^{(2)}_I\} (M^{-1})^{IJ} \{\Phi^{(2)}_J, g\}$$

- Dirac bracket modified Poisson bracket
- encodes dependencies of variables due to second class constraints
- By construction $\{f, \Phi^{(2)}_I\}_* = 0$
  - ↳ if we use Dirac brackets we can implement second class constraints strongly $\Phi^{(2)}_I = 0$ also inside $\{\cdot,\cdot\}_*$
- following Dirac the set $\{\Phi^{(1)}_I\}$ is called the set of generators of gauge-trafos
- with this we can introduce the notion of weak Dirac observables

Def.

A smooth function is called weak Dirac observable provided that

$$\{f, \Phi^{(1)}_I\} \approx 0 \quad \text{for all } I$$

It is called strong Dirac observable if the condition above holds without a weak but strong equality, that is everywhere on phase space.

- For $\Phi^{(1)}_I$ we have $\{f, \Phi^{(1)}_I\}_* \approx \{f, \Phi^{(1)}_I\}$

2.2 Canonical Quantisation of constrained systems

- Let us consider canonical quantisation in systems without constraints first

2.2.1 Quantum mechanics in $D=3$:

- **Step 1:** Given a classical phase space with elementary variables (q^j, p_j) we consider the corresponding Heisenberg algebra

$$\{q^j, p_k\} = \delta^j{}_k \quad \{q^j, q^k\} = 0 \quad \{p_j, p_k\} = 0$$

↳ choice of suitable algebra at the classical level.

- **Step 2:** Given the classical choice we construct an abstract $*$-algebra of operators

Algebra $\mathcal{A}$: vector space equipped with a bilinear product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$

$*$-algebra has in addition an involution map $*: \mathcal{A} \to \mathcal{A} \quad a \mapsto a^*$ with

$$(ca_1 + c'a_2)^* = \bar{c}a_1^* + \bar{c}'(a_2^*) \qquad c, c' \in \mathbb{C}$$
$$(a_1 a_2)^* = a_2^* a_1^* \qquad a_1, a_2 \in \mathcal{A}$$
$$(a^*)^* = a$$

- Since we aim at relating commutators and Poisson brackets we consider the $*$-algebra generated by

$$\hat{q}^j, \hat{p}_j, \mathbb{1}_{\mathcal{A}}$$

and set

$$[\hat{q}^j, \hat{p}_k] = i\hbar \widehat{\{q^j, p_k\}} \qquad (\hat{q}^j)^* = \hat{q}^j \qquad (\hat{p}_j)^* = \hat{p}_j$$

- **Step 3:**

Given the abstract $*$-algebra we look for representations thereof that is a map

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$$

($\mathcal{L}(\mathcal{H})$: linear operators on Hilbert space $\mathcal{H}$)

with the following properties

$$\pi(c\hat{a}_1 + c'\hat{a}_2) = c\pi(\hat{a}) + c'\pi(\hat{a}_2)$$
$$\pi(\hat{a}_1\hat{a}_2) = \pi(\hat{a}_1)\pi(\hat{a}_2)$$
$$\pi(\hat{a}_1^*) = (\pi(\hat{a}_1))^\dagger \quad + \text{adjoint in } \mathcal{H}$$

$\hat{a}_1, \hat{a}_2$ are elements of the algebra generated by $(\hat{q}^j, \hat{p}_j, \mathbb{1}_{\mathcal{A}})$

We further require

$$[\pi(\hat{q}^j), \pi(\hat{p}_k)] = i\hbar\pi(\mathbb{1}_{\mathcal{A}}) = i\hbar\mathbb{1}_{\mathcal{H}}$$

- **Lecture Madhavan Varadarajan:**

Method to construct π: GNS construction

- In the case of QM the representation normally used is the Schrödinger representation

  Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, d^3x)$

$$(\pi(\hat{q}_j)\psi)(x) = x_j \psi(x)$$
$$(\pi(\hat{p}_j)\psi)(x) = -i\hbar \frac{\partial}{\partial x_j}\psi(x)$$

- final quantum theory depends on choice of classical algebra + choice of representation
- In general even for a given classical algebra more than one representation exist → QM — Stone-von Neumann Thm
- Physically equivalence: if representations are unitary equivalent, that is

  $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$

  unitary operator $\hat{U}: \mathcal{H}_1 \to \mathcal{H}_2$
  s.t. $\hat{U}\pi_1(\hat{a})\hat{U}^{-1} = \pi_2(\hat{a})$ for all $\hat{a} \in \mathcal{A}$.
- There is no Stone-von Neumann theorem in case of field theory

2.2.2 Reduced and Dirac quantisation

For systems with constraints we have two options:

A.) Reduced phase space quantisation:

(i) Reduce the constraints at the classical level.

a) <u>for 2nd class constraints:</u>
Construct the corresponding Dirac bracket.
↳ in general yields more complicated algebra

b.) <u>for remaining system with 1st-class constraints:</u>

(ii) Find a suitable set of Dirac observables $\{O_f\}$ s.t.

$$\{O_f, \Phi_I^{(1)}\} \approx 0 \quad \text{for all 1st class } \Phi_I^{(1)}$$

- Given the set $\{O_f\}$ derive their Poisson algebra

$$\{O_f, O_g\} \approx O_{\{f,g\}^*}$$

↳ usually more complicated than original kinematical algebra

(iii) Derive the Hamiltonian that generates the EOM $\{O_f\}$

↳ so-called physical Hamiltonian

Then dynamics wrt physical time τ

$$\frac{d}{d\tau} O_f = \dot{O}_f = \{ O_f, H_{phys} \}$$

(iv) Quantisation:

- Find a representation of the observable algebra:

  $\mathcal{H}_{phys}$: physical Hilbert space

  $\hat{O}_f$: operators for observables

- Dynamics:

  Only those representations where $\hat{H}_{phys}$ can be implemented, then

$$\frac{d}{dt} \hat{O}_f = \frac{1}{i\hbar} [\hat{O}_f, \hat{H}_{phys}]$$

Heisenberg equations

↳ quantum dynamics directly formulated in $\mathcal{H}_{phys}$.

3) Dirac quantisation

- Solve the constraints in the quantum theory.

1.) Quantise the unconstrained (kinematical) phase space (Q^J, P_J)

$\mathcal{H}_{kin}$ kinematical Hilbert space

- elementary operators $\hat{Q}^J, \hat{P}_J$
- Classical constraints $\Phi_I(Q^J, P_J) \approx 0$ become operators

$$\hat{\Phi}_I = \Phi_I(\hat{Q}^J, \hat{P}_J)$$

choose operator ordering

- Select physical states:

$$\hat{\Phi}_I |\psi_{phys}\rangle \doteq 0 \quad \text{for all } I$$

yields $\mathcal{H}_{phys}$

- Quantum Dirac observable

$$[\hat{O}, \hat{\Phi}_I] \approx 0 \quad \text{for all } I$$

- Remark:
Dirac quantisation can only be applied to 1st class constraints

Given $\Phi_I |\psi_{phys}\rangle = 0$ for all I $\rightsquigarrow$ $\hat{\Phi}_J \hat{\Phi}_I |\psi_{phys}\rangle = 0$

$\rightsquigarrow$ $(\hat{\Phi}_I \hat{\Phi}_J - \hat{\Phi}_J \hat{\Phi}_I)|\psi_{phys}\rangle = 0$

$\rightsquigarrow$ $[\hat{\Phi}_I, \hat{\Phi}_J]|\psi_{phys}\rangle = 0$

- Now if the classical constraints are first class:

$$\{\Phi_I^{(1)}, \Phi_J^{(1)}\} = f_{IJ}{}^K \Phi_K^{(1)}$$

in Q.T:

$$[\hat{\Phi}_I^{(1)}, \hat{\Phi}_J^{(1)}] = f_{IJ}{}^K \hat{\Phi}_K^{(1)}$$

then consistent quantisation.

- However for second class constraints

$$\{\Phi_I^{(2)}, \Phi_J^{(2)}\} \neq f_{IJ}{}^K \Phi_K$$

thus contradiction

$$[\hat{\Phi}_I^{(2)}, \hat{\Phi}_J^{(2)}]\,|\psi_{phys}\rangle \neq 0$$

- Way out:

(i) Consider Dirac bracket for 2nd class constraints in kinematical phase space

(ii) Weak implementation of the constraints

$$\langle \psi | \hat{\Phi}_I^{(2)} | \psi' \rangle = 0 \quad \text{for all } |\psi\rangle, |\psi'\rangle \in \mathcal{H}_{weak}$$

- As we will discuss reduced quantisation in case of LQG for most of the constraints

Example: Dirac quantisation

2 dimensional classical system

$$\{x^j, p_k\} = \delta^j{}_k \qquad j,k = 1,2$$

primary Hamiltonian: $H = \frac{p_2^2}{2m} + \lambda \phi$, $\phi = p_1 \approx 0$

- Obviously ϕ stable

final set of constraints $\phi_1 = p_1 \approx 0$.

- Kinematical Hilbert space

$$\mathcal{H} = L_2(\mathbb{R}^2)$$

elements $\psi(x,y)$ with $\|\psi\| < \infty$

$$\langle \psi, \psi \rangle = \|\psi\|^2 = \int_{\mathbb{R}} dx^1 \int_{\mathbb{R}} dx^2 \, |\psi|^2 (x^1, x^2)$$

- Constraint operator

$$\hat{\phi}_1 = \hat{p}_1 = -i\hbar \frac{\partial}{\partial x^1}$$

- physical states

$$(\hat{p}_1 \psi)(x^1, x^2) \stackrel{!}{=} 0 \Leftrightarrow -i\hbar \frac{\partial}{\partial x^1} \psi(x^1, x^2) \stackrel{!}{=} 0$$

Solution: physical states do not depend on x^1

$$\psi_{phys} = \varphi(x^2)$$

- With respect to kinematical inner product Ψ_{phys} is not square-integrable

$$\|\Psi_{phys}\|^2_{\mathcal{H}_{kin}} = \int_{\mathbb{R}} dx^1 \int_{\mathbb{R}} dx^2 \, |\varphi|^2(x^2)$$
$$= \underbrace{\left(\int_{\mathbb{R}} dx^1\right)}_{=\infty} \left(\int_{\mathbb{R}} dx^2 \, |\varphi|^2(x^2)\right) \not< \infty$$

- these physical states are no elements of $\mathcal{H}_{kin}$
- Can be understood as linear functionals on $\mathcal{H}_{kin}$ (generalised eigenvectors)
- Consider linear functional

$$\ell_{phys} : \underbrace{\mathcal{D}(\hat{\Phi}_1)}_{\text{dense domain in } \mathcal{H}_{kin} \text{ here } \mathcal{S}(\mathbb{R}^2) \text{ (Schwartz space)}} \longrightarrow \mathbb{C}$$

$$\varphi \in \mathcal{D}(\hat{\Phi}_1) \longmapsto \ell_{phys}(\varphi) \in \mathbb{C}$$

To construct ℓ_{phys} explicitely, we consider family of unitary operators

$$\{\hat{U}_\alpha\}_{\alpha \in \mathbb{R}} = \{e^{i\alpha \hat{p}_1}\}_{\alpha \in \mathbb{R}} = \{e^{i\alpha \hat{\Phi}_1}\}_{\alpha \in \mathbb{R}}$$

and can define for any $\Psi \in \mathcal{D}(\hat{\Phi}_1)$

$$\ell^{\Psi}_{phys} := \int_{\mathbb{R}} d\alpha \, \langle \hat{U}(\alpha)\Psi, \cdot \rangle_{\mathcal{H}_{kin}}$$

- then

$$\varphi \mapsto \ell^{\psi}_{phys}(\varphi) = \int_{\mathbb{R}} d\alpha \, \langle \hat{U}(\alpha)\psi, \varphi \rangle_{\mathcal{H}_{kin}}$$

here we consider not infinitesimal but finite action of constraint operator

- infinitesimal:

$$(\hat{\Phi}_1 \psi_{phys})(x^1, x^2) \overset{!}{=} 0$$

- finite: $\hat{U}(\beta) = e^{i\beta\hat{\Phi}_1}$

Action on linear functionals ℓ

$$(\hat{U}'(\beta)\ell)(\varphi) := \ell(\hat{U}^{\dagger}(\beta)\varphi)$$

- condition on ℓ^{ψ}_{phys} reads:

$$(\hat{U}'(\beta)\ell^{\psi}_{phys})(\varphi) = \ell^{\psi}_{phys}(\hat{U}^{\dagger}(\beta)\varphi) \overset{!}{=} \ell^{\psi}_{phys}(\varphi)$$

- Explicitly we obtain:

$$(\hat{U}'(\beta)\ell^{\psi}_{phys})(\varphi) = \int_{\mathbb{R}} d\alpha \, \langle \hat{U}(\alpha)\psi, \hat{U}^{\dagger}(\beta)\varphi \rangle_{\mathcal{H}_{kin}}$$

$$= \int_{\mathbb{R}} d\alpha \, \langle \hat{U}(\beta)\hat{U}(\alpha)\psi, \varphi \rangle_{\mathcal{H}_{kin}}$$

$$= \int_{\mathbb{R}} d\alpha \, \langle \hat{U}(\beta+\alpha)\psi, \varphi \rangle_{\mathcal{H}_{kin}} \qquad \tilde{\alpha} := \beta + \alpha$$

$$= \int_{\mathbb{R}} d\tilde{\alpha} \, \langle \hat{U}(\tilde{\alpha})\psi, \varphi \rangle_{\mathcal{H}_{kin}} = \ell^{\psi}_{phys}(\varphi) \quad \checkmark$$

2.3. Hamiltonian formulation of General Relativity in terms of ADM -variables (Summary)

- We want to apply the Dirac algorithm to GR now (lectures by Yongge Ma)

Plan:

1.) Summary application of Dirac algorithm in ADM variables

2.) Afterwards introduce Ashtekar-Barbero variables

2.3.1. Brief discussion on ADM formalism in general relativity

- Starting point is Einstein Hilbert action (restrict to vacuum for simplicity)

$$S[g] = \frac{1}{\kappa}\int_{\mathcal{M}} d^4x \sqrt{-\det(g)}\, R \qquad \kappa = 16\pi G_N$$

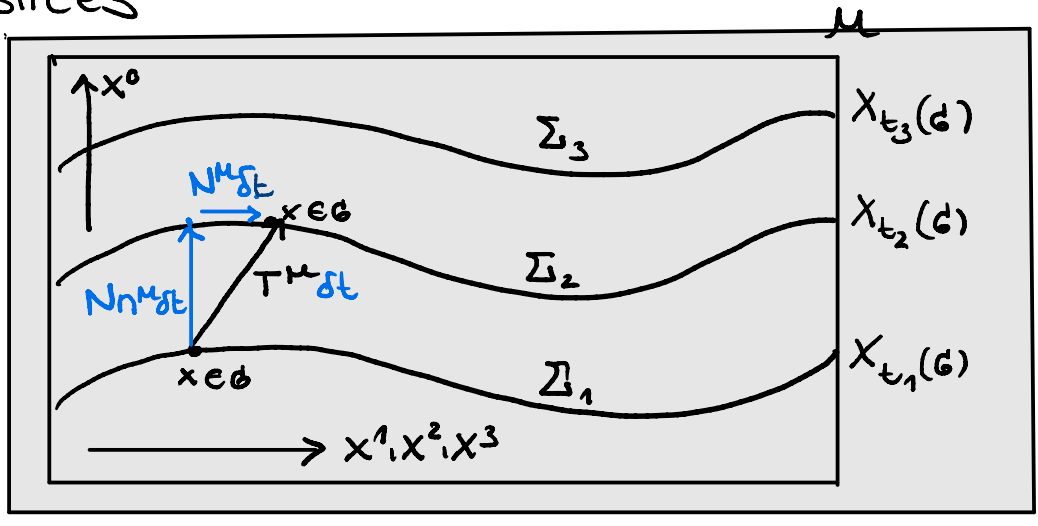
$\mathcal{M}$ spacetime coord. X^μ, $\mu = 0, \dots, 3$

Consider globally hyperbolic manifolds

then $\mathcal{M} \simeq \mathbb{R} \times \mathcal{C}$ topologically equivalent

$\mathcal{C}$: 3-dim manifold, x^a $a = 1, 2, 3$ coord. $\mathcal{C}$

Foliation of 4D $\mathcal{M}$ into three-dim. slices



- 3-dim spatial slices $X_t(\mathcal{C})$, $t \in \mathbb{R}$
- Σ_t subsets of $\mathcal{M}$; $\mathcal{C}$ is mapped to Σ_t via the embedding map X_t
- The evolution is described along a time-like vector field T^μ

- We decompose T^μ into normal and tangential components

$$T^\mu = N n^\mu + N^\mu = \partial_t X_t^\mu$$

- N lapse function, $n_\mu n^\mu = -1$
  N^μ shift vector with $N^\mu n_\mu = 0$
- embedding map $x^a \mapsto X_t^\mu(x^a)$ coord.

- This allows to consider Hamiltonian evolution along T^μ
- T^μ fixed by chosen N, N^μ $\rightsquigarrow$ determines slicing
- consider generic slicing do not fix N, N^μ

- Strategy:

  Rewrite integral in action $\int_{\mathcal{M}} d^4x$ in terms of coordinates adapted to the slicing $\int_{\mathbb{R}} dt \int_{\Sigma} d^3x$

  needed: metric $g_{\mu\nu}$ and derivatives thereof as tensors on Σ

- For this purpose we consider the first & second fundamental form: ADM metric q_{ab}, extrinsic curvature K_{ab}
- In Hamiltonian formulation

$$(q_{ab}, p^{ab}),\ (N, \Pi),\ (N^a, \Pi_a)$$

- primary constraints

$$\Pi \approx 0 \quad \text{and} \quad \Pi_a \approx 0$$

- Secondary Constraints: Hamiltonian and spatial diffeomorphism constraint

$$C_a = -2 q_{ac} D_b P^{bc}$$

$$C = \frac{1}{\sqrt{q}} G_{abcd} P^{ab} P^{cd} - \sqrt{q}\, R^{(3)}$$

with $\sqrt{q} := \sqrt{\det(q)}$, $G_{abcd} := q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd}$

- Then we consider the reduced ADM phase space:
primary constraints are satisfied:

$$\Pi \approx 0 \,, \quad \Pi_a \approx 0$$

↳ consequence: N, N^a become Lagrange multipliers

hypersurface deformation algebra:

$$\{\vec{C}(\vec{N}), \vec{C}(\vec{M})\} = -\kappa \vec{C}(\mathcal{L}_{\vec{M}} \vec{N})$$

$$\{C(N), \vec{C}(\vec{M})\} = -\kappa C(\mathcal{L}_{\vec{M}} N)$$

$$\{C(N), C(M)\} = -\kappa \vec{C}(f(q, N, M))$$

$f^a(q,N,M) := q^{ab}(M_{,b} N - M N_{,b})$ structure functions

$C(N) := \int_G d^3x\, (NC)(x)$ $\quad \vec{C}(\vec{N}) := \int_G d^3x\, (N^a C_a)(x)$

- Reduction wrt primary constraints
  ↳ can treat N, N^a as Lagrange multipliers

- Reduced ADM action

$$S^{ADM}_{red} = \frac{1}{\kappa} \int_\sigma d^3x \, (\dot{q}_{ab} p^{ab} - (N^a C_a + N C))(x)$$

Canonical ADM Hamiltonian:

$$H_{can} = \int_\sigma d^3x \, (NC + N^a C_a)(x)$$

Question: How can we rediscover Einstein's equations?

Hamilton's equation of motion

$$\dot{q}_{ab}(t,x) = \{ q_{ab}(t,x), H_{can} \}$$
$$\dot{p}_{ab}(t,x) = \{ p_{ab}(t,x), H_{can} \}$$
$$C(t,x) \approx 0 \quad \text{and} \quad C_a(t,x) \approx 0$$

- these are equivalent to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

- Remark: adding matter:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- then additional dof $\phi(x), \Pi_\phi(x)$ etc and constraints involve additional contributions

$$C^{tot} = C^{geo} + C^{\phi} \quad , \quad C_a^{tot} = C_a^{geo} + C_a^{\phi}$$ etc

- Counting degrees of freedom:

- Started with 10 dof $g_{\mu\nu}$ in $D=4$ + matter dof
- full ADM setup: (q_{ab}, N, N^a) + matter dof
- Constraints: 4 primary + 4 secondary all first class
- Each 1st class constraints reduces 2 dof in phase space

(q_{ab}, p^{ab}), (N, Π), (N^a, Π_a) 20 dof + matter dof

$8 \times 2 = 16$ reduced

↳ 4 physical dof in phase space + matter dof

↳ 2 physical dof for vacuum gravity ✓

2.4. Formulation of GR in terms of Ashtekar-Barbero variables

- Geometrodynamics: Applying a Schrödinger quantisation to GR in ADM variables
  ↳ Wheeler-DeWitt eqn

  open questions: underlying representation, Hilbert space; non-polynomial operators

- **Idea:** Considers a different choice of variables at the classical level
  ↳ Ashtekar-Barbero variables

- Recall in GR tetrads $D=4$
- For a given $\mathcal{M}$ at each $p \in \mathcal{M}$ $\rightsquigarrow T_p\mathcal{M}$
- If $(\mathcal{M}, g)$ is equipped with a metric we can choose a pseudo-orth. basis of $T_p\mathcal{M}$

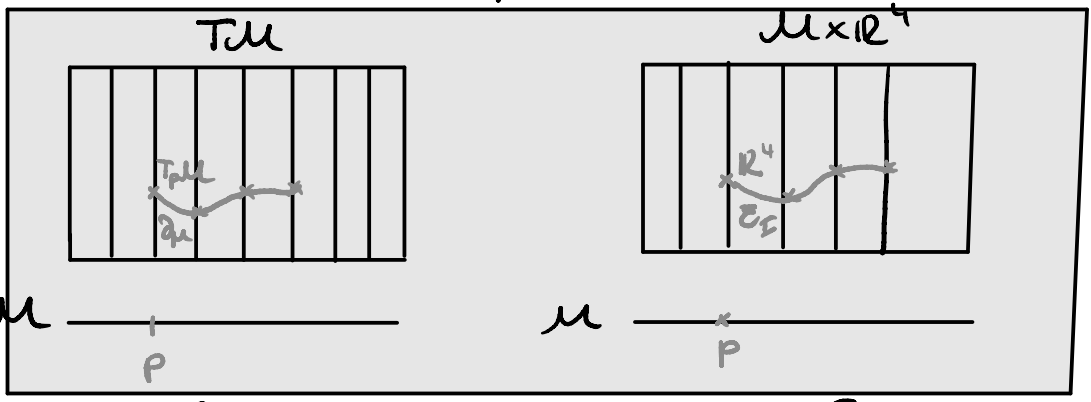
  $e_0(p), \dots, e_3(p)$ $\quad \{e_I\}$ $I = 0, \dots, 3$

- choose one of the e_I's time-like, others spacelike then

$$g(e_I, e_J) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{IJ}$$

the vector fields $\{e_I\}$ $I = 0, \dots, 3$ are denoted as tetrads

- Geometrical interpretation



- basis of section in $M \times \mathbb{R}^4$: e_I

$$e_0(p) = (1,0,0,0)^T \qquad e_1(p) = (0,1,0,0)^T$$
$$e_2(p) = (0,0,1,0)^T \qquad e_3(p) = (0,0,0,1)^T$$

then we can understand e as a map

$$e: M \times \mathbb{R}^4 \to TM$$
$$\text{with } e_I \mapsto e(e_I) = e_I = e_I^\mu \partial_\mu$$

- e_I^μ are components of e_I wrt coord. basis ∂_μ
- e is uniquely determined once $\{\partial_\mu\}$ and $\{e_I\}$ are chosen.
- Often one denotes not only $\{e_I\}$ but also the map e or e_I^μ as tetrads
- Inverse map exists:

$$e^{-1}: TM \to M \times \mathbb{R}^4$$
$$e^{-1}(\partial_\mu) = e_\mu = e_\mu^I e_I$$

I internal indices

e^{-1}, e_μ^I co-tetrads

- The co-tetrads define a spacetime metric

$$g(e_I, e_J) = g(e(\xi_I), e(\xi_J)) = \eta(\xi_I, \xi_J) = \eta_{IJ}$$

written in more detail

- Given this

$$\begin{aligned} g_{\mu\nu} &= g(\partial_\mu \partial_\nu) = \eta(e^{-1}(\partial_\mu), e^{-1}(\partial_\nu)) \\ &= \eta(e^I_\mu \xi_I, e^J_\nu \xi_J) \\ &= e^I_\mu e^J_\nu \eta(\xi_I, \xi_J) \\ &= e^I_\mu e^J_\nu \eta_{IJ} \end{aligned}$$

- If we know $e^I_\mu(p)$ for each $p \in \mathcal{M}$
  ↳ determined $g_{\mu\nu}$ uniquely
- On the other hand:
  Given a metric does not fix tetrads:

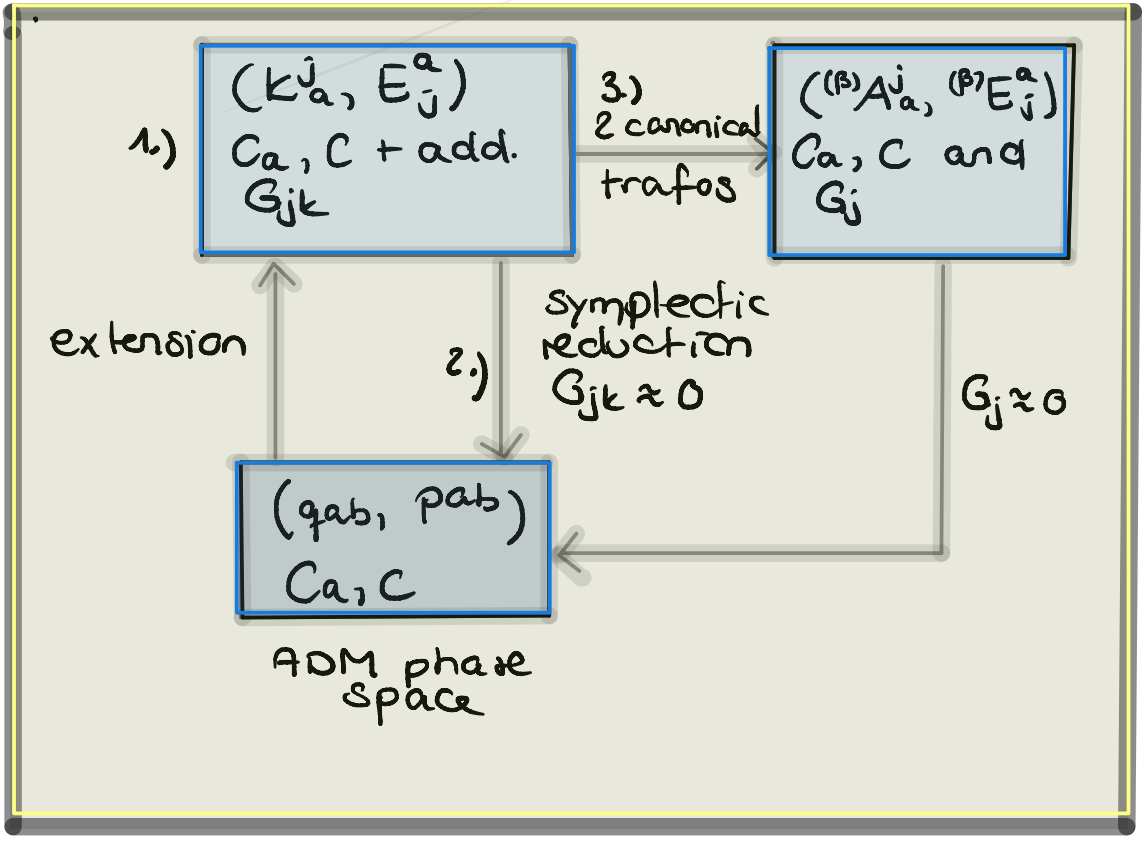
  Consider a Lorentz-transformation

$$\tilde{e}_I(p) = \Lambda_I{}^J e_J(p)$$

then $\tilde{g}^{\mu\nu} = \tilde{e}^\mu_I \tilde{e}^\nu_J \eta^{IJ}$

$$\begin{aligned} &= \Lambda^K{}_I \Lambda^L{}_J e^\mu_K e^\nu_L \eta^{IJ} \\ &= (\Lambda \eta^{-1} \Lambda^T)^{KL} e^\mu_K e^\nu_L = g^{\mu\nu} \end{aligned}$$

- Hence, tetrad determines metric only up to $SO(1,3)$ trafos.
- Going back to ADM phase space (q_{ab}, p_{ab})
- Strategy to introduce new variables



1) First step: Introduce in $D = 3$ triads & co-triads

$$q_{ab} = e^i_a e^j_b \delta_{ij}$$

e^i_a co-triad

i,j internal indices $\mathbb{R}^3$ $\quad i,j = 1,2,3$, $a = 1,2,3$ spatial indices

$$q(e_i, e_j) = \delta_{ij}$$

- here metric determined up to $SO(3)$-trafos (rotations)

$$\tilde{e}^i_a = O^i_j e^j_a$$ yields same q_{ab}

↳ 3 add. dof in e^i_a not encoded in q_{ab}

- co-triads are $\mathrm{Lie}(SU(2))$-valued one-forms
- Use map between $\mathbb{R}^3$ and $\mathrm{Lie}(SU(2))$

$v \mapsto v^i \tau_i$ $\quad \tau_i$ basis of $\mathrm{Lie}(SU(2))$

$$e = e^i_a \tau_i dx^a$$ 1-form with values in $\mathrm{Lie}(SU(2))$

- spatial indices are pulled with q_{ab} internal indices are pulled with δ_{ij}

- to have a canonical pair need add indep. phase space variable K^j_a

$$K_{ab} = K^j_a e^k_b \delta_{jk}$$

K^j_a: indep. Lie(SU(2))-valued 1-form

- We know K_{ab} is symmetric:

$$K_{ab} = K^j_{(a} e^k_{b)} \delta_{jk} + \underbrace{K^j_{[a} e^k_{b]} \delta_{jk}}_{\text{needs to vanish}}$$

- Hence constraint:

$$G_{ab} := K^j_{[a} e^k_{b]} \approx 0$$

for all a, b (3 conditions)

- We consider 'densitised' triads

$$E^a_j = \sqrt{\det(q)}\, e^a_j$$

$\det(q)$ detn of co-triads

- for triads we have

$$e^a_j e^k_a = \delta^k_j \quad \text{and} \quad e^a_j e^j_b = \delta^a_b$$

using $E^j_a = \frac{1}{\sqrt{\det(q)}} e^j_a$

$$E^a_j E^k_a = \delta^k_j \quad \text{and} \quad E^a_j E^j_b = \delta^a_b$$

- Thus we can rewrite the rotational constraint

$$G_{ab} \approx 0 \quad (a,b = 1,2,3) \iff G_{ab} E^a_j E^b_k \approx 0 \quad (j,k = 1,2,3)$$

- Then easy to show

$$G_{ab} E^a_j E^b_k = \frac{1}{\sqrt{\det(q)}} K_{a[k} E^a_{j]}$$

equivalent form:

$$G_{jk} := K_{a[j} E^a_{k]} \approx 0$$

- Extended phase space

- elementary variables (K^j_a, E^a_j)

- Poisson algebra

$$\{K^j_a(x), E^b_k(y)\} = \frac{k}{2} \delta^{(3)}(x,y) \delta^b_a \delta^j_k$$

$$\{K^j_a(x), K^k_b(y)\} = 0 \quad \{E^a_j(x), E^b_k(y)\} = 0$$

- Constraints: in addition to G_{jk}:

$$C_a = -2 D_b (K^j_a E^b_j - \delta^b_a E^c_j K^j_c)$$

$$C = \frac{1}{\sqrt{\det(q)}} (K^l_a K^j_b - K^j_a K^l_b) E^a_j E^b_l - \sqrt{\det(q)}\, R^{(3)}$$

- here $R^{(3)}$ and $\sqrt{det(q)}$ are understood as a function of (K, E)

$$q_{ab} = E^i_a E^j_b \, det(E^a_i) \, \delta_{ij}$$

$$det(q) = det(E^a_i) \quad \text{using} \quad det(E^i_a) = \frac{1}{det(E^a_i)}$$

2.) **Symplectic reduction wrt $G_{jk} \approx 0$**

Strategy: express q_{ab}, p^{ab} in terms of (K, E) variables

then compute their Poisson brackets and show that same result as ADM phase space on $G_{jk} \approx 0$ hypersurface

$$q_{ab}(E) = det(E^a_i) \, E^j_a E^k_b \, \delta_{jk}$$

$$p^{ab}(K,E) = \frac{2}{det(E^a_i)} \left(E^a_i E^{[c}_j E^{b]}_m K^m_c \right) \delta^{ij}$$

- using the elementary Poisson brackets of (K^i_a, E^a_j) compute those of $q_{ab}(E)$ and $p^{ab}(K,E)$

- **Result:**

(i) Since $q_{ab}(E)$ only a fctn of E^a_i:

$$\{q_{ab}(E)(x), q_{cd}(E)(y)\} = 0$$ ✔

(ii) Not hard to compute

$$\{q_{ab}(E)(x), P^{cd}(K,E)(y)\} = k\,\delta^{c}_{(a}\delta^{d}_{b)}\,\delta^{(3)}(x,y)$$ ✔

(iii) A bit more complicated to compute

$$\{P^{ab}(K,E)(x), P^{cd}(K,E)(y)\}$$
$$= -\frac{\sqrt{\det(E^a_j)}}{4}\left(q^{bc}G^{ad} + q^{bd}G^{ac} + q^{ac}G^{bd} + q^{ad}G^{bc}\right)$$

where $G^{ab} := q^{ac}q^{bd}G_{cd} = q^{ac}q^{bd}G_{jk}e^j_a e^k_b$

- Thus G^{ab} are combinations of the rotational constraints

↳ rediscover standard PB on $G_{jk} \approx 0$ ✔

- Both formulations describe the same physics ⟶ general relativity

- Therefore choosing (K^i_a, E^a_j) we can rewrite the action for vacuum GR as:

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \, (2K^i_a \dot{E}^a_j - (\Lambda^{jk} G_{jk} + N^a C_a + NC))(x)$$

- Geometrical interpretation:
  C_a, C generate spatial diffeos within Σ_t and orthog. to Σ_t only if $G_{jk} \neq 0$
- Extended constraint algebra:
  - additional $G(\Lambda) := \int_\Sigma d^3x \, (\Lambda^{jk} G_{jk})(x)$
  - We have:

$$\{ q_{ab}(E)(x), G(\Lambda) \} = 0, \quad \{ p^{ab}(K,E)(x), G(\Lambda) \} = 0$$

thus mixed Poisson brackets with $C(N)$, $\vec{C}(\vec{N})$ & $G(\Lambda)$ all vanish

- Rotational constraints

$$\{ G(\Lambda_1), G(\Lambda_2) \} = \frac{\kappa}{2} G([\Lambda_1, \Lambda_2])$$

expected; infinitesimal $SO(3)$ rotations

- Hypersurface deformation algebra stays the same up to term involving G_{jk}

3.) From (K^i_a, E^a_j) to Ashtekar-Barbero variables

- To perform the two canonical trafos we need notion of a spin connection
- Discussed already tetrad basis $\{e_I = e^\mu_I \partial_\mu\}$ coord. basis $\{\partial_\mu\}$
- Consider a VF we can expand it wrt $\{\partial_\mu\}$ or $\{\varepsilon_I\}$

$$V = V^\mu \partial_\mu \quad \text{or} \quad V = V^I \varepsilon_I$$

showed already

$$\varepsilon_I = e^\mu_I \partial_\mu \quad \text{and hence}$$

$$V = V^I \varepsilon_I = \underbrace{V^I e^\mu_I}_{V^\mu} \partial_\mu = V^\mu \partial_\mu$$

- we can extend this to tensor

$$T^I{}_J = e^I_\mu e^\nu_J T^\mu{}_\nu$$
$$T^\mu{}_\nu = e^\mu_I e^J_\nu T^I{}_J$$

- for tensors with spacetime indices:

$$\nabla_\mu \partial_\nu = \Gamma^\rho_{\mu\nu} \partial_\rho$$

$\Gamma^\rho_{\mu\nu}$ Christoffel symbols

- here $\nabla_\mu e^I = \omega_\mu{}^I{}_J e^J$

  $\omega_\mu{}^I{}_J$ spin connection coefficients

- then for tensors with internal indices

$$\nabla_\mu T^I{}_J = \partial_\mu T^I{}_J + \omega_\mu{}^I{}_K T^K{}_J - \omega_\mu{}^K{}_J T^I{}_K$$

and for mixed indices

$$\nabla_\mu T^I{}_\nu = \partial_\mu T^I{}_\nu - \Gamma^\rho_{\mu\nu} T^I{}_\rho + \omega_\mu{}^I{}_J T^J{}_\nu$$

- up to now $\Gamma^\rho_{\mu\nu}$ & $\omega_\mu{}^I{}_J$ are completely independent.
- Using that $V^I = e^I_\mu V^\mu$ etc we can compare

$$\nabla V = (\nabla_\mu V^\nu)\, dx^\mu \otimes \partial_\nu$$
$$= (\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho)\, dx^\mu \otimes \partial_\nu$$

and

$$\nabla V = (\nabla_\mu V^I)\, dx^\mu \otimes e_I$$
$$= (\partial_\mu V^I + \omega_\mu{}^I{}_J V^J)\, dx^\mu \otimes e_I \quad \big| V^I = e^I_\mu V^\mu$$
$$= (\partial_\mu V^\rho + e^\rho_I ((\partial_\mu e^I_\nu) + \omega_\mu{}^I{}_J e^J_\nu) V^\nu)\, dx^\nu \otimes \delta_\rho$$

- Comparing the two yields

$$\Gamma^{\rho}_{\mu\nu} = e^{\rho}_{I}\left(\partial_\mu e^{I}_{\nu} + \omega_\mu{}^{I}{}_{J}\, e^{J}_{\nu}\right)$$

and solving for the spin connection

$$\boxed{\omega_\mu{}^{I}{}_{J} = \Gamma^{\rho}_{\mu\nu} e^{\nu}_{J} e^{I}_{\rho} - e^{\nu}_{J}\left(\partial_\mu e^{I}_{\nu}\right)}$$

- Relation for spin connection can also be obtained

$$\boxed{\nabla_\mu e^{I}_{\nu} \overset{!}{=} 0}$$

carrying over metric compatibility $\nabla_\mu g_{\nu\rho} = 0$ to tetrads

- metric compatibility in tetrad basis:

$$\nabla_\mu g(e_I, e_J) = \nabla_\mu \eta_{IJ}$$

$$= \underbrace{\partial_\mu \eta_{IJ}}_{=0} - \omega_\mu{}^{K}{}_{I}\,\eta_{KJ} - \omega_\mu{}^{K}{}_{J}\,\eta_{IK}$$

$$= -\omega_{\mu JI} - \omega_{\mu IJ} \overset{!}{=} 0$$

$$\hookrightarrow \omega_{\mu IJ} = -\omega_{\mu JI}$$

thus $(\omega_\mu)^{I}{}_{J}$ is antisymmetric matrix

- Now back to $D=3$ and triads

$$\omega_\mu{}^{IJ} \rightsquigarrow \Gamma_a{}^i{}_j \qquad \begin{matrix} a=1,2,3 \\ j=1,2,3 \end{matrix}$$

then we have

$$\boxed{\Gamma_a{}^i{}_j = \Gamma^c_{ab} e^j_c e^b_k - e^b_k (\partial_\mu e^\nu_a)}$$

here also $\Gamma_{aij} = -\Gamma_{aji}$

$(\Gamma_a)^i{}_j$ is an antisymmetric matrix.

- We can associate to an anti-symmetric matrix in $D=3$ a vector
  - $\boxed{N_{ij} = -N_{ji} \rightsquigarrow u^k = -\frac{1}{2}\varepsilon^{kem} N_{em}}$

Check:

$$N_{ij} = -\varepsilon_{ijk} u^k = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{kem} N_{em}$$
$$= \frac{1}{2}(\delta_i{}^k \delta_j{}^e - \delta_{ij}\delta^{ke}) N_{em} = N_{ij} \checkmark$$

- <u>for spin connection:</u>

$$\boxed{\Gamma_a^i = -\frac{1}{2}\varepsilon^{ijk}\Gamma_{ajk}} \rightsquigarrow \Gamma_{ajk} = -\varepsilon_{jke}\Gamma_a^e$$

and from this

$$\boxed{\Gamma_a^i = \frac{1}{2}\varepsilon^{ijk} e^b_k (\partial_b e_{aj} - \partial_a e_{bj} + e^c_j e^\ell_a \partial_b e_{c\ell})}$$

- Expressed in terms of densitised triads:

$$\Gamma^i_a = \frac{1}{2}\varepsilon^{ijk} E^b_k \left(E^j_{a,b} - E^j_{b,a} + E^c_j E^l_a E^l_{c,b} \right) + \frac{1}{4}\varepsilon^{ijk} E^b_k \left(2 E^i_a \frac{\det(E)_{,b}}{\det(E)} - E^i_b \frac{\det(E)_{,a}}{\det(E)} \right)$$

$\rightarrow \Gamma^i_a(E)$

- Now discuss two canonical trafos:

(i) Rescaling by $\beta \in \mathbb{C}$

$$^{(\beta)}K^j_a := \beta K^j_a$$
$$^{(\beta)}E^a_j := \frac{1}{\beta} E^a_j$$

β Immirzi parameter

obviously canonical transformation

(ii) Affine transformation

Introduce new connection

$$^{(\beta)}A^j_a := \Gamma^j_a + {}^{(\beta)}K^j_a$$

Ashtekar–Barbero connection

here $^{(\beta)}\Gamma^j_a(E) = \Gamma^j_a({}^{(\beta)}E) = \Gamma^j_a$

invariant under rescaling.

- Need to show that $({}^{(\beta)}A_a^j, {}^{(\beta)}E_j^a)$ canonical conjugate pair

(a) $\{ {}^{(\beta)}E_j^a(x), {}^{(\beta)}E_k^b(y) \} = 0$ ✔ only β rescaling

(b) $\{ {}^{(\beta)}A_a^j(x), {}^{(\beta)}E_k^b(y) \}$

$= \{ \Gamma_a^j(E)(x) + {}^{(\beta)}K_a^j(x), {}^{(\beta)}E_k^b(y) \}$

$= \{ {}^{(\beta)}K_a^j(x), {}^{(\beta)}E_k^b(y) \} = \frac{k}{2} \delta_k^j \delta_a^b \delta^{(3)}(x,y)$ ✔

(c) non-trivial computation

$\{ {}^{(\beta)}A_a^j(x), {}^{(\beta)}A_b^k(y) \}$

$= \{ \Gamma_a^j(x) + {}^{(\beta)}K_a^j(x), {}^{(\beta)}\Gamma_b^k(x) + {}^{(\beta)}K_k^b(y) \}$

$= 0$ ✔

- Constraints in terms of Ashtekar-Barbero variables

First observation:

Rotational constraint can be written as a standard SU(2) Gauß constraint

- From $\mathcal{D}_a e^a_j = 0 \quad \rightsquigarrow \quad \mathcal{D}_a {}^{(\beta)}E^a_j = 0$

$$\mathcal{D}_a {}^{(\beta)}E^a_j = \partial_a {}^{(\beta)}E^a_j - \Gamma_a{}^i{}_j {}^{(\beta)}E^a_i$$
$$= \partial_a {}^{(\beta)}E^a_j + \varepsilon^l{}_{jk} \Gamma^k_a {}^{(\beta)}E^a_l = 0$$

- thus we can rewrite the rotational constraint as

$$ {}^{(\beta)}G_j = \varepsilon_{jk}{}^l \left({}^{(\beta)}K^k_a \right) \left({}^{(\beta)}E^a_l \right) = G_j$$
$$= 0 + \varepsilon_{jk}{}^l \, {}^{(\beta)}K^k_a \, {}^{(\beta)}E^a_l$$
$$= \mathcal{D}_a {}^{(\beta)}E^a_j + \varepsilon_{jk}{}^l \, {}^{(\beta)}K^k_a \, {}^{(\beta)}E^a_l$$
$$= \partial_a {}^{(\beta)}E^a_j + \varepsilon_{jk}{}^l \underbrace{\left(\Gamma^k_a + {}^{(\beta)}K^k_a \right)}_{= {}^{(\beta)}A^k_a} {}^{(\beta)}E^a_l$$
$$=: \mathcal{D}_a {}^{(\beta)}E^a_j$$

- Thus we obtain a standard SU(2) Gauß law with connection ${}^{(\beta)}A^j_a$

$$G_j = \mathcal{D}_a {}^{(\beta)}E^a_j \approx 0$$

- Remaining constraints:

  Spatial diffeomorphism:

$$C_a = {}^{(\beta)}F^j_{ab}\,{}^{(\beta)}E^b_j - {}^{(\beta)}K^j_a G_j$$

- here $${}^{(\beta)}F^j_{ab} = \partial_a {}^{(\beta)}A^j_b - \partial_b {}^{(\beta)}A^j_a - \varepsilon_{jke}\,{}^{(\beta)}A^k_a\,{}^{(\beta)}A^e_b$$

- Hamiltonian constraint

$$C = \frac{\beta^2}{\sqrt{\det({}^{(\beta)}E\cdot\beta)}}\left(\varepsilon_j{}^{ke}\,{}^{(\beta)}E^a_k\,{}^{(\beta)}E^b_e\,{}^{(\beta)}F^j_{ab} + 2\,{}^{(\beta)}E^a_j D_a G_j\right)$$
$$+\frac{(\beta^2+1)}{\sqrt{\det({}^{(\beta)}E\cdot\beta)}}\left(({}^{(\beta)}E^a_j\,{}^{(\beta)}K^j_b)({}^{(\beta)}E^b_j\,{}^{(\beta)}K^j_a) - ({}^{(\beta)}K^k_c\,{}^{(\beta)}E^c_k)^2\right)$$

here $${}^{(\beta)}K^j_a = -\Gamma^j_a(E) + {}^{(\beta)}A^j_a$$

understood as a fctn of Ashtekar-Barbero variables.

- One can neglect G_j-terms in C, C_a since those modified constraints together with G_j define equivalent constraint hypersurface

- the Einstein-Hilbert action in terms of Ashtekar-Barbero variables

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left(2 {}^{(\beta)}\dot{A}^j_a {}^{(\beta)}E^a_j - (\Lambda^j G_j + N C + N^a C_a)(x) \right)$$

β Immirzi parameter $\beta \neq 0 \in \mathbb{C}$ parametrises the canonical transf.

- for $\beta = \pm i$ second part in C vanishes

but

$$ {}^{(\pm i)}A^j_a = \Gamma^j_a \pm i K^j_a $$ complex connection

- can work with complex connection but need to implement add. reality condition

$$ {}^{(\beta)}E^a_j / \beta \overset{!}{=} \overline{{}^{(\beta)}E^a_j / \beta} $$

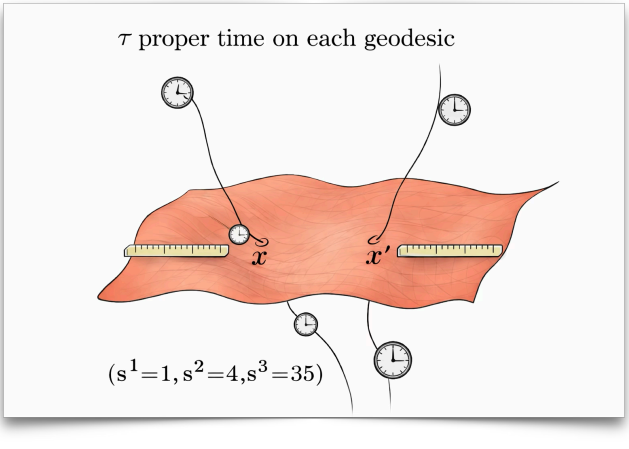
$$ \frac{({}^{(\beta)}A^j_a - \Gamma^j_a)}{\beta} \overset{!}{=} \overline{\frac{{}^{(\beta)}A^j_a - \Gamma^j_a}{\beta}} $$

- either complicated Ham. constraint or complicated reality condition.

Part 3

Reduced phase space Quantisation of LQG:

Step 1: Reduced phase space using Brown-Kuchař dust



Introduce dynamical reference frames

3. Reduced Phase space quantisation for L QG

- GR in Ashtekar-Barbero variables $({}^{(\beta)}A_a^j(x), {}^{(\beta)}E^a_j)$ drop label (β) from now on
- Constraints
  - $G_j(A,E)$ SU(2) - Gauß constraint
  - $C_a(A,E)$ spatial diffeom. constraint
  - $C(A,E)$ Hamilton constraint
- $G_j(A,E)$ will be solved via Dirac quantisation
- $C_a(A,E)$, $C(A,E)$ via reduced quantisation.

Relational formalism

- **Idea:** Formulate dynamics in GR in a relational way wrt a chosen dynamical reference frame

↳ circumvents problem of time in GR

$$H_{can} = \int_\Sigma d^3x \, (NC + N^a C_a + \lambda^j G_j)(x) \approx 0$$

Dirac observables: O_f:

$$\frac{d}{dt} O_f = \{O_f, H_{can}\} \approx 0$$

since they commute at least weakly with all constraints

- **Relational formalism:**
- Choose a dynamical reference frame
- Construct Dirac observables wrt dynamical reference frame allows notion of physical temporal & spatial coord.
- Given this one can obtain 'true' Hamiltonian $H_{phys} \neq 0$ that generates the dynamics.

- **Step 1:** Choose dynamical reference frame

Strategy: Choose 4 reference fields in a given phase space

$C_I = (C(N), \vec{C}(\vec{N}))$ ← reference frame
$(A_a^j, E_j^a) + (T^I, P_J)$ $I = 0, \ldots, 3$

s.t. $$M^I{}_J(x,y) = \{T^I(x), C_J(y)\}$$

inverse $(M^{-1})^I{}_J$ exists

$$\int d^3z\, (M^{-1})^I{}_J(x,z) M^J{}_K(z,y) \approx \delta^I{}_K \delta^{(3)}(x,y)$$

- then one can define an equivalent set of constraint

$$C'_I(x) = \int dy (M^{-1})_I{}^J(x,y)\, C_J(y)$$

that satisfy

$$\{T^I(x), C'_J(y)\} \approx \delta^I_J \delta^{(3)}(x,y)$$

- Reference fields and C'_J are weakly canonically conjugate
- further C'_J are weakly Abelian

$$\{C'_J(x), C'_I(y)\} \approx 0$$

- General idea: Use combinations of (A^j_a, E^a_j) and T^I's to construct Dirac observables for (A^j_a, E^a_j)
- Consider general formula for generic phase space fctn f introduce: $G^{I_1} := \tau^{I_1} - T^{I_1}$ $I_1 = 0, \dots, 3$ $\tau^{I_1} = (\tau, \sigma^a)$ ↑ values of I_1

$$O_{f,T}(\tau,\sigma) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C d^3y_1 \cdots \int_C d^3y_n \left(G^{I_1}(y_1) \cdots G^{I_n}(y_n)\right) \{\dots, \{f, C'_{I_1}(y_1)\}, \dots, C'_{I_n}(y_n)\}$$

- let us look at lowest orders

$n=0$ $\boxed{O_{f,T} = f}$

$n \geq 1$ $\boxed{O_{f,T} = f + \int d^3y\, G^I(y) \{ f, C'_I(y) \}}$

Consider Poisson bracket with C_J

$$\{ f + \int d^3y\, G^I(y) \{ f, C'_I(y) \}, C_J(x) \}$$

$$= \{ f, C_J(x) \} + \int d^3y \underbrace{\{ G^I(y), C_J(x) \}}_{-M^{-1}{}^I_J(y,x)} \{ f, C'_I(y) \}$$

$$+ \int d^3y\, G^I(y) \{ \{ f, C'_I(y) \}, C_J(x) \}$$

$$= \{ f, C_J(x) \}$$

$$- \int d^3y \int d^3z \, \underbrace{M^I{}_J(y,x) (M^{-1})^K_I(y,z)}_{= C'_I(y)} \{ f, C_K(z) \}$$

$$- \int d^3y \int d^3z \, M^I{}_J(y,x) \underbrace{C_I(x)}_{\approx 0} \{ f, \underbrace{M_I{}^K(x,z)}_{\approx 0} \}$$

$$+ \int d^3y\, G^I(y) \{ \{ f, C'_I(y) \}, G_J(x) \}$$

$$\approx \{ f, C_J(x) \} - \underbrace{\int d^3z\, \delta_J{}^K \delta^{(3)}(x,z) \{ f, C_K(z) \}}_{\{ f, C_J(x) \}}$$

$$+ \boxed{\int d^3y\, G^I(y) \{ \{ f, C'_I(y) \}, G_J(x) \}}$$

- **Realize:** first two terms exactly cancel
  term with $\{f_1 C_I(y) y_1 C_J(x) y\}$ survives would be cancelled by high order n.
- Need $\sum_{n=0}^{\infty}$ to construct Dirac observable.
- **Properties of Dirac observables**
  - Commute with all constraints weakly for which reference frames have been introduced

$$\{O_f, C_I\} \approx 0$$

  - Algebra of Dirac observables

$$\{O_{f_1}, O_g\} \simeq O_{\{f_1, g\}^*}$$ ← Dirac bracket

$$\{f_1 g\}^* := \{f_1 g\} - \{f_1 C_I\}(M^{-1})^I{}_J \{T^J, g\} + \{g_1 C_I\}(M^{-1})^I{}_J \{T^J, f\}$$

here slightly different since $M^I{}_J = \{T^I, C_J\}$ and not all comb. of second class constraints

- Since $\{\cdot,\cdot\}^* \neq \{\cdot,\cdot\}$ in general observable algebra more complicated.

Question: What reference frame to choose?

- No preferred choice but some reference frames might be technically simpler in quantisation step (Algebra!)

Example: Brown-Kuchar dust

Idea: Choose dust as dynamical reference frame

$$S^{dust} = -\frac{1}{2}\int d^4x \sqrt{-\det(g)}\,\rho\,(g^{\alpha\beta}U_\alpha U_\beta + 1)$$

$$U_\alpha = -T_{,\alpha} + W_j S^j{}_{,\alpha} \qquad j = 1,2,3 \text{ internal dust field index}$$

- Considers gravity coupled to dust

$$S = S^{EH} + S^{dust}$$

- S^{dust} involves ρ, W_j, T, S^j 8 scalar fields
- System with second class constraints

  ρ: energy density of dust

  W_j: related to momenta of T, S^j
- (T, S^j) reference frame $T^0 = T,\ T^j = S^j$
- U_α satisfies geodesic eqn

  $U^\beta D_\beta U_\alpha = 0$ ⇝ freely falling observer

- Now idea of BK model:

$$(T, P_T) \;,\; (S^j, P_j)$$

Reduce wrt 2nd class constraints remaining first class constraints

$$C^{tot} = C + C^{dust}$$
$$C_a^{tot} = C_a + C^{dust}$$

$$C^{dust} = -\sqrt{P_T^2 + q^{ab}(P_T T_{,a} + P_j S^j_{\ a})(P_T T_{,b} + S^j_{\ ,b} P_j)}$$
$$C_a^{dust} = P T_{,a} + P_j S^j_{\ ,a}$$

- Strategy: Solve C, C_a for P_T and P_j obtain equivalent set of constraints

$$\tilde{C}^{tot} = P_T + h(A, E)$$
$$\tilde{C}_j^{tot} = P_j + h_j(A, E, S^j, T)$$

with

$$h = \sqrt{C^2 + q^{ab} C_a C_b}$$
$$h_j = S^a_j(-h T_{,a} + C_a) \;; \quad S^a_j S^j_{\ ,b} = \delta^a_b$$

- Now we have $\tilde{C}_I^{tot} = (\tilde{C}^{tot}, \tilde{C}_j^{tot})$

$$\{\tilde{C}_I^{tot}, \tilde{C}_J^{tot}\} = 0$$

• Construction of Dirac observables

Reference frame $T^I = (T, S^j)$

T : Hamiltonian constraint $\tilde{C}^{tot}$

S^j : Diffeomorphism constraint $\tilde{C}^{tot}_j$

- Since $C_J = (\tilde{C}^{tot}, \tilde{C}^{tot}_j)$ linearly in $P_J = (P_1 P_j)$

$$\{T^I(x), C_J(y)\} = \underbrace{\delta^I_J \delta^{(3)}(x,y)}_{M^I{}_J(x,y) \text{ here}}$$

thus

$$C'_J(x) = \int_\sigma d^3y \; \delta^I{}_J \delta^{(3)}(x,y) C_I(y) = C_J(x)$$

↳ can replace C'_J by C_J in observable formula.

- Then construct Dirac observables

$(O_{A^j_a}(\tau, G), O_{E^a_j}(\tau, G))$

- Observable algebra

$$\{O_{A^j_a}(\tau, G), O_{E^k_b}(\tau, G')\} = O_{\{A^j_a, E^k_b\}^*}$$

$$= O_{\{A^j_a, E^k_b\}} = \frac{k}{2} \delta^{(3)}(G, G')$$

↳ simple observable algebra.

- Construct observables of C_i, C_j and q^{ik} then the physical Hamiltonian reads

$$H_{phys} = \int_S d^3\sigma \; H(\sigma) \qquad H_{phys} \neq 0$$

$$\text{with} \quad H(\sigma) = \sqrt{O_C^2 - (O_q)^{ab} O_{C_a} O_{C_b}}$$

↑ includes only gravitational contrib

- **Question:** What happens for other choices of reference matter?

- In general algebra of observables & physical Hamiltonian changes
- Other dust and scalar field models have been considered in case of GR and context of LQG
- In these cases quantisation program can be completed
- Reference frame dependent QFT

- **Next step:**
  Apply loop quantum gravity quantisation

Part 4: Reduced phase space quantisation of LQG Step 2: Kinematics

4.1. Choice of the classical algebra (holonomy-flux algebra)

4.2. Representation of the holonomy-flux algebra

4.3. Spin network functions

4. Kinematics: Canonical quantisation of (reduced) LQG

4.1. Choice of the classical algebra (Holonomy-flux algebra)

- To simplify notation (abuse of notation)

$$\mathcal{O}_{A^i_a} = A^i_a$$
$$\mathcal{O}_{E^a_i} = E^a_i$$

- Keeping in mind (A^i_a, E^a_j) are elementary variables of the physical phase space (still need to solve Gauß constraint later)

We have

$$\{ A^i_a(G), E^b_k(G') \} = \frac{\kappa}{2} \delta^{(3)}(G, G') \delta^i_k \delta^b_a$$

- introduce smeared quantities
- Choice of smearing guided by
  - (i) background independence
  - (ii) observables (gauge-inv. quantities) variables should transform simple under SU(2) gauge trafos

- **Configuration variables:**

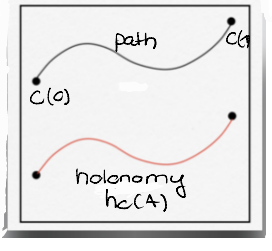
$A^j_a(x) \rightsquigarrow$ holonomies $h_c(A)$

Given

$A = A^j_a \tau_j \otimes dx^a$ Lie(SU(2)) valued 1-form

- background indep. smearing along 1-dim curves

$$\int_c A = \int_0^1 dt\, A^j_a(c(t))\, \dot{c}^a(t)\, \tau_j$$



- Holonomy $h_c(A) \in SU(2)$ of the connection A along c is defined as the unique solution of the differential eqn

$$\frac{d}{ds} h_{c_s}(A) - h_{c_s}(A)\, A(c(s)) = 0$$

$$h_{c_0}(A) = \mathbb{1}_2 \,, \quad h_c(A) := h_{c_1}(A)$$

with $c_s(t) := c(st)$ with $s \in [0,1]$

and $A(c(s)) := A^j_a(c(s))\, \dot{c}^a(s)\, \tau_j$

The solution can be expressed as the path-ordered exponential

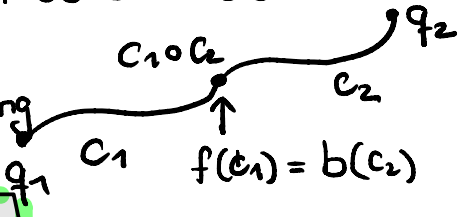
$$h_C(A) = \overrightarrow{\mathcal{P}}\exp\left(\int_C A\right)$$
$$= \mathbb{1}_2 + \sum_{n=1}^{\infty} \int_0^1 dt_1 \int_{t_1} dt_2 \cdots \int_{t_{n-1}} dt_n \, A(c(t_1)) \cdots A(c(t_n))$$

smallest parameter to the most left position.

- Properties of holonomies

Consider composition of two curves c_1, c_2 with $f(c_1) = b(c_2)$

↑ final ↑ beginning



then we have

$$h_{c_1 \circ c_2}(A) = h_{c_1}(A)\, h_{c_2}(A)$$

- Consider for a given curve c from q_1 to q_2 its inverse then we have

$$h_{c^{-1}}(A) = (h_c(A))^{-1}$$

- Given the trafo of connection A under $SU(2)$-trafo, $g \in SU(2)$

$$Ag = gAg^{-1} + dg\,g^{-1}$$

then

$$h_c^g(A) := h_c(Ag) = g(b(c))\, h_c(A)\, g^{-1}(f(c))$$

- In particular this means so-called Wilson loop is gauge-invariant
  - Let α be a loop



$$\boxed{\mathrm{tr}\left(h_\alpha(A)\right)}$$ (Lattice gauge theory)

- Apply SU(2)-trafo

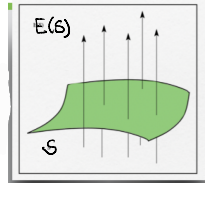
$$\begin{aligned}\mathrm{tr}\left(h^g_\alpha(A)\right) &= \mathrm{tr}\left(h_\alpha(Ag^{-1})\right)\\ &= \mathrm{tr}\left(g(b(\alpha))\,h_\alpha(A)\,g^{-1}(f(\alpha))\right)\\ &= \mathrm{tr}\left(g^{-1}(f(\alpha))\,g(b(\alpha))\,h_\alpha(A)\right)\\ &= \mathrm{tr}\left(h_\alpha(A)\right)\end{aligned}$$

cyclic permutation

- Momenta variables

$E^a_j(x) \rightsquigarrow$ flux variables $E_n(S)$

- Given $E = E^a_j \tau^j \otimes \partial_a$ Lie-SU(2) valued vector density



- Consider 2-dim surface S in $\mathcal{B}$ and define

$$E_n(S) := \int_S n^j E^a_j \,\varepsilon_{abc}\, dx^b \wedge dx^c$$

n^j Lie-algebra valued smearing fctn

- $E_n = E^a_n \partial_a$ is vector density of weight +1
- ε_{abc} has density weight -1

↳ integrand in $E_n(S)$ is a 2-form

- $\boxed{E^a_j n^j \varepsilon_{abc} dx^b \wedge dx^c = E^a_j n^j n^S_a}$ (n^S_a: co-normal)

is a 2-form

↳ can be integrated against 2-surfaces in background indep. way.

- **Question** How does the Poisson algebra of $h_c(A)$ & $E_n(S)$ look like?

Realize: $h_c(A)$ smeared in 1 dimension
$E_n(S)$ smeared in 2 dimensions

$\{h_c(A), E_n(S)\}$ a priori ill-defined

- Explicit form

For this purpose choose an embedding of the surface S into $\mathcal{C}$

$$X_S : [(u_{10}, u_{20}), (u_{11}, u_{21})] \to \mathcal{C}$$

$$(u_1, u_2) \mapsto X_S(u_1, u_2)$$

co-normal $n^S_a = \varepsilon_{abc} dx^b \times dx^c$

choose $n^j = \delta^j_K \longrightarrow E_n(S) \rightsquigarrow E_j(S)$

$$\boxed{E_j(S) = \int_{u_{10}}^{u_{11}} du_1 \int_{u_{20}}^{u_{21}} du_2 \; n^S_b(X_S(u^1, u^2)) E^b_j(X_S(u_1, u_2))}$$

- holonomy

$$h_c(A) = \mathcal{P}\exp\left(\int_c A\right)$$
$$= \mathcal{P}\exp\left(\int_{t_0}^{t_1} ds\, A_b^k(c(s))\,\dot{c}^b(s)\,\tau_k\right)$$

- then we can compute the Poisson bracket

$$\{h_c(A), E_j(S)\}$$
$$= \int_{t_0}^{t_1} ds \int_{u_{10}}^{u_{11}} du_1 \int_{u_{20}}^{u_{21}} du_2 \Big(n_a^S(X_S(u_1,u_2))\,\dot{c}^a(s)\, h_{c(t_0,s)}(A)\,\tau_j\, h_{c(s,t_1)}(A)\, \delta^{(3)}(c(s), X_S(u_1,u_2)) \Big)$$

($c(s)$: points on c; $X_S(u_1,u_2)$: points on S)

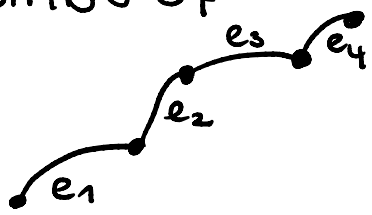
- Problem: If curves c lies entirely in S ill-defined.
- To circumvent this we consider regularisation.
- First introduce notion of path: equivalence class of curves

$c \sim c'$ $\quad b(c) = b(c')$ and $f(c) = f(c')$

$h_c(A) = h_{c'}(A)$ for all $A \in \mathcal{A}$

$\mathcal{A}$: set of smooth connections

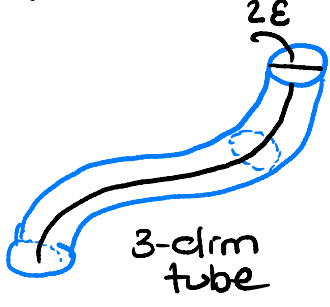
- path consists of finite number of edges $\{e_1, \dots, e_4\}$ at end and beginning of e_k $C^{(1)}$.



- finite set of edges is independent provided that e_k intersects in $b(e_k)$, $f(e_k)$ at most.
- **Aim:** Compute $\{h_e(A), E_j(S)\}$

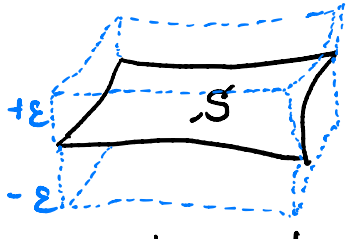
1.) Regularised holonomy

$$h_e^\varepsilon(A) := \mathcal{P}\exp\left(\int_{\mathbb{R}^2} d^2s\, \delta^\varepsilon(s) \int_e A\right)$$



2.) Regularised flux

$$E_j^\varepsilon(S) := \int_{\mathbb{R}} ds\, \delta^\varepsilon(s)\, E_j(S)$$



regularised with 3 dim smearing

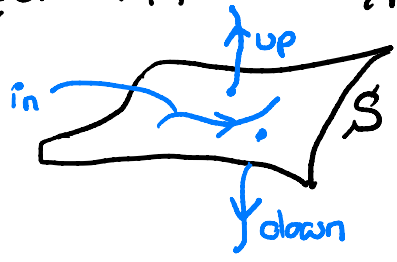
- Given regularised quantities we can map them into a sub-algebra of $C^\infty(\mathcal{A}) \times V^\infty(A)$ (smooth fctns of connections and vector fields thereon)

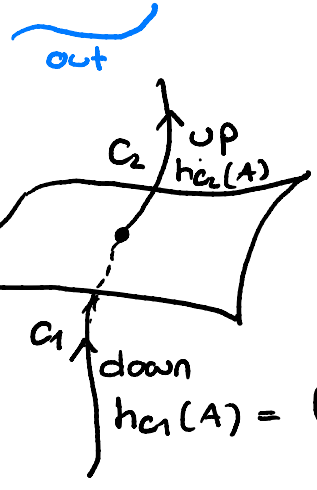
↳ then now action of VF on $h_e^\varepsilon(A)$

- Strategy: Compute the result of the regularised quantities

$$\lim_{\varepsilon \to 0} Y_n^\varepsilon(S)[h_e^\varepsilon(A)] := \{h_e^\varepsilon(A), E_n^\varepsilon(S)\}$$
$$= Y_n(S)[h_e(A)]$$

- Result can be formulated in terms of four different type of edges wrt surface S





example:
split $c = c_1 \circ c_2$
$h_{c_1^{-1}}(A)$ is outgoing since orientation of c_1 is inverted.

- sufficient to know the results of all for basic edge types

Result:

$$Y_n(S)(h_e(A))$$
$$= \frac{1}{2} \varepsilon(e,S) \frac{n^j(b(e))}{2} \tau_j h_e(A)$$

where $\varepsilon(e,S) = \begin{cases} +1 & \text{for } e \text{ up} \\ -1 & \text{for } e \text{ down} \\ 0 & \text{for } e \text{ out} \gtrless \text{in} \end{cases}$

- So far considered only action of $Y_n(S)$ on holonomy at single edge e
- **Aim:**

  Generalize this to fctns of holonomies of several edges

  ↳ notion of graphs
  notion of cylindrical fctns

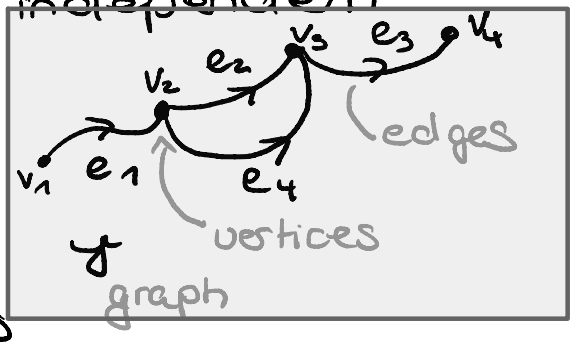
- Consider oriented graphs:

  defined by a set of independent edges $\{e_1, \dots, e_N\}$

  for a given graph γ:

  $E(\gamma)$: set of edges

  $V(\gamma)$: set of vertices



- Each graph γ depends on a finite number of edges only

  ↳ need to consider fctns not only on one γ but on all γ's that can be embedded into G.

- For a given graph γ:

  $\mathcal{A}$: set of smooth connections

  $\mathcal{A}_\gamma$: subset of $\mathcal{A}$ with connections associated with graph γ contains all A_{e_i} associated with $\{e_i\}$ of γ.

- There exists a map

$$I_E : \mathcal{A}_\gamma \longrightarrow SU(2)^n \text{ with}$$
$$A \in \mathcal{A}_\gamma \mapsto I_E(A) := (he_1(A), \ldots, he_n(A))$$

- We use I_E to define cylindrical functions wrt a given graph γ with $E(\gamma) = \{e_1, \ldots, e_n\}$

$$f : \mathcal{A}_\gamma \to \mathbb{C}, \quad F_\gamma : SU(2)^n \to \mathbb{C}$$
$$A \mapsto f_\gamma(A) := F_\gamma(I_E(A))$$
$$= F_\gamma(he_1(A), \ldots, he_n(A))$$

- A fctn is called cylindrical fctns wrt a given graph γ if it can be written in the above form.

Notation: $f \in Cyl_\gamma$.

- Remark:
A given cylindrical fctn is cylindrical wrt many graphs

Example:

$$f_\gamma(A) = F_\gamma(he(A)) \qquad E(\gamma) = \{e\}$$

Consider $\gamma' = \{e_1, e_2, e_3\gamma$ with $e = e_1 \circ e_2$ and e_3 indep. of e.

- then f is also cylindrical wrt γ' since

$$f_{\gamma'}(A) = F_{\gamma'}(he_1(A), he_2(A))$$

- A graph γ' is larger than a graph γ if every edge in γ can be written as a linear combination of e_i''s, that is

$$e = e_1'^{s_1} \circ e_2'^{s_2} \circ \dots \circ e_\ell'^{s_n} \qquad s_i = \pm 1$$

for some set $\{e_1', \dots, e_\ell'\}$ of $E(\gamma)$

- Realise:

Any f that is cylindrical wrt γ will also be cylindrical wrt any larger graph γ'

- This allows to define an equivalence relation on $\bigcup_\gamma Cyl_\gamma$

Given f if $f' \in \bigcup_\gamma Cyl_\gamma$ we can find γ, γ' such that $f \in Cyl_\gamma$, $f' \in Cyl_{\gamma'}$

equivalence relation

$f \sim f'$ if f, f' agree on all larger graphs $\gamma'' > \gamma, \gamma'$.

- We define the space of smooth cylindrical fctns

$$Cyl^\infty(A) := \bigcup_\gamma Cyl_\gamma/\sim$$

abelian C*-algebra with pointwise operations and supremum norm.

- In order to work with that algebra need to extend the action of flux vector fields $Y_n(S)$ to cylindrical fctn:

$$Y_n(S)[f_\gamma] = \frac{1}{2}\sum_{k=1}^{N} \varepsilon(e_k, S)\left[\frac{n^j(b(e_k))}{2}\tau_j h_{e_k}(A)\right]_{AB} \frac{\partial f_\gamma}{\partial(h_{e_k}(A))_{AB}}(h_{e_1}(A), \ldots, h_{e_N}(A))$$

here A,B are SU(2) indices and it is assumed that any of edges e_k can be classified as being of type up, down, in or out.

- Starting point for LQG:

The Lie-*-subalgebra of $Cyl^\infty(\mathcal{A}) \times V^\infty(\mathcal{A})$ generated by smooth cylindrical functions $Cyl^\infty(\mathcal{A})$ and the flux vector fields $Y_n(S)$ on $Cyl^\infty(\mathcal{A})$, called holonomy-flux algebra.

- Involution: complex conjugation

$$(h_e(A))^* = \overline{h_e(A)} = (h_{e^{-1}}(A))^T$$

$$Y_n^*(S) = \overline{Y_n(S)} = Y_n(S)$$

4.2 : Representation of the holonomy flux algebra

- 'Physical' Hilbert space

$$\mathcal{H}_{phys} = L_2(\bar{\mathcal{A}}, d\mu_{AL})$$

here $\bar{\mathcal{A}}$: space of generalized connections; not necessarily smooth can even be distributional

- To briefly discuss the construction of $\mathcal{H}_{phys}$ we first consider

$$\mathcal{H}_\gamma = \bigotimes_{e \in E(\gamma)} \mathcal{H}_e$$

$\mathcal{H}_\gamma$: Hilbert space associated to graph γ

$\mathcal{H}_e$: Hilbert space for one edge in γ

for each edge:

$$\mathcal{H}_e = L_2(SU(2), d\mu_H)$$

- $SU(2)$ compact group
- there exist a unique Haar measure on $SU(2)$ with the following properties

(i) $\int_{SU(2)} d\mu_H(g) = 1$

- invariant under left- and right transl.

(ii) $\int_{SU(2)} d\mu_H(g) f(hg) = \int_{SU(2)} d\mu_H(g) f(g)$ (iii) $\int_{SU(2)} d\mu_H(g) f(gh) = \int_{SU(2)} d\mu_H(g) f(g)$

- explicit form for $SU(2)$ expressed in terms spherical coord. $\mathbb{R}^4$

$$x^1 = \sin\psi \sin\varphi \cos\theta \qquad \psi, \varphi \in [0, \pi], \ \theta \in [0, 2\pi]$$
$$x^2 = \sin\psi \sin\varphi \sin\theta$$
$$x^3 = \sin\psi \cos\varphi$$
$$x^4 = \cos\psi$$

then

$$d\mu_H(\psi, \theta, \varphi) = \frac{1}{2\pi^2} \sin^2\psi \sin\varphi \, d\psi d\varphi d\theta$$

- Considering cylindrical fctns wrt a chosen graph γ, $E(\gamma) = \{e_1, \dots, e_N\}$

↳ associate to each edge a copy of $\mathcal{H}_e = L_2(SU(2), d\mu_H)$

Then we can define an inner product on Cyl_γ:

$$\langle f_\gamma, g_\gamma \rangle := \int_{SU(2)^N} \prod_{i=1}^{N} d\mu(h_{e_i}) \, \overline{F}(h_{e_1}(A), \dots, h_{e_N}(A)) \, G(h_{e_1}(A), \dots, h_{e_N}(A))$$

here $F, G : SU(2)^N \to \mathbb{C}$

Cauchy completion $\mathcal{H}_\gamma = \overline{Cyl_\gamma}$

- In case f_{γ_1} and g_{γ_2} are cylindrical wrt different graphs γ_1 and γ_2

↳ we can find always γ_3 s.t. f, g are cylindrical wrt γ_3

ex.: $\gamma_3 = \gamma_1 \cup \gamma_2$

Aim: Inner product should not depend on the choice of γ_3

- this is an additional requirement for measure.
- one construct a family of measures that is cylindrically consistent
- means: Given f cylindrical wrt γ_1 and γ_2 then we have

γ_1, $E(\gamma_1) = \{e_1, \dots, e_{N_1}\}$

γ_2, $E(\gamma_2) = \{e_1, \dots, e_{N_2}\}$

then

$$\int_{SU(2)^{N_1}} d\mu_{\gamma_1} F_{\gamma_1} = \int_{SU(2)^{N_2}} d\mu_{\gamma_2} F_{\gamma_2}$$

$F_{\gamma_1}: SU(2)^{N_1} \to \mathbb{C}$; $F_{\gamma_2}: SU(2)^{N_2} \to \mathbb{C}$

here normalisation as well as left- and right invariance crucial.

- Given cylindrically consistent family of measures $\{\mu_\gamma\}_\gamma$ then one can construct μ_{AL} on $\bar{\mathcal{A}}$ and get $\mathcal{H}_{phys} = L_2(\bar{\mathcal{A}}, d\mu_{AL})$

- Action of basic operators

$$(\pi(f_{\vec{y}})\, f_g)[A] = f_{\vec{y}}[A]\, f_g[A] \quad \text{multiplication}$$

$$(\pi(E_n(S))\, f_g)[A] = (Y_n(S)\, f_g)[A]$$

with

$$(Y_n(S)\, f_g)[A] = \frac{\beta \ell_p^2}{4} \sum_{v \in V(g)} n^{(v)}(v) \left[\sum_{e \text{ at } v} \varepsilon(e,S)\, J_j^{(v|e)}\, F \right](h_{e_1}(A), \ldots, h_{e_N}(A))$$

with $J_j^{(v|e)} := id \otimes id \otimes \ldots \otimes \begin{cases} L_j^e \\ R_j^e \end{cases} \otimes id \otimes \ldots$

when $\begin{cases} e \text{ is ingoing} \\ e \text{ is outgoing} \end{cases}$

- here L_j^e and R_j^e denote the left-/right invariant vector fields on $SU(2)$ associated with each edge

$$(R_j f) := \frac{d}{dt}\Big|_{t=0} f(e^{t\tau_j} h)$$

$$(L_j f) := \frac{d}{dt}\Big|_{t=0} f(h e^{t\tau_j})$$

- more explicit form of action of multiplication operator in spin-network basis

4.3 Spin network functions: an orthonormal basis of $\mathcal{H}_{phys}$

Aim: express $\mathcal{H}_{phys}$ as orthogonal sum over individual $\mathcal{H}_\gamma$'s.

Spin network functions

- Denote by $\{\pi^j, j=\frac{n}{2} \text{ with } n>0 \in \mathbb{N}\}$ set of irred. representation of $SU(2)$
- Given graph γ we associate to each edge

  $e \in E(\gamma)$: $\pi_e^{j_e} \in \{\pi^j, j=\frac{n}{2}, n>0 \in \mathbb{N}\}$

  notation $\vec{\pi} := \{\pi^{j_e}, e \in E(\gamma)\}$

- (Gauge-variant) spin network functions

$$T_{\gamma,\vec{\pi},\vec{m},\vec{n}} : \bar{\mathcal{A}} \to \mathbb{C}$$

$$A \mapsto \prod_{e \in E(\gamma)} \sqrt{2j_e+1}\, [\pi^{j_e}(h_e(A))]_{m_e n_e}$$

with $\vec{m} := \{m_e, e \in E(\gamma)\}$, $\vec{n} := \{n_e, e \in E(\gamma)\}$ where m_e, n_e label the corresponding matrix elements $m_e, n_e \in \{1, \dots, 2j_e+1\}$

- In order to understand why SNF provide an ONB of $\mathcal{H}_{phys}$ we consider the matrix functions

$$\pi^j_{mn} : SU(2) \to \mathbb{C}$$
$$g \mapsto \pi^j_{mn}(g) := [\pi^j(g)]_{mn}$$

with $m,n = 1, \dots, 2j+1$.

- $\{\pi^j_{mn}\}$ build on ONB of $L_2(SU(2), d\mu_H) = \mathcal{H}_e$

$$\langle \pi^j_{mn}, \pi^{j'}_{m'n'} \rangle$$
$$= \int_{SU(2)} d\mu_H(g)\, \overline{\pi^j_{mn}(g)}\, \pi^{j'}_{m'n'}(g)$$
$$= \delta_{j,j'}\, \delta_{m,m'}\, \delta_{n,n'}\, \frac{1}{2j+1}$$

can be shown by Peter & Weyl thm for compact Lie groups

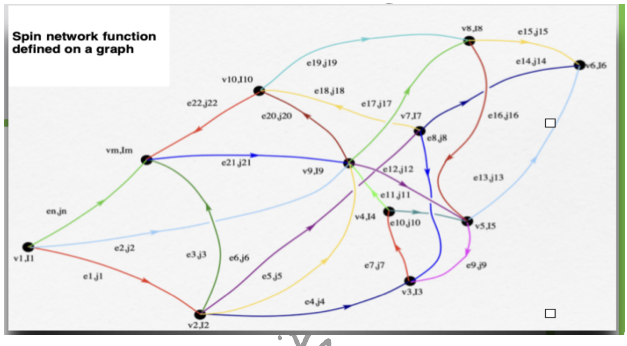
- Given that we immediately obtain that SNF are an ONB of $\mathcal{H}_{phys}$
- Action of flux & holonomy operators

$$\hat{\pi}^{je}_{m'n'}(h_e(A))\, \pi^{je}_{mn}(h_e(A))$$
$$= \sum_{k,m'',n''} C^{(j,e,j,e,k,e)}_{m,m',m''}\, C^{(j,e,j,e,k,e)}_{n,n',n''}\, \pi^{ke}_{m'',n''}(h_e(A))$$

$$\left(\hat{Y}_n(S)\, \pi^{je}_{mn}\right)(h_e(A)) = \frac{\beta \ell_P^2}{4} n^j(b(e)) \left[\tau_j^{(je)}\, \pi^{je}(h_e(A))\right]_{mn}$$

- Spin network

$T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$



- Now we want to proceed to express $\mathcal{H}_{phys}$ in terms of $\mathcal{H}_\gamma$'s.
- Problem:
  Consider two graphs $\gamma, \tilde{\gamma}$ that differ only by one edge only
  Let $\vec{\tilde{\pi}} = \lambda\vec{\pi}$, trivial representation at e_γ
  then we have $\vec{\tilde{m}} = \vec{m}$ and $\vec{\tilde{n}} = \vec{n}$ and

$$\langle T_{\gamma,\vec{\pi},\vec{m},\vec{n}}, T_{\tilde{\gamma},\vec{\tilde{\pi}},\vec{\tilde{m}},\vec{\tilde{n}}}\rangle = 1$$

  and thus $T_{\tilde{\gamma},\vec{\tilde{\pi}},\vec{\tilde{m}},\vec{\tilde{n}}} \not\perp T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$

- We do not allow trivial representations associated to edges.

$$\mathcal{H}_{phys} = \bigoplus_\gamma \mathcal{H}_\gamma$$

- $\mathcal{H}_\gamma$ are separable Hilbert space
- Since we consider direct sum over all $\mathcal{H}_\gamma$, $\mathcal{H}_{phys}$ non-separable

Part 5:
Reduced phase space quantisation of LQG
Step 3: Dynamics

5.1. Brief discussion on SU(2) Gauß constraint operator

5.2. Quantisation of the physical Hamiltonian



5.1. Quantisation of the Gauß constraint

Classical expression of Gauß constraint

$$G(\Lambda) = \int_S d^3\sigma (\Lambda^j D_a E^a_j)(\sigma) = -\int_S d^3\sigma (D_a \Lambda^j) E^a_j$$
$$= E(D_a \Lambda^j)$$

- **Aim:** want to compute $\{h_e(A), G(\Lambda)\}$

  again we introduce the regularised holonomy $h^\varepsilon_e(A)$ smeared in 3 dimensions

- Computing $\{h^\varepsilon_e(A), G(\Lambda)\}$ and removing the regulator yields:

$$\{h_e(A), G(\Lambda)\} = -\frac{\beta k}{2} \int_0^1 dt\, (D_a \Lambda^j)(e(t))\, \dot{e}^a(t)\, h_{e(0,t)}(A)\, \tau_j\, h_{e(t,1)}(A)$$

- Realise: $(D_a \Lambda^j)$ is phase space dep. since it involves connection A^j_a

One can show that the integrand above can be written as a total time derivative.

- To do this we consider

$$\frac{d}{dt}\left(he(0,t) \wedge he(t,1)\right)$$

From the holonomy differential eqn we have

$$\frac{d}{dt} he(0,t) = he(0,t)\, A(e(t))$$
$$\frac{d}{dt} he(t,1) = -\, A(e(t))\, he(t,1)$$

- Using this and the product rule we get:

$$\frac{d}{dt}\left(he(0,t) \wedge he(t,1)\right) = he(0,t)\left(\frac{d\Lambda}{dt} + [A(e(t)), \Lambda]\right) he(t,1)$$

- Considering the explicit form of the covariant derivative:

$$(D_a \Lambda)\dot{e}^a(t) = (\partial_a \Lambda + \varepsilon^j{}_{ke} A^k_a \Lambda^e \tau_j)\dot{e}^a(t)$$

$$= \frac{d\Lambda}{dt} + A^k_a \Lambda^e [\tau_k, \tau_e]\dot{e}^a(t)$$

$$= \frac{d\Lambda}{dt} + [A(e(t)), \Lambda]$$

used
$\tau_j = -\, i G_j / 2$
$[\tau_j, \tau_k] = \varepsilon_{jk}{}^{\ell} \tau_\ell$

- Combining these two results we obtain for the Poisson bracket:

$$\{h_e(A), G(1)\} = \frac{\beta k}{2}\left(1_{(e(0))} h_e(A) - h_e(A) 1_{(e(1))}\right)$$

- this can be extended to cylindrical functions

$$\{f_\gamma(A), G(1)\} = \frac{\beta k}{2} \sum_{e \in E(\gamma)} \left(1^j_{(b(e))} R^e_j - 1^j_{(f(e))} L^e_j\right) f_\gamma(A)$$

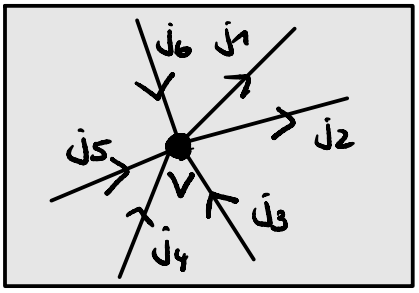
- We can rewrite the sum over all edges as a sum over all vertices $v \in V(\gamma)$ and then sum at each vertex of all edges that start or end there.
- Considering this $\hat{G}(1)$ can be written as

$$\hat{G}(1) = i\hbar\,[f_\gamma(A), G(1)] = \frac{i\beta\ell_p^2}{2} \sum_{v \in V(\gamma)} 1^j_{(v)} \left(\sum_{\substack{e \in E(\gamma) \\ v = b(e)}} R^e_j - \sum_{\substack{e \in E(\gamma) \\ v = f(e)}} L^e_j\right) f_\gamma(A)$$

- Solutions to the Gauß constraint:

Dirac quantisation

$$\hat{G}(\Lambda) f_g(A) \stackrel{!}{=} 0$$



those cylindrical fctns where the ingoing & outgoing edges couple to same spin $J^{out} = J^{in}$ so that total coupled J at V is

$$J^{out} \otimes J^{in} = 0$$

↳ subspace of $\mathcal{H}_{phys}$

$\mathcal{H}^G_{phys}$ SU(2) gauge-inv.

$$\mathcal{H}^G_{phys} \subset \mathcal{H}_{phys}$$

- For spin network function we can obtain the gauge-invariant set by introducing so called intertwiners at each vertex!

5.2. Quantisation of the physical Hamiltonian H_{phys}

- Physical Hamiltonian Brown-Kuchar model

$$H_{phys} = \int_S d^3\sigma\, H(\sigma)$$

$$H(\sigma) = \sqrt{C^2(\sigma) - q^{ab}(E)\, C_a C_b\,(\sigma)}$$

- In order to quantize H_{phys} we need operators corresponding to

$C(\sigma)$ and $q^{ab}(E) C_a C_b$

- Question: Why not $\hat{C}_j$ directly
  - ↳ not possible in LQG representation only finite diffeos

- Compare to Dirac quantisation without dust:

physical states

$$\hat{C}(N)\,\psi_{phys} \overset{!}{=} 0$$
$$\hat{U}(\varphi)\,\psi_{phys} \overset{!}{=} \psi_{phys}$$

finite diffeos

- techniques to quantize these operators useful in both approaches

- Start with quantisation of $C(6)$:
- For simplicity we restrict discussion of the Euclidian part here

$$C(N) = \frac{1}{\kappa\mu} \int_S d^3x\, N(6) \left[\frac{\varepsilon_{ij}{}^{kl} F^i_{ab} E^a_k E^b_l}{\sqrt{|\det(E)|}} + \frac{(1+\beta^2)}{\sqrt{|\det(E)|}} \left((K^i_b E^a_j)(K^k_a E^b_k) - (K^i_c E^c_i)^2 \right) \right]$$

$$=: C_{Eucl}(N) + (1+\beta^2) T(N)$$

Lorentzian part

further $K^i_a = A^i_a - \Gamma^i_a$ understood as fctns of A, E

- Euclidian part depends non-polynomial on E and involves inverse of $\sqrt{\det(E)}$
  ↳ problematic
- Idea: Rewrite $\frac{\varepsilon_{ijk} E^a_k E^b_l}{\sqrt{|\det(E)|}}$ by using a classical identity such that $\sqrt{|\det(E)|}$ is no longer in the denominator
- We have in $D=1$ class. mechanics

$$\frac{1}{\sqrt{p}} = 2\{q, \sqrt{p}\}$$

- Follow similar idea here.

- For this purpose we consider the classical volume fctnal of a region R

$$V(R) = \int_R d^3\sigma \sqrt{|\det(E)|}(\sigma)$$

$$\det(E) = \frac{1}{3!}\,\epsilon^{ijk}\,\epsilon_{abc}\,E^a_i E^b_j E^c_k$$

↳ third order polynomial in E^a_i

- The classical relation that we obtain is

$$e^i_c(\sigma) = \frac{1}{2}\,\mathrm{sgn}(\det(e))\,\frac{\epsilon_{abc}\,\epsilon^{ijk}\,E^b_j E^c_k}{\sqrt{|\det(E)|}} = \frac{4}{k}\,\{A^i_a(\sigma), V(R)\}$$

- Using this we obtain

$$\frac{\epsilon^{ijk}\,E^a_j E^b_k}{\sqrt{|\det(E)|}} = \frac{4}{k}\,\epsilon^{abc}\,\{A^i_c(\sigma), V(R)\}$$

- Volume operator $\hat{V}(R)$ exists in LQG. Now to be able to promote $C_{Euc}(N)$ to an operator we need to express

$F^i_{ab}(\sigma)$ as well as $A^i_c(\sigma)$

in terms of holonomies

Question: How can this be done?

1.) We rewrite $C^{euc}(N)$ in term of a trace

- We have $\tau_j = -iG_j$ then

$$\mathrm{Tr}(\tau_j \tau_k) = -2\delta_{jk}$$

- further $F_{ab} = F^j_{ab}\frac{\tau_j}{2}$, $A_c = A^j_c \frac{\tau_j}{2}$ then we have

$$C^{euc}(N) = C_1 \int d^3b \underbrace{(N \,\mathrm{sgn}(\det(e)))(b)}_{=\tilde{N}(b)} \int_S \varepsilon^{abc}\, \mathrm{Tr}(F_{ab}\{A_c(b), V(R)\})$$

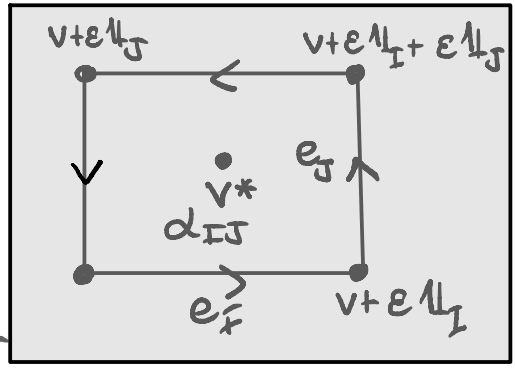
- C_1: includes all involved constants

2.) Rewrite F^j_{ab} in terms of holonomies we have

$$F^j_{ab} = \partial_a A^j_b - \partial_b A^j_a + [A_a, A_b]^j$$

- Consider a loop of edge length $\varepsilon \ll 1$

$$h_{\alpha_{IJ}} = h_{e_{I,v}} h_{e_{J,v+\varepsilon 1\!\!1_I}} h^{-1}_{e_{I,v+\varepsilon 1\!\!1_I+\varepsilon 1\!\!1_J}} h^{-1}_{e_{J,v+\varepsilon 1\!\!1_J}}$$



with $v^* := v + \frac{\varepsilon}{2}1\!\!1_I + \frac{\varepsilon}{2}1\!\!1_J$

- We consider a Taylor expansion of the holonomies up to the order ε^2

$$h_{e_I}(A) = \mathbb{1} + \int_0^1 dt\, A(c(t)) + \frac{1}{2}\int_0^1 dt_1 \int_{t_1}^1 dt_2\, A(c(t_1))\, A(c(t_2)) \approx \mathbb{1} + \varepsilon A(v + \xi_I \mathbb{1}_I) + \frac{1}{2}\varepsilon^2 A^2(v + \xi_I \mathbb{1}_I) + O(\varepsilon^2)$$

- we do this for all edges involved and then Taylor expand $A(v+\dots)$ around $v^* = v + \xi_I \mathbb{1}_I + \xi_J \mathbb{1}_J$.

- We obtain

$$h_{\alpha_{IJ}}(A) = \mathbb{1} + \varepsilon^2 F_{IJ}(A) + O(\varepsilon^2)$$

$$\rightsquigarrow\ h_{\alpha_{IJ}} - h^{-1}_{\alpha_{IJ}} = \varepsilon^2 (F_{IJ} - F_{JI}) = 2\varepsilon^2 F_{IJ}$$

using $F_{IJ} = -F_{JI}$.

- F^j_{ab} can be approximated by loops of holonomies.
- Next we need to express $\{A^j_a(e), V(R)\}$ in terms of holonomies

$$\{A_I, V(R)\} = h_{e_I}\{h^{-1}_{e_I}, V(R)\}$$

where $A_I := \int_{e_I} A$

- to show this one expands the holonomies up to linear order in ε.
- Given this we are able to express

$$\frac{F^j_{ab} E^a_k E^b_e \varepsilon_{ij}{}^{ke}}{\sqrt{|\det(E)|}}$$

in terms of holonomies and fluxes.

- As the next step we choose regularisation of $C_{EUCL}(N)$
- For this purpose we choose a triangulation of $\mathcal{G}$ in terms of tetrahedra $T(\varepsilon)$, where ε is a parameter of the fineness of the triangulation.
- Then we obtain

$$\int_{\mathcal{G}} d^3\mathcal{G} = \sum_{\Delta \in T(\varepsilon)} \int_\Delta d^3\mathcal{G} = \lim_{\varepsilon \to 0} \sum_{\Delta \in T(\varepsilon)} \mathrm{vol}(\Delta)$$

Riemann sum approximation

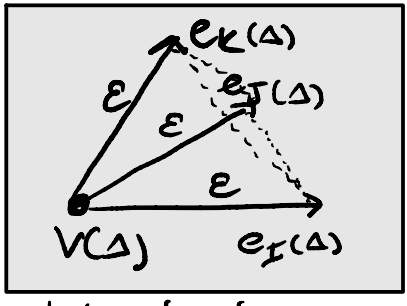
- If we consider a function $f(x)$ then we can use

$$\int_{\mathcal{G}} d^3\mathcal{G}\, f(\mathcal{G}) = \sum_{\Delta \in T(\varepsilon)} \int_\Delta d^3x\, f(\mathcal{G}_\Delta)$$

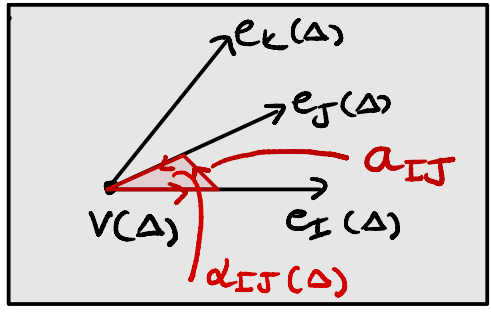
$$= \lim_{\varepsilon \to 0} \sum_{\Delta \in T(\varepsilon)} C_\Delta \varepsilon^3 f(\mathcal{G}_\Delta), \quad \mathcal{G}_\Delta \in \Delta$$

$$\mathrm{vol}(\Delta) = \frac{\sqrt{2}}{12}\varepsilon^3, \quad C_\Delta = \frac{\sqrt{2}}{12}$$

- Let e_I, e_J, e_K be 3 edges with a common intersection point $v(\Delta)$



tetrahedron



- Now we consider a loop α_{IJ}
- Then the regularized expression of $C_{cub}(N)$ can be written as

$$C_{cub}(N) = \tilde{C}_1 \lim_{\varepsilon \to 0} \sum_{\Delta \in T(\varepsilon)} N(v(\Delta)) \varepsilon^3 \varepsilon^{abc} \mathrm{Tr}\left(F_{ab}\{A_c, V(R_{v(\Delta)})\}\right)$$

($F_{ab} \sim \varepsilon^{-2}$, $\{A_c, V(R_{v(\Delta)})\} \sim \varepsilon^{-1}$)

$$= \tilde{C}_1 \lim_{\varepsilon \to 0} \sum_{\Delta \in T(\varepsilon)} N(v(\Delta)) \varepsilon^{IJK} \mathrm{Tr}\left(h_{\alpha_{IJ}(\Delta)} h_{e_K(\Delta)} \{h^{-1}_{e_K(\Delta)}, V(R_{v(\Delta)})\}\right)$$

- regularized expression of $C_{cub}(N)$ which only involves holonomies & fluxes

$\tilde{C}_1$ involves again all constants involved.

- Strategy to quantise now is

(i) replace $\{\cdot,\cdot\}$ by $\frac{1}{i\hbar}[\cdot,\cdot]$ in regularised expression and

$h_e \longrightarrow \hat{h}_e$ and $V(R) \to \hat{V}(R)$

thus we obtain

$$C^{\varepsilon}_{\text{Euclid}}(N) = \tilde{\tilde{C}}_1 \sum_{\Delta \in T(\varepsilon)} N(v(\Delta))\, \varepsilon^{IJK}\, \mathrm{Tr}\left(\hat{h}_{\alpha_{IJ}(\Delta)} \hat{h}_{e_K(\Delta)} \left[\hat{h}^{-1}_{e_K(\Delta)}, \hat{V}(R_{v(\Delta)}) \right] \right)$$

where the volume operator is given by

$$\hat{V}(R) = \left(\frac{\ell_p}{8}\right)^3 \sum_{v \in V(\gamma) \cap R} \sqrt{\left| \frac{i}{3!} \sum_{e,e',e'' \cap v} \varepsilon(e,e',e'')\, \varepsilon_{jkl}\, R^j_e R^k_{e'} R^l_{e''} \right|}$$

where $\varepsilon(e,e',e'')$ is $+1, -1, 0$ if the tangent vectors of e,e',e'' positive, negative or vanishing determinant.

- The final operator is then defined as

$$\hat{C}_{Euc_E}(N) = \lim_{\epsilon \to 0} \hat{C}^{\epsilon}_{Euc_E}(N)$$

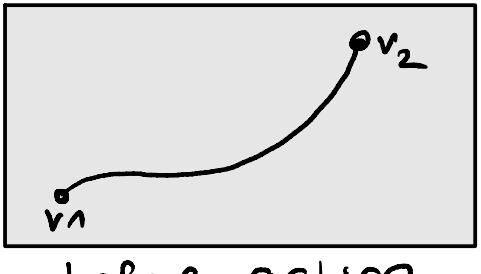
- We can rewrite the sum over all tetrahedra as a sum over all vertices when applied to spin networks

Reason:

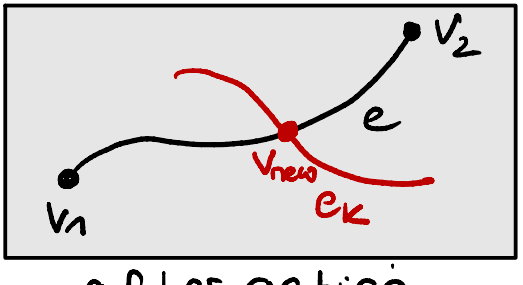
- Action of $\hat{C}_{Euc_E}(N)$ involves combinations of operators of the form

$$\hat{h}_{e_k}\left[\hat{h}^{-1}_{e_k}, \hat{V}(R)\right] = \hat{V}(R) - \hat{h}_{e_k}\hat{V}(R)\hat{h}^{-1}_{e_k}$$

suppose it acts on some interior point of some edge e



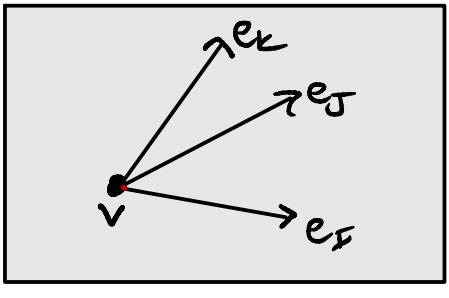
before action



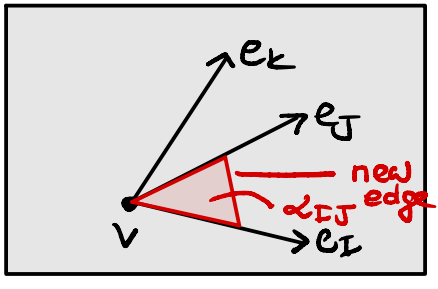
after action

- Thus those contribution vanish as the action of the holonomy operator creates a 4-valent vertex with at least 2 planar edges
  ↳ $\varepsilon(e,e',e'')$ vanishes
- If $\hat{V}$ acts directly on an interior point of an edge $\varepsilon(e,e',e'')=0$ since only 2-valent vertex
  ↳ that is why $\sum_{v \in V(\gamma) \cap R}$ in $\hat{V}(R)$

- Action of the operator:
  Graph-modifying operator it creates new edges



before action



after action

- further it modified the representations at the already existing edges

- Now going back to the physical Hamiltonian

$$H_{phys} = \int_S d^3\sigma\, H(\sigma) \qquad H(\sigma) = \sqrt{C^2 - q^{ab} C_a C_b}$$

- Also here we choose a regularisation of the classical expression
- Here we choose a partition of S into cubes then

$$H_{phys} = \sum_{\square \in \mathcal{P}} \int_{\square} d^3\sigma \sqrt{|C^2 - q^{ab} C_a C_b|}(\sigma)$$

$$= \lim_{\mathcal{P} \to \infty} \sum_{\square \in \mathcal{P}} V(\square) \sqrt{|C^2 - q^{ab} C_a C_b|}(\sigma(\square))$$

- Also here we need to express C^2 and $q^{ij} C_j C_k$ in terms of holonomies & fluxes
- For this purpose we use the following classical identities

$$q^{ab} = \frac{E^a_j E^b_k \delta^{jk}}{\det(E)} \qquad \mathcal{B}^a_j := \frac{1}{2} \varepsilon^{abc} F^j_{bc}$$

with $\mathcal{B}^a = \mathcal{B}^a_j \tau^j$, $e_a = e^j_a \tau_j$, $\mathcal{B} = \mathcal{B}^a e_a$

- then we obtain:

$$C^2 = [\mathrm{Tr}(B)]^2 \qquad q^{ab} C_a C_b = \left[\frac{\mathrm{Tr}(B\tau_j)}{4}\right]^2 =: C_j^2$$

then we can express H_{phys} in terms of C^2 and C_E^2 as

$$H_{phys} = \lim_{\mathcal{P}\to S} \sum_{\square \in S} \sqrt{|C^2(\square) - \delta^{jk} C_j(\square) C_k(\square)|}$$

where

$$C(\square) := \int_{\square} d^3\sigma\, C(\sigma) \qquad C_j(\square) := \int_{\square} d^3\sigma\, C_j(\sigma)$$

strategy to quantize H_{phys}:

$$\hat{H}_{phys} = \lim_{\mathcal{P}\to S} \sum_{\square \in S} \sqrt{|\hat{C}(\square)^\dagger \hat{C}(\square) - \delta^{jk} C_j(\square) C_k^\dagger(\square)|}$$

- $C(\square)$ ← quantisation we already discussed

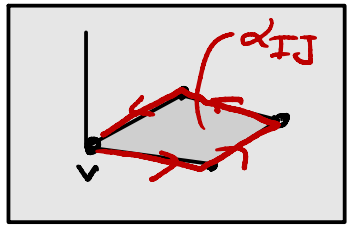
for $C_j(\square)$ similar procedure using

$$\tau_\mu := (\mathbb{1}_{2^1}, \tau_j)$$

$$\int_{\square} d^3\sigma\, \mathrm{Tr}(B\tau_\mu) = \int_{\square} \mathrm{Tr}(F \wedge e \tau_\mu) = \frac{1}{\kappa} \int_{\square} \mathrm{Tr}(F \wedge \{V(\square), A\}\tau_\mu)$$

↳ quantised as $C(\square)$ with additional τ_j involved.

- here it is quantised in a graph-preserving way to be consistent with classical symmetries of $\mathcal{H}_{phys}$
- Consequence the loop is only attached to already existing edges
- Consider $\mathcal{H}_{phys} = \bigoplus_\gamma \mathcal{H}_\gamma$



then define $\hat{C}_{\mu,\gamma,v}$

$$\hat{C}_{\mu,\gamma,v} := \frac{1}{\ell_p^2 |Tv(\gamma)|} \sum_{\substack{(e_1,e_2,e_3) \\ \in Tv(\gamma)}} \varepsilon^{IJK} \frac{1}{|L_{\gamma,v,e_I,e_J}|} \sum_{\alpha \in L_{\gamma,v,e_I,e_J}} Tr\left(\tau_\mu \hat{h}_\alpha \hat{h}_{e_K} [\hat{h}_{e_K}^{-1}, \hat{V}_{\gamma,v}]\right)$$

- $Tv(\gamma)$: set of ordered triples at vertex v
  L_{γ,v,e_I,e_J} : set of minimal loops
- then we can define $\hat{H}_{phys}$ for each graph γ

$$\hat{H}_{phys,\gamma} := \sum_{v \in \gamma} \sqrt{|P_\gamma (\hat{C}^\dagger_{\gamma,v} \hat{C}_{\gamma,v} - \tfrac{1}{4} \hat{C}^\dagger_{\gamma,v} \hat{C}_{\gamma,v}) P_\gamma|}$$

with $P_\gamma : \mathcal{H} \to \mathcal{H}_\gamma$ necessary to ensure graph preserving property.

- $\hat{H}_{phys}$ on $\mathcal{H}_{phys}$ is then given by

$$\hat{H}_{phys} = \bigoplus_g \hat{H}_{physig}$$

↳ which finalides the quantisation of the dynamics in the Brown-Kuchar model.

Hence one arrives at a form of the Quantum Einstein equations of this model wrt to a chosen dynamical reference frame. (dust)

- Full Quantum Einstein equations technically challenging
- However motivation for symmetry reduced models for instance in the context of cosmology & black holes.

Literature:

Books:

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- Introductory lectures to loop quantum gravity, Pietro Dona, Simone Speziale arXiv: 1007.0402 (gr-qc)
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Thank you!
感谢您



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Thank you!
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