Operator Algebra Methods in LQG

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Motivation for these lectures: Mathematical tools allow us to view a single formalism of physics in different ways. Understanding these different viewpoints can inspire us, in our research, both to find new applications of the formalism , as well as to generalise the formalism. The formalism of interest in these lectures is that of quantum theory i.e. of representations of quantum operators on Hilbert spaces of quantum states. The mathematical tools we will use are those devloped in the context of $*$-Algebras and C^*-Algebras. This will allow us to develop a powerful Algebraic Viewpoint for quantum theory.

PLAN

Lecture 1-2: Quantum Operators as elements of $*$-Algebras and the Gelfand Naimark Segal (GNS) Representation.

Lecture 3-4: Quantum Operators as elements of abelian C^*-Algebras and Gelfand construction of quantum configuration space.

Discussion Sessions: If interest we can discuss:
Day 1 GNS algebraic viewpoint to QFT in CS
Day 2 Structural similarities between LQC and LQG from C^* algebraic viewpoint.

LECTURE 1-2:

I. Some definitions:

(i) Algebra $\mathbf{A}$: An algebra $\mathbf{A}$ over the field of complex numbers $\mathbf{C}$ is a set of elements such that following properties hold:

1.Addition: '+', $\forall A, B \in \mathbf{A}$, $\exists\ A + B \in \mathbf{A}$ such that:

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) There exists zero element $0_{\mathbf{A}} \in \mathbf{A}$ s.t. $\forall A \in \mathbf{A},\ A + 0_{\mathbf{A}} = A$.

(d) $\forall A \in \mathbf{A}$, there exists an element $(-A)$ s.t. $A + (-A) = 0_{\mathbf{A}}$

2. Scalar multiplication: $\forall A \in \mathbf{A}$, $\alpha \in \mathbf{C}$, $\exists \alpha A \in \mathbf{A}$ s.t.:

(a) $\beta(\alpha A) = (\beta\alpha)A$

(b) $1A = A$

(c) $\alpha(A + B) = \alpha A + \alpha B$

(d) $(\alpha + \beta)A = \alpha A + \beta A$

1,2 imply that **A** is a vector space, we also need multiplication of $A, B \in \mathbf{A}$ for **A** to be algebra.

3. Multiplication:$\forall A, B \in \mathbf{A}$, $\exists\ AB \in \mathbf{A}$ such that:

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(\alpha A)(\beta B) = (\alpha\beta)AB$

(d) $\exists\ \mathbf{1} \in \mathbf{A}$ s.t. $\mathbf{1}A = A\mathbf{1} = A\ \forall A \in \mathbf{A}$.

1-3 define the notion of an Algebra.
You can think of 1-3 as properties of quantum operators.

We also need to include notion of adjoint of operator, for this we need to define $*$ Algebra.

(ii) *-Algebra $\mathcal{S}$: A *-Algebra $\mathcal{S}$ is an Algebra equipped with an operation $*$, which maps $\mathcal{S}$ to itself: $* : \mathcal{S} \to \mathcal{S}$.
For any $A \in \mathcal{S}$, denote $*(A)$ by A^*.
Then we require that $\forall A, B \in \mathcal{S}, \alpha \in \mathbf{C}$ the $*$ operation is s.t.:
(a) $(A^*)^* = A$
(b) $(A + B)^* = A^* + B^*$
(c) $(\alpha A)^* = \bar{\alpha} A^*$
(d) $(AB)^* = B^* A^*$
NOTE: 3 implies that $\mathbf{1}^* = \mathbf{1}$.
H.W: Show that $\mathbf{1}^* = \mathbf{1}$.

The GNS construction uses the notion of a Postive Linear Functional on a * Algebra...

(iii) Positive Linear Functional ω on a *- Algebra $\mathcal{S}$: A PLF is a map ω from $\mathcal{S}$ to **C**.

Thus for any $A \in \mathcal{S}$ $\omega(A)$ is a complex number.

The map ω is linear, positive and normalised i.e. it satisfies:

1. Linearity: $\omega(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \omega(A_1) + \lambda_2 \omega(A_2)$
2. Positivity: $\omega(A^* A) \geq 0$
3. Normalization: $\omega(\mathbf{1}) = 1$.

1.-3. imply that $\omega(A^*) = \overline{\omega(A)}$

Proof: $\omega((\mathbf{1} + A))^*(\mathbf{1} + A)) = 1 + \omega(A^* A) + \omega(A) + \omega(A^*) \geq 0$

$\Rightarrow \omega(A) + \omega(A^*)$ must be real.

GNS Construction: Given a * Algebra $\mathcal{S}$ and a PLF ω on it, GNS construct a representation of the algebra $\mathcal{S}$ on a Hilbert space. Thus every element of the algebra is represented as a quantum operator on states on this Hilbert space and the Hilbert space inner product is constructed in such a way that the $*$ operation on an algebra element is implemented as the operation of adjointness on the corresponding quantum operator. The PLF turns out then to be equal to the expectation value function i.e. $\omega(A) = \langle\Omega|\hat{A}|\Omega\rangle$ for a particular state $|\Omega\rangle$ in the Hilbert space.

The GNS construction only needs ω to be positive i.e $\omega(A^*A) \geq 0$. We shall provide a proof for the simpler case where ω is positive definite i.e.
$\omega(A^*A) \geq 0$, together with the condition that:
$\omega(A^*A) = 0 \Rightarrow A = 0$. If there is interest we can discuss the general case in the tutorial time.

Steps which we will follow:

(a) We will define a vector space V on which we will look for a representation of $\mathcal{S}$.

(b) We will represent every $A \in \mathcal{S}$ as a linear operator $\Lambda(A)$ on V s.t. $\Lambda(A)\Lambda(B)|v\rangle = \Lambda(AB)|v\rangle, \forall |v\rangle \in V$

(c) We will define a scalar product $(,)$ on V s.t. $\overline{(|v_2\rangle, \Lambda(A)|v_1\rangle)} = (|v_1\rangle, \Lambda(A^*)|v_2\rangle)$.

The Cauchy completion of V wrto $(,)$ will yield the GNS Hilbert space.

GNS Construction for positive definite ω:

Step (a)Choice of Repn space V:Since the algebra $\mathcal{S}$ is itself a vector space, we will choose $V \equiv \mathcal{S}$. Just for notational purposes when we think of $A \in \mathcal{S} = V$ as a vector we will denote it by $A = |A\rangle$. When we think of $A \in \mathcal{S}$ as an element of the algebra to be represented as an operator, we will denote it as A.
The representation of A as a linear operator on V will be denoted as $\Lambda(A)$.

Step (b) Representation of elements of $\mathcal{S}$ **as operators on** V:
Given any algebra element $B \in \mathcal{S}$ and a state $|A\rangle \in \mathcal{S} = V$, we define the action of the linear operator $\Lambda(B)$ on $|A\rangle$ as:
$\Lambda(B)|A\rangle = |BA\rangle$
(i.e. $\Lambda(B)$ maps the state $|A\rangle \in V$ to the state $|BA\rangle \in V$.
$\Rightarrow \Lambda(C)\Lambda(B)|A\rangle = \Lambda(C)|BA\rangle = |CBA\rangle = \Lambda(CB)|A\rangle$
$\Rightarrow \Lambda(C)\Lambda(B) = \Lambda(CB)$.

Step (c) Definition of Inner Product:

Define $(|A\rangle, |B\rangle) = \omega(A^* B)$. Note that:

(i)$(\lambda|A\rangle, \mu|B\rangle) = \omega(\bar{\lambda}A^* \mu B) = (\bar{\lambda}\mu)\omega(A * B) = \bar{\lambda}\mu(|A\rangle, |B\rangle)$

(ii) $(|A\rangle, |A\rangle) = \omega(A^* A) > 0$ for $A \neq 0$.

(i)-(ii) imply that $(,)$ is an inner product on V.

Finally , note that:

$$\begin{aligned} \overline{(|A\rangle, \Lambda(C)|B\rangle)} &= \overline{(|A\rangle, |CB\rangle)} \\ &= \overline{\omega(A^* CB)} \\ &= \omega(B^* C^* A) = (|B\rangle, \Lambda(C^*)|A\rangle) \end{aligned}$$

Thus the * operation is implemented as operator adjointness!

Thus the GNS construction provides a Hilbert space representation of $\mathcal{S}$. We now show that this representation is cyclic i.e. every state in V can be obtained through the action of some element of $\mathcal{S}$ on a fixed state $|\Omega\rangle$, $|\Omega\rangle$ is referred to as a cyclic state. We shall also see that the PLF is just the operator expectation value function in this cyclic state.

We set $|\mathbf{1}\rangle = |\Omega\rangle$.
Clearly any $|A\rangle \in V$ can be obtained as $\Lambda(A)|\Omega\rangle = |A\mathbf{1}\rangle = |A\rangle$.
Thus V is generated by action of every element of $\mathcal{S}$ on $|\Omega\rangle$.
It is also the case that:

$$(|\Omega\rangle, A|\Omega\rangle) = (|\mathbf{1}\rangle A|\mathbf{1}\rangle) = (|\mathbf{1}\rangle, |A\rangle) := \omega(\mathbf{1}A) = \omega(A)$$

so that the PLF is just the exp value function in the cyclic state.

We shall illustrate the GNS construction through two examples.

For both example the * Algebra $\mathcal{S}$ will be the same but the two PLFs we use will be different and lead to two *unitarily inequivalent* representations of the same operator algebra.

As we might discuss in the tutorial session, one physically important context in which the dynamics often takes a state in one representation and evolves it to a state in a unitarily inequivalent representation of the *same operator algebra* is that of QFT in CS.

The GNS algebraic viewpoint of states as PLFs on the algebra provides a powerful way to handle unitarily inequivalent states.

III. Examples: In both the examples we shall fix $\mathcal{S}$ to be the standard Weyl Algebra for QM.

Definition of the Weyl Algebra:

Consider classical phase space for a system in 1d with position-momentum (x, p), $\{x, p\} = 1$.

In QM we want $[\hat{x}, \hat{p}] = i\hbar\mathbf{1}$.

Define $W(a, b) = e^{i(a\hat{x}+b\hat{p})}$. Since x, p are classically real, the * operation on W is defined to be:

$W^*(a, b) = W(-a, -b)$.

Using the commutation relations between $\hat{x}, \hat{p}$, one can show that the product of a pair of Weyl operators is:

$W(a_1, b_1)W(a_2, b_2) = e^{\frac{i\hbar}{2}(a_1 b_2 - a_2 b_1)} W(a_1 + a_2, b_1 + b_2)$.

To summarise: the individual Weyl operators satisfy:
1.Product Rule:
$W(a_1, b_1)W(a_2, b_2) = e^{\frac{i\hbar}{2}(a_1 b_2 - a_2 b_1)} W(a_1 + a_2, b_1 + b_2)$.
2. The * operation:
$W^*(a, b) = W(-a, -b)$.

The Weyl algebra is the free algebra generated by taking arbitrary complex linear combinations of products of individual Weyl operators, simplifying products of arbitary number of individual Weyl operators using 1 and inducing the *-operation on the result using 2. The resulting *-Algebra is called the Weyl Algebra.

Since product of 2 W's is a single W all multiple products can be simplified to a complex phase times a single W. Hence elements of the Weyl algebra reduce to arbitrary complex linear combinations of individual Weyl operators for which the * operation is defined through 2.

For the GNS construction it suffices to specify the PLF ω on each W operator because all products of such operators reduce to a single one due to the product rule.

Example 1: A PLF for Standard QM: For concreteness, consider a Simple Harmonic Oscillator of mass m and frequency ω. For simplicity fix units such that $m = \omega = \hbar = 1$ and define the PLF ω to be:
$\omega(W(a, b)) = e^{-\frac{1}{4}(a^2+b^2)}$.

ω is linear by definition so that:
$\omega(\sum_i c_i W(a_i, b_i)) := \sum_i c_i \omega(W(a_i, b_i))$.

To establish positivity of ω i.e. we need to check that:
$\omega((\sum_{j=1}^n c_j W(a_j, b_j)^*(\sum_{i=1}^n c_i W(a_i, b_i)))\geq 0$.
Since this is not a linear condition, establishing positivity of a candidate PLF is usually non-trivial. The LHS expands to:

$$\sum_{i,j} \bar{c}_j c_i \omega(W(a_j, b_j)^* W(a_i, b_i))$$

$$= \sum_{i,j} \bar{c}_j c_i \omega(W(-a_j, -b_j) W(a_i, b_i))$$

$$= \sum_{i,j} c_i \bar{c}_j e^{-\frac{i}{2}(a_j b_i - b_j a_i)} \omega(W(a_i - a_j, b_i - b_j)$$

$$
\begin{aligned}
&= \sum_{i,j} c_i \bar{c}_j \omega(W(a_i - a_j, b_i - b_j)) e^{-\frac{i}{2}(a_j b_i - b_j a_i)} \\
&= \sum_{i,j} c_i \bar{c}_j e^{-\frac{1}{4}((a_i - a_j)^2 + (b_i - b_j)^2)} e^{-\frac{i}{2}(a_j b_i - b_j a_i)} \\
&= \sum_{i,j} c_i \bar{c}_j e^{-\frac{1}{4}(a_i^2 + a_j^2 + b_i^2 + b_j^2 - 2a_i a_j - 2b_i b_j)} e^{-\frac{i}{2}(a_j b_i - b_j a_i)} \\
\text{Set } d_i &= c_i e^{-\frac{1}{4}(a_i^2 + b_i^2)} \\
LHS &= \sum_{i,j} d_i \bar{d}_j e^{\frac{1}{2}(a_i a_j + b_i b_j - i a_j b_i + i b_j a_i)} = \sum_{i,j} d_i \bar{d}_j e^{\frac{1}{2}(a_i - i b_i)(a_j + i b_j)} \\
&= \sum_{i,j} d_i \bar{d}_j e^{\alpha_i \bar{\alpha}_j} = \sum_{i,j} d_i \bar{d}_j \sum_{m=0}^{\infty} \frac{(\alpha_i)^m (\bar{\alpha}_j)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \Big(\sum_i (d_i \alpha_i^m)\Big)\Big(\sum_j (\bar{d}_j \bar{\alpha}_j)^m\Big) =: \sum_{m=0}^{\infty} \frac{1}{m!} D_m \bar{D}_m \geq 0
\end{aligned}
$$

Note: We could reorder the sum because the series in the last line can be shown to be absolutely convergent.

H.W. Show this by showing that $|D_m| \leq C \sum_i |\alpha_i|^m$ for some $C > 0$.

H.W. Check that:
$\omega(W(a,b)) = e^{-\frac{1}{4}(a^2+b^2)} = \langle\Omega|\hat{W}(a,b)|\Omega\rangle$ where $|\Omega\rangle$ is the SHO vacuum state for an oscillator where we have used units such that $m = 1 = \omega = \hbar$.

H.W. In the GNS construction for this PLF which of the standard oscillator states correspond to $|W(a,b)\rangle$?

Note that the PLF we have chosen is *continuous* with respect to a, b. This property is what allows us to conclude , via the Stone von Neumann uniqueness theorem, that the GNS representation is unitarily equivalent to the standard representation in terms of wave functions in $L^2(R, dx)$.

Example 2: A PLF for LQC: Define ω through:

$\omega(W(a,b)) = 1$ if $a = 0$, $\quad \omega(W(a,b)) = 0$ if $a \neq 0$.

For showing positivity, it is useful to use an alternate notation:
$W(a = x_1, b = x_2) = W(x_1, x_2) := W(\vec{x})$,
so that in this notation we have:
$\omega(W(\vec{x})) = \delta_{x_1,0}$.

We denote a linear combination of Weyl operators by:
$\sum_{\vec{x}} c_{\vec{x}} W(\vec{x})$ where $\vec{x}$ varies over some set $S = \{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$

For positivity of ω we need to show:
$\omega(\ (\sum_{\vec{y}} c_{\vec{y}} W(\vec{y})\ )^*(\ \sum_{\vec{x}} c_{\vec{x}} W(\vec{x})\ )\ ) \ \geq \ 0.$
In this notation the * reln, product rule are:

$(W(\vec{x}))^* = W(-\vec{x})$ and $W(\vec{y})\ W(\vec{x}) \ = \ e^{\frac{i}{2}(x_1 y_2 - x_2 y_1)}\ W(\vec{x} + \vec{y})$
Using these, the positivity condition reduces to:
$\omega(\ \sum_{\vec{y},\vec{x} \in S}\ \bar{c}_{\vec{y}} c_{\vec{x}}\ W(\vec{x} - \vec{y})\ e^{-\frac{i}{2}(x_1 y_2 - x_2 y_1)}\ ) \ \geq \ 0$
The proof which we display next is simpler than for the previous example of standard QM.

Proof that $\omega(\sum_{\vec{y},\vec{x}\in S} \bar{c}_{\vec{y}} c_{\vec{x}} W(\vec{x}-\vec{y}) e^{-\frac{i}{2}(x_1 y_2 - x_2 y_1)}) \geq 0$

$$
\begin{aligned}
LHS &= \sum_{\vec{y},\vec{x}\in S} \bar{c}_{\vec{y}} c_{\vec{x}} \delta_{x_1,y_1} e^{-\frac{i}{2}(x_1 y_2 - x_2 y_1)} \\
&= \sum_{y_2,x_2} \sum_{x_1} \bar{c}_{(x_1,y_2)} c_{(x_1,x_2)} e^{-\frac{i}{2}(x_1(y_2-x_2))} \\
&= \sum_{y_2,x_2} \sum_{x_1} \bar{c}_{(x_1,y_2)} c_{(x_1,x_2)} e^{-\frac{i}{2}x_1 y_2} e^{\frac{i}{2}x_1 x_2} \\
\text{Set } \ d_{(x_1,x_2)} &:= c_{(x_1,x_2)} e^{\frac{i}{2}x_1 x_2} \quad \forall (x_1,x_2) \in S \\
LHS &= \sum_{x_1} \sum_{y_2,x_2} \bar{d}_{(x_1,y_2)} d_{(x_1,x_2)} \\
&= \sum_{x_1} (\sum_{x_2} d_{(x_1,x_2)})(\sum_{y_2} \bar{d}_{(x_1,y_2)}) := \sum_{x_1} D^{(x_1)} \overline{D^{(x_1)}} \geq 0
\end{aligned}
$$

Note that the PLF in this example, namely,

$$\begin{aligned}\omega(W(a,b)) &= 1 \quad \text{if} \quad a = 0 \\ &= 0 \quad \text{if} \quad a \neq 0\end{aligned}$$

is *not* continuous at $a = 0$.

This implies that $\hat{x}$ is not a well defined operator.

To see this try to define $\omega(\hat{x})$ through:

$\lim_{a\to 0} \omega(\frac{e^{ia\hat{x}} - \mathbf{1}}{a}) = \lim_{a\to 0} \omega(\frac{W(a,0) - \mathbf{1}}{a})$,

The above limit does not exist!

While this is not a complete proof, it does given an idea as to why $\hat{x}$ is not a well defined operator (only its exponential is).

As you have learnt, in LQC x is like the connection, so the connection operator is not well defined in LQC.

It turns out that in LQG also the connection operator is not well defined only its (path ordered) exponential (i.e. its holonomy) is well defined. This property of the LQG representation is what creates profound open issues in defining the LQG dynamics. We can discuss this further in the tutorial if there is interest.

H.W. Show that the GNS representation from the above PLF is exactly the LQC representation. What does the GNS cyclic state correspond to in the LQC representation?

H.W. For tomorrow's lectures please revise the definition of continuity of a real function on R^3 which you know from calculus.

Lectures 3-4

I. Some Mathematical Definitions:

(i) Normed Vector Space:

Let V be a vector space. A norm of a vector can be thought of as its length. We denote the norm of a vector $v \in V$ by $||v||$. Formally, a norm $||\ \ ||$ is a map from V to R i.e. $||\ \ || : V \to R$ satisfying the following properties:

1. $||x + y|| \leq ||x|| + ||y||\ \forall x, y \in V$.
2. $||\lambda x|| = |\lambda| ||x||\ \forall x \in V, \lambda \in \mathbf{C}$.
3. $||x|| \geq 0, = 0$ only when $x = 0$.

A vector space with a norm is called a Normed Vector Space

NOTE: We can view R^3 as a normed vector space as follows. A point p in R^3 with Cartesian coordinates (x_1, x_2, x_3) can be thought of as defining a vector $\vec{x}$ from the origin to p, with the components of the vector being x_1, x_2, x_3. Thus we can coordinatize each point p by a vector $\vec{x}$ and view R^3 as a vector space. The norm of a vector $\vec{x}$ is just its length $\sqrt{x_1^2 + x_2^2 + x_3^2}$ Similarly distance between two points p, q with coordinates $\vec{x}$, $\vec{y}$ is the norm of the vector $\vec{x} - \vec{y}$.

(ii) Continuity:

1. Continuous functions on R^3: View R^3 as a vector space as above. A complex function f is continuous at a point $\vec{x}_0 \in R^3$ if: for any $\epsilon > 0$ $\exists\, \delta > 0$ s.t. $\forall \vec{x}$ s.t. $||\vec{x} - \vec{x}_0|| < \delta$, we have that $|f(\vec{x}) - f(\vec{x}_0)| < \epsilon$.

2. Continuity of Linear functions on a normed vector space V:
Linearity: f, is a map $f : V \to \mathbf{C}$ s.t.
$f(av_1 + bv_2) = af(v_1) + bf(v_2), a, b \in \mathbf{C}, v_1, v_2 \in V$.

Continuity at $v_0 \in V$: For any $\epsilon > 0$ $\exists \delta > 0$ s.t.
$\forall v$ s.t. $||v - v_0|| < \delta$ we have that $|f(v) - f(v_0)| < \epsilon$.

$\Rightarrow |f(v - v_0)| < \epsilon$, $\forall v$ s.t. $||v - v_0|| < \delta$
Setting $v - v_0 = w$, $|f(w)| < \epsilon$, $\forall ||w|| < \delta$
Reversing the steps it is easy to see that Continuity at origin implies continuity everywhere!

3. An important fact: Any complex linear function on a vector space is contnuous if and only if the function is BOUNDED. (Boundedness: $|f(v)| < K||v||$, for some $K > 0$. Proof is straightforward, can discuss later if interest.)

(iii) Norm on $*$ Algebra $\mathcal{S}$:

A norm on $\mathcal{S}$ is a map from $\mathcal{S}$ to R which satisfies certain properties. Recall that a * Algebra is also a vector space. A norm on $\mathcal{S}$ satisfies, in addition to the properties of a norm on a $\mathcal{S}$ as a vector space, two additional properties related to the two additional operations AB and A^* on elements of $\mathcal{S}$:

(a) $||AB|| \leq ||A|| \, ||B||$

(b) $||A^*A|| = ||A||^2$

H.W Show that (a), (b) imply $||A|| = ||A^*||$.

(iv) C^* Algebra: A C^* Algebra $\mathcal{C}$ is a normed $*$ Algebra which is Cauchy complete with respect to the norm

NOTE:
A Cauchy sequence in $\mathcal{C}$ is a set $\{A_i \in \mathcal{C}, i = 1, 2...\}$ such that given any $\epsilon > 0$, $\exists$ N such that:
$\forall i, j > N$, we have that $||A_i - A_j|| < \epsilon$.

A Cauchy sequence in $\mathcal{C}$ admits a limit in $\mathcal{C}$ if there exists $A \in \mathcal{C}$ such that $\forall \epsilon > 0$, $\exists$ N such that
$\forall i > N$, we have that $||A - A_i|| < \epsilon$.

Cauchy completeness of $\mathcal{C}$ is the property that every Cauchy sequence in $\mathcal{C}$ converges to a limit in $\mathcal{C}$.

(v) Abelian C^* Algebra: An Abelian C^* algebra $\mathcal{C}$ is a C^* Algebra in which $\forall A, B \in \mathcal{C}$ we have that $AB = BA$.
Example of Abelian C^* Algebra: The algebra of continuous functions on a compact Hausdorff space X.

Digression on Topology: Topology is the study of the concept of continuity. The basic objects which determine if a function is continuous on a space X are the open subsets of X:
A complex function f is continuous if for every $x \in X$, and every $\epsilon > 0$, there is an open set U containing x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$.
So if we know all the open sets in X, we can check if a function $f : X \to \mathbf{C}$ is continuous

Thus, specifying the topology of X is the same as specifiying all the open subsets of X.
For e.g. the topology of R^n is determined by open sets corresponding to open balls of arbitrarily small sizes.
Two key properties which the topology of a space may or may not satisfy are the Hausdorff property and the compactness property.

The topology of a space X is said to be Hausdorff if any 2 points in X can be separated by disjoint open neighborhoods (e.g. in R^3, small enough open balls around any p, q seperate them)

Compactness of X is a certain property of the collection of open sets contained in X which captures a certain notion of 'finiteness' X. For example, the line is non-compact and "infinitely long" , the open interval is non-compact as it has no "endings" but the circle is compact and "finite".
It is difficult to describe an intuituve notion of compactness as it is quite a subtle concept.

Two key consequences of a space X being Hausdorff and compact:

(a) Any continuous complex valued function on X is bounded

(b) X always admits integration measures so that we can integrate functions on X.

This ends our discussion of Topology, let us continue with our example of an Abelian C^* Algebra.

Example of Abelian C^* Algebra: The algebra of continuous functions, $\mathcal{F}_X$, on a compact Hausdorff space X.
$f : X \to \mathbf{C}$ so $f(x)$ is a complex number for each $x \in X$.

We can define addition, multiplication, * operation for the algebra in the obvious "pointwise way":
$f(x) := f_1(x) + f_2(x)$, $f(x) := f_1(x)f_2(x)$, $f^*(x) := \bar{f}(x)$.

Since functions are bounded we can define the following norm:
$||f|| = \sup_{x \in X} |f(x)|$ where sup means the least upper bound of $|f|$ which in this case equals its maximum value.

It is straightforward to show that Cauchy sequences of continuous functions in this norm converge also to continuous functions.

Thus $\mathcal{F}_X$ is Cauchy complete and hence an Abelian C^* algebra

Couple of H.W. Exercises:

H.W.: Check that sup norm satisfies properties of a norm for the * algebra $\mathcal{F}_X$.
H.W Try to show Cauchy limit of sequence of elements in $\mathcal{F}_X$ is also in $\mathcal{F}_X$.
(Hint:Show that convergence in the norm implies pointwise convergence. Use continuity of each function in sequence at every point $x \in X$)

Gel'fand showed that *every* Abelian C^* Algebra $\mathcal{C}$ can be realised as the Abelian C^* algebra $\mathcal{F}_\Delta$ of continuous functions on a certain compact, Hausdorff space Δ.

This suggests that we can represent elements of $\mathcal{C}$ as operators in a 'multiplication' representation on Δ.
To see this, let us define wave functions $\psi(x), x \in \Delta$ to be continuous on Δ.
Since any element $A \in \mathcal{C}$ corresponds to some function f_A on Δ we can define $\hat{A}\psi(x) = f_A(x)\psi(x)$.
Also since Δ is compact,Hausdorff we can define measures on Δ and integrate functions on Δ and define an inner product
$\int_\Delta d\mu(x)\bar{\phi}(x)\psi(x)$

II. Gel'fand Results on Abelian C^* Algebras

1. Any Abelian C^* algebra $\mathcal{C}$ is isomorphic to the Abelian C^* algebra $\mathcal{F}_\Delta$ of continuous functions on a compact, Hausdorff space Δ (Δ is referred to as the Spectrum of the C^* algebra $\mathcal{C}$).

2. Elements of Δ are in 1-1 correspondence with homomorphisms h from $\mathcal{C}$ to $\mathbf{C}$.
More in detail $h(A \in \mathcal{C})$ is a complex number such that:

(a) $h(A+B) = h(A) + h(B)\ \forall A, B \in \mathcal{C}$

(b) $h(\lambda A) = \bar{\lambda} h(A),\ \forall \lambda \in \mathbf{C}, A \in \mathcal{C}$.

(c) $h(AB) = h(BA) = h(A)h(B) = h(B)h(A),\ \forall A, B \in \mathcal{C}$

(d) $h(A^*) = \bar{h}(A)\ \forall A \in \mathcal{C}$

(e) $h(\mathbf{1}) = 1$

Thus we have identified the space $X \equiv \Delta$ as the set of all $h : \mathcal{C} \to \mathbf{C}$.
It remains to specify which function $f_A(h) \in \mathcal{F}_\Delta$ corresponds to the elment $A \in \mathcal{C}$.

The relevant correspondence is:

3. $f_A(h) = h(A)$,

This looks confusing so let us say it in words:
$f_A(h)$ is a function on Δ so for every $h \in \Delta$, $f_A(h)$ is a complex number.
On the other hand h is map from $\mathcal{C}$ to the complex numbers.
Hence $h(A)$ is also a complex number.

Gel'fand tells us that these 2 complex numbers are the same!

Note for Students familar with Topology: The Topology of Δ is defined to be the weakest topology in which the functions $f_A, \forall A \in \mathcal{C}$ (where f_A is defined by **3.** above) are all continuous.

Application of these ideas to LQG: As you have learnt in the first part of this School, the classical phase space before we go to quantum theory is that of triads and $SU(2)$ connections. Thus the classical configuration variable is a connection. The relevant classical functions of connection which we want to represent as operators are Wilson Loops (traces of holonomies).

It turns out that these holonomy traces generate an abelian C^* algebra. Then the spectrum of this algebra, Δ, can be thought of as the 'quantum configuration space'. It can be seen to be a certain extension of the classical space of connections.

As we discussed, one can then look for a 'multiplication' repn where holonomy traces act by multiplication and can also look for a suitable measure on this space. This is called the Ashtekar-Lewandowski measure and it also supports a nice representation of suitable triad operators. This is the LQG representation!

These ideas were first brought into the field by Ashtekar and Isham in 1990 or so.

III.Example: LQC
(Ref: J.M. Velhinho,C.Q.G. 24 (2007) 3745-3758)

Connection variable is like x, triad variable p. Focus on functions of x. Similar to holonomy of connection which is (path ordered) exponential, consider the functions $e^{ikx}, k \in R$. Since their Poisson brackets with each other vanish, we expect that the corresponding quantum operators to commute so that their algebra is abelian.

Accordingly, we now construct the abelian C^* algebra generated by these functions and apply the Gelfand results. This will be done in following steps:

1. We will construct the * algebra $\mathcal{S}$ generated by these functions.
2. We will define a norm on $\mathcal{S}$ and thereby convert into a C^* algebra $\mathcal{C}$.
3. We will identify the spectrum Δ of $\mathcal{C}$.
4. We will characterize elements of Δ using the notion of algebraically independent real numbers
5. We will discuss the measure on Δ relevant to LQC.

1. Construction of Abelian * algebra $\mathcal{S}$

Denote abstract elment corresponding to e^{ikx} by a_k.

Since $e^{ik_1x}e^{ik_2x} = e^{i(k_1+k_2)x}$, we define multplication rule on $\mathcal{S}$ as:

$a_{k_1}a_{k_2} = a_{k_1+k_2}$

Since $\overline{e^{ikx}} = e^{-ikx}$, we define the * operation as:

$a_k^* = a_{(-k)}$

Similar to case of Weyl Algebra, $\mathcal{S}$ is generated by arbitrary linear combinations of products of the 'generators' a_k. As in that case, product rule reduces these to linear combinations of single generators on which we impose the * relation.

Thus any element $a \in \mathcal{S}$ is of the form

$a = \sum_{i=1}^{n} c_i a_{k_i}$

where $c_i \in \mathbf{C}$ and $k_i, i = 1..n$ are a set of n real numbers.

It follows that:

$a^* = \sum_{i=1}^{n} \bar{c}_i a_{(-k_i)}$

NOTE: The identity element is $\mathbf{1} = a_{k=0}$.

It follows that $a_k a_k^* = a_k a_{(-k)} = a_0 = \mathbf{1}$.

2.Construction of Abelian C^* algebra $\mathcal{C}$

Define the norm of $a = \sum_{i=1}^{n} c_i a_{k_i}$ as:

$||a|| = ||\sum_{i=1}^{n} c_i a_{k_i}|| := \sup_{x \in R} |\sum_{i=1}^{n} c_i e^{ik_i x}|$

The supremum is well defined because e^{ikx} is a bounded function.

Note that $||a_k|| = 1$

$\mathcal{C}$ is obtained by Cauchy completion of $\mathcal{S}$ in the sup norm.

3.Gel'fand Spectrum Δ:

Gel'fand tells us that the spectrum Δ is compact and Hausdorff.

Elements of Δ are homomorphisms from $\mathcal{C}$ to $\mathbf{C}$:

$h : \mathcal{C} \to \mathbf{C}$.

In particular we have for $a_k \in \mathcal{C}$ that:

$h(a_k a_k^*) = h(a_0) = h(\mathbf{1}) = 1 = h(a_k) h^*(a_k)$

$\Rightarrow |h(a_k)| = 1$

$\Rightarrow h(a_k) = e^{i\phi_k}$ for some real phase $\phi_k \Rightarrow h(a_k) \in U(1)$

In what follows, it useful to sometimes denote the element a_k by the symbol $[k]$.

In this notation we have that:
$h([p])h([q]) = h([p+q])$
Note that the real line R has the group structure of an abelian group under the group composition law given by addition i.e. for $p, q \in R$, the composition of $p \circ q := p + q = q + p \in R$.

Since we have shown that $h([k]) \in U(1)$, it then follows that every $h \in \Delta$ defines a group homomorphism from $R, +$ to $U(1)$. Thus, if we restrict the action of every $h \in \Delta$ to the set $\{a_k, k \in R\}$, such a restriction defines a group homomorphism from the additive group of reals, $R, +$ to $U(1)$.
Note also that every $h \in \Delta$ is completely determined by its restriction to the set $\{a_k, k \in R\}$.
These results will be useful in characterizing elements of Δ.

4. Characterization of elements of Δ

We have seen that any element of Δ defines a group homomorphism from the additive group of real numbers to $U(1)$. We will show that converse is also true.

Let g be such a homomorphism so that $g(k \in R) \in U(1)$. Setting $h_g(a_k) := g(k)$ it follows immediately that:
$h_g(a_p)h_g(a_q) = g(p)g(q) = g(p+q) = h_g(p+q) \in U(1)$.

We can extend h_g to all of $\mathcal{S}$ by linearity so that:
$h_g(a = \sum_i c_i a_{k_i}) := \sum_i c_i h_g(a_{k_i})$.

We need to show that h_g extends also to elements in $\mathcal{C}$ (recall that $\mathcal{C}$ is the completion of $\mathcal{S}$ and $\mathcal{S}$ contains finite linear combinations a of elements a_k).
It is not difficult to see, after some thought, that this is true if we can show that h_g is bounded on $\mathcal{S}$ i.e. $|h(a)| \leq C||a|| \forall a \in \mathcal{S}$.

To show boundedness, we need the concept of algebraically independent real numbers.

Definition: A set of real numbers $\{k_1, k_2, ..k_n\}$ is said to be algebraically independent if the only set of integer coefficients $m_i, i = 1, .., n$ for which the equation $\sum_i m_i k_i = 0$ holds, is $\{m_i = 0, i = 1..n\}$.

It is not difficult to show that given any set of n real numbers $k_i, i = 1..n$, there always exists a set of algebraically independent real numbers $l_J, J = 1..M$ such that every k_i can be expressed as

$k_i = \sum_J m_{iJ} l_J$

where $m_{iJ} \in \mathbf{Z} \ \ \forall i = 1..n, J = 1..M$.

We shall say that the set $l_J, J = 1..M$ spans the set $k_i, i = 1..n$

Let us return to characterization of $h_g \in \Delta$ in terms of group homomorphisms g from $R, +$ to $U(1)$.

Recall, that we need to show that $|h_g(a)| \leq C||a||\forall a \in \mathcal{S}$

We have that $h_g(a) = \sum_i c_i h_g(a_{k_i})$.
Let $l_J, J = 1..M$ be a set of algebraically indep numbers which span $k_i = 1..n$ so that
$k_i = \sum_J m_{iJ} l_J$.

$$\begin{aligned} \Rightarrow h_g(a_{k_i}) &= h_g(\sum_J m_{iJ} l_J) = \prod_J h_g(m_{iJ} l_J) \\ &= \prod_J h_g(l_J + l_J + \ldots + l_J) = \prod_J (h_g(l_J))^{m_{iJ}}) \\ \Rightarrow h_g(a) &= \sum_i c_i \prod_J (h_g(l_J))^{m_{iJ}}) \end{aligned}$$

Next, recall that
$||a|| = \sup_{x\in R} |\sum_i c_i e^{ik_i x}| = \sup_{x\in R} |\sum_i c_i e^{i\sum_J m_{iJ} l_J x}|$

$\Rightarrow ||a|| = \sup_{x\in R} |\sum_i c_i \prod_J (e^{il_J x})^{m_{iJ}}|$

We shall show that

$|h_g(a)| = |\sum_i c_i \prod_J (h_g(l_J))^{m_{iJ}})|$
$\leq ||a|| = sup_{x\in R} |\sum_i c_i \prod_J (e^{il_J x})^{m_{iJ}}|$

To show that:

$|h_g(a)| = |\sum_i c_i \prod_J (h_g(l_J))^{m_{iJ}})|$
$\leq ||a|| = sup_{x \in R} |\sum_i c_i \prod_J (e^{il_J x})^{m_{iJ}}|$:

We note that:

1. Each $h_g(l_J) \in U(1)$ and so describes a point on a circle S^1. Hence together the set $h_g(l_J), J = 1..M$ defines a point in $S^1 \times S^1 .. \times S^1 = (S^1)^M$.

2. The number l_J in $e^{il_J x}$ for each J can be thought of as a 'frequency'. Algebraic independence of these frequencies implies that they are incommensurate.
The set $\{e^{il_J x}, J = 1..M\}$ for each x also gives a point in $S^1 \times S^1 .. \times S^1 = (S^1)^M$.
Because the frequencies are incommensurate one can show that as x varies over R the point traverses a dense set in $(S^1)^M$ i.e. it comes arbitrarily close to every point in $(S^1)^M$.

3. Hence for appropriate value of x, the point in $(S^1)^M$ defined by 1. can be approached arbitrarily closely. This establishes the desired result! Hence $|h_g(a)| \leq ||a|| \; \forall a \in \mathcal{S}$.

Thus elements of the spectrum are in 1-1 correspondence with homomorphisms from the additive group of reals to $U(1)$.

In the case of LQG, one finds a characterization of the spectrum in terms of homomorphisms of a certain group of 'holonomically equivalent loops' to $SU(2)$.

Just as algebraically independent numbers play a key role here, in LQG also there is a well defined notion of independent 'holonomically equivalent loops'.

Just as each set of algebraically indep numbers $l_J, j = 1...M$ can be associated with $(S^1)^M$, each set of M independent 'holonomically equivalent loops' can be associated with $SU(2)^M$. The Haar measure on $SU(2)^M$ can then be used to define the Ashtekar-Lewandowski measure on the Spectrum of the C^* algebra generated by Wilson loop functions.

While it is possible to define the LQC measure in a similar way from Haar measure on $U(1)^M$, we will adopt a different and somewhat quicker method.

5.Measure on Δ:

We will only sketch the argumentation. Details can be discussed in tutorial if there is interest.

Recall that we had defined a PLF ω for LQC on Weyl algebra elements $W(k, b)$. We can restrict this PLF to elements $W(k, b = 0)$. These elements are exactly our a_k so that we have a PLF $\omega(a_k)$.

By linearity we can extend this PLF to all elements of $\mathcal{S}$ i.e. all finite linear combinations $a = \sum_i c_i a_{k_i}$. Using methods similar to previous slides we can show that:

$|\omega(a)| \leq ||a||$

so that ω is a bounded linear map from $\mathcal{S}$ to $\mathbf{C}$.

Since ω is a bounded linear map from $\mathcal{S}$ to $\mathbf{C}$, one can then show that ω extends to all of $\mathcal{C}$ (i.e. to elements in the completion of $\mathcal{S}$.)

Gel'fand's theorem then tells us that ω defines a PLF on the C^* algebra $\mathcal{F}_\Delta$ of continuous functions on the Spectrum.
Since the Spectrum is Compact and Hausdorff, there is a theorem due to Riesz and Markov which implies that there is a measure on Δ such that $\omega(a) = \int_{\vec{x}\in\Delta} f_a(x) d\mu(x)$.

This turns out be exactly the measure for which LQC can be expressed as an $L^2(d\mu, \Delta)$ representation in which the operators corresponding to $a \in \mathcal{C}$ act by multiplication.