

Effective quantum black hole model: note for Loops' 26 summer school

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This is the note for Loops' 26 summer school on effective quantum black hole. This note mainly presents the detailed calculations underlying the following works of mine. For completeness, some paragraphs are adapted from, or directly reproduced from, these works. This note consists of two parts. The first part explains how general covariance constrains the form of the effective Hamiltonian constraint. The second part discusses how to determine the coordinate system when a dust shell is present in the dynamics.

Works covered in this note.

1, **CZ**, J. Lewandowski, Y. Ma, J. Yang, *Black holes and covariance in effective quantum gravity*, Phys. Rev. D 111, L081504 (2025)

2, **CZ**, J. Lewandowski, Y. Ma, J. Yang, *Black holes and covariance in effective quantum gravity: A solution without Cauchy horizons*, Phys. Rev. D 112, 044054 (2025).

3, **CZ**, Z. Cao, *Covariant dynamics from static spherically symmetric geometries*, Phys. Rev. Lett. 135, 261401 (2025).

4, J. Lewandowski, Y. Ma, J. Yang, **CZ**, *Quantum Oppenheimer-Snyder and Swiss Cheese Models*, Phys. Rev. Lett. 130, 101501 (2023).

5, D. Qu, **CZ**, *Consistent Gauge Conditions for Dust-Shell Dynamics in Effective Quantum Gravity*, arXiv:2603.24168.

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I. SPHERICALLY SYMMETRIC QUANTUM BLACK HOLE AND ITS GENERAL COVARIANCE

A. canonical formulation of classical spherically symmetric GR

Consider the spatial manifold Σ satisfying $\Sigma \cong \mathbb{R} \times S^2$ topologically. Let (x, θ, φ) be the coordinate compatible to the topology. Then the spherically symmetric Ashtekar variables (A_a^i, E_i^a) takes the form [R. Gambini, J. Olmedo, and J. Pullin, *CQG* 31, 095009 (2014)]

$$\begin{aligned} A_a^i dx^a \tau_i &= A_1 \tau_3 dx + (A_2 \tau_1 + A_3 \tau_2) d\theta + \sin(\theta)(A_2 \tau_2 - A_3 \tau_1) d\varphi + \cos(\theta) \tau_3 d\varphi, \\ E_i^a \partial_a \tau^i &= E^1 \sin(\theta) \tau_3 \partial_x + (E^2 \tau_1 + E^3 \tau_2) \sin(\theta) \partial_\theta + (E^2 \tau_2 - E^3 \tau_1) \partial_\varphi. \end{aligned} \quad (1.1)$$

In our course, we adopt the convention $\tau_j = -i/2\sigma_j$ with σ_j being the Pauli matrix. These spherically symmetric Ashtekar variables form a sub-space of the entire phase space of GR. The symplectic form on the symmetry-reduced phase space is thus given by the restriction of the complete one, denoted by Ω , on this subspace. Let δ_1 and δ_2 be two vector tangent to the symmetry-reduced phase space at some (A_a^i, E_i^a) given in (1.1)¹. We have

$$\begin{aligned} \Omega(\delta_1, \delta_2) &= \frac{1}{8\pi G \gamma} \int_{\Sigma} d^3\sigma \delta_1 A_a^i(\sigma) \wedge \delta_2 E_i^a(\sigma) \\ &= \frac{1}{2G\gamma} \int_{\mathbb{R}} (\delta_1 A_1(x) \wedge \delta_2 E^1(x) + 2\delta_1 A_2(x) \wedge \delta_2 E^2(x) + 2\delta_1 A_3(x) \wedge \delta_2 E^3(x)) dx \end{aligned} \quad (1.3)$$

where γ is the Immirzi Barbaro parameter, and we defined $\delta_1 F \wedge \delta_2 G = \delta_1 F \delta_2 G - \delta_2 F \delta_1 G$. This result implies the following Poisson bracket

$$\{A_1(x), E^1(y)\} = 2G\gamma\delta(x, y), \quad \{A_2(x), E^2(y)\} = G\gamma\delta(x, y) = \{A_3(x), E^3(y)\}. \quad (1.4)$$

With the Ashtekar variables, we can now calculate the spin connection Γ_a^i and thus the extrinsic curvature 1-form K_a^i . To begin with, we have the triad e_a^i and the metric given by

$$e_a^i = \frac{E_a^i}{\sqrt{\det E}}, \quad q_{ab} = e_a^i \delta_{ij} e_b^j, \quad (1.5)$$

¹ More precisely, a vector on the phase space is defined as follows: Let $p = (\hat{A}_a^i, \hat{E}_j^b)$ be a point on the phase space. Consider a curve $t \mapsto (A_a^i(x; t), E_i^a(x; t))$ on the phase space which passes through p at $t = 0$. We will use δ to denote the tangent vector $\partial/\partial t$ to the curve at p , i.e.,

$$(\delta \triangleright F)(\hat{A}_a^i, \hat{E}_b^j) = \left. \frac{d}{dt} \right|_{t=0} F(A_a^i(\cdot; t), E_j^b(\cdot; t)), \quad (1.2)$$

for any phase space dependent function F . Here $\delta \triangleright F$ denotes the action of the vector δ on the function F , which will map the function to a number according to the definition of a vector [C. Liang, B. Zhou and W. Jia, *Differential Geometry and General Relativity: Volume 1*]

where E_a^i is the inverse of E_i^a and $\det(E)$ is the determinant of E_i^a . We assumed that we have already fixed the orientation so that we only consider the triad with $\det(E) > 0$. We have

$$q_{ab}dx^a dx^b = \frac{(E^2)^2 + (E^3)^2}{E^1} dx^2 + E^1 d\Omega^2 \quad (1.6)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. With the expression of g_{ab} we can calculate the connection Γ_{bc}^a as

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_a g_{bc}). \quad (1.7)$$

Then, Γ_{aj}^i is defined by

$$\partial_a e_i^b + \Gamma_{ad}^b e_i^d = \Gamma_{ai}^j e_j^b \quad (1.8)$$

Then, we have Γ_a^i given by

$$\Gamma_a^i \tau_i = \frac{1}{4} \Gamma_{aj}^k \tau_k \tau^j. \quad (1.9)$$

A straightforward calculation shows

$$\begin{aligned} \Gamma_a^i \tau_i dx^a = & -\frac{E^3 \partial_x E^2 - E^2 \partial_x E^3}{(E^2)^2 + (E^3)^2} \tau_3 dx + \left(\frac{E^3 \partial_x E^1}{2[(E^2)^2 + (E^3)^2]} \tau_1 - \frac{E^2 \partial_x E^1}{2[(E^2)^2 + (E^3)^2]} \tau_2 \right) d\theta \\ & + \sin(\theta) \left(\frac{E^3 \partial_x E^1}{2[(E^2)^2 + (E^3)^2]} \tau_2 + \frac{E^2 \partial_x E^1}{2[(E^2)^2 + (E^3)^2]} \tau_1 \right) d\varphi + \cos(\theta) \tau_3 d\varphi \end{aligned} \quad (1.10)$$

Then, we can obtain

$$K_a^i = \frac{1}{\gamma} (A_a^i - \Gamma_a^i) \quad (1.11)$$

Substituting the result into the full expression of the Gauss constraint, we get

$$\tilde{G}_i = \frac{1}{8\pi G\gamma} (\partial_a E_i^a + \epsilon_{il}^k A_a^l E_k^a) = \frac{\delta_{3i}}{8\pi G\gamma} (\partial_x E^1 + 2A_2 E^3 - 2A_3 E^2) \sin\theta. \quad (1.12)$$

Then, we define \mathcal{G} as

$$\mathcal{G} = \int_{\mathbb{S}^2} \tilde{G}_3 d\theta d\phi = \frac{1}{2G\gamma} (\partial_x E^1 + 2A_2 E^3 - 2A_3 E^2), \quad (1.13)$$

which will be the remaining Gauss constraint in the spherically symmetric model. Now let us investigate the gauge transformation generated by \mathcal{G} . We have to following Poisson bracket

$$\begin{aligned} \{A_1(x), \mathcal{G}[\Lambda]\} &= -\frac{1}{G\gamma} \partial_x \Lambda, \quad \{E^1(x), \mathcal{G}[\Lambda]\} = 0, \\ \{A_2(x) + iA_3(x), \mathcal{G}[\Lambda]\} &= \frac{i\Lambda}{G\gamma} (A_2 + iA_3), \quad \{E^2 + iE^3, \mathcal{G}[\Lambda]\} = \frac{i\Lambda}{G\gamma} (E^2 + iE^3). \end{aligned} \quad (1.14)$$

This results imply the following gauge transformation generated by $\mathcal{G}[\Lambda]$ (Homework)

$$A_1 \rightarrow A_1 - \frac{1}{G\gamma} \partial_x \Lambda, \quad E^1 \rightarrow E^1, \quad A_2 + iA_3 \rightarrow \exp\left(\frac{i\Lambda}{G\gamma}\right) (A_2 + iA_3), \quad E^2 + iE^3 \rightarrow \exp\left(\frac{i\Lambda}{G\gamma}\right) (E^2 + iE^3) \quad (1.15)$$

Here, the definition of the gauge transformation generated by a constraint G is

$$\mathcal{L}_G(F) := e^{\{F, G[\Lambda]\}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\{\dots \{F, G[\Lambda]\}, G[\Lambda]\} \dots}_{n \text{ folds}}, \quad (1.16)$$

where F is any phase space dependent function and λ is any smearing function. The above gauge transformation implies that A_1 performs as the U(1) gauge field, and $A_2 + A_3$ perform as the U(1) complex scalar field. E^1 and $E^2 + iE^3$, as their conjugate momentum, are transformed accordingly.

In loop quantum BH model, at this stage, we usually solve the Gauss constraint classically by introducing a gauge fixing condition. Here we impose the condition

$$E^3 = 0 \quad (1.17)$$

so that the Gauss constraint $\mathcal{G} = 0$ lead to

$$A_3 = \frac{1}{2} \frac{\partial_x E^1}{E^2}. \quad (1.18)$$

Substituting these results into (1.10) and (1.11), we finally get

$$K_a^i dx^a \tau_i = \frac{1}{\gamma} (A_1 \tau_3 dx + A_2 \tau_1 d\theta + \sin(\theta) A_2 \tau_2 d\varphi) \quad (1.19)$$

Moreover, the triad E becomes

$$E_i^a \partial_a \tau^i = E^1 \sin(\theta) \tau_3 \partial_x + E^2 \tau_1 \sin(\theta) \partial_\theta + E^2 \tau_2 \partial_\varphi. \quad (1.20)$$

Let us define

$$K_I = \frac{1}{\gamma} A_I, \forall I = 1, 2. \quad (1.21)$$

We then get

$$\{K_1(x), E^1(y)\} = 2G\delta(x, y), \quad \{K_2(x), E^2(y)\} = G\delta(x, y). \quad (1.22)$$

In what follows, rather than the variables A , we prefer to using K as our canonical variables.

B. The dynamics and classical constraint algebra

Substituting the expressions (1.19) and (1.20) into the full expression of the diffeomorphism and Hamiltonian constraint, we get the diffeomorphism constraint as

$$H_x(x) = \frac{1}{2G} (-K_1(x) \partial_x E^1(x) + 2E^2(x) \partial_x K_2(x)), \quad (1.23)$$

and the Hamiltonian constraint as

$$H(x) = \frac{1}{2G\sqrt{E^1(x)}} \left(-E^2(x) - 2E^1(x)K_1(x)K_2(x) - E^2(x)K_2(x)^2 - \frac{3\partial_x E^1(x)^2}{4E^2(x)} + \partial_x \left(\frac{E^1(x)\partial_x E^1(x)}{E^2(x)} \right) \right). \quad (1.24)$$

The constraint algebra is

$$\begin{aligned} \{H_x[N_1^x], H_x[N_2^x]\} &= H_x[N_1^x \partial_x N_2^x - N_2^x \partial_x N_1^x], \\ \{H[N], H_x[M^x]\} &= -H[M^x \partial_x N], \\ \{H[N_1], H[N_2]\} &= H_x[E^1(E^2)^{-2}(N_1 \partial_x N_2 - N_2 \partial_x N_1)]. \end{aligned} \quad (1.25)$$

1. Gauge transformation by the $H_x[N^x]$

To check the gauge transformation generated by the $H_x[N^x]$, we calculate the Poisson bracket as follows,

$$\{E^1(x), H_x[N^x]\} = N^x(x) \partial_x E^1(x) \quad (1.26)$$

Let

$$t \mapsto (E^1(x, t), K_1(x, t), E^2(x, t), K_2(x, t)) \quad (1.27)$$

be the gauge transformation orbit generated by $H_x[N^x]$, passing through the initial point

$$(\mathring{E}^1(x), \mathring{K}_1(x), \mathring{E}^2(x), \mathring{K}_2(x)) \quad (1.28)$$

in the phase space. We have the equation

$$\partial_t E^1(x, t) = N^x(x) \partial_x E^1(x, t), \quad E^1(x, 0) = E_o^1(x). \quad (1.29)$$

whose solution is

$$E^1(x, t) = C \left(t + \int_{x_o}^x \frac{1}{N^x(y)} dy \right) \quad (1.30)$$

for a general function C on \mathbb{R} . The initial condition needs us to have

$$C \left(\int_{x_o}^x \frac{1}{N^x(y)} dy \right) = \mathring{E}^1(x), \quad (1.31)$$

leading to

$$C = \mathring{E}^1 \circ X^{-1}, \quad \text{with } X(x) = \int_{x_o}^x \frac{1}{N^x(y)} dy. \quad (1.32)$$

By definition of $X(x)$, its inverse function satisfies

$$\frac{dX^{-1}(t)}{dt} = \frac{1}{X'(X^{-1}(t))} = N^x(X^{-1}(t)). \quad (1.33)$$

Thus $t \rightarrow X^{-1}(t)$ is just the integration curve of the vector field $N^x \partial_x$. Moreover, we have

$$X(x_o) = 0 \Rightarrow X^{-1}(0) = x_o \Rightarrow \text{the curve } X^{-1} \text{ passes through } x_o \text{ at } t = 0. \quad (1.34)$$

Then, we get

$$E^1(x, t) = \mathring{E}^1 \circ X^{-1} \left(t + \int_{x_o}^x \frac{1}{N^x(y)} dy \right) \quad (1.35)$$

where by (1.34),

$$t \rightarrow X^{-1} \left(t + \int_{x_o}^x \frac{1}{N^x(y)} dy \right) \text{ is the curve passing through } x \text{ at } t = 0.$$

Therefore,

$$E^1(x, t) = [(\phi_t^{(x)})^* \mathring{E}^1](x) \quad (1.36)$$

where $\phi_t^{(x)}$ denotes the diffeomorphism transformation generated by the vector field $N^x \partial_x$, i.e.,

$$\phi_t^{(x)}(x) = X^{-1} \left(t + \int_{x_o}^x \frac{1}{N^x(y)} dy \right). \quad (1.37)$$

Computing the Poisson bracket between K_1 and $H_x[N^x]$, we get

$$\{K_1(x), H_x[N^x]\} = N^x(x) \partial_x K_1(x) + K_1(x) \partial_x N^x. \quad (1.38)$$

leading to

$$\partial_t K_1(x, t) = N^x(x) \partial_x K_1(x, t) + K_1(x, t) \partial_x N^x. \quad (1.39)$$

Its solution reads

$$K_1(x, t) = \frac{1}{N^x(x)} C \left(t + \int_{x_o}^x \frac{1}{N^x(y)} dy \right), \quad (1.40)$$

for some function C on \mathbb{R} . The initial condition $K_1(x, 0) = \mathring{K}_1(x)$ implies

$$C = (\mathring{K}_1 \circ X^{-1}) \times (N^x \circ X^{-1}), \quad (1.41)$$

i.e.

$$K_1(x, t) = \frac{N^x(\phi_t^{(x)}(x))}{N^x(x)} \mathring{K}_1(\phi_t^{(x)}(x)). \quad (1.42)$$

To see what is the meaning of $\frac{N^x(\phi_t^{(x)}(x))}{N^x(x)}$, let us consider the coordinate transformation

$$x \mapsto \phi_t^{(x)}(x) = X^{-1}(t + \int_{x_o}^x \frac{1}{N^x(y)} dy). \quad (1.43)$$

Then, we have

$$\frac{d\phi_t^{(x)}(x)}{dx} = (X^{-1})'(t + \int_{x_o}^x \frac{1}{N^x(y)} dy) \frac{1}{N^x(x)} = \frac{1}{X'(X^{-1}(t + \int_{x_o}^x \frac{1}{N^x(y)} dy))} \frac{1}{N^x(x)} = \frac{N^x(\phi_t^{(x)}(x))}{N^x(x)} \quad (1.44)$$

Therefore, we get

$$K_1(x, t) = \frac{d\phi_t^{(x)}(x)}{dx} \mathring{K}_1(\phi_t^{(x)}(x)). \quad (1.45)$$

If we regard $K_1(x, t)$ for each fix t as a one-form $K_1(x, t)dx$, the above equation tells us

$$K_1(x, t)dx = ((\phi_t^{(x)})^*[\mathring{K}_1 dx])(x). \quad (1.46)$$

Similarly, we get that

$$E^2(x, t)dx = ((\phi_t^{(x)})^*[\mathring{E}^2 dx])(x), \quad K_2(x, t) = ((\phi_t^{(x)})^* \mathring{K}_2)(x). \quad (1.47)$$

Indeed, if we treat N^x as the vector field $N^x \partial_x$, K_1 as $K_1(x)dx$ and $E^2(x)$ as $E^2(x)dx$, we get

$$\begin{aligned} \{K_1(x)dx, H_x[N^x]\} &= \mathcal{L}_{N^x \partial_x}(K_1 dx)(x), \\ \{E^1(x), H_x[N^x]\} &= \mathcal{L}_{N^x \partial_x}(E^1)(x), \\ \{K_2(x), H_x[N^x]\} &= \mathcal{L}_{N^x \partial_x}(K_2)(x), \\ \{E^2(x)dx, H_x[N^x]\} &= \mathcal{L}_{N^x \partial_x}(E^2 dx)(x). \end{aligned} \quad (1.48)$$

which explain the push-forwards in the gauge transformation generated by $H_x[N^x]$.

C. construct spacetime from a physical state

Let us return to the full theory where the phase space contains (q_{ab}, P^{ab}) to illustrate how to construct the spacetime from the Hamiltonian theory.

Solving the diffeomorphism and Hamiltonian constraints, we get the space of physical states Γ_{phy} . Each element Γ_{phy} is an equivalence class with the equivalence relation given by the gauge transformation generated by the constraints. Given a state $[\mathring{q}_{ab}, \mathring{P}^{cd}] \in \Gamma_{\text{phy}}$, the corresponding spacetime (M, g_{ab}) can be constructed as follows.

Choose a lapse function N and a shift vector N^a , with which we have a Hamiltonian

$$\mathbf{H} = H[N] + \vec{H}[\vec{N}]. \quad (1.49)$$

Taking \mathring{q} and \mathring{P} as the initial data, we solve the Hamilton equation

$$\dot{q}_{ab} = \{q_{ab}, \mathbf{H}\}, \quad \dot{P}^{ab} = \{P^{ab}, \mathbf{H}\}, \quad q_{ab}(t=0, x) = \mathring{q}_{ab}(x), \quad P^{ab}(t=0, x) = \mathring{P}^{ab}(x), \quad (1.50)$$

to get the solve $q_{ab}(t, x)$ and $P^{ab}(t, x)$.

We now consider a 4-manifold $M \cong \mathbb{R} \times \Sigma$, with the diffeomorphism

$$\Psi : \mathbb{R} \times \Sigma \rightarrow M.$$

Applying $\Psi(t, \cdot) \equiv \Psi_t : \Sigma \rightarrow M$, we push forward $q_{ab}(t), N^a(t), N(t)$ from Σ to the slice $\Psi_t[\Sigma] \subset M$. Note that N^a and N could depend on t because one can choose them to be phase space dependent. *Since the push forward $\Psi_{t*}q_{ab}$ at this moment can only act on vector tangent to $\Psi_t[\Sigma]$, it is not a 4-dimensional tensor field on M .* To promote $\Psi_{t*}q_{ab}$ to an 4-D tensor field, we need to fix once for all a vector fields ∂_t on M satisfying

$$(\partial_t)^a(dt)_a = 1. \quad (1.51)$$

With ∂_t we define

$$\Psi_t^*q_{ab}(\partial_t)^a = (\Psi_t^*q_{ab})(\Psi_t^*N^a). \quad (1.52)$$

Then, we can define

$$g_{ab} = -N^2(dt)_a(dt)_b + 2q_{c(a}N^c(dt)_{b)} + q_{ab} \quad (1.53)$$

as the 4-D metric on M , where N, N^c and q_{ab} are abbreviations for $\Psi_t^*N, \Psi_t^*N^c$ and $\Psi_t^*q_{ab}$, and q_{ab} has been promoted as a 4-dimensional tensor.

D. Spacetime construction in the classical model

With the procedure introduce in the above section Sec. IC, we get the 4-D metric in the BH model as

$$ds^2 = -N^2dt^2 + \frac{(E^2)^2}{E^1}(dx + N^x dt)^2 + E^1 d\Omega^2. \quad (1.54)$$

We choose a lapse function N and a shift vector $N^x\partial_x$, with which we have a Hamiltonian

$$\mathbf{H} = H[N] + \vec{H}[\vec{N}]. \quad (1.55)$$

Taking $(\hat{K}_1, \hat{E}_1, \hat{K}_2, \hat{E}^2) \in \Gamma_{\text{phy}}$ as the initial data, we solve the Hamilton equation

$$\partial_t K_I = \{K_I, \mathbf{H}\}, \quad \partial_t E^J = \{E^J, \mathbf{H}\}, \quad K_I(t=0, x) = \hat{K}_I(x), \quad E^J(t=0, x) = \hat{E}^J(x), \quad (1.56)$$

to get the solution of $K_I(t, x)$ and $E^J(t, x)$, with which we can rewrite down the metric (1.54) on the 4-manifold M . The Hamilton equation leads to

$$\begin{aligned} \partial_t E^1 &= \{E^1, H[N] + H_x[N^x]\} = 2N\sqrt{E^1}K_2 + N^x\partial_x E^1, \\ \partial_t E^2 &= \{E^2, H[N] + H_x[N^x]\} = N\sqrt{E^1}K_1 + \frac{NE^2K_2}{\sqrt{E^1}} + N^x\partial_x E^2 + \partial_x N^x E^2, \\ \partial_t K_1 &= \{K_1, H[N] + H_x[N^x]\} = \frac{1}{2\sqrt{E^1}}\partial_x \left(\frac{N\partial_x E^1}{E^2} \right) + \sqrt{E^1}\partial_x \left(\frac{\partial_x N}{E^2} \right) \\ &\quad + \frac{N}{\sqrt{E^1}} \left(-\frac{(\partial_x E^1)^2}{8E^1 E^2} + \frac{E^2(K_2)^2}{2E^1} + \frac{E^2}{2E^1} - K_1 K_2 \right) + N^x\partial_x K_1 + K_1\partial_x N^x \\ \partial_t K_2 &= \{K_2, H[N] + H_x[N^x]\} = \frac{\partial_x N\sqrt{E^1}\partial_x E^1}{2(E^2)^2} + \frac{N}{2\sqrt{E^1}} \left(\frac{(\partial_x E^1)^2}{4(E^2)^2} - (K_2)^2 - 1 \right) + N^x\partial_x K_2 \end{aligned} \quad (1.57)$$

Now let us treat E^I and K_I as 4-dimensional object. According these equations, if we treat the fields $E^1(t, x)$ and $K_2(t, x)$ as scalar field on M , but treat the fields $E^2(t, x)$ and $K_1(t, x)$ as two forms $E^2 dx \wedge dt$ and $K_1 dx \wedge dt$ on M ,

we could rewrite the above equations as

$$\begin{aligned}
\mathcal{L}_{\mathfrak{N}}E^1 &= 2N\sqrt{E^1}K_2, \\
\mathcal{L}_{\mathfrak{N}}(E^2 dx \wedge dt) &= N \left(\sqrt{E^1}K_1 + \frac{E^2K_2}{\sqrt{E^1}} \right) dx \wedge dt, \\
\mathcal{L}_{\mathfrak{N}}(K_1 dx \wedge dt) &= \frac{1}{2\sqrt{E^1}}\partial_x \left(\frac{N\partial_x E^1}{E^2} \right) + \sqrt{E^1}\partial_x \left(\frac{\partial_x N}{E^2} \right) \\
&\quad + \frac{N}{\sqrt{E^1}} \left(-\frac{(\partial_x E^1)^2}{8E^1E^2} + \frac{E^2(K_2)^2}{2E^1} + \frac{E^2}{2E^1} - K_1K_2 \right) dx \wedge dt \\
\mathcal{L}_{\mathfrak{N}}K_2 &= \left[\frac{\partial_x N \sqrt{E^1} \partial_x E^1}{2(E^2)^2} + \frac{N}{2\sqrt{E^1}} \left(\frac{(\partial_x E^1)^2}{4(E^2)^2} - (K_2)^2 - 1 \right) \right]
\end{aligned} \tag{1.58}$$

where

$$\mathfrak{N}^a = (\partial_t)^a - N^x (\partial_x)^a \tag{1.59}$$

is a vector field we introduced on M . Here, we used the fact

$$\begin{aligned}
\mathcal{L}_v f &= v^a \partial_a f, \\
\mathcal{L}_v(g(dt)_a \wedge (dx)_b) &= (v^c \partial_c g)(dt)_a \wedge (dx)_b + g(dt)_a \partial_b v^c + g(dx)_b \partial_a v^c - a \leftrightarrow b \\
&= (v^c \partial_c g)(dt)_a \wedge (dx)_b + g(\partial_c v^c)(dt)_a \wedge (dx)_b
\end{aligned} \tag{1.60}$$

We now solve the dynamics concretely. Precisely, we need to solve the constraint equation and (1.58) with a fixed lapse function and shift vector. However, in practice, we usually do not do like that. The point is that, since N and N^x are freely chosen, we could get any $E^I(t, x)$ we want by suitable choosing N and N^x . Thus, we could solve the dynamics by specifying for instance $E^1 = x^2$ and $E^2 = x$, which will be called the gauge condition. Then, we must choose N and N^x such that they satisfy the EOMs, i.e.,

$$\begin{aligned}
0 = \dot{E}^1 &= \{E^1, H[N] + H_x[N^x]\} = N^x \partial_x E^1 + 2N\sqrt{E^1}K_2, \\
0 = \dot{E}^2 &= \{E^2, H[N] + H_x[N^x]\} = \partial_x(N^x E^2) + \frac{NE^2K_2}{\sqrt{E^1}} + N\sqrt{E^1}K_1.
\end{aligned} \tag{1.61}$$

To solve these two equations for N and N^x , we have to know K_1 and K_2 . We choose the constraints to solve them. Substituting the gauge condition, we have

$$\begin{aligned}
-xK_1 + x\partial_x K_2 &= 0 \\
2xK_1K_2 + (K_2)^2 &= 0
\end{aligned} \tag{1.62}$$

leading to

$$0 = 2x\partial_x K_2 K_2 + (K_2)^2 = \partial_x(x(K_2)^2) \Rightarrow K_2 = \frac{c}{\sqrt{x}}, \quad K_1 = -\frac{c}{2x^{3/2}}. \tag{1.63}$$

Substituting the gauge condition and the results of K_I into (1.61), we get a differential equation for N and N^x , which finally leads to

$$N = c', \quad N^x = -\frac{cc'}{\sqrt{x}}. \tag{1.64}$$

Substituting the result into the definition of metric, we get

$$ds^2 = -(c')^2 dt^2 + \left(dx - \frac{cc'}{\sqrt{x}} dt \right)^2 + x^2 d\Omega^2. \tag{1.65}$$

c' is a rescaling of the time coordinate and thus can be chosen as $c' = 1$. Comparing it with the Schwarzschild metric in PG coordinates, we get that $c = \pm\sqrt{2M}$ so that the metric becomes

$$ds^2 = -dt^2 + \left(dx \pm \sqrt{\frac{2M}{x}} dt \right)^2 + x^2 d\Omega^2. \tag{1.66}$$

Another gauge can be chose such that $E^1 = x^2$ and $K_2 = 0$. Under this gauge condition, we can solve the constraints to get

$$K_1 = 0, \quad 0 = (E^2)^2 \left(\frac{2x^3 \partial_x E^2}{(E^2)^3} + 1 - \frac{3x^2}{(E^2)^2} \right) = (E^2)^2 \partial_x \left(x - \frac{x^3}{(E^2)^2} \right) \quad (1.67)$$

leading to

$$E^2 = \pm \frac{x^{3/2}}{\sqrt{x+c}} \quad (1.68)$$

Substituting all the above results into the evolution equation of E^1 and K_2 , we have

$$N^x = 0, \quad \partial_x N + \frac{c \partial_x N}{x} + \frac{cN}{2x^2} = 0 \Rightarrow N = \frac{c' \sqrt{x+c}}{\sqrt{x}}. \quad (1.69)$$

Substituting the results into the expression of the metric, we get

$$ds^2 = - \left(1 + \frac{c}{x} \right) (c')^2 dt^2 + \left(1 + \frac{c}{x} \right)^{-1} dx^2 + x^2 d\Omega^2. \quad (1.70)$$

We again choose $c' = 1$ and choose $c = 2M$. Then we obtain the standard Schwarzschild metric

$$ds^2 = - \left(1 + \frac{2M}{x} \right) dt^2 + \left(1 + \frac{2M}{x} \right)^{-1} dx^2 + x^2 d\Omega^2. \quad (1.71)$$

According to the above discussion, we choose two different gauges, but get the same metric up to diffeomorphism transformation. This result shows that the two different choices of gauge can get the metric up to a diffeomorphism transformation. Then, a nature question results, is this generally true? Equivalently, *does the metric (1.54) depend on the choice of N and N^x or not? The theory is general covariance if the answer is negative.*

To examine this issue, one direct strategy is to choose another lapse function and shift vector, denoted by \tilde{N} and \tilde{N}^x , solve the equations of motion again to get the variation of K_I and E^I , and then show that the resulting metric \tilde{g} is related to the original metric g by a coordinate transformation. However, in practice, we adopt an alternative approach. We first prescribe variations of K_I and E^I , and then determine the corresponding lapse and shift, \tilde{N} and \tilde{N}^x . The reason is that any variation of K_I and E^I must be generated by the constraints.

E. Variation of K_I , E^I , N and N^x

To investigate the general covariance of the model, we do not restrict ourselves to the classical theory. Instead, we consider a more general model whose phase space is still coordinatized by K_I and E^I , as in the classical case. Moreover, the diffeomorphism constraint is kept in its classical form (1.23), since it generates spatial diffeomorphisms. The generalization is introduced through the Hamiltonian constraint. We assume that the Hamiltonian constraint is no longer given by its classical expression, but is replaced by an effective constraint, denoted by H_{eff} . The constraint algebra becomes

$$\begin{aligned} \{H_x[N_1^x], H_x[N_2^x]\} &= H_x[N_1^x \partial_x N_2^x - N_1^x \partial_x N_2^x], \\ \{H_{\text{eff}}[N], H_x[M^x]\} &= -H_{\text{eff}}[M^x \partial_x N], \\ \{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} &= -H_x[\mu E^1 (E^2)^{-2} (N_2 \partial_x N_1 - N_1 \partial_x N_2)]. \end{aligned} \quad (1.72)$$

with the modification factor $\mu(K_I, E^I)$ which is a general function of K_I and E^I .

This is still a first class constraint system, so we can solve the constraints with the procedure introduced above to get the physical phase space Γ_{phy} . Then, given a physical state $[\overset{\circ}{K}_I, \overset{\circ}{E}^I] \in \Gamma_{\text{phy}}$, we solve the Hamilton equations

$$\begin{aligned} \dot{\overset{\circ}{K}}_I &= \{\overset{\circ}{K}_I, H_{\text{eff}}[N]\} + \{\overset{\circ}{K}_I, H_x[N^x]\} \\ \dot{\overset{\circ}{E}}^I &= \{\overset{\circ}{E}^I, H_{\text{eff}}[N]\} + \{\overset{\circ}{E}^I, H_x[N^x]\} \end{aligned} \quad (1.73)$$

with taking $(\overset{\circ}{K}_I, \overset{\circ}{E}^I)$ as the initial data. Then, we push-forward $(K_I(t), E^I(t), N[t], N^x[t])$ from Σ to $\Psi_t[\Sigma] \subset M$ as did in Sec. I C. Treating K_I, E^I, N^x and N as 4-D fields on M , as did in (1.58), we can rewrite the Hamilton equation

(1.73) as

$$\begin{aligned}
\mathcal{L}_{\mathfrak{N}}E^1 &= \{E^1, H_{\text{eff}}[N]\}, \\
\mathcal{L}_{\mathfrak{N}}(E^2 dx \wedge dt) &= \{E^2, H_{\text{eff}}[N]\}dx \wedge dt, \\
\mathcal{L}_{\mathfrak{N}}(K_1 dx \wedge dt) &= \{K_1, H_{\text{eff}}[N]\}dx \wedge dt, \\
\mathcal{L}_{\mathfrak{N}}K_2 &= \{K_2, H_{\text{eff}}[N]\}.
\end{aligned} \tag{1.74}$$

Here the EOM can still be written in terms of the Lie derivative since we did not change the diffeomorphism constraint.

Let α be a scalar and β^x be a vector. As mentioned before, a general variation of K_I and E^I must be generated by the constraints. We thus have

$$\delta K_I = \{K_I, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}, \quad \delta E^I = \{E^I, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}. \tag{1.75}$$

Here we add an addition factor N in H_{eff} for further convenience. After transformation, $X \equiv E^I$, K_I becomes $X + \epsilon \delta X$ with δX given above. As the variation generated by the constraint, they still satisfies the constraints. Then, if they are still solutions to the EOMs (1.74), with respect to some new lapse function and shift vector, we should be

$$\mathcal{L}_{\mathfrak{N} + \epsilon \delta \mathfrak{N}}(X + \epsilon \delta X) = \{X, H[N]\} \Big|_{\substack{K_I \rightarrow K_I + \epsilon \delta K_I \\ E^I \rightarrow E^I + \epsilon \delta E^I \\ N \rightarrow N + \epsilon \delta N}} + O(\epsilon^2). \tag{1.76}$$

For the left hand side, it can be expand easily. Let us see how to calculate the right hand side. Since we are only concerned with the terms up to $O(\epsilon^2)$, we have

$$\begin{aligned}
\{X, H_{\text{eff}}[N]\} \Big|_{\substack{K_I \rightarrow K_I + \epsilon \delta K_I \\ E^I \rightarrow E^I + \epsilon \delta E^I \\ N \rightarrow N + \epsilon \delta N}} &= \{X, H_{\text{eff}}[N]\} \Big|_{\substack{K_I \rightarrow K_I \\ E^I \rightarrow E^I \\ N \rightarrow N}} + \epsilon \{X, H_{\text{eff}}[\delta N]\} \Big|_{\substack{K_I \rightarrow K_I \\ E^I \rightarrow E^I}} \\
&+ \epsilon \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left(\{X, H_{\text{eff}}[N]\} \Big|_{\substack{K_I \rightarrow K_I + \epsilon \delta K_I \\ E^I \rightarrow E^I + \epsilon \delta E^I \\ N \rightarrow N}} \right) + O(\epsilon^2).
\end{aligned} \tag{1.77}$$

For the last term, we have

$$\begin{aligned}
&\frac{d}{d\epsilon} \Big|_{\epsilon=0} \left(\{X, H_{\text{eff}}[N]\} \Big|_{\substack{K_I \rightarrow K_I + \epsilon \delta K_I \\ E^I \rightarrow E^I + \epsilon \delta E^I \\ N \rightarrow N}} \right) \\
&= \sum_{O \in \{K_I, E^I\}} \frac{\partial}{\partial O} \{X, H_{\text{eff}}[N]\} \delta O \\
&= \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}.
\end{aligned} \tag{1.78}$$

where (1.75) is considered. Combining all the results, (1.76) can be simplified up to $O(\epsilon^2)$ as

$$\mathcal{L}_{\delta \mathfrak{N}}X + \mathcal{L}_{\mathfrak{N}}\delta X = \{X, H_{\text{eff}}[\delta N]\} + \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}. \tag{1.79}$$

namely,

$$\begin{aligned}
\mathcal{L}_{\delta \mathfrak{N}}E^I + \mathcal{L}_{\mathfrak{N}}\delta E^I &= \{E^I, H_{\text{eff}}[\delta N]\} + \{\{E^I, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}, \\
\mathcal{L}_{\delta \mathfrak{N}}K_I + \mathcal{L}_{\mathfrak{N}}\delta K_I &= \{K_I, H_{\text{eff}}[\delta N]\} + \{\{K_I, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}.
\end{aligned} \tag{1.80}$$

For $X \equiv E^I$, K_I , we have

$$\begin{aligned}
\mathcal{L}_{\mathfrak{N}}\delta X &= \mathcal{L}_{\mathfrak{N}}\{X, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\} \\
&= \{\{X, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}, H_{\text{eff}}[N]\} + \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N) - \{\alpha N, H_{\text{eff}}[N]\}]\} \\
&\quad + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x - \{\beta^x, H_{\text{eff}}[N]\}]\} \\
&= -\{\{H_{\text{eff}}[\alpha N] + H_x[\beta^x], H_{\text{eff}}[N]\}, X\} + \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\} \\
&\quad + \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N) - \{\alpha N, H_{\text{eff}}[N]\}]\} + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x - \{\beta^x, H_{\text{eff}}[N]\}]\}
\end{aligned} \tag{1.81}$$

where we consider the general case with α and β being phase space-dependent. In the second equality, $\{\{X, H_{\text{eff}}[\alpha N] + H_x[\beta^x], H_{\text{eff}}[N]\}$ contains the Lie derivatives for all phase space variables, and the other terms include the Lie derivatives for those phase-space-independent variable. Since α and β can be in general phase space dependent, we get

$$\begin{aligned} & \{H_{\text{eff}}[\alpha N] + H_x[\beta^x], H_{\text{eff}}[N]\} \\ &= -H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha] + H_{\text{eff}}[\{\alpha N, H_{\text{eff}}[N]\}] + H_{\text{eff}}[\beta^x\partial_x N] + H_x[\{\beta^x, H_{\text{eff}}[N]\}] \end{aligned} \quad (1.82)$$

Thus, we get

$$\begin{aligned} \mathcal{L}_{\mathfrak{N}}\delta X &= \{H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha], X\} - \{H_{\text{eff}}[\{\alpha N, H_{\text{eff}}[N]\}], X\} - \{H_{\text{eff}}[\beta^x\partial_x N], X\} - \{H_x[\{\beta^x, H_{\text{eff}}[N]\}], X\} \\ &+ \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\} + \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N) - \{\alpha N, H_{\text{eff}}[N]\}]\} \\ &+ \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x - \{\beta^x, H_{\text{eff}}[N]\}]\} \\ &= \{H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha], X\} - \{H_{\text{eff}}[\beta^x\partial_x N], X\} + \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\} \\ &+ \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N)]\} + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x]\}. \end{aligned} \quad (1.83)$$

With this result, (1.80) can be simplified as

$$\begin{aligned} \mathcal{L}_{\delta\mathfrak{N}}X &+ \{H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha], X\} - \{H_{\text{eff}}[\beta^x\partial_x N], X\} + \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\} \\ &+ \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N)]\} + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x]\} = \{X, H_{\text{eff}}[\delta N]\} + \{\{X, H_{\text{eff}}[N]\}, H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}, \end{aligned} \quad (1.84)$$

i.e.

$$\begin{aligned} & \mathcal{L}_{\delta\mathfrak{N}}X + \{H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha], X\} - \{H_{\text{eff}}[\beta^x\partial_x N], X\} \\ &+ \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N)]\} + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x]\} = \{X, H_{\text{eff}}[\delta N]\}. \end{aligned} \quad (1.85)$$

Since $\mathfrak{N}^a = (\partial_t)^a - N^x(\partial_x)^a$, we get $\delta\mathfrak{N}^a(dt)_a = 0$, leading to

$$\mathcal{L}_{\delta\mathfrak{N}}X = \{X, H_x[\delta\mathfrak{N}^x]\} - H_x[\{X, \delta\mathfrak{N}^x\}] \quad (1.86)$$

Thus (1.85) can be simplified as

$$\begin{aligned} & \{X, H_x[\delta\mathfrak{N}^x]\} - H_x[\{X, \delta\mathfrak{N}^x\}] + \{H_x[\mu E^1(E^2)^{-2}N^2\partial_x\alpha], X\} - \{H_{\text{eff}}[\beta^x\partial_x N], X\} \\ &+ \{X, H_{\text{eff}}[\mathcal{L}_{\mathfrak{N}}(\alpha N)]\} + \{X, H_x[(\mathcal{L}_{\mathfrak{N}}\beta)^x]\} = \{X, H_{\text{eff}}[\delta N]\}. \end{aligned} \quad (1.87)$$

i.e.,

$$\begin{aligned} & -H_x[\{X, \delta\mathfrak{N}^x\}] + \left\{X, H_x\left[\delta\mathfrak{N}^x - \mu E^1(E^2)^{-2}N^2\partial_x\alpha + (\mathcal{L}_{\mathfrak{N}}\beta)^x\right]\right\} \\ &= \left\{X, H_{\text{eff}}\left[\delta N - \beta^x\partial_x N - \mathcal{L}_{\mathfrak{N}}(\alpha N)\right]\right\}. \end{aligned} \quad (1.88)$$

This equation lead to

$$\begin{aligned} \delta\mathfrak{N}^x &= N^2\mu E^1(E^2)^{-2}\partial_x\alpha - (\mathcal{L}_{\mathfrak{N}}\beta)^x \\ \delta N &= \beta^x\partial_x N + \mathcal{L}_{\mathfrak{N}}(\alpha N) = \mathcal{L}_{\alpha\mathfrak{N}+\beta}N + N\mathfrak{N}^a\partial_a\alpha, \end{aligned} \quad (1.89)$$

where we used $\beta = \beta^x\partial_x$

F. general covariance of the model

Now let us calculate the variation of the metric g . Since H_{eff} could depend on $\partial_x^n K_1$, we have

$$\begin{aligned}
\{E^1(x), H_{\text{eff}}[\alpha N]\} &= -\alpha(x)N(x)\frac{\partial H_{\text{eff}}(x)}{\partial K_1(x)} - \sum_n (-1)^n \frac{\partial^n}{\partial x^n} \left(\alpha(x)N(x)\frac{\partial H_{\text{eff}}(x)}{\partial(\partial_x^n K_1(x))} \right) \\
&= -\alpha(x)N(x)\frac{\partial H_{\text{eff}}(x)}{\partial K_1(x)} - \alpha(x)\sum_n (-1)^n \frac{\partial^n}{\partial x^n} \left(N(x)\frac{\partial H_{\text{eff}}(x)}{\partial(\partial_x^n K_1(x))} \right) \\
&\quad - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x)\frac{\partial H_{\text{eff}}(x)}{\partial(\partial_x^n K_1(x))} \right) \right] \\
&= \alpha(x)\{E^1(x), H_{\text{eff}}[N]\} - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x)\frac{\partial H_{\text{eff}}(x)}{\partial(\partial_x^n K_1(x))} \right) \right].
\end{aligned} \tag{1.90}$$

Then, we have

$$\delta E^1 \approx \mathcal{L}_{\alpha\mathfrak{N}+\beta} E^1 - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \frac{\delta H_{\text{eff}}[N]}{\delta(\partial_x^n K_1(x))} \right] \tag{1.91}$$

where \approx means ‘‘equal on the constraint surface’’.

Moreover, we have for E^2

$$\{E^2(x), H_{\text{eff}}[\alpha N]\} = \alpha(x)\{E^2(x), H_{\text{eff}}[N]\} + \Delta \tag{1.92}$$

where we define Δ as

$$\Delta = \{E^2(x), H_{\text{eff}}[\alpha N]\} - \alpha(x)\{E^2(x), H_{\text{eff}}[N]\} = - \sum_{n \geq 1} (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x)\frac{\partial H_{\text{eff}}(x)}{\partial(\partial_x^n K_2(x))} \right) \right] \tag{1.93}$$

Therefore, we have

$$\delta E^2 = \mathcal{L}_{\alpha\mathfrak{N}+\beta} E^2 - E^2 \mathfrak{N}^a \partial_a \alpha + \Delta. \tag{1.94}$$

With this result, we have

$$\delta(E^2(dx)_a \wedge (dt)_b \mathfrak{N}^b) = \mathcal{L}_{\alpha\mathfrak{N}+\beta} [E^2(dx)_a \wedge (dt)_b \mathfrak{N}^b] - N^2 \mu E^1 (E^2)^{-1} \partial_x \alpha (dt)_a + \Delta(dx)_a \wedge (dt)_b \mathfrak{N}^b \tag{1.95}$$

Let us define Q_{ab} as

$$Q_{ab} = (E^2(dx)_a \wedge (dt)_c \mathfrak{N}^c) (E^2(dx)_b \wedge (dt)_d \mathfrak{N}^d). \tag{1.96}$$

We get

$$\begin{aligned}
\delta Q_{ab} &= \mathcal{L}_{\alpha\mathfrak{N}+\beta} Q_{ab} - 2N^2 \mu E^1 \partial_x \alpha (dt)_{(a} [(dx)_{b)} \wedge (dt)_d \mathfrak{N}^d] \\
&\quad + 2E^2 \Delta [(dx)_a \wedge (dt)_c \mathfrak{N}^c] [(dx)_b \wedge (dt)_d \mathfrak{N}^d] \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} Q_{ab} - 2N^2 \mu E^1 \partial_x \alpha (dt)_{(a} [(dx)_{b)} \wedge (dt)_d \mathfrak{N}^d] + \frac{2\Delta}{E^2} Q_{ab}
\end{aligned} \tag{1.97}$$

Moreover, according to (1.89), we get

$$\begin{aligned}
\delta N^2 (dt)_a (dt)_b &= 2N \delta N (dt)_a (dt)_b = (\mathcal{L}_{\alpha\mathfrak{N}+\beta} N^2) (dt)_a (dt)_b + 2N^2 \mathfrak{N}^a \partial_a \alpha (dt)_a (dt)_b \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - N^2 \mathcal{L}_{\alpha\mathfrak{N}+\beta} [(dt)_a (dt)_b] + 2N^2 \mathfrak{N}^a \partial_a \alpha (dt)_a (dt)_b \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - 2N^2 (dt)_{(a} \partial_{b)} \alpha + 2N^2 \mathfrak{N}^a \partial_a \alpha (dt)_a (dt)_b \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - 2N^2 \partial_x \alpha (dt)_{(a} (dx)_{b)} - 2N^2 N^x \partial_x \alpha (dt)_a (dt)_b \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - 2N^2 \partial_x \alpha (dt)_{(a} (dx + N^x dt)_{b)}.
\end{aligned} \tag{1.98}$$

With these result, we have

$$\begin{aligned}
\delta g_{ab} &= -\delta N^2 (dt)_a (dt)_b + \frac{\delta Q_{ab}}{E^1} - \frac{Q_{ab}}{(E^1)^2} \delta E^1 + \delta E^1 d\Omega_{ab} \\
&= -\left(\mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - 2N^2 \partial_x \alpha (dt)_{(a} (dx + N^x dt)_{b)} \right) \\
&\quad + \frac{1}{E^1} \mathcal{L}_{\alpha\mathfrak{N}+\beta} Q_{ab} - 2N^2 \partial_x \alpha (dt)_{(a} [(dx)_{b)} \wedge (dt)_d \mathfrak{N}^d] + \frac{2\Delta}{E^1 E^2} Q_{ab} \\
&\quad - \frac{Q_{ab}}{(E^1)^2} \delta E^1 + \left(\mathcal{L}_{\alpha\mathfrak{N}+\beta} E^1 - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x) \frac{\partial H_{\text{eff}}(x)}{\partial (\partial_x^n K_1(x))} \right) \right] \right) d\Omega_{ab} \quad (1.99) \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (g_{ab}) + \frac{Q_{ab}}{(E^1)^2} \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \frac{\delta H_{\text{eff}}[N]}{\delta (\partial_x^n K_1(x))} \right] + \frac{2\Delta}{E^1 E^2} Q_{ab} \\
&\quad - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x) \frac{\partial H_{\text{eff}}(x)}{\partial (\partial_x^n K_1(x))} \right) \right] d\Omega_{ab} \\
&\quad + 2(1-\mu)N^2 \partial_x \alpha (dt)_{(a} (dx + N^x dt)_{b)}
\end{aligned}$$

According to this result, we can see why the classical theory is generally covariant. In the classical theory, we have H_{eff} is independent of derivatives of K_I , implying $\Delta = 0 = \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x) \frac{\partial H_{\text{eff}}(x)}{\partial (\partial_x^n K_1(x))} \right) \right]$. Moreover, we have $\mu = 1$ in the classical theory, leading to

$$\delta g_{ab} = \mathcal{L}_{\alpha\mathfrak{N}+\beta} (g_{ab}). \quad (1.100)$$

The right hand side is the infinitesimal transformation of diffeomorphisms. Thus, the resulting metrics for different N and N^x will be the same up to a diffeomorphism transformation. For general case with $\mu \neq 1$, the most problematic term in the above result is the one proportional to $(1-\mu)$. If we want $\delta g_{ab} = \mathcal{L}_{\alpha\mathfrak{N}+\beta} (g_{ab})$, this term cannot be 0, even we given any requirements on H_{eff} . This motivates us to change the definition of the metric. We define the new metric \tilde{g} via the line element

$$d\tilde{s}^2 = -N^2 dt^2 + \frac{(E^2)^2}{\mu E^1} (dx + N^x dt)^2 + E^1 d\Omega^2. \quad (1.101)$$

Then, we get

$$\begin{aligned}
\delta \tilde{g}_{ab} &= -\delta N^2 (dt)_a (dt)_b + \frac{\delta Q_{ab}}{\mu E^1} - \frac{Q_{ab}}{(\mu E^1)^2} \delta(\mu E^1) + \delta E^1 d\Omega_{ab} \\
&= -\left(\mathcal{L}_{\alpha\mathfrak{N}+\beta} (N^2 (dt)_a (dt)_b) - 2N^2 \partial_x \alpha (dt)_{(a} (dx + N^x dt)_{b)} \right) \\
&\quad + \frac{1}{\mu E^1} \mathcal{L}_{\alpha\mathfrak{N}+\beta} Q_{ab} - 2N^2 \partial_x \alpha (dt)_{(a} [(dx)_{b)} \wedge (dt)_d \mathfrak{N}^d] + \frac{2\Delta}{\mu E^1 E^2} Q_{ab} \\
&\quad - \frac{Q_{ab}}{(\mu E^1)^2} \delta(\mu E^1) + \left(\mathcal{L}_{\alpha\mathfrak{N}+\beta} E^1 - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x) \frac{\partial H_{\text{eff}}(x)}{\partial (\partial_x^n K_1(x))} \right) \right] \right) d\Omega_{ab} \\
&= \mathcal{L}_{\alpha\mathfrak{N}+\beta} (\tilde{g}_{ab}) + \frac{Q_{ab}}{(\mu E^1)^2} (\mathcal{L}_{\alpha\mathfrak{N}+\beta} (\mu E^1) - \delta(\mu E^1)) + \frac{2\Delta}{\mu E^1 E^2} Q_{ab} \\
&\quad - \sum_n (-1)^n \left[\sum_{m=1}^n \binom{n}{m} (\partial_x^m \alpha) \frac{\partial^n}{\partial x^{n-m}} \left(N(x) \frac{\partial H_{\text{eff}}(x)}{\partial (\partial_x^n K_1(x))} \right) \right] d\Omega_{ab}. \quad (1.102)
\end{aligned}$$

According to this result, to have $\delta \tilde{g}_{ab} = \mathcal{L}_{\alpha\mathfrak{N}+\beta} (\tilde{g}_{ab})$, we need i) H_{eff} is independent of $\partial_x^n K_1$ and ii)

$$\mathcal{L}_{\alpha\mathfrak{N}+\beta} \mu - \delta \mu + \frac{2\mu}{E^2} \Delta = 0 \quad (1.103)$$

Because μ is a spacetime scalar, we have

$$\mathcal{L}_{\alpha\mathfrak{N}+\beta} \mu = \alpha \mathcal{L}_{\mathfrak{N}} \mu + \mathcal{L}_{\beta} \mu = \alpha \{ \mu, H_{\text{eff}}[N] \} + \{ \mu, H_x[\beta^x] \}. \quad (1.104)$$

Using (1.104), we finally have

$$\alpha\{\mu, H_{\text{eff}}[N]\} - \{\mu, H_{\text{eff}}[\alpha N]\} + \frac{2\mu}{E^2}\{E^2(x), H_{\text{eff}}[\alpha N]\} - \alpha(x)\frac{2\mu}{E^2}\{E^2(x), H_{\text{eff}}[N]\} = 0 \quad (1.105)$$

leading to

$$\begin{aligned} \alpha\left(\frac{1}{(E^2)^2}\{\mu, H_{\text{eff}}[N]\} - \frac{2\mu}{(E^2)^3}\{E^2(x), H_{\text{eff}}[N]\}\right) &= \left(\frac{1}{(E^2)^2}\{\mu, H_{\text{eff}}[\alpha N]\} - \frac{2\mu}{(E^2)^3}\{E^2(x), H_{\text{eff}}[\alpha N]\}\right) \\ \Rightarrow \alpha\left\{\frac{\mu E^1}{(E^2)^2}, H_{\text{eff}}[N]\right\} &= \left\{\frac{\mu E^1}{(E^2)^2}, H_{\text{eff}}[\alpha N]\right\} \end{aligned} \quad (1.106)$$

where in the last step, we used

$$\alpha\{E^1, H_{\text{eff}}[N]\} = \{E^1, H_{\text{eff}}[\alpha N]\} \quad (1.107)$$

resulting from the requirement i) that H_{eff} is independent of $\partial_x^n K_1$.

In summary, we conclude that

Theorem I.1. *In the Hamiltonian framework for spherically symmetric gravity, where H_{eff} denotes the Hamiltonian constraint, suppose the following conditions hold:*

(1) *the constraint algebra is given by (1.72) for some phase space function μ .*

The theory is general covariant, i.e., $\delta g_{ab} = \mathcal{L}_{\alpha\mathfrak{N}+\beta}g_{ab}$, with g_{ab} given by

$$g_{ab} = -N^2(dt)_a(dt)_b + \frac{(E^2)^2}{\mu E^1}((dx)_a + N^x(dt)_a)((dx)_b + N^x(dt)_b) + E^1 d\Omega_{ab}, \quad (1.108)$$

if and only if i) H_{eff} is independent of $\partial_x^n K_1$ for all $n \geq 1$ and ii) for all phase space independent smeared functions α , the following equation is satisfied:

$$\{\mu(x)E^1(x)/E^2(x)^2, H_{\text{eff}}[\alpha N]\} = \alpha(x)\{\mu(x)E^1(x)/E^2(x)^2, H_{\text{eff}}[N]\}. \quad (1.109)$$

This result will also be true once matter fields are included. In such cases, H_{eff} becomes the total Hamiltonian constraint including the matter part.

II. GENERAL FORM OF THE COVARIANT HAMILTONIAN CONSTRAINT

A. Poisson bracket between the Hamiltonian constraints

Since the Hamiltonian constraint is a scalar with density weight 1, it in general can be rewritten as $E^2 F$ with F being a scalar. Since F is a scalar, it can be rewritten as a general function of basic scalars given by K_I , E^I and their derivatives. Due to the condition (i), the derivatives of K_1 will be excluded. Similarly, it is natural to omit the derivatives of K_2 as K_1 and K_2 play analogous roles in the model. This ansatz aligns with the cases in classical GR and several widely studied quantum BH models. Furthermore, derivatives of E^I higher than the second order are not allowed, since the third equation in (1.72) is expected to hold with the right-hand side proportional to $N_1\partial_x N_2 - N_2\partial_x N_1$ while K_I must be involved in H_{eff} . Suppose that $\partial_x^n E^I$ is included in F . Then the result of $\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\}$ will contain terms of the form $\int dx f(N_1\partial_x^n N_2 - N_2\partial_x^n N_1)$ with f depending on K_I and E^I . These terms are proportional to $(N_1\partial_x N_2 - N_2\partial_x N_1)$ for $n \leq 2$, but not for $n > 2$. For $n = 2$, the integration by parts can be used to simplify the expression to $\int dx (\partial_x f)(N_1\partial_x N_2 - N_2\partial_x N_1)$. With these considerations, the only remaining fundamental scalars are as follows:

$$s_1 = E^1, s_2 = K_2, s_3 = \frac{K_1}{E^2}, s_4 = \frac{\partial_x E^1}{E^2}, s_5 = \frac{\partial_x E^1}{K_1}, s_6 = \frac{1}{E^2}\partial_x\left(\frac{\partial_x E^1}{E^2}\right), s_7 = \frac{1}{K_1}\partial_x\left(\frac{\partial_x E^1}{E^2}\right). \quad (2.1)$$

Then, H can be written as

$$H = E^2 F(s_a) \quad (2.2)$$

with a general function $F(s_1, \dots, s_7)$. Now let us consider the Poisson bracket $\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\}$,

$$\begin{aligned}
& \{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = \left\{ \int N_1(x) E^2(x) F(s_a(x)), \int N_1(y) E^2(y) F(s_a(y)) \right\} \\
& = \iint (N_1(x) N_2(y) F(s_a(x)) E^2(y) \{E^2(x), F(s_a(y))\} - N_1(x) N_2(y) F(s_a(y)) E^2(x) \{E^2(y), F(s_a(x))\}) \\
& \quad + \iint N_1(x) N_2(y) E^2(x) E^2(y) \{F(s_a(x)), F(s_a(y))\} \\
& = \iint (N_1(x) N_2(y) - N_1(y) N_2(x)) F(s_a(x)) E^2(y) \{E^2(x), F(s_a(y))\} \\
& \quad + \iint N_1(x) N_2(y) E^2(x) E^2(y) \{F(s_a(x)), F(s_a(y))\}
\end{aligned} \tag{2.3}$$

Since $\partial_x K_I$ are not involved in s_a , we get

$$\{E^2(x), F(s_a(y))\} \propto \delta(x, y), \tag{2.4}$$

leading to that the first term in (2.3) vanishes. Thus, we have

$$\begin{aligned}
& \{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = \left\{ \int N_1(x) E^2(x) F(s_a(x)), \int N_1(y) E^2(y) F(s_a(y)) \right\} \\
& = \iint N_1(x) N_2(y) E^2(x) E^2(y) \{F(s_a(x)), F(s_a(y))\} \\
& = \sum_{a,b=1}^7 \iint N_1(x) N_2(y) E^2(x) E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_b(y)} \{s_a(x), s_b(y)\} \\
& = \sum_{a=1}^7 \iint N_1(x) N_2(y) E^2(x) E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_a(y)} \{s_a(x), s_a(y)\} \\
& \quad + \sum_{a < b=1}^7 \iint N_1(x) N_2(y) E^2(x) E^2(y) \left(\frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_b(y)} \{s_a(x), s_b(y)\} + \frac{\partial F(\vec{s}(x))}{\partial s_b(x)} \frac{\partial F(\vec{s}(y))}{\partial s_a(y)} \{s_b(x), s_a(y)\} \right) \\
& = \sum_{a=1}^7 \iint N_1(x) N_2(y) E^2(x) E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_a(y)} \{s_a(x), s_a(y)\} \\
& \quad + \sum_{a < b=1}^7 \iint (N_1(x) N_2(y) - N_1(y) N_2(x)) E^2(x) E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_b(y)} \{s_a(x), s_b(y)\}
\end{aligned} \tag{2.5}$$

To simplify the first term, we observe

$$\{s_a(x), s_a(y)\} = 0, \quad \forall a = 1, 2, 3, 4, 6. \tag{2.6}$$

To calculate $\{s_a(x), s_a(y)\}$ for $a = 5, 7$, we consider a general case with $s = G(\partial_x E^1, \partial_x^2 E^1)/K_1$ and calculate $\{s(x), s(y)\}$. We have

$$\begin{aligned}
& \left\{ \frac{G(\partial_x E^1(x), \partial_x^2 E^1(x))}{K_1(x)}, \frac{G(\partial_y E^1(y), \partial_y^2 E^1(y))}{K_1(y)} \right\} \\
& = \frac{G(\partial_x E^1(x), \partial_x^2 E^1(x))}{K_1(y)} \left\{ \frac{1}{K_1(x)}, G(\partial_y E^1(y), \partial_y^2 E^1(y)) \right\} - (x \leftrightarrow y) \\
& = - \frac{G(\partial_x E^1(x), \partial_x^2 E^1(x))}{K_1(y)} \frac{1}{K_1(x)^2} \{K_1(x), G(\partial_y E^1(y), \partial_y^2 E^1(y))\} + (x \leftrightarrow y) \\
& = - 2G \frac{G(\partial_x E^1(x), \partial_x^2 E^1(x))}{K_1(y)} \frac{1}{K_1(x)^2} \left(G^{(1,0)}(\partial_y E^1(y), \partial_y^2 E^1(y)) \partial_y \delta(x, y) \right. \\
& \quad \left. + G^{(0,1)}(\partial_y E^1(y), \partial_y^2 E^1(y)) \partial_y^2 \delta(x, y) \right) + (x \leftrightarrow y)
\end{aligned} \tag{2.7}$$

Then, we get

$$\{s_5(x), s_5(y)\} = -2G \frac{s_5(x)}{K_1(x)K_1(y)} \partial_y \delta(x, y) + (x \leftrightarrow y), \quad (2.8)$$

and

$$\{s_7(x), s_7(y)\} = -2G \frac{s_7(x)}{K_1(x)K_1(y)} \left(\frac{-\partial_y E^2(y)}{E^2(y)^2} \partial_y \delta(x, y) + \frac{1}{E^2(y)} \partial_y^2 \delta(x, y) \right) + (x \leftrightarrow y). \quad (2.9)$$

With these results, we have

$$\begin{aligned} & \sum_{a=1}^7 \iint N_1(x)N_2(y)E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_a(x)} \frac{\partial F(\vec{s}(y))}{\partial s_a(y)} \{s_a(x), s_a(y)\} \\ &= \iint N_1(x)N_2(y)E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(y))}{\partial s_5(y)} \{s_5(x), s_5(y)\} \\ & \quad + \iint N_1(x)N_2(y)E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(y))}{\partial s_7(y)} \{s_7(x), s_7(y)\} \\ &= \iint N_1(x)N_2(y)E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(y))}{\partial s_5(y)} \left(-2G \frac{s_5(x)}{K_1(x)K_1(y)} \partial_y \delta(x, y) + (x \leftrightarrow y) \right) \\ & \quad + \iint N_1(x)N_2(y)E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(y))}{\partial s_7(y)} \left\{ -2G \frac{s_7(x)}{K_1(x)K_1(y)} \times \right. \\ & \quad \left. \left(\frac{-\partial_y E^2(y)}{E^2(y)^2} \partial_y \delta(x, y) + \frac{1}{E^2(y)} \partial_y^2 \delta(x, y) \right) + (x \leftrightarrow y) \right\} \\ &= -2G \iint [N_1(x)N_2(y) - N_1(y)N_2(x)] E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(y))}{\partial s_5(y)} \frac{s_5(x)}{K_1(x)K_1(y)} \partial_y \delta(x, y) \\ & \quad - 2G \iint [N_1(x)N_2(y) - N_1(y)N_2(x)] E^2(x)E^2(y) \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(y))}{\partial s_7(y)} \times \\ & \quad \frac{s_7(x)}{K_1(x)K_1(y)} \left(\frac{-\partial_y E^2(y)}{E^2(y)^2} \partial_y \delta(x, y) + \frac{1}{E^2(y)} \partial_y^2 \delta(x, y) \right) \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] E^2(x)E^2(x) \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{s_5(x)}{K_1(x)K_1(x)} \\ & \quad - 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] E^2(x)E^2(x) \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{s_7(x)}{K_1(x)K_1(x)} \frac{\partial_y E^2(x)}{E^2(x)^2} \\ & \quad - 4G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \frac{E^2(x)s_7(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \partial_x \left\{ \frac{1}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right\} \\ & \quad - 2G \int [N_1(x)\partial_x^2 N_2(x) - \partial_x^2 N_1(x)N_2(x)] E^2(x)s_7(x) \left(\frac{1}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] s_5(x) \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right)^2 \\ & \quad - 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \left(\frac{1}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 s_7(x) \partial_x E^2(x) \\ & \quad + 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \left(\frac{1}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 \partial_x (E^2(x)s_7(x)) \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] s_5(x) \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right)^2 \\ & \quad + 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \left(\frac{1}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 E^2(x) \partial_x s_7(x) \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] s_5(x) \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right)^2 \\ & \quad + 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \left(\frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 \frac{E^2(x)}{K_1(x)} \frac{\partial_x s_7(x)}{K_1(x)} \end{aligned} \quad (2.10)$$

Now, let us go to the Poisson bracket for $a \neq b$. For $a \neq b$, each term in the result (2.5) takes the form

$$B_{ab} = \iint (N_1(x)N_2(y) - N_1(y)N_2(x))\mathcal{P}_a(x)\mathcal{P}_b(y)\{s_a(x), s_b(y)\}. \quad (2.11)$$

with

$$\mathcal{P}_a = E^2 \frac{\partial F(\vec{s})}{\partial s_a} \quad (2.12)$$

Let us use $S_{ab}(x, y)$ to denote $\{s_a(x), s_b(y)\}$. Then we have

$$S_{ab}(x, y) \propto \delta(x, y), \quad (a, b) \in \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3), (2, 4), (2, 5), (4, 6)\} \equiv V \quad (2.13)$$

leading to

$$B_{ab} = 0, \forall (a, b) \in V. \quad (2.14)$$

For $(a, b) \notin V$, let us rewrite B_{ab} as

$$\begin{aligned} B_{ab} &= \iint N_1(x)N_2(y)\mathcal{P}_a(x)\mathcal{P}_b(y)\{s_a(x), s_b(y)\} - \iint N_2(x)N_1(y)\mathcal{P}_a(x)\mathcal{P}_b(y)\{s_a(x), s_b(y)\} \\ &= \left\{ \int N_1(x)s_a(x), \int N_2(y)s_b(y) \right\} \Big|_{\substack{N_1(x) \rightarrow N_1(x)\mathcal{P}_a(x) \\ N_2(y) \rightarrow N_2(y)\mathcal{P}_b(y)}} - (N_1 \leftrightarrow N_2) \end{aligned} \quad (2.15)$$

Then, a straightforward calculation gives

(1) For $(a, b) = (2, 6)$, we have

$$B_{26} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)}{2E^2(x)} \quad (2.16)$$

(2) For $(a, b) = (2, 7)$, we have

$$B_{27} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial_x E^1(x)}{2K_1(x)} \quad (2.17)$$

(3) For $(a, b) = (3, 4)$, we have

$$B_{34} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \left(-\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \right) \quad (2.18)$$

(4) For $(a, b) = (3, 5)$, we have

$$B_{35} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{-E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \quad (2.19)$$

(4) For $(a, b) = (3, 6)$, we have

$$B_{36} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{1}{E^2(x)} \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \right) \right) \quad (2.20)$$

(5) For $(a, b) = (3, 7)$, we have

$$\begin{aligned} B_{37} &= 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \partial_x \left(\frac{2}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) \right. \\ &\quad \left. - \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{E^2(x)}{K_1(x)} \partial_x \left(\frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \right) \right\} \end{aligned} \quad (2.21)$$

(6) For $(a, b) = (4, 5)$, we have

$$B_{45} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{E^2(x)\partial_x E^1}{K_1(x)^2} \quad (2.22)$$

(7) For $(a, b) = (4, 7)$, we have

$$B_{47} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \partial_x \left(\frac{\partial_x E^1(x)}{E^2(x)} \right) \frac{E^2(x)}{K_1(x)^2} \quad (2.23)$$

(8) For $(a, b) = (5, 6)$, we have

$$B_{56} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)^2} \right) - \frac{E^2(x)\partial_x E^1(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) + \frac{\partial_x E^1(x)}{K_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right) \right\} \quad (2.24)$$

(9) For $(a, b) = (5, 7)$, we have

$$B_{57} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{E^2(x)}{K_1(x)^2} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)} \right) + \frac{E^2(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right) - \frac{\partial_x E^1(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) \right\} \quad (2.25)$$

(10) For $(a, b) = (6, 7)$, we have

$$B_{67} = 2G \int (N_1(x)\partial_x N_2(x) - N_2(x)\partial_x N_1(x)) \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{1}{E^2(x)} \partial_x \left(\frac{E^2(x)\partial_x^2 E^1(x) - \partial_x E^2(x)\partial_x E^1(x)}{E^2(x)K_1(x)^2} \right) + \frac{E^2(x)\partial_x^2 E^1(x) - \partial_x E^2(x)\partial_x E^1(x)}{E^2(x)^2 K_1(x)^2} \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) \right\} \right\}. \quad (2.26)$$

Combining all the contribution together, we get

$$\begin{aligned} & \{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \times \\ & \left\{ s_5 \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right)^2 + \left(\frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right)^2 \frac{E^2(x)}{K_1(x)} \frac{\partial_x s_7(x)}{K_1(x)} + \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)}{2E^2(x)} + \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial_x E^1(x)}{2K_1(x)} \right. \\ & - \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} - \frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} + \frac{1}{E^2(x)} \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \right) \right) \\ & + \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \partial_x \left(\frac{2}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) - \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{E^2(x)}{K_1(x)} \partial_x \left(\frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \right) + \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{E^2(x)\partial_x E^1}{K_1(x)^2} \\ & + \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \partial_x \left(\frac{\partial_x E^1(x)}{E^2(x)} \right) \frac{E^2(x)}{K_1(x)^2} + \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)^2} \right) \\ & - \frac{E^2(x)\partial_x E^1(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) + \frac{\partial_x E^1(x)}{K_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right) \\ & + \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{E^2(x)}{K_1(x)^2} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)} \right) + \frac{E^2(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{\partial_x E^1(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \right) \\ & - \frac{\partial_x E^1(x)}{K_1(x)^2} \frac{\partial F(\vec{s}(x))}{\partial s_5(x)} \partial_x \left(\frac{E^2(x)}{K_1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) + \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{1}{E^2(x)} \partial_x \left(\frac{E^2(x)\partial_x^2 E^1(x) - \partial_x E^2(x)\partial_x E^1(x)}{E^2(x)K_1(x)^2} \right) \\ & \left. + \frac{E^2(x)\partial_x^2 E^1(x) - \partial_x E^2(x)\partial_x E^1(x)}{E^2(x)^2 K_1(x)^2} \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_7(x)} \right) \right\} \right\}. \quad (2.27) \end{aligned}$$

The classical Hamiltonian constraint can be written in terms of s_a as

$$H_{cl} = \frac{E^2}{2G} \left(-\frac{1}{\sqrt{s_1}} - 2\sqrt{s_1}s_2s_3 - \frac{s_2^2}{\sqrt{s_1}} + \frac{s_4^2}{4\sqrt{s_1}} + \sqrt{s_1}s_6 \right) \quad (2.28)$$

At first, let us omit the terms involving s_5 and s_7 in (2.27), because H_{cl} does not contain them. Moreover, in s_5 and s_7 , K_1 appears in the denominator, which we do not like. Then, we have

$$\begin{aligned} \{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \times \\ &\quad \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)}{2E^2(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \right. \\ &\quad \left. + \frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \right) - \frac{1}{E^2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \partial_x \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \right) \right\} \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \frac{E^1(x)}{E^2(x)^2} \times \\ &\quad \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)E^2(x)}{2E^1(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{E^2(x)^2}{E^1(x)} \right. \\ &\quad \left. + \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \left(\sum_{a \in \mathcal{A}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_a(x)} \partial_x s_a(x) \right) - \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \left(\sum_{b \in \mathcal{A}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_b(x)} \partial_x s_b(x) \right) \right\} \\ &= 2G \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \frac{E^1(x)}{E^2(x)^2} \times \\ &\quad \left\{ \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)E^2(x)}{2E^1(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{E^2(x)^2}{E^1(x)} \right. \\ &\quad \left. + \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \left(\sum_{a \in \mathcal{A} \setminus \{2\}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_a(x)} \partial_x s_a(x) \right) - \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \left(\sum_{b \in \mathcal{A} \setminus \{2\}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_b(x)} \partial_x s_b(x) \right) \right. \\ &\quad \left. + \frac{1}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_2(x)} E^2(x) \partial_x K_2(x) - \frac{1}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_2(x)} E^2(x) \partial_x K_2(x) \right\} \quad (2.29) \end{aligned}$$

with $\mathcal{A} = \{1, 2, 3, 4, 6\}$. Notice that

$$H_x(x) = \frac{E^2(x)\partial_x K_2(x)}{G} - \frac{K_1(x)\partial_x E^1(x)}{2G},$$

and that all of the terms but the last two in (2.29) do not contain $\partial_x K_2$. Thus, if we want

$$\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = \int [N_1(x)\partial_x N_2(x) - \partial_x N_1(x)N_2(x)] \frac{E^1(x)}{E^2(x)^2} \mu(x) H_x(x), \quad (2.30)$$

we have

$$\begin{aligned} \mu &= \frac{2G^2}{E^1(x)} \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_2(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_2(x)} \right), \\ &= \frac{2G^2}{s_1(x)} \left(\frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_2(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_2(x)} \right) \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} -\frac{\mu(x)K_1(x)\partial_x E^1(x)}{4G^2} &= \frac{\partial F(\vec{s}(x))}{\partial s_2(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \frac{\partial_x E^1(x)E^2(x)}{2E^1(x)} - \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \frac{\partial F(\vec{s}(x))}{\partial s_4(x)} \frac{E^2(x)^2}{E^1(x)} \\ &+ \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_3(x)} \left(\sum_{a \in \mathcal{A} \setminus \{2\}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_6(x) \partial s_a(x)} \partial_x s_a(x) \right) - \frac{E^2(x)}{E^1(x)} \frac{\partial F(\vec{s}(x))}{\partial s_6(x)} \left(\sum_{b \in \mathcal{A} \setminus \{2\}} \frac{\partial^2 F(\vec{s}(x))}{\partial s_3(x) \partial s_b(x)} \partial_x s_b(x) \right). \end{aligned} \quad (2.32)$$

B. Poisson bracket between μ and H_{eff}

Under our assumption on H_{eff} , (1.109) becomes

$$\{\mu(x), H_{\text{eff}}[\alpha N]\} = \alpha(x)\{\mu(x), H_{\text{eff}}[N]\}. \quad (2.33)$$

Then, we have

$$\begin{aligned} 0 &= \alpha_v \{\mu(x), H_{\text{eff}}[N]\} - \{\mu(x), H_{\text{eff}}[\alpha_v N]\} \\ &= \sum_{a,b} \left(\alpha_v(x) \frac{\partial \mu(x)}{\partial s_a(x)} \int N(y) E^2(y) \frac{\partial F}{\partial s_b(y)} \{s_a(x), s_b(y)\} - \frac{\partial \mu(x)}{\partial s_a(x)} \int \alpha_v(y) N(y) E^2(y) \frac{\partial F(y)}{\partial s_b(y)} \{s_a(x), s_b(y)\} \right) \\ &\quad + \sum_a \left(\alpha_v(x) \frac{\partial \mu(x)}{\partial s_a(x)} \int N(y) F(y) \{s_a(x), E^2(y)\} - \frac{\partial \mu(x)}{\partial s_a(x)} \int \alpha_v(y) N(y) F(y) \{s_a(x), E^2(y)\} \right) \\ &= \sum_{a,b} \left(\alpha_v(x) \frac{\partial \mu(x)}{\partial s_a(x)} \int N(y) E^2(y) \frac{\partial F(y)}{\partial s_b(y)} \{s_a(x), s_b(y)\} - \frac{\partial \mu(x)}{\partial s_a(x)} \int \alpha_v(y) N(y) E^2(y) \frac{\partial F(y)}{\partial s_b(y)} \{s_a(x), s_b(y)\} \right) \end{aligned} \quad (2.34)$$

where we used the fact s_a does not involve $\partial_x K_2$, implying

$$\{s_a(x), E^2(y)\} \propto \delta(x, y). \quad (2.35)$$

Since μ is a scalar, it is a function of basic scalars s_a . According to (2.31), μ is also a function of s_1, s_2, s_3, s_4 and s_6 if we focus on the case where H_{eff} is independent of s_5 and s_7 . For further simplification, we need the following trick. For any function X which could depend on phase space variable, we have

$$\int X(y) \{s_a(x), s_b(y)\} = \left\{ s_a(x), \int \lambda(y) s_b(y) \right\} \Big|_{\lambda=X} \quad (2.36)$$

where on the right hand λ is treated as a function independent of phase space variable. Taking advantage of this trick, we have

$$\begin{aligned} &\{\mu(x), H_{\text{eff}}[\alpha_v N]\} - \alpha_v(x)\{\mu(x), H_{\text{eff}}[N]\} \\ &= GN \frac{\partial_x \alpha_v}{E^2} \mu_2 F_6 s_4 - 2GN \frac{\partial_x \alpha_v}{E^2} \mu_3 F_4 \\ &\quad - 6GN \frac{\partial_x \alpha_v}{E^2} \mu_3 F_6 \frac{\partial_x E^2}{(E^2)^2} + 4G \frac{\partial_x \alpha_v}{E^2} \mu_3 \frac{1}{(E^2)^2} \frac{\partial}{\partial x} (NE^2 F_6) + 2GN \frac{\partial_x^2 \alpha_v}{(E^2)^2} \mu_3 F_6 \\ &\quad - 2GN \frac{\partial_x \alpha_v}{E^2} \mu_4 F_3 \\ &\quad + GN \frac{\partial_x \alpha_v}{E^2} \mu_6 F_2 s_4 + 6GN \frac{\partial_x \alpha_v}{E^2} \mu_6 F_3 \frac{\partial_x E^2}{(E^2)^2} - 4G \frac{\partial_x \alpha_v}{E^2} \mu_6 \frac{1}{(E^2)^2} \frac{\partial}{\partial x} (NE^2 F_3) - 2GN \frac{\partial_x^2 \alpha_v}{(E^2)^2} \mu_6 F_3 \end{aligned} \quad (2.37)$$

where μ_a denotes

$$\mu_a = \frac{\partial \mu}{\partial s_a} \quad (2.38)$$

Then, $0 = \alpha_v(x)\{\mu(x), H_{\text{eff}}[N]\} - \{\mu(x), H_{\text{eff}}[\alpha_v N]\}$ for all α_v leads to

$$\begin{aligned} \mu_3 F_6 - \mu_6 F_3 &= 0 \\ \mu_2 F_6 s_4 - 2\mu_3 F_4 - 2\mu_4 F_3 + \mu_6 F_2 s_4 + 4\mu_3 \frac{1}{E^2} \partial_x F_6 - 4\mu_6 \frac{1}{E^2} \partial_x F_3 &= 0. \end{aligned} \quad (2.39)$$

C. Solve the covariance conditions to get the covariant Hamiltonian constraint

To solve the covariance equation, let us introduce

$$F_a = \frac{\partial F}{\partial s_a}. \quad (2.40)$$

Then, (2.31) can be rewritten as

$$\frac{\mu s_1}{2G^2} = F_3 \partial_{s_2} F_6 - F_6 \partial_{s_2} F_3 \quad (2.41)$$

and (2.32) can be rewritten as

$$\begin{aligned} \frac{-\mu s_3 s_4}{4G^2} &= F_3 F_6 \frac{s_4}{2s_1} - F_3 F_4 \frac{1}{s_1} + \frac{1}{s_1} \sum_{a \in \mathcal{A} \setminus \{2\}} (\partial_{s_a} F_6 F_3 - F_6 \partial_{s_a} F_3) \frac{\partial_x s_a}{E^2} \\ \Rightarrow \frac{-\mu s_1 s_3 s_4}{4G^2} &= \frac{1}{2} F_3 F_6 s_4 - F_3 F_4 + \sum_{a \in \mathcal{A} \setminus \{2\}} (\partial_{s_a} F_6 F_3 - F_6 \partial_{s_a} F_3) \frac{\partial_x s_a}{E^2} \end{aligned} \quad (2.42)$$

i.e.

$$\begin{aligned} \frac{-\mu s_1 s_3 s_4}{4G^2} &= \frac{1}{2} F_2 F_6 s_4 - F_3 F_4 + (\partial_{s_1} F_6 F_3 - F_6 \partial_{s_1} F_3) s_4 + (\partial_{s_3} F_6 F_3 - F_6 \partial_{s_3} F_3) \frac{\partial_x s_3}{E^2} \\ &\quad + (\partial_{s_4} F_6 F_3 - F_6 \partial_{s_4} F_3) s_6 + (\partial_{s_6} F_6 F_3 - F_6 \partial_{s_6} F_3) \frac{\partial_x s_6}{E^2} \end{aligned} \quad (2.43)$$

where we used $\partial_x s_1 = E^2 s_4$ and $\partial_x s_4 = E^2 s_6$.

According to (2.41), μ can be a function of s_a for $a \in \mathcal{A}$. In other words, there is no $\partial_x s_6/E^2$ and $\partial_x s_3/E^2$ contained in μ . Therefore, (2.43) can be simplified as

$$\begin{aligned} 0 &= F_3 \partial_{s_6} F_6 - F_6 \partial_{s_6} F_3 \\ 0 &= F_3 \partial_{s_3} F_6 - F_6 \partial_{s_3} F_3 \\ \frac{-\mu s_1 s_3 s_4}{4G^2} &= \frac{1}{2} F_2 F_6 s_4 - F_3 F_4 + (F_3 \partial_{s_1} F_6 - F_6 \partial_{s_1} F_3) s_4 + (F_3 \partial_{s_4} F_6 - F_6 \partial_{s_4} F_3) s_6 \end{aligned} \quad (2.44)$$

From the first two equations of (2.44), we get

$$F_6(\{s_a\}_{a \in \mathcal{A}}) = C_1(\{s_a\}_{a \in \mathcal{A} \setminus \{3,6\}}) F_3(\{s_a\}_{a \in \mathcal{A}}). \quad (2.45)$$

Substituting (2.45) into (2.41), we get

$$\frac{\mu s_1}{2G^2} = (F_3)^2 \partial_{s_2} C_1. \quad (2.46)$$

Equation (2.45) implies that the function F should take the form

$$F = X(s_1, s_2, s_4, s_3 + C_1(s_1, s_2, s_4) s_6), \quad (2.47)$$

for arbitrary function $X(x, y, w, z)$. Then, according to (2.46), we have

$$\mu = \frac{2G^2}{s_1} \partial_{s_2} C_1 (\partial_z X)^2. \quad (2.48)$$

With the expressions for F and μ , it can be checked that

$$\mu_3 F_6 - \mu_6 F_3 = 0, \quad \mu_3 \partial_{s_6} F_6 - \mu_6 \partial_{s_6} F_3 = 0, \quad \mu_3 \partial_{s_3} F_6 - \mu_6 \partial_{s_3} F_3 = 0. \quad (2.49)$$

Thus only the last equation in (2.39) needs to be solved. The last equation in (2.39) can be simplified as

$$\begin{aligned} \mu_2 F_6 s_4 - 2\mu_3 F_4 - 2\mu_4 F_3 + \mu_6 F_2 s_4 + 4(\mu_3 \partial_{s_1} F_6 - \mu_6 \partial_{s_1} F_3) s_4 + 4(\mu_3 \partial_{s_4} F_6 - \mu_6 \partial_{s_4} F_3) s_6 \\ + 4(\mu_3 \partial_{s_2} F_6 - \mu_6 \partial_{s_2} F_3) \frac{\partial_x K_2}{E^2} = 0. \end{aligned} \quad (2.50)$$

Except the last term, there is no other terms involving $\partial_x K_2$. Thus, we expect that the last term should vanish. *It is noted that here one may want to applying the Diffeomorphism constraint to replace $\partial_x K_2$ by $\partial_x E^1 K_1 / (2E^2)$. However, this replacement is only valid in the vacuum case. If this replacement is applied, the resulting theory cannot be coupled to matter fields.* Then, we have

$$\mu_3 \partial_{s_2} F_6 - \mu_6 \partial_{s_2} F_3 = \frac{4G^2 (\partial_{s_2} C_1)^2 (\partial_z X)^2 \partial_z^2 X}{s_1} = 0, \quad (2.51)$$

leading to the non-trivial result

$$\partial_z^2 X = 0. \quad (2.52)$$

Therefore, we have

$$X(s_1, s_2, s_4, z) = A(s_1, s_2, s_4) + B(s_1, s_2, s_4)z. \quad (2.53)$$

Consequently, we have that F takes the form

$$F = A(s_1, s_2, s_4) + F_3(s_1, s_2, s_4)s_3 + F_6(s_1, s_2, s_4)s_6, \quad F_3 = B, F_6 = BC_1. \quad (2.54)$$

Substituting (2.53) into (2.48), we have

$$\mu = \frac{2G^2}{s_1} (F_3 \partial_{s_2} F_6 - \partial_{s_2} F_3 F_6) \quad (2.55)$$

following which we obtain $\partial_{s_3} \mu = 0 = \partial_{s_6} \mu$. Actually (2.55) is just (2.41), the original covariant equation.

As a consequence of (2.54), we get

$$F_2 = \partial_{s_2} A + \partial_{s_2} F_3 s_3 + \partial_{s_2} F_6 s_6, \quad F_4 = \partial_{s_4} A + \partial_{s_4} F_3 s_3 + \partial_{s_4} F_6 s_6. \quad (2.56)$$

Therefore, the last equation in (2.44) becomes

$$\begin{aligned} \frac{-\mu s_1 s_3 s_4}{4G^2} &= \frac{1}{2} (\partial_{s_2} A + \partial_{s_2} F_3 s_3 + \partial_{s_2} F_6 s_6) F_6 s_4 - F_3 (\partial_{s_4} A + \partial_{s_4} F_3 s_3 + \partial_{s_4} F_6 s_6) \\ &\quad + (F_3 \partial_{s_1} F_6 - F_6 \partial_{s_1} F_3) s_4 + (F_3 \partial_{s_4} F_6 - F_6 \partial_{s_4} F_3) s_6 \\ &= \frac{1}{2} (\partial_{s_2} A) F_6 s_4 + \frac{1}{2} \partial_{s_2} F_3 F_6 s_4 s_3 + \frac{1}{2} (\partial_{s_2} F_6) F_6 s_4 s_6 - F_3 (\partial_{s_4} A) - F_3 (\partial_{s_4} F_3) s_3 - F_3 (\partial_{s_4} F_6) s_6 \\ &\quad + F_3 (\partial_{s_1} F_6) s_4 - F_6 (\partial_{s_1} F_3) s_4 + (F_3 \partial_{s_4} F_6 - F_6 \partial_{s_4} F_3) s_6 \\ &= \frac{1}{2} (\partial_{s_2} A) F_6 s_4 - F_3 (\partial_{s_4} A) + \left(\frac{1}{2} (\partial_{s_2} F_3) F_6 s_4 - F_3 (\partial_{s_4} F_3) \right) s_3 \\ &\quad + F_3 (\partial_{s_1} F_6) s_4 - F_6 (\partial_{s_1} F_3) s_4 + \left(\frac{1}{2} (\partial_{s_2} F_6) F_6 s_4 - F_6 \partial_{s_4} F_3 \right) s_6 \end{aligned} \quad (2.57)$$

i.e.

$$\begin{aligned} 0 &= \frac{1}{2} (\partial_{s_2} A) F_6 s_4 - F_3 (\partial_{s_4} A) + \left(\frac{\mu s_1 s_4}{4G^2} + \frac{1}{2} (\partial_{s_2} F_3) F_6 s_4 - F_3 (\partial_{s_4} F_3) \right) s_3 \\ &\quad + F_3 (\partial_{s_1} F_6) s_4 - F_6 (\partial_{s_1} F_3) s_4 + \left(\frac{1}{2} (\partial_{s_2} F_6) F_6 s_4 - F_6 \partial_{s_4} F_3 \right) s_6 \end{aligned} \quad (2.58)$$

Note that K_1 is only contained in s_3 , leading to

$$\frac{\mu s_1 s_4}{4G^2} + \frac{1}{2} (\partial_{s_2} F_3) F_6 s_4 - F_3 \partial_{s_4} (F_3) = 0. \quad (2.59)$$

$\partial_x^2 E^1$ is contained only in s_6 , resulting in

$$F_6 \left(\frac{1}{2} (\partial_{s_2} F_6) s_4 - \partial_{s_4} (F_3) \right) = 0. \quad (2.60)$$

With these two conditions, we get

$$\frac{1}{2} F_6 s_4 (\partial_{s_2} A) - F_3 (\partial_{s_4} A) + [F_3 (\partial_{s_1} F_6) - F_6 (\partial_{s_1} F_3)] s_4 = 0 \quad (2.61)$$

According to (2.41), we get

$$F_3 \partial_{s_2} F_6 = F_6 \partial_{s_2} F_3 + \frac{\mu s_1}{2G^2} \quad (2.62)$$

With this equation, (2.60) can be simplified as

$$0 = \frac{1}{2}F_3(\partial_{s_2}F_6)s_4 - F_3\partial_{s_4}(F_3) = \frac{\mu s_1 s_4}{4G^2} + \frac{1}{2}s_4F_6\partial_{s_2}F_3 - F_3\partial_{s_4}F_3 \quad (2.63)$$

which is the same as (2.59). That is to say, once (2.41) is satisfied, (2.59) and (2.60) are equivalent.

According to (2.60), we get that there exist a function $Q(s_1, s_2, s_4)$ such that

$$F_3 = \frac{1}{2}\frac{\partial Q}{\partial s_2}, \quad F_6 = \frac{1}{s_4}\frac{\partial Q}{\partial s_4}, \quad (2.64)$$

In addition (2.61) becomes

$$\begin{aligned} & (\partial_{s_2}A)\partial_{s_4}Q - (\partial_{s_4}A)\partial_{s_2}Q + \partial_{s_2}Q(\partial_{s_1}\partial_{s_4}Q) - \partial_{s_4}Q(\partial_{s_1}\partial_{s_2}Q) = 0 \\ \implies & \partial_{s_4}Q(\partial_{s_2}A - \partial_{s_1}\partial_{s_2}Q) = \partial_{s_2}Q(\partial_{s_4}A - \partial_{s_1}\partial_{s_4}Q), \\ \implies & \partial_{s_4}Q\partial_{s_2}(A - \partial_{s_1}Q) = \partial_{s_2}Q\partial_{s_4}(A - \partial_{s_1}Q), \end{aligned} \quad (2.65)$$

This equation leads to

$$A - \partial_{s_1}Q = C_2(s_1, Q) \Rightarrow A = C_2(s_1, Q) + \partial_{s_1}Q. \quad (2.66)$$

Now we have the Hamiltonian $H = E^2F$ taking

$$\begin{aligned} H &= C_2(s_1, Q)E^2 + \partial_{s_1}QE^2 + \frac{\partial_{s_2}Q}{2}s_3E^2 + \frac{\partial_{s_4}Q}{s_4}s_6E^2 \\ &= C_2(s_1, Q)E^2 + \partial_{s_1}QE^2 + \frac{\partial_{s_2}Q}{2}K_1 + \frac{\partial_{s_4}Q}{s_4}\partial_x s_4 \end{aligned} \quad (2.67)$$

Taking advantage of (2.64), (2.55) can be simplified as

$$\mu = \frac{G^2}{s_1 s_4} (\partial_{s_2}Q\partial_{s_2}\partial_{s_4}Q - \partial_{s_4}Q\partial_{s_2}^2Q) \quad (2.68)$$

With μ replaced by (2.68) and applying (2.64), we can check that (2.59) is satisfied automatically. Thus the only remaining equation that is not satisfied is (2.50). By (2.64), we can simplify (2.50) as

$$\partial_{s_2}\mu\partial_{s_4}Q - \partial_{s_4}\mu\partial_{s_2}Q = 0 \Rightarrow \mu = Y(s_1, Q). \quad (2.69)$$

Considering the diffeomorphism constraint

$$GH_x = E^2\partial_x K_2 - \frac{1}{2}K_1\partial_x E^1 \quad (2.70)$$

we gave

$$\begin{aligned} H + \frac{G\partial_{s_2}Q}{\partial_x E^1}H_x &= C_2(s_1, Q)E^2 + \partial_{s_1}QE^2 + \frac{\partial_{s_2}Q}{2}K_1 + \frac{\partial_{s_4}Q}{s_4}\partial_x s_4 + \frac{\partial_{s_2}Q}{\partial_x E^1} \left(E^2\partial_x K_2 - \frac{1}{2}K_1\partial_x E^1 \right) \\ &= C_2(s_1, Q)E^2 + \partial_{s_1}QE^2 + \frac{\partial_{s_2}Q}{s_4}\partial_x s_2 + \frac{\partial_{s_4}Q}{s_4}\partial_x s_4 \\ &= \frac{1}{s_4} (C_2(s_1, Q)\partial_x s_1 + \partial_{s_1}Q\partial_x s_1 + \partial_{s_2}Q\partial_x s_2 + \partial_{s_4}Q\partial_x s_4) \\ &= \frac{1}{s_4} C_2(s_1, Q)\partial_x s_1 + \frac{1}{s_4}\partial_x Q \end{aligned} \quad (2.71)$$

We thus get that for $C_2(s_1, Q) = 0$, the theory carries a constant of motion which is just Q . In the classical theory, the Hamiltonian constraint can be rewritten as

$$H - \frac{2G\partial_{s_2}M}{\partial_x E^1}H_x = -\frac{2}{s_4}\partial_x M \quad (2.72)$$

with

$$M = \frac{\sqrt{s_1}}{2G} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4} \right) \quad (2.73)$$

Thus, $Q = -2M$. We now arrive at the conclusion:

Theorem II.1. *In the Hamiltonian framework for spherically symmetric gravity, the requirements for covariance and the assumption on H_{eff} imply that $H_{\text{eff}} = E^2 F$ can only depend on the following seven basic scalars*

$$s_1 = E^1, s_2 = K_2, s_3 = \frac{K_1}{E^2}, s_4 = \frac{\partial_x E^1}{E^2}, s_5 = \frac{\partial_x E^1}{K_1}, s_6 = \frac{1}{E^2} \partial_x \left(\frac{\partial_x E^1}{E^2} \right), s_7 = \frac{1}{K_1} \partial_x \left(\frac{\partial_x E^1}{E^2} \right). \quad (2.74)$$

Excluding s_5 and s_7 from our consideration, we can solve the constraint algebra (1.72) and (1.109) to get

$$H_{\text{eff}} = -2E^2 \left(\partial_{s_1} M_{\text{eff}} + \frac{\partial_{s_2} M_{\text{eff}}}{2} s_3 + \frac{\partial_{s_4} M_{\text{eff}}}{s_4} s_6 + \mathcal{R}(s_1, M_{\text{eff}}) \right). \quad (2.75)$$

where $\mathcal{R}(s_1, M_{\text{eff}})$ is an arbitrary function, and $M_{\text{eff}}(s_1, s_2, s_4)$ satisfies the following equations:

$$\frac{\mu s_1 s_4}{4G^2} = \partial_{s_2} M_{\text{eff}} \partial_{s_2} \partial_{s_4} M_{\text{eff}} - \partial_{s_4} M_{\text{eff}} \partial_{s_2}^2 M_{\text{eff}}, \quad (2.76a)$$

$$\partial_{s_2} \mu \partial_{s_4} M_{\text{eff}} - \partial_{s_4} \mu \partial_{s_2} M_{\text{eff}} = 0. \quad (2.76b)$$

The equation (2.76b) implies that μ depends only on s_1 and M . Additionally, in the vacuum case where $H_{\text{eff}} = 0$ should be satisfied, M_{eff} is a constant of motion, representing the BH mass, provided $\mathcal{R} = 0$.

Equation (2.76a) and (2.76a) will be referred to as the covariance equations.

III. SPECIAL SOLUTIONS TO THE COVARIANCE EQUATIONS AND THEIR DYNAMICS

A. Get the classical theory

In the classical theory, we have

$$M_{\text{cl}} = \frac{\sqrt{s_1}}{2G} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4} \right) \quad (3.1)$$

which satisfies

$$\partial_{s_2} M_{\text{cl}} \partial_{s_2} \partial_{s_4} M_{\text{cl}} - \partial_{s_4} M_{\text{cl}} \partial_{s_2}^2 M_{\text{cl}} = -\frac{s_1}{G^2} \frac{-2 \times 2s_4}{4} = \frac{s_1 s_4}{4G^2}. \quad (3.2)$$

This implies

$$\mu = 1$$

Then, we have

$$\partial_{s_2} M = \frac{\sqrt{s_1} s_2}{G}, \quad \partial_{s_1} M = \frac{(s_2)^2}{4G\sqrt{s_1}} - \frac{(s_4)^2}{16G\sqrt{s_1}} + \frac{1}{4G\sqrt{s_1}}, \quad \partial_{s_4} M = -\frac{\sqrt{s_1} s_4}{4G} \quad (3.3)$$

leading to

$$H_{\text{cl}} = -2E^2 \left(\frac{1}{4G\sqrt{E^1}} + \frac{(K_2)^2}{4G\sqrt{E^1}} - \frac{1}{16G\sqrt{E^1}} \left(\frac{\partial_x E^1}{E^2} \right)^2 + \frac{\sqrt{E^1} K_2 K_1}{2GE^2} - \frac{\sqrt{E^1}}{4G} \frac{1}{E^2} \partial_x \left(\frac{\partial_x E^1}{E^2} \right) \right) \quad (3.4)$$

B. Solutions with polymerization

Another solution is obtained by considering the polymerization of M_{cl} . Naively, we have

$$\tilde{M}_{\text{eff}} = \frac{1}{2} \left(\frac{\sqrt{s_1}}{G} + \frac{\sqrt{s_1}^3}{G\zeta^2} \sin^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right) - \frac{\sqrt{s_1} (s_4)^2}{4G} \right) \quad (3.5)$$

Then, we get the value of μ associated with \tilde{M}_{eff} being

$$\mu = \cos\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right) \quad (3.6)$$

which is not a function of s_1 and \tilde{M}_{eff} . Thus \tilde{M}_{eff} cannot leads a covariant theory with metric defined by (1.108). To get a covariant theory, let us define

$$M_{\text{eff}} = \frac{\sqrt{s_1}}{2G} + \frac{\sqrt{s_1}^3}{2G\zeta^2} \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right) - \frac{\sqrt{s_1}(s_4)^2}{8G} X(s_2) \quad (3.7)$$

Then, we have

$$\mu = -\frac{\sqrt{s_1}X'(s_2) \sin\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right)}{2\zeta} + X(s_2) \cos\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right) - \frac{1}{8}s_4^2 X(s_2)X''(s_2) + \frac{1}{8}s_4^2 X'(s_2)^2 \quad (3.8)$$

1. solution with $\mu = 1$ and its dynamics

A solution for $\mu = 1$ can be solved with

$$X(s_2) = \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \quad (3.9)$$

i.e.

$$M_{\text{eff}} = \frac{\sqrt{s_1}}{2G} + \frac{\sqrt{s_1}^3}{2G\zeta^2} \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right) - \frac{\sqrt{s_1}(s_4)^2}{8G} \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \quad (3.10)$$

Then, we get

$$\begin{aligned} \partial_{s_1} M_{\text{eff}} &= \frac{1}{4G\sqrt{s_1}} + \frac{3\sqrt{s_1} \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right)}{4\zeta^2 G} - \frac{s_2 \sin\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right)}{4\zeta G} + \frac{i\zeta s_2 s_4^2 \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right)}{8G s_1} - \frac{s_4^2}{16G\sqrt{s_1}} \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \\ \partial_{s_2} M_{\text{eff}} &= \frac{s_1 \sin\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right)}{2\zeta G} - \frac{i\zeta s_4^2 \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right)}{4G} \\ \partial_{s_6} M_{\text{eff}} &= -\frac{\sqrt{s_1} s_4}{4G} \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \end{aligned} \quad (3.11)$$

Then, we have

$$\begin{aligned} H_{\text{eff}} &= -2E^2 \left\{ \frac{1}{4G\sqrt{s_1}} + \frac{3\sqrt{s_1} \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right)}{4\zeta^2 G} - \frac{s_2 \sin\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right)}{4\zeta G} + \frac{i\zeta s_2 s_4^2 \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right)}{8G s_1} - \frac{s_4^2}{16G\sqrt{s_1}} \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \right. \\ &\quad \left. + \frac{s_1 \sin\left(\frac{2\zeta s_2}{\sqrt{s_1}}\right)}{4\zeta G} s_3 - \frac{i\zeta s_4^2 \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right)}{8G} s_3 - \frac{\sqrt{s_1}}{4G} \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) s_6 \right\} \\ &= -\frac{3\sqrt{E^1} E^2}{2\zeta^2 G} \sin^2\left(\frac{\zeta K_2}{\sqrt{E^1}}\right) + \frac{K_2 E^2}{2G\zeta} \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right) - \frac{E^1 K_1}{2G\zeta} \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right) \\ &\quad - \frac{E^2}{2G\sqrt{E^1}} + \frac{(\partial_x E^1)^2}{8G\sqrt{E^1} E^2} \exp\left(\frac{2i\zeta K_2}{\sqrt{E^1}}\right) + \frac{\sqrt{E^1}}{2G} \partial_x \left(\frac{\partial_x E^1}{E^2}\right) \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) \\ &\quad + \frac{i\zeta E^2}{4G} \left(\frac{\partial_x E^1}{E^2}\right)^2 \left(\frac{K_1}{E^2} - \frac{K_2}{E^1}\right) \exp\left(\frac{2i\zeta K_2}{\sqrt{E^1}}\right) \end{aligned} \quad (3.12)$$

solve the dynamics

To solve the dynamics, let us still choose the areal gauge

$$E^1(x) = x^2. \quad (3.13)$$

Then, the diffeomorphism constraint leads to

$$K_1 = \frac{\partial_x K_2 E^2}{x}. \quad (3.14)$$

Substituting these equations into the effective Hamiltonian constraint, we get

$$\begin{aligned} H_{\text{eff}} &= -\frac{3xE^2}{2\zeta^2 G} \sin^2\left(\frac{\zeta K_2}{x}\right) + \frac{K_2 E^2}{2G\zeta} \sin\left(\frac{2\zeta K_2}{x}\right) - \frac{x\partial_x K_2 E^2}{G\zeta} \sin\left(\frac{2\zeta K_2}{x}\right) \\ &\quad - \frac{E^2}{2Gx} + \frac{x}{2GE^2} \exp\left(\frac{2i\zeta K_2}{x}\right) + \frac{x}{G} \partial_x \left(\frac{x}{E^2}\right) \exp\left(\frac{2i\zeta s_2}{\sqrt{s_1}}\right) + \frac{i\zeta E^2}{G} \left(\frac{x}{E^2}\right)^2 \left(\frac{\partial_x K_2}{x} - \frac{K_2}{x^2}\right) \exp\left(\frac{2i\zeta K_2}{x}\right) \\ &= -\frac{E^2}{x} \partial_x \left(\frac{x^3}{2G\zeta^2} \sin^2\left(\frac{\zeta K_2}{x}\right) + \frac{x}{2G} - \frac{x^3}{(E^2)^2} \exp\left(\frac{2i\zeta s_2}{2G\sqrt{s_1}}\right)\right) \\ &= -\frac{E^2}{x} \partial_x M_{\text{eff}} \end{aligned} \quad (3.15)$$

Vanishing the Hamiltonian constraint leads to

$$x^3 \sin^2\left(\frac{\zeta K_2}{x}\right) + \zeta^2 x - \frac{\zeta^2 x^3}{(E^2)^2} e^{\frac{2i\zeta K_2}{x}} = 2GC_2 \zeta^2 = 2GM_{\text{eff}} \zeta^2, \quad (3.16)$$

leading to

$$\begin{aligned} &\frac{x^3}{-4} \left(e^{\frac{2i\zeta K_2}{x}} + e^{-\frac{2i\zeta K_2}{x}} - 2 \right) + \zeta^2 x - \frac{\zeta^2 x^3}{E^2(x)^2} e^{\frac{2i\zeta K_2}{x}} = 2GC_2 \zeta^2 \\ \Rightarrow &\left(\frac{x^3}{-4} - \frac{\zeta^2 x^3}{E^2(x)^2} \right) \left[e^{\frac{2i\zeta K_2}{x}} \right]^2 + \left(\frac{x^3}{2} + \zeta^2 x - 2GC_2 \zeta^2 \right) e^{\frac{2i\zeta K_2}{x}} - \frac{x^3}{4} = 0 \\ \Rightarrow &e^{\frac{2i\zeta K_2}{x}} = \frac{-\left(\frac{x^3}{2} + \zeta^2 x - 2GC_2 \zeta^2\right) \pm \sqrt{\left(\frac{x^3}{2} - \zeta^2 x + 2GC_2 \zeta^2\right)^2 - x^3 \left(\frac{x^3}{4} + \frac{\zeta^2 x^3}{E^2(x)^2}\right)}}{-2\left(\frac{x^3}{4} + \frac{\zeta^2 x^3}{E^2(x)^2}\right)}. \end{aligned} \quad (3.17)$$

Since we are interested in the stationary metric, we need to choose the lapse function $N(x)$ and the shift vector $N^x(x)$ such that

$$\begin{aligned} \{E^1(x), H_{\text{eff}}[N] + H_x[N^x]\} &= 0 \\ \{E^2(x), H_{\text{eff}}[N] + H_x[N^x]\} &= 0. \end{aligned} \quad (3.18)$$

As a consequence of the first equation, we get

$$N^x(x) = \left(\frac{i\zeta x e^{\frac{2\zeta K_2(x)}{x}}}{E^2(x)^2} - \frac{x \sin\left(\frac{2\zeta K_2(x)}{x}\right)}{2\zeta} \right) N(x). \quad (3.19)$$

Substituting this result into the second equation, we have

$$N(x) = \frac{C_1 x}{E^2(x)}. \quad (3.20)$$

Combing (3.19) with (3.17), we get

$$N_x(x) = \mp \frac{iC_1}{xE^2(x)^2} \sqrt{(E^2(x)^2(x - 2GC_2)(x^3 + x\zeta^2 - 2GC_2\zeta^2) - x^6)} \quad (3.21)$$

Then, we get the metric, with only the minus sign considered,

$$ds^2 = -\left(\frac{x}{E^2(x)}\right)^2 dt^2 + \frac{E^2(x)^2}{x^2} \left(dx - \frac{iC_1}{xE^2(x)^2} \sqrt{(E^2(x)^2(x - 2GC_2)(x^3 + x\zeta^2 - 2GC_2\zeta^2) - x^6)} dt \right)^2 + x^2 d\Omega^2. \quad (3.22)$$

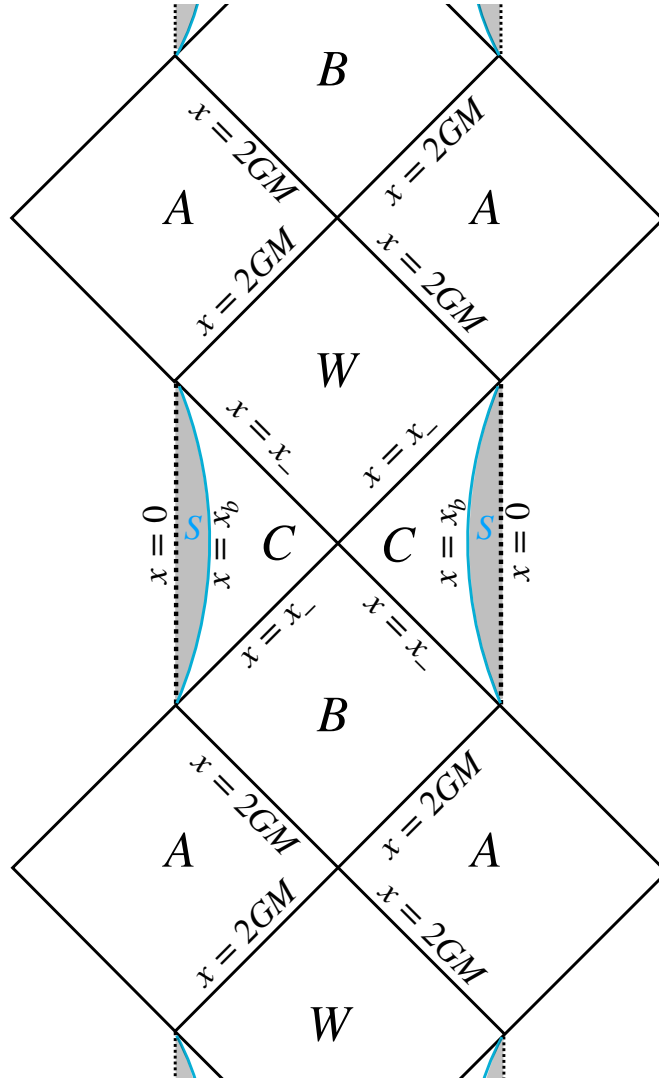


FIG. 1: The Penrose diagram of the spacetime given by (3.25)

- (1) Consider the PG gauge where we choose $E^2(x) = x$. Then, we get the metric with $C_2 = M$

$$ds^2 = -dt^2 + \left(dx + \sqrt{\frac{2GM}{x} - \frac{\zeta^2}{x^2} \left(\frac{2GM}{x} - 1 \right)^2} dt \right)^2 + x^2 d\Omega^2 \quad (3.23)$$

- (2) Consider the Schwarzschild gauge with $N_x(x) = 0$. Solving $N^x(x) = 0$, we get

$$E^2(x) = \pm x^3 \sqrt{\frac{1}{(x - 2GC_2)(x^3 + x\zeta^2 - 2G\zeta^2 C_2)}}. \quad (3.24)$$

Substituting this result into the metric, we get with $C_2 = M$

$$ds^2 = - \left(1 - \frac{2M}{x} + \frac{\zeta^2}{x^2} \left(\frac{2M}{x^2} - 1 \right)^2 \right) dt^2 + \left(1 - \frac{2M}{x} + \frac{\zeta^2}{x^2} \left(\frac{2M}{x^2} - 1 \right)^2 \right)^{-1} dx^2 + x^2 d\Omega^2. \quad (3.25)$$

The spacetime structure is shown as in Fig. (1).

2. solution with $\mu = 1 + O(\zeta)$ and its dynamics

Another solution is obtained by setting

$$X = \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right), \quad (3.26)$$

leading to

$$\mu = 1 + \frac{\zeta^2}{\sqrt{s_1}^3} (\sqrt{s_1} - 2GM_{\text{eff}}) \quad (3.27)$$

with

$$M_{\text{eff}} = \frac{s_1^{3/2} \sin^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{2\zeta^2 G} - \frac{\sqrt{s_1} s_4^2 \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{8G} + \frac{\sqrt{s_1}}{2G} \quad (3.28)$$

Then, we have

$$\begin{aligned} \partial_{s_1} M_{\text{eff}} &= \frac{3\sqrt{s_1} \sin^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{4\zeta^2 G} - \frac{s_2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{4\zeta G} - \frac{s_4^2 \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{16G\sqrt{s_1}} - \frac{\zeta s_2 (s_4)^2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{16G s_1} + \frac{1}{4G\sqrt{s_1}} \\ \partial_{s_2} M_{\text{eff}} &= \frac{s_1 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{2\zeta G} + \frac{\zeta s_4^2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{8G} \\ \partial_{s_6} M_{\text{eff}} &= - \frac{\sqrt{s_1} s_4 \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{4G} \end{aligned} \quad (3.29)$$

Then, we get

$$\begin{aligned} H_{\text{eff}} &= -2E^2 \left\{ \frac{3\sqrt{s_1} \sin^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{4\zeta^2 G} - \frac{s_2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{4\zeta G} - \frac{s_4^2 \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{16G\sqrt{s_1}} - \frac{\zeta s_2 s_4^2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{16G s_1} + \frac{1}{4G\sqrt{s_1}} \right. \\ &\quad \left. + \frac{s_1 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{4\zeta G} s_3 + \frac{\zeta s_4^2 \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \right)}{16G} s_3 - \frac{\sqrt{s_1} s_6 \cos^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \right)}{4G} \right\} \\ &= - \frac{3\sqrt{E^1} E^2}{2\zeta^2 G} \sin^2 \left(\frac{\zeta K_2}{\sqrt{E^1}} \right) + \frac{K_2 E^2}{2\zeta G} \sin \left(\frac{2\zeta K_2}{\sqrt{E^1}} \right) - \frac{E^1 K_1}{2\zeta G} \sin \left(\frac{2\zeta K_2}{\sqrt{E^1}} \right) \\ &\quad - \frac{E^2}{2G\sqrt{E^1}} + \frac{(\partial_x E^1)^2}{8G\sqrt{E^1} E^2} \cos^2 \left(\frac{\zeta K_2}{\sqrt{E^1}} \right) + \frac{\sqrt{E^1}}{2G} \partial_x \left(\frac{\partial_x E^1}{E^2} \right) \cos^2 \left(\frac{\zeta K_2}{\sqrt{E^1}} \right) \\ &\quad + \frac{\zeta E^2}{8G} \left(\frac{\partial_x E^1}{E^2} \right)^2 \left(\frac{K_2}{E^1} - \frac{K^1}{E^2} \right) \sin \left(\frac{2\zeta K_2}{\sqrt{E^1}} \right) \end{aligned} \quad (3.30)$$

Solving the dynamics

We still choose the areal gauge to get

$$E^1(x) = x^2, \quad K_1(x) = \frac{E^2(x) \partial_x K_2(x)}{x}. \quad (3.31)$$

Then, we have

$$H_{\text{eff}} = - \frac{E^2(x)}{x} \partial_x M_{\text{eff}} \quad (3.32)$$

Vanishing the Hamiltonian constraint leads to

$$\begin{aligned} GM_{\text{eff}} &= \frac{x}{2} + \frac{x^3}{4\zeta^2} - \frac{x^3}{4\zeta^2} \cos \left(\frac{2\zeta K_2(x)}{x} \right) - \frac{x^3}{2E^2(x)^2} \cos^2 \left(\frac{\zeta K_2(x)}{x} \right) = GC_2 \\ \Rightarrow \cos \left(\frac{2\zeta K_2(x)}{x} \right) &= \frac{-4GC_2 \zeta^2 E^2(x)^2 + x^3 E^2(x)^2 + 2\zeta^2 x E^2(x)^2 - \zeta^2 x^3}{x^3 (E^2(x)^2 + \zeta^2)} \end{aligned} \quad (3.33)$$

Since we are interested in the stationary metric, we need to choose the lapse function $N(x)$ and the shift vector $N^x(x)$ such that

$$\begin{aligned} \{E^1(x), H_{\text{eff}}[N] + H_x[N^x]\} &= 0 \\ \{E^2(x), H_{\text{eff}}[N] + H_x[N^x]\} &= 0. \end{aligned} \quad (3.34)$$

As a consequence of the first equation, we get

$$N^x(x) = -\frac{(4x^2\zeta^2 + 4x^2E^2(x)^2)}{8x\zeta E^2(x)^2} \sin\left(\frac{2\zeta K_2(x)}{x}\right) N(x). \quad (3.35)$$

Substituting this result into the second equation, we have

$$N(x) = \frac{C_1 x}{E^2(x)}. \quad (3.36)$$

Substituting (3.33) into (3.35), we get

$$N^x(x) = \pm \frac{C_1 \sqrt{(x^3 + x\zeta^2 - 2GC_2\zeta^2)(x^3 - (x - 2GC_2)E^2(x)^2)}}{xE^2(x)^2} \quad (3.37)$$

Moreover, we have

$$\mu = 1 + \frac{\zeta^2}{\sqrt{s_1}^3} (\sqrt{s_1} + GM_{\text{eff}}) = 1 + \frac{\zeta^2}{x^2} - \frac{2\zeta^2 C_2}{x^3} \quad (3.38)$$

We thus have the metric, with the minus sign in N^x considered,

$$\begin{aligned} ds^2 &= -N^2 dt^2 + \frac{(E^2)^2}{\mu E^1} (dx + N^x dt)^2 + E^1 d\Omega^2 \\ &= -\frac{C_1 x}{(E^2)^2} dt^2 + \frac{(E^2)^2}{\left(1 + \frac{\zeta^2}{x^2} - \frac{2G\zeta^2 C_2}{x^3}\right) x^2} \left(dx - \frac{C_1 \sqrt{(x^3 + x\zeta^2 - 2GC_2\zeta^2)(x^3 - (x - 2GC_2)E^2(x)^2)}}{xE^2(x)^2} dt\right)^2 + x^2 d\Omega^2 \end{aligned} \quad (3.39)$$

In the Schwarzschild gauge with $g_{01} = 0 = g_{10}$, we have with $C_2 = M$,

$$ds^2 = -\left(1 - \frac{2GM}{x}\right) dt^2 + \left(1 + \frac{\zeta^2}{x^2} - \frac{2GM\zeta^2}{x^3}\right)^{-1} \left(1 - \frac{2GM}{x}\right)^{-1} dx^2 + x^2 d\Omega^2. \quad (3.40)$$

In the PG gauge, we have

$$ds^2 = -dt^2 + \left(1 + \frac{\zeta^2}{x^2} - \frac{2G\zeta^2 M}{x^3}\right)^{-1} \left(dx + \sqrt{\frac{2GM}{x} + \frac{2GM\zeta^2}{x^3} \left(1 - \frac{2GM}{x}\right)} dt\right)^2 + x^2 d\Omega^2 \quad (3.41)$$

It can be checked that, under the coordinate transformation

$$dt_{\text{PG}} \mapsto dt_{\text{sch}} - \frac{\sqrt{2}x^2 \sqrt{\frac{GM}{\zeta^2(x-2GM)+x^3}}}{2GM-x} dx_{\text{sch}}, \quad dx_{\text{PG}} \mapsto dx_{\text{sch}}, \quad (3.42)$$

the metric ds_{PG}^2 can be transformed into ds_{sch}^2 .

Then spacetime structure of the spacetime is shown in Fig. 2. In this diagram, there exist those homogeneous region $B \cup W$. In what follows, we will show how to solve the dynamics in these region by choosing the homogeneous gauge.

Solving the dynamics: homogeneous gauge

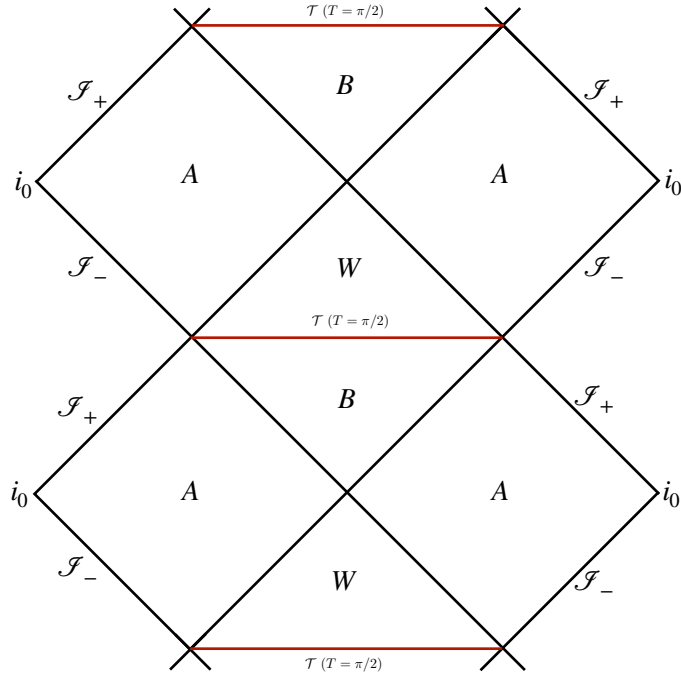


FIG. 2: The Penrose diagram of the metric given in (3.40).

Let us consider the homogeneous solution, where consider the solution such that

$$\partial_x E^I = 0 = \partial_x K_I. \quad (3.43)$$

Then, the diffeomorphism constraint is solve automatically. For the Hamiltonian constraint, we have

$$H_{\text{eff}} = -\frac{3\sqrt{E^1}E^2}{2\zeta^2 G} \sin^2\left(\frac{\zeta K_2}{\sqrt{E^1}}\right) + \frac{K_2 E^2}{2\zeta G} \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right) - \frac{E^1 K_1}{2\zeta G} \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right) - \frac{E^2}{2G\sqrt{E^1}} \quad (3.44)$$

The constant of motion M_{eff} becomes

$$M_{\text{eff}} = \frac{\sqrt{E^1}}{2G} + \frac{\sqrt{E^1}^3}{2G\zeta^2} \sin^2\left(\frac{\zeta K_2}{\sqrt{E^1}}\right). \quad (3.45)$$

The first solution is obtained by choosing $E^1 = t^2$

We have

$$M = M_{\text{eff}} = \frac{t}{2G} + \frac{t^3}{2G\zeta^2} \sin^2\left(\frac{\zeta K_2}{t}\right) \Rightarrow \sin^2\left(\frac{\zeta K_2}{t}\right) = \frac{G\zeta^2}{t^3} \left(2M - \frac{t}{G}\right) \quad (3.46)$$

On the other hand, we have

$$2t = \dot{E}^1 = \{E^1, H_{\text{eff}}[N]\} = \frac{E^1 N}{\zeta} \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right) = \pm 2 \frac{E^1 N}{\zeta} \sqrt{\frac{G\zeta^2}{t^3} \left(2M - \frac{t}{G}\right)} \sqrt{1 - \frac{G\zeta^2}{t^3} \left(2M - \frac{t}{G}\right)} \quad (3.47)$$

leading to

$$N = \pm \frac{1}{\sqrt{\frac{2GM}{t} - 1} \sqrt{1 + \frac{\zeta^2}{t^2} \left(1 - \frac{2GM}{t}\right)}}. \quad (3.48)$$

To get E^2 , we need to consider

$$\dot{E}^2 = N \left(E^2 \left(\frac{\sin\left(\frac{2\zeta K_2}{t}\right)}{\zeta} - \frac{K_2 \cos\left(\frac{2\zeta K_2}{t}\right)}{t} \right) + t K_1 \cos\left(\frac{2\zeta K_2}{t}\right) \right) \quad (3.49)$$

and the constraint $H_{\text{eff}} = 0$ to get

$$K_1 = - \frac{E^2 \left(3t^2 \sin^2\left(\frac{\zeta K_2}{t}\right) \csc\left(\frac{2\zeta K_2}{t}\right) + \zeta^2 \csc\left(\frac{2\zeta K_2}{t}\right) - \zeta t K_2 \right)}{t^3 \zeta} \quad (3.50)$$

Combining (3.49), (3.50), (3.48) and the gauge fixing condition $E^1 = t^2$, we finally get

$$E^2 = \frac{\sqrt{2GM} - t \sqrt{-2\zeta^2 GM + t^3 + \zeta^2 t}}{t}. \quad (3.51)$$

In this case,

$$\mu = 1 + \frac{\zeta^2}{\sqrt{E^1}^3} (\sqrt{E^1} - 2GM) = 1 + \frac{\zeta^2}{t^3} (t - 2GM) = \frac{-2GM\zeta^2 + \zeta^2 t + t^3}{t^3}. \quad (3.52)$$

We finally have the metric

$$ds^2 = - \frac{1}{\mu \left(\frac{2GM}{t} - 1 \right)} dt^2 + \left(\frac{2GM}{t} - 1 \right) dx^2 + t^2 d\Omega^2. \quad (3.53)$$

The second solution is obtained by choosing $\frac{\zeta K_2}{\sqrt{E^1}} = t$.

We have

$$M = M_{\text{eff}} = \frac{\sqrt{E^1}}{2G} + \frac{\sqrt{E^1}^3}{2G\zeta^2} \sin^2(t) \quad (3.54)$$

leading to

$$\sqrt{E^1} = R(t) = \frac{1}{\sin(t)} \left(\frac{\zeta^2}{\beta(t)} - \frac{\beta(t)}{3} \right), \quad \beta(t)^3 = 3\zeta^2 \left(-9GM \sin(t) + \sqrt{3\zeta^2 + 81G^2 M^2 \sin^2(t)} \right). \quad (3.55)$$

On the other hands, we have

$$\dot{R}(t) = \{ \sqrt{E^1}, H_{\text{eff}}[N] \} = \frac{N}{2\zeta} R(t) \sin(2t). \quad (3.56)$$

To get N , let is consider

$$0 = \dot{M}_{\text{eff}} = - \frac{3R(t)^2 \sin^2(t) \dot{R}(t)}{\zeta^2 G} - \frac{\dot{R}(t)}{G} - \frac{2R(t)^3 \sin(t) \cos(t)}{\zeta^2 G} = - \frac{\zeta^2 \dot{R}(t) + 3R(t)^2 \sin^2(t) \dot{R}(t) + R(t)^3 \sin(2t)}{\zeta^2 G} \quad (3.57)$$

leading to

$$\dot{R}(t) = - \frac{2R(t)^3 \sin(t) \cos(t)}{\zeta^2 + 3R(t)^2 \sin^2(t)}, \quad N = - \frac{2\zeta R(t)^2}{\zeta^2 + 3R(t)^2 \sin^2(t)}. \quad (3.58)$$

Now let us consider E^2 . To this end, we need

$$\begin{aligned} \dot{E}^2 &= N \left(\frac{(E^1 K_1 - E^2 K_2) \cos\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right)}{\sqrt{E^1}} + \frac{E^2 \sin\left(\frac{2\zeta K_2}{\sqrt{E^1}}\right)}{\zeta} \right) \\ &= \frac{N(E^2(\sin(2t) - t \cos(2t)) + \zeta K_1 R \cos(2t))}{\zeta} \end{aligned} \quad (3.59)$$

and $H_{\text{eff}} = 0$ leading to

$$K_1 = -\frac{E^2 \left(\zeta^2 \csc \left(\frac{2\zeta K_2}{\sqrt{E^1}} \right) - \zeta \sqrt{E^1} K_2 + 3E^1 \sin^2 \left(\frac{\zeta K_2}{\sqrt{E^1}} \right) \csc \left(\frac{2\zeta K_2}{\sqrt{E^1}} \right) \right)}{\zeta (E^1)^{3/2}} \quad (3.60)$$

We thus get

$$\frac{\dot{E}^2}{E^2} = N \left(-\frac{\zeta \cot(2t)}{R(t)^2} - \frac{(\sin(3t) - 5 \sin(t)) \sec(t)}{4\zeta} \right) \quad (3.61)$$

According to (3.56), we get

$$N = \frac{2\zeta \dot{R}(t)}{R(t) \sin(2t)} \quad (3.62)$$

resulting in

$$\frac{\dot{E}^2}{E^2} = \frac{2\zeta \dot{R}(t)}{R(t) \sin(2t)} \left(-\frac{\zeta \cot(2t)}{R(t)^2} - \frac{(\sin(3t) - 5 \sin(t)) \sec(t)}{4\zeta} \right) \quad (3.63)$$

Moreover, (3.54) leads to

$$\sin^2(t) = \frac{G\zeta^2}{R(t)^3} \left(2M - \frac{R(t)}{G} \right) = \frac{2GM\zeta^2}{R(t)^3} - \frac{\zeta^2}{R(t)^2} \quad (3.64)$$

Substituting this result into (3.61), we get

$$\frac{\dot{E}^2}{E^2} = \frac{\dot{R}(t) (4\zeta^2 G^2 M^2 - 2\zeta^2 GMR(t) + GMR(t)^3 - R(t)^4)}{R(t)(2GM - R(t))(-2\zeta^2 GM + \zeta^2 R(t) + R(t)^3)} \quad (3.65)$$

leading to

$$E^2 = \frac{\sqrt{2GM - R(t)} \sqrt{-2\zeta^2 GM + \zeta^2 R(t) + R(t)^3}}{R(t)}. \quad (3.66)$$

In this case, we have

$$\mu = \frac{-2GM\zeta^2 + \zeta^2 R(t) + R(t)^3}{R(t)^3}. \quad (3.67)$$

Thus, we have the metric

$$ds^2 = -\frac{4\zeta^2 R(t)^4}{(\zeta^2 + 3R(t)^2 \sin^2(t))^2} dt^2 + \left(\frac{2GM}{R(t)} - 1 \right) dx^2 + R(t)^2 d\Omega^2. \quad (3.68)$$

The transition surface is located at $t = \frac{\pi}{2}$.

To see that (3.68) is the same as the one under PG coordinate, let us rewrite (3.68) as

$$ds^2 = -\frac{4\zeta^2 R(T)^4}{(\zeta^2 + 3R(T)^2 \sin^2(T))^2} dT^2 + \left(\frac{2GM}{R(T)} - 1 \right) dX^2 + R(T)^2 d\Omega^2. \quad (3.69)$$

Then, we consider the coordinate transformation $(t, x) \mapsto (T, X)$ so that

$$x = R(T), \quad dt = dX + \alpha dT \quad (3.70)$$

To rewrite the metric under PG coordinate as

$$\begin{aligned}
ds^2 &= - (dX + \alpha dT)^2 + \mu^{-1} \left(\dot{R}(T) dT \pm \sqrt{\mu \frac{2M}{R(T)}} (dX + \alpha dT) \right)^2 \\
&= - dX^2 - 2\alpha dX dT - \alpha^2 dT^2 + \mu^{-1} \left(\left(\dot{R}(T) \pm \alpha \sqrt{\mu \frac{2M}{R(T)}} \right) dT \pm \sqrt{\mu \frac{2M}{R(T)}} dX \right)^2 \\
&= \left(-\alpha^2 + \mu^{-1} \left(\dot{R}(T) \pm \alpha \sqrt{\mu \frac{2M}{R(T)}} \right)^2 \right) dT^2 + \left(\frac{2M}{R(T)} - 1 \right) dX^2 \\
&\quad + 2 \left(-\alpha \pm \mu^{-1} \left(\dot{R}(T) \pm \alpha \sqrt{\mu \frac{2M}{R(T)}} \right) \sqrt{\mu \frac{2M}{R(T)}} \right) dX dT
\end{aligned} \tag{3.71}$$

In order to get the metric (3.69), we need

$$\begin{aligned}
0 &= -\alpha \pm \mu^{-1} \left(\dot{R}(T) \pm \alpha \sqrt{\mu \frac{2M}{R(T)}} \right) \sqrt{\mu \frac{2M}{R(T)}} \\
\Rightarrow 0 &= -\alpha \pm \mu^{-1} \dot{R}(T) \sqrt{\mu \frac{2M}{R(T)}} + \alpha \frac{2M}{R(T)} \\
\Rightarrow \alpha &= \pm \mu^{-1} \dot{R}(T) \sqrt{\mu \frac{2M}{R(T)}} \left(1 - \frac{2M}{R(T)} \right)^{-1}
\end{aligned} \tag{3.72}$$

Then, we get the value of the coefficient of dT^2 in (3.71),

$$\begin{aligned}
-\alpha^2 + \mu^{-1} \left(\dot{R}(T) \pm \alpha \sqrt{\mu \frac{2M}{R(T)}} \right)^2 &= \frac{R(T)^4 \dot{R}(T)^2}{(R(T) - 2M) (-2\zeta^2 M + \zeta^2 R(T) + R(T)^3)} \\
&= \frac{R(T)^{10} \sin^2(2T)}{(R(T) - 2M) (-2\zeta^2 M + \zeta^2 R(T) + R(T)^3) (\zeta^2 + 3R(T)^2 \sin^2(T))^2} \\
&= - \frac{4\zeta^2 R(T)^4}{(\zeta^2 + 3R(T)^2 \sin^2(T))^2}.
\end{aligned} \tag{3.73}$$

where(3.58) is applied and we used (3.54) to get

$$\sin(2T)^2 = - \frac{4\zeta^2(2M - R(T)) (2\zeta^2 M - R(T) (\zeta^2 + R(T)^2))}{R(T)^6} \tag{3.74}$$

C. New solution without Cauchy horizon

The new solution is given by

$$M_{\text{eff}}^{(3)} = h(s_1) + \frac{\mathcal{F}(s_1) \sqrt{s_1}}{2G\lambda(s_1)} \sin \left(\lambda(s_1) \left[1 + (s_2)^2 - \frac{(s_4)^2}{4} + \psi(s_1) \right] \right) \tag{3.75}$$

for arbitrary functions \mathcal{F} , h , λ and ψ . Plugging $M_{\text{eff}}^{(3)}$ into Eqs. (2.76a) and (2.76b), we get

$$\mu \equiv \mu_3 = \mathcal{F}(s_1)^2 \left\{ 1 - \left[\frac{2G\lambda(s_1) [M_{\text{eff}}^{(3)} - h(s_1)]}{\sqrt{s_1}} \right]^2 \right\}. \tag{3.76}$$

The resulting effective Hamiltonian constraint reads

$$H_{\text{eff}}^{(3)} = -\mathcal{F}(E^1) \left[\frac{\sqrt{E^1} E^2}{G\lambda(E^1)} \frac{\partial \mathfrak{F}}{\partial E^1} + \frac{\sqrt{E^1} K_1 K_2}{G} - \frac{\sqrt{E^1}}{2G} \partial_x \left(\frac{\partial_x E^1}{E^2} \right) \right] \cos(\mathfrak{F}) - \frac{E^2}{G} \frac{\partial}{\partial E^1} \left(\frac{\sqrt{E^1} \mathcal{F}(E^1)}{\lambda(E^1)} \right) \sin(\mathfrak{F}) + E^2 \left(\frac{\partial h}{\partial E^1} + \mathcal{R}(E^1, M_{\text{eff}}^{(3)}) \right), \quad (3.77)$$

where

$$\mathfrak{F} \equiv \lambda(E^1) \left[1 + (K_2)^2 - \frac{(\partial_x E^1)^2}{4(E^2)^2} + \psi(E^1) \right], \quad (3.78)$$

and hence

$$\frac{\partial \mathfrak{F}}{\partial E^1} = \lambda'(E^1) \left[1 + (K_2)^2 - \frac{(\partial_x E^1)^2}{4(E^2)^2} + \psi(E^1) \right] + \psi'(E^1) \lambda(E^1). \quad (3.79)$$

1. *The spacetime metric from $H_{\text{eff}}^{(3)}$ with $\mathcal{R} = 0$*

To get the metric, it is convenient to choose the areal gauge

$$E^1(x) = x^2. \quad (3.80)$$

Then, solving the diffeomorphism constraint, we get

$$K_1(x) = \frac{E^2(x) \partial_x K_2(x)}{x}. \quad (3.81)$$

By the gauge fixing conditions (3.80) and (3.81), the effective Hamiltonian constraint $H_{\text{eff}}^{(3)}$ becomes

$$H_{\text{eff}}^{(3)}(x) = -\frac{E^2(x)}{x} \partial_x \hat{M}_{\text{eff}}^{(3)}(x), \quad (3.82)$$

where

$$\hat{M}_{\text{eff}}^{(3)}(x) = h(x^2) + \frac{x \mathcal{F}(x^2) \sin(\lambda(x^2) F(x))}{2G\lambda(x^2)},$$

with

$$F(x) = 1 + K_2(x)^2 - \frac{x^2}{E^2(x)^2} + \psi(x^2) \quad (3.83)$$

is just the value of $M_{\text{eff}}^{(3)}$ under the gauge fixing conditions (3.80) and (3.81).

Taking account of the stationary condition

$$\begin{aligned} 0 &= \{E^1(x), H_{\text{eff}}^{(3)}[N] + H_x[N^x]\}, \\ 0 &= \{E^2(x), H_{\text{eff}}^{(3)}[N] + H_x[N^x]\}, \end{aligned} \quad (3.84)$$

we have

$$\begin{aligned} N(x) &= \frac{x}{E^2(x)}, \\ \frac{N^x(x)}{N(x)} &= -\cos(\lambda(x^2) F(x)) K_2(x) \mathcal{F}(x^2). \end{aligned} \quad (3.85)$$

In addition, the vanishing of $H_{\text{eff}}^{(3)}$ yields

$$\sin(\lambda(x^2) F(x)) = \frac{2G\lambda(x^2)}{x\mathcal{F}(x^2)} (M - h(x^2)). \quad (3.86)$$

To get the metric in the Schwarzschild-like coordinate, we choose $K_2(x) = 0$ so that $N^x(x) = 0$. Then, we have

$$\frac{x^2}{E^2(x)^2} = f_3^{(n)}(x) \equiv 1 - \frac{n\pi + (-1)^n \arcsin\left(\frac{2G\lambda(x^2)(M-h(x^2))}{x\mathcal{F}(x^2)}\right)}{\lambda(x^2)} + \psi(x^2), \quad (3.87)$$

where $n \in \mathbb{Z}$ is an arbitrary integer. We can also get

$$\mu_3 = \mathcal{F}(x^2)^2 - \frac{4G^2\lambda(x^2)^2}{x^2} (M - h(x^2))^2. \quad (3.88)$$

Thus, line element in this case reads

$$ds_{(3)}^2 = -f_3^{(n)} dt^2 + \mu_3^{-1} (f_3^{(n)})^{-1} dx^2 + x^2 d\Omega^2. \quad (3.89)$$

To get the line element of the metric in the Painlevé-Gullstrand-like coordinates, one needs to choose the gauge fixing condition $N(x) = 1$, i.e., $E^2(x) = x$. This condition, together with the constraint equation of $H_{\text{eff}}^{(3)}$, leads to

$$N^x = \pm \sqrt{\mu_3(1 - f_3^{(n)})}. \quad (3.90)$$

Thus, the line element in the Painlevé-Gullstrand-like coordinates, denoted by $(t_{\pm}, x, \theta, \phi)$, reads

$$ds_{(3)}^2 = -dt_{\pm}^2 + \frac{1}{\mu_3} \left(dx \pm \sqrt{\mu_3(1 - f_3^{(n)})} dt_{\pm} \right)^2 + x^2 d\Omega^2. \quad (3.91)$$

where the sign of $+$ corresponds to the ingoing Painlevé-Gullstrand-like coordinates and the sign of $-$ corresponds to the outgoing Painlevé-Gullstrand-like coordinates. It is easy to check that the metric (3.89) is the same as that of Eq. (3.91) up to a coordinate transformation, as a consequence of the fact that the theory described by $H_{\text{eff}}^{(3)}$ is covariant.

2. Spacetime structure of $ds_{(3)}^2$ in a concrete example

It should be noted that the arguments of the sine function in $H_{\text{eff}}^{(3)}$ involve $K_2(x)^2$. Inspired by the effective loop quantum black hole models, where the arguments of the sine function typically include $K_2(x)$, we set $\lambda(s_1)$ as the square of the corresponding term in the $\bar{\mu}$ -scheme of those models. Thus, we will consider the spacetime structure of $ds_{(3)}^2$ with λ and ψ chosen as

$$\lambda(s_1) = \frac{\zeta^2}{s_1}, \quad \psi(s_1) = 0. \quad (3.92)$$

For simplicity, the other two free functions are chosen as

$$\mathcal{F}(s_1) = 1, \quad h(s_1) = 0. \quad (3.93)$$

Substituting Eqs. (3.92) and (3.93) into Eq. (3.77), we get the specific expression of $H_{\text{eff}}^{(3)}$ as

$$H_{\text{eff}}^{(3)} = - \left[- \frac{E^2}{G\sqrt{E^1}} \left(1 + (K_2)^2 - \frac{(\partial_x E^1)^2}{4(E^2)^2} \right) + \frac{\sqrt{E^1} K_1 K_2}{G} - \frac{\sqrt{E^1}}{2G} \partial_x \left(\frac{\partial_x E^1}{E^2} \right) \right] \\ \times \cos \left(\frac{\zeta^2}{E^1} \left[1 + (K_2)^2 - \frac{(\partial_x E^1)^2}{4(E^2)^2} \right] \right) - \frac{3\sqrt{E^1} E^2}{2G\zeta^2} \sin \left(\frac{\zeta^2}{E^1} \left[1 + (K_2)^2 - \frac{(\partial_x E^1)^2}{4(E^2)^2} \right] \right)$$

The line element of the metric in the Schwarzschild-like coordinates becomes

$$ds_{(3)}^2 = -\bar{f}_3^{(n)} dt^2 + \bar{\mu}_3^{-1} (\bar{f}_3^{(n)})^{-1} dx^2 + x^2 d\Omega^2, \quad (3.94)$$

with

$$\bar{f}_3^{(n)}(x) = 1 - (-1)^n \frac{x^2}{\zeta^2} \arcsin \left(\frac{2GM\zeta^2}{x^3} \right) - \frac{n\pi x^2}{\zeta^2}, \quad (3.95) \\ \bar{\mu}_3(x) = 1 - \frac{4G^2\zeta^4 M^2}{x^6}.$$

The metric can return to the Schwarzschild metric as x approaches ∞ only if $n = 0$. Hence we set $n = 0$ initially to analyze the metric.

To ensure that the argument of the arcsine function remains within its valid domain, one needs to impose the condition

$$x \geq (2GM\zeta^2)^{1/3} \equiv x_{\min}. \quad (3.96)$$

For x within this range, we get

$$0 \leq \bar{\mu}_3 \leq 1. \quad (3.97)$$

The number of horizons in the spacetime of $ds_{(3)}^2$ is determined by the number of real roots of $\bar{f}_3^{(0)}$. Observing that as $x \rightarrow \infty$, $\bar{f}_3^{(0)}$ approaches 1, a positive number, the number of roots of $\bar{f}_3^{(0)}$ is therefore determined by the sign of its value at $x = (2GM\zeta^2)^{1/3}$, which reads

$$\bar{f}_3^{(0)} = 1 - \frac{\pi}{2} \left(\frac{2GM}{\zeta} \right)^{2/3}. \quad (3.98)$$

Let us introduce

$$m_o = \frac{\zeta}{2} \left(\frac{2}{\pi} \right)^{3/2}. \quad (3.99)$$

It is easy to see from Eq. (3.98) that there is no real root of $\bar{f}_3^{(0)}$ for $GM < m_o$, while there is one real root of $\bar{f}_3^{(0)}$ for $GM \geq m_o$. Now we can explore the spacetime structure of $ds_{(3)}^2$ case by case.

Spacetime structure for $GM < m_o$

In the case of $GM < m_o$, the coordinates employed in Eq. (3.94) are well-defined for the region of $x > x_{\min}$. To extend the spacetime beyond $x = x_{\min}$, we introduce a new coordinate $z(x)$, as suggested in [?] for a similar context, defined by

$$\left(\frac{dz}{dx} \right)^2 = \mu_3^{-1}. \quad (3.100)$$

In the coordinates (t, z, θ, ϕ) , the metric reads

$$ds_{(3)}^2 = -\bar{f}_3^{(0)}(z)dt^2 + (\bar{f}_3^{(0)}(z))^{-1}dz^2 + x(z)^2d\Omega^2, \quad (3.101)$$

where $\bar{f}_3^{(0)}(z) := \bar{f}_3^{(0)}(x(z))$ and $x(z)$ denotes the inverse function of $z(x)$ determined by Eq. (3.100). The metric (3.101) is well defined at $x(z) = x_{\min}$. Since $x = x_{\min}$ is a root of $\mu_3(x)$, the function $x(z)$ exhibits a turning point at $x(z) = x_{\min}$. This implies that the extension beyond $x = x_{\min}$ turns back to $x > x_{\min}$. However, it should be noted that, due to the arcsine function in $\bar{f}_3^{(n)}$, after the turning point, the line element should transition to the branch of the arcsine function corresponding to $n = 1$. Then the metric, after the turning point, becomes

$$ds_{(3)}^2 = -\bar{f}_3^{(1)}(z)dt^2 + (\bar{f}_3^{(1)}(z))^{-1}dz^2 + x(z)^2d\Omega^2. \quad (3.102)$$

The choice in Eq. (3.102) ensures the smoothness of the metric, i.e.,

$$\lim_{z \rightarrow z_o^+} \frac{d^n}{dz^n} f_{\text{eff,III}}^{(0)} = \lim_{z \rightarrow z_o^-} \frac{d^n}{dz^n} f_{\text{eff,III}}^{(1)}. \quad (3.103)$$

It is obvious that, for $GM < m_o$, $\bar{f}_3^{(1)}$ has a real root, denoted by x_h , satisfying the inequality $x_h > x_{\min}$. Hence, there is a horizon in the spacetime region associated with $\bar{f}_3^{(1)}$. The expression of $\bar{f}_3^{(1)}$ in Eq. (3.95) indicates that, as $x \rightarrow \infty$, the spacetime region associated with $\bar{f}_3^{(1)}$ asymptotically approaches to a de-Sitter spacetime. Thus, by gluing the spacetime regions associated with $\bar{f}_3^{(0)}$ and $\bar{f}_3^{(1)}$, we get the Penrose diagram of the spacetime $ds_{(3)}^2$ as shown in Fig. 3, where A is the region associated with $\bar{f}_3^{(0)}$, and $A' \cup B$ and $A' \cup W$ are those associated with $\bar{f}_3^{(1)}$.

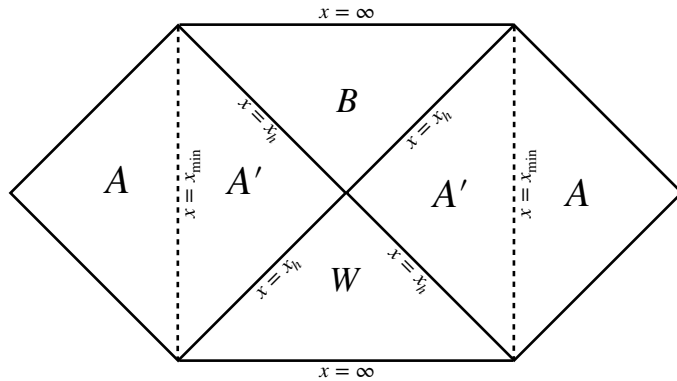


FIG. 3: The Penrose diagram of the spacetime $ds_{(3)}^2$ for $GM < m_o$. The diagram contains the wormhole region $A \cup A'$ with throat occurring at $x = x_{\min}$, the BH region B and the WH region W .

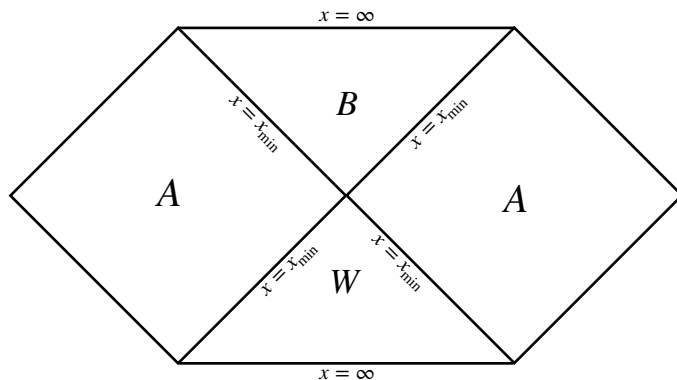


FIG. 4: The Penrose diagram of the spacetime $ds_{(3)}^2$ for $GM = m_o$. The diagram contains the asymptotically flat regions A , the BH region B and the WH region W .

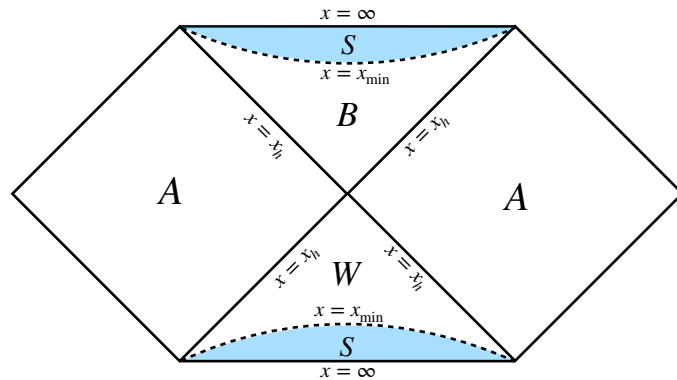


FIG. 5: The Penrose diagram of the spacetime $ds_{(3)}^2$ for $GM > m_o$. In our convention, the shaded regions S are not considered as part of either B or W . The diagram contains asymptotically flat regions A , the BH region B , the WH region W , and the Schwarzschild-de Sitter-like regions S .

For the region $B \cup A'$ or $W \cup A'$, the Painlevé-Gullstrand-like coordinates, denoted by $(t_{\pm}, x, \theta, \phi)$, can be employed to cover it. The line element in the region $B \cup A'$ is

$$ds_{(3)}^2 = -dt_+^2 + \mu_3^{-1} \left(dx + \sqrt{\mu_3(1 - \bar{f}_3^{(1)})} dt_+ \right)^2 + x^2 d\Omega^2. \quad (3.104)$$

and in the region $W \cup A'$ is

$$ds_{(3)}^2 = -dt_-^2 + \mu_3^{-1} \left(dx - \sqrt{\mu_3(1 - \bar{f}_3^{(1)})} dt_- \right)^2 + x^2 d\Omega^2. \quad (3.105)$$

Besides the coordinates (t, z, θ, ϕ) introduced in Eq. (3.102), to cover the entire wormhole region $A \cup A'$, we can also introduce the new coordinates (t, X, θ, ϕ) as follows:

$$x^3 = \frac{2GM\zeta^2}{\sin(X)}, \quad (3.106)$$

and hence

$$dX = \mp \frac{6\zeta^2 GM}{x^4} \frac{dx}{\sqrt{\mu_3}}. \quad (3.107)$$

Thus the metric in the new coordinates becomes

$$ds_{(3)}^2 = -\Phi(X)dt^2 + \frac{\Xi(X)^2}{9\Phi(X)\sin^2(X)}dX^2 + \Xi(X)^2d\Omega^2, \quad (3.108)$$

with

$$\begin{aligned} \Xi(X) &= \left[\frac{2GM\zeta^2}{\sin(X)} \right]^{1/3}, \\ \Phi(X) &= 1 - \left[\frac{2GM}{\zeta \sin(X)} \right]^{2/3} X. \end{aligned} \quad (3.109)$$

The wormhole throat is located at $X = \pi/2$, and the horizon appears at $X = X_o$, where X_o , greater than $\pi/2$, is the root of $\Phi(X)$. The range of X reads $0 < X < X_o$.

For the cases of $GM \geq m_o$, the procedure to construct the Penrose diagram of the corresponding metrics is similar to the case of $GM < m_o$. Hence, in the following cases, we will introduce the results directly.

Spacetime structure for $GM = m_o$

In the case of $GM = m_o$, $x = x_{\min} = \zeta\sqrt{2/\pi}$ becomes a horizon in the spacetime of $ds_{(3)}^2$. The Penrose diagram, as shown in Fig. 4, includes the asymptotically flat regions A , the BH region B and the WH region W . It is convenient to choose the Painlevé-Gullstrand-like coordinates $(t_{\pm}, x, \theta, \phi)$ to cover any of the regions A , B and W individually. Then the line element in the regions A reads

$$ds_{(3)}^2 = -dt^2 + \frac{1}{\mu_3} \left(dx + \sqrt{\mu_3(1 - \bar{f}_3^{(0)})} dt \right)^2 + x^2 d\Omega^2, \quad (3.110)$$

and the line element in the regions B and W reads

$$ds_{(3)}^2 = -dt_{\pm}^2 + \frac{1}{\mu_3} \left(dx + \sqrt{\mu_3(1 - \bar{f}_3^{(0)})} dt_{\pm} \right)^2 + x^2 d\Omega^2, \quad (3.111)$$

where t_+ and t_- are the time coordinate in the region B and W respectively.

Due to the vanishing of μ at $x = x_{\min}$, the Painlevé-Gullstrand-like coordinates $(t_{\pm}, x, \theta, \phi)$ cannot cross the horizon. To avoid the coordinate singularity, we can introduce the coordinate z defined by Eq. (3.100) or the coordinate X defined by Eq. (3.106) to replace the x coordinate in the Painlevé-Gullstrand-like coordinates. With the coordinates $(t_{\pm}, z, \theta, \phi)$ covering $A \cup B$ or $A \cup W$, the line elements respectively read

$$ds_{(3)}^2 = -dt_{\pm}^2 + \left(dz + \sqrt{\mu_3(z)(1 - \bar{f}_3^{(0)}(z))} dt_{\pm} \right)^2 + x(z)^2 d\Omega^2. \quad (3.112)$$

With the coordinates $(t_{\pm}, X, \theta, \phi)$, the line elements become

$$ds_{(3)}^2 = -dt_{\pm}^2 + \Xi(X)^2 \left[\frac{dX}{3\sin(X)} + \frac{\sqrt{X}}{\zeta} dt_{\pm} \right]^2 + \Xi(X)^2 d\Omega^2. \quad (3.113)$$

Spacetime structure for $GM > m_o$

In this case, $\bar{f}_3^{(0)}$ has a real root denoted by x_h where the horizons form. The Penrose diagram, as shown in Fig. 5, consists of the asymptotically flat regions A , the BH region B , the WH region W and the Schwarzschild-de Sitter-like region S . In the current scenario, the surface $x = x_{\min}$ occurs inside the horizon and is spacelike.

In either the region $A \cup B$ or the region $A \cup W$, one can choose the Painlevé-Gullstrand-like coordinates $(t_{\pm}, x, \theta, \phi)$ to cover it. Then, the line element reads

$$ds_{(3)}^2 = -dt_{\pm}^2 + \frac{1}{\bar{\mu}_3} \left(dx + \sqrt{\bar{\mu}_3(1 - \bar{f}_3^{(0)})} dt_{\pm} \right)^2 + x^2 d\Omega^2, \quad (3.114)$$

where as in Eqs. (3.104) and (3.105), the signs \pm correspond to the regions $A \cup B$ and $A \cup W$ respectively.

In the shadow region S , the Schwarzschild-like coordinate can be chosen, in which the line element reads

$$ds_{(3)}^2 = -\bar{f}_3^{(1)} dt^2 + \bar{\mu}_3^{-1} (\bar{f}_3^{(1)})^{-1} dx^2 + x^2 d\Omega^2. \quad (3.115)$$

It is easy to see that $\bar{f}_3^{(1)} < 0$ for all $x > x_{\min}$ in the current case. Moreover, the metric in region S asymptotically approaches the Schwarzschild-de Sitter one with negative mass in the far future.

To define the coordinates covering the entire region $A \cup B \cup S$ or $A \cup W \cup S$, we again use the coordinates $(t_{\pm}, z, \theta, \phi)$ with z define by Eq. (3.100), or the coordinates $(t_{\pm}, X, \theta, \phi)$ with X given by Eq. (3.106). Then, the metric in these two coordinate systems will take the same forms as those in Eq. (3.112) and Eq. (3.113) respectively.

IV. GENERAL SOLUTION OF THE COVARIANCE EQUATION

Now let us consider the general solution to the covariance equation (2.76a) and (2.76b).

(2.76b) implies that μ is a function of M_{eff} and s_1 . To solve (2.76a), let us introduce $\tilde{s}_4 = (s_4)^2$. Then, we have

$$\frac{\partial M_{\text{eff}}}{\partial \tilde{s}_4} = \frac{\partial M_{\text{eff}}}{\partial s_4} \frac{ds_4}{d\tilde{s}_4} = \frac{1}{2s_4} \frac{\partial M_{\text{eff}}}{\partial s_4}. \quad (4.1)$$

Therefore (2.76a) can be simplified to

$$(\partial_{M_{\text{eff}}} F)^{-1} = \partial_{s_2} M_{\text{eff}} \partial_{s_2} \partial_{\tilde{s}_4} M_{\text{eff}} - \partial_{\tilde{s}_4} M_{\text{eff}} \partial_{s_2}^2 M_{\text{eff}} \quad (4.2)$$

where $F(s_1, M_{\text{eff}})$ is defined by

$$(\partial_{M_{\text{eff}}} F)^{-1} = \frac{\mu s_1}{8}. \quad (4.3)$$

Let us introduce $Q = \partial_{s_2} M_{\text{eff}}$. Then, (4.2) leads to

$$1 = \partial_{M_{\text{eff}}} F \partial_{s_2} M_{\text{eff}} \partial_{\tilde{s}_4} Q - \partial_{M_{\text{eff}}} F \partial_{\tilde{s}_4} M_{\text{eff}} \partial_{s_2} Q = \partial_{s_2} F \partial_{\tilde{s}_4} Q - \partial_{\tilde{s}_4} F \partial_{s_2} Q = \det \begin{pmatrix} \partial_{s_2} F & \partial_{\tilde{s}_4} F \\ \partial_{s_2} Q & \partial_{\tilde{s}_4} Q \end{pmatrix}. \quad (4.4)$$

As we know,

$$\begin{pmatrix} \partial_{s_2} F & \partial_{\tilde{s}_4} F \\ \partial_{s_2} Q & \partial_{\tilde{s}_4} Q \end{pmatrix} \begin{pmatrix} \partial_F s_2 & \partial_Q s_2 \\ \partial_F \tilde{s}_4 & \partial_Q \tilde{s}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.5)$$

i.e.,

$$\begin{pmatrix} \partial_{s_2} F & \partial_{\tilde{s}_4} F \\ \partial_{s_2} Q & \partial_{\tilde{s}_4} Q \end{pmatrix}^{-1} = \begin{pmatrix} \partial_F s_2 & \partial_Q s_2 \\ \partial_F \tilde{s}_4 & \partial_Q \tilde{s}_4 \end{pmatrix} \quad (4.6)$$

According to (4.4),

$$\begin{pmatrix} \partial_{s_2} F & \partial_{\tilde{s}_4} F \\ \partial_{s_2} Q & \partial_{\tilde{s}_4} Q \end{pmatrix}^{-1} = \begin{pmatrix} \partial_{\tilde{s}_4} Q & -\partial_{\tilde{s}_4} F \\ -\partial_{s_2} Q & \partial_{s_2} F \end{pmatrix} \quad (4.7)$$

Therefore, we have

$$\begin{pmatrix} \partial_F s_2 & \partial_Q s_2 \\ \partial_F \tilde{s}_4 & \partial_Q \tilde{s}_4 \end{pmatrix} = \begin{pmatrix} \partial_{\tilde{s}_4} Q & -\partial_{\tilde{s}_4} F \\ -\partial_{s_2} Q & \partial_{s_2} F \end{pmatrix} \quad (4.8)$$

Then, we have

$$\frac{\partial \tilde{s}_4}{\partial Q} = \partial_{s_2} F = \partial_{M_{\text{eff}}} F \partial_{s_2} M_{\text{eff}} = Q \partial_{M_{\text{eff}}} F \quad (4.9)$$

leading to

$$\tilde{s}_4 = \frac{1}{2} Q^2 \partial_{M_{\text{eff}}} F - \mathcal{Z}(s_1, M_{\text{eff}}) \quad (4.10)$$

for arbitrary function \mathcal{Z} . By definition of Q and F , we finally obtain

$$\frac{\mu s_1}{4} (\tilde{s}_4 + \mathcal{Z}(s_1, M_{\text{eff}})) = (\partial_{s_2} M_{\text{eff}})^2. \quad (4.11)$$

This equation lead to

$$\partial_{s_2} M_{\text{eff}} = \pm \frac{\sqrt{s_1}}{2} \sqrt{\mu(s_1, M_{\text{eff}}) (\tilde{s}_4 + \mathcal{Z}(s_1, M_{\text{eff}}))} = \pm \frac{\sqrt{s_1}}{2} \sqrt{\mu(s_1, M_{\text{eff}}) ((s_4)^2 + \mathcal{Z}(s_1, M_{\text{eff}}))} \quad (4.12)$$

Substituting this result back into (2.76a), we can prove that (2.76a) is satisfied. Thus, (4.12) is equivalent to (2.76a). Solving (2.76a), we get

$$\pm \frac{\sqrt{s_1}}{2} s_2 + \Xi(s_1, s_4) = \int^{M_{\text{eff}}} \frac{dx}{\sqrt{\mu(s_1, x) [(s_4)^2 + \mathcal{Z}(s_1, x)]}}. \quad (4.13)$$

for arbitrary function Ξ .

A. General dynamics of the covariant model

To determine the metric, we need to solve for the lapse function N and the shift vector N^x from the following equations:

$$\begin{aligned} \dot{E}^1 &= \{E^1, H_{\text{eff}}[N] + H_x[N^x]\} = -2 \left(N \frac{\partial H_{\text{eff}}}{\partial K_1} + N^x \frac{H_x}{\partial K_1} \right) = -\frac{2N}{E^2} \frac{\partial H_{\text{eff}}}{\partial s_3} + N^x \partial_x E^1 \\ \dot{E}^2 &= \{E^2, H_{\text{eff}}[N] + H_x[N^x]\} = - \left(N \frac{\partial H_{\text{eff}}}{\partial K_2} - \partial_x \left(N^x \frac{H_x}{\partial(\partial_x K_2)} \right) \right) = -N \frac{\partial H_{\text{eff}}}{\partial s_2} + \partial_x (N^x E^2) \end{aligned} \quad (4.14)$$

By the relation between H_{eff} and M_{eff} given by (2.75), we have

$$\frac{\partial H_{\text{eff}}}{\partial s_3} = -E^2 \partial_{s_2} M_{\text{eff}} \quad (4.15)$$

and

$$\frac{\partial H_{\text{eff}}}{\partial s_2} = -2E^2 \left[\partial_{s_1} \partial_{s_2} M_{\text{eff}} + \frac{\partial_{s_2}^2 M_{\text{eff}}}{2} s_3 + \frac{\partial_{s_4} \partial_{s_2} M_{\text{eff}}}{s_4} s_5 + \partial_{M_{\text{eff}}} \mathcal{R} \partial_{s_2} M_{\text{eff}} \right] \quad (4.16)$$

Applying (4.15) and (4.16), we have

$$\begin{aligned} \dot{E}^1 &= 2N \partial_{s_2} M_{\text{eff}} + N^x \partial_x E^1 \\ \dot{E}^2 &= N \frac{2E^2}{\partial_x E^1} \partial_x (\partial_{s_2} M_{\text{eff}}) + 2NE^2 \partial_{M_{\text{eff}}} \mathcal{R} \partial_{s_2} M_{\text{eff}} + \partial_x (N^x E^2) \end{aligned} \quad (4.17)$$

According to following relation

$$H_{\text{eff}} - 2 \frac{G \partial_{s_2} M_{\text{eff}}}{\partial_x E^1} H_x = -\frac{2}{s_4} \mathcal{R} \partial_x s_1 - \frac{2}{s_4} \partial_x M_{\text{eff}}, \quad (4.18)$$

we could have that $H_{\text{eff}} = 0$ leads to

$$\partial_x M_{\text{eff}} + \partial_x E^1 \mathcal{R} = 0. \quad (4.19)$$

Moreover, we have

$$\dot{M}_{\text{eff}} + 2N \partial_{s_2} M_{\text{eff}} \mathcal{R} = N^x \partial_x M_{\text{eff}} \quad (4.20)$$

Applying (4.17), we have

$$\dot{M}_{\text{eff}} + \dot{E}^1 \mathcal{R} = N^x (\partial_x M_{\text{eff}} + \partial_x E^1 \mathcal{R}) = 0 \quad (4.21)$$

where we used $H_x = 0$ for the first equality and (4.19) for the last equality. According to the last equation, the integration constant in (4.19) must be time independent which can be seen as follows: according to (4.19) and (4.21), we have $M_{\text{eff}}(t, x) = M(E^1(t, x))$ with

$$\frac{dM}{dE^1} = -\mathcal{R}(E^1, M). \quad (4.22)$$

1. dynamics for arbitrary gauge with $\partial_{s_2} M_{\text{eff}} \neq 0$

Considering an arbitrary gauge by $E^I = E^I(t, x)$, we have according to (4.17),

$$N^x = \frac{\dot{E}^1 - 2N \partial_{s_2} M_{\text{eff}}}{\partial_x E^1} \quad (4.23)$$

and N satisfy the equation

$$\partial_x N = N \left(\partial_{M_{\text{eff}}} \mathcal{R} \partial_x E^1 + \partial_x \ln \left(\frac{\partial_x E^1}{E^2} \right) \right) + \frac{1}{2\partial_{s_2} M_{\text{eff}}} \left(\partial_x \partial_t E^1 - \frac{\partial_t E^1 \partial_x^2 E^1}{\partial_x E^1} + \frac{\partial_t E^1 \partial_x E^2}{E^2} - \frac{\partial_t E^2 \partial_x E^1}{E^2} \right) \quad (4.24)$$

whose solution is

$$N = C \frac{\partial_x E^1}{E^2} \exp \left(\int \partial_{M_{\text{eff}}} \mathcal{R} \partial_x E^1 dx \right) \quad (4.25)$$

with $C(x)$ satisfying

$$\partial_x C = \frac{\exp \left(- \int \partial_{M_{\text{eff}}} \mathcal{R} \partial_x E^1 dx \right)}{2\partial_x E^1 \partial_{s_2} M_{\text{eff}}} \left(E^2 \partial_x \partial_t E^1 - \frac{E^2 \partial_t E^1 \partial_x^2 E^1}{\partial_x E^1} + \partial_t E^1 \partial_x E^2 - \partial_t E^2 \partial_x E^1 \right) \quad (4.26)$$

2. dynamics for arbitrary gauge with $\partial_{s_2} M_{\text{eff}} = 0$

Now let us consider the case where we choose a gauge such that $\partial_{s_2} M_{\text{eff}} = 0$. In this case, (4.17) becomes

$$\begin{aligned} \dot{E}^1 &= N^x \partial_x E^1 \\ \dot{E}^2 &= \partial_x (N^x E^2) \end{aligned} \quad (4.27)$$

from which we cannot solve N . To get the result of N , we need to consider

$$\begin{aligned} 0 &= \partial_t (\partial_{s_2} M_{\text{eff}}) = \{ \partial_{s_2} M_{\text{eff}}, H_{\text{eff}}[N] + H_x[N^x] \} \\ &= \frac{\mu E^1 \left(\partial_x N - N \left(\partial_x E^1 \partial_{M_{\text{eff}}} \mathcal{R} + \partial_x \ln \left(\frac{\partial_x E^1}{E^2} \right) \right) \right)}{2(E^2)^3} + \frac{N \partial_x \left[(\partial_{s_2} M_{\text{eff}})^2 \right]}{\partial_x E^1} + N^x \partial_x \partial_{s_2} M_{\text{eff}} \\ &= \frac{\mu E^1 \left(\partial_x N - N \left(\partial_x E^1 \partial_{M_{\text{eff}}} \mathcal{R} + \partial_x \ln \left(\frac{\partial_x E^1}{E^2} \right) \right) \right)}{2(E^2)^3} \end{aligned} \quad (4.28)$$

where we used the constraint equations $H_{\text{eff}} = 0 = H_x$ and the covariance equation. This result leads to

$$N = C \frac{\partial_x E^1}{E^2} \exp \left(\int \partial_{M_{\text{eff}}} \mathcal{R} \partial_x E^1 dx \right) \quad (4.29)$$

with $\partial_x C = 0$.

3. Metric in Schwarzschild coordinates

On any open coordinate patch where the gauge choice $E^1 = x^2$ and $\partial_{s_2} M_{\text{eff}} = 0$ is admissible, the line element reads

$$ds^2 = \frac{1}{4} N_o^2 \mathcal{Z} dt^2 - \frac{4}{\mu \mathcal{Z}} dx^2 + x^2 d\Omega^2, \quad (4.30)$$

where N_o is

$$N_o = \exp\left(2 \int x(\partial_{M_{\text{eff}}} \mathcal{R}) dx\right). \quad (4.31)$$

Here we used $(E^2)^2 = -4x^2/\mathcal{Z}$ obtained by evaluating Eq. (4.11) in the gauge. In addition, since E^1 in the gauge is time independent, the line element (4.30) is static.

B. Reconstruction algorithm

Let us consider BH with metric of the form

$$ds^2 = -F(x; m) dt^2 + H(x; m)^{-1} dx^2 + x^2 d\Omega^2. \quad (4.32)$$

In our covariant model, the constant m entering the metric as the integration constant of

$$\partial_x M + \mathcal{R}(E^1, M) \partial_x E^1 = 0 \quad (4.33)$$

where the result should be identified with M_{eff} on shell. More precisely, $M_{\text{eff}} = M$ solves the constraints. Comparing (4.30) with (4.32), we get that if the model takes the given metric (4.32) as its vacuum solution, we should have

$$\frac{1}{4} N_o^2 \mathcal{Z} = -F, \quad \frac{\mu \mathcal{Z}}{4} = -H \quad (4.34)$$

where we need to substitute the evaluation $E^1 = x^2$ and $M_{\text{eff}} = M(x, m)$ to the LHSs so that both sides of the equations take x and m and their argument. In (4.33), we have relate \mathcal{R} to the mass function $M(x, m)$. To further express the function \mathcal{Z} and μ in terms of $M(x, m)$, let us consider the derivative of (4.33) with respect to m , leading to

$$\partial_x \partial_m M + 2x(\partial_{M_{\text{eff}}} \mathcal{R}) \partial_m M = 0 \quad (4.35)$$

where we have evaluated $E^1 = x^2$. In addition, by the expression of N_o in (4.31), it also satisfies the equation

$$\partial_x N_o - 2x(\partial_{M_{\text{eff}}} \mathcal{R}) N_o = 0 \implies \partial_x \left(\frac{1}{N_o} \right) + 2x(\partial_{M_{\text{eff}}} \mathcal{R}) \frac{1}{N_o} = 0. \quad (4.36)$$

Combining (4.36) with (4.35), we find that

$$\partial_m M N_o = C(m) \quad (4.37)$$

for some function $C(m)$. This function is convention-dependent and can therefore be fixed, for convenience, by choosing

$$C(m) = 1. \quad (4.38)$$

Thus, we have

$$N_o = \frac{1}{\partial_m M}. \quad (4.39)$$

Then, by defining

$$\alpha = \frac{F}{H}, \quad (4.40)$$

we can relate all the function \mathcal{R} , μ and \mathcal{Z} to $M(x, m)$ as

$$\mathcal{R} = -\frac{\partial_x M}{2x}, \quad \mu = \frac{1}{\alpha(\partial_m M)^2} = \frac{H}{F(\partial_m M)^2}, \quad \mathcal{Z} = -\frac{4H}{\mu} = -4F(\partial_m M)^2 \quad (4.41)$$

where \mathcal{R} , μ , and \mathcal{Z} are first expressed in terms of x and m , and then reinterpreted as functions of (s_1, M_{eff}) using $s_1 = x^2$ and $M_{\text{eff}} = M(x, m)$.

As the first, we consider the loop quantum black hole spacetime with

$$H = F = 1 - \frac{2m}{x} \left(1 - \frac{2\zeta^2 m}{x^3} \right). \quad (4.42)$$

Note that this metric was previously derived in various works on loop quantum black hole, using different approaches, but always in the absence of a covariant dynamical framework. Choosing $M(x, m) = 1$, we finally get

$$M_{\text{eff}} = \frac{\sqrt{s_1}^3}{2\zeta^2} \sin^2 \left(\frac{\zeta s_2}{\sqrt{s_1}} \pm \frac{2\zeta \Xi}{s_1} \right) \mp \frac{s_1 \sqrt{(s_4)^2 - 4} \sin \left(\frac{2\zeta s_2}{\sqrt{s_1}} \pm \frac{4\zeta \Xi}{s_1} \right)}{4\zeta}.$$

To ensure the correct classical limit, i.e., that M_{eff} returns to the expression in Einstein's GR as ζ approaches 0, we should choose $\Xi = \sqrt{s_1(s_4)^2 - 4s_1}/4$.

For the Hayward metric, we have

$$F(x; m) = H(x; m) \equiv F_H(x; m) = 1 - \frac{2mx^2}{x^3 + 2\zeta^2 m}. \quad (4.43)$$

With choosing $M(x, m) = m$, we finally have

$$-\frac{\left(2\zeta^2 M_{\text{eff}} + \sqrt{s_1}^3 \right) \sqrt{(s_4)^2 - 4F_H(\sqrt{s_1}; M_{\text{eff}})}}{8s_1 + 2\zeta^2 ((s_4)^2 - 4)} = \frac{2\sqrt{s_1}^5 \operatorname{arctanh} \left(\frac{\zeta \sqrt{(s_4)^2 - 4F_H(\sqrt{s_1}; M_{\text{eff}})}}{\sqrt{4s_1 + \zeta^2 ((s_4)^2 - 4)}} \right)}{\zeta [4s_1 + \zeta^2 ((s_4)^2 - 4)]^{3/2}} \mp \frac{1}{2} \sqrt{s_1} s_2,$$

which defines M_{eff} as a function of s_1 , s_2 and s_4 implicitly. In the Hayward case, we have chosen $\Xi = 0$ to ensure the classical limit.

V. FORMATION OF BH BY DUST COLLAPSE

A. Quantum OS model

In LQC, we have the spacetime \mathcal{M}_1 which is diffeomorphism to \mathbb{R}^4 . The metric is given by

$$ds^2 = -d\tilde{t}^2 + a(t)^2(d\tilde{r}^2 + r^2 d\Omega^2). \quad (5.1)$$

A geodesic in this spacetime is

$$\tau \mapsto (\tilde{t}(\tau), \tilde{r}(\tau), \tilde{\theta}(\tau), \tilde{\phi}(\tau)) = (\tau, r_0, \theta_0, \phi_0). \quad (5.2)$$

Then, we have the second spacetime \mathcal{M}_2 with a chart $\varphi : U \rightarrow \mathbb{R}^4$ which endows U with a coordinate $(t, r, \mathfrak{h}, \phi)$ with $r \in I \subset \mathbb{R}$. With this coordinate, the metric takes the form

$$ds^2 = -(1 - F(r))dt^2 + (1 - F(r))^{-1}dr^2 + r^2 d\Omega^2. \quad (5.3)$$

Let $I_\tau \ni \tau \mapsto (t(\tau), r(\tau), \mathfrak{h}_0, \phi_0) \in U$ be a geodesic of a freely falling ball. We have

$$\begin{aligned} g_{ab} \left(\frac{\partial}{\partial \tau} \right)^a \left(\frac{\partial}{\partial \tau} \right)^b &= -[1 - F(r(\tau))] \dot{t}(\tau)^2 + [1 - F(r(\tau))]^{-1} \dot{r}(\tau)^2 = -1 \\ g_{ab} \left(\frac{\partial}{\partial \tau} \right)^a \left(\frac{\partial}{\partial t} \right)^b &= -[1 - F(r(\tau))] \dot{t}(\tau) = -E \end{aligned} \quad (5.4)$$

which leads to

$$\begin{aligned} \dot{t}(\tau) &= \frac{E}{1 - F(r(\tau))}, \\ \dot{r}(\tau)^2 &= E^2 - 1 + F(r(\tau)). \end{aligned} \quad (5.5)$$

Consider the sphere $S_1 = \{(0, \tilde{r}_0, \tilde{\theta}, \tilde{\phi}), \theta \in (0, \pi), \phi \in (0, 2\pi)\}$ in \mathcal{M}_1 . Its freely falling forms a hypersurface $H_1 \subset \mathcal{M}_1$, i.e.,

$$H_1 = \{(\tau, \tilde{r}_0, \tilde{\theta}, \tilde{\phi}) \in \mathcal{M}_1, \tau \in \mathbb{R}, \tilde{\theta} \in (0, \pi), \tilde{\phi} \in (0, 2\pi)\}. \quad (5.6)$$

Similarly, we consider the sphere $S_2 = \{(t(\tau_0), r(\tau_0), \mathfrak{h}, \phi), \mathfrak{h} \in (0, \phi), \phi \in (0, 2\pi)\}$ whose freely falling forms a hypersurface $H_2 \subset U \subset \mathcal{M}_2$, i.e.,

$$H_2 = \{(t(\tau), r(\tau), \mathfrak{h}, \phi) \in U, \tau \in I', \theta \in (0, \phi), \phi \in (0, 2\pi)\}. \quad (5.7)$$

We want to glue U with a subset of \mathcal{M}_1 via identifying H_2 with a subset of H_1 , such that the extrinsic curvature and the reduced metric change continuously cross the gluing surface. We do the identification

$$(\tau, \tilde{r}, \tilde{\theta}_0, \tilde{\phi}_0) \sim (t(\tau), r(\tau), \mathfrak{h}_0, \phi_0), \forall \tau \in I_\tau. \quad (5.8)$$

We first take into account the reduced metric. We have, on H_1

$$ds^2 \Big|_{H_1} = -d\tau^2 + a(\tau)^2 \tilde{r}_0^2 d\Omega^2. \quad (5.9)$$

On H_2 , one has, taking advantage of (5.5),

$$\begin{aligned} ds^2 \Big|_{H_2} &= -[1 - F(r(\tau))] \dot{t}(\tau)^2 d\tau^2 + [1 - F(r(\tau))]^{-1} \dot{r}(\tau)^2 d\tau^2 + r(\tau)^2 d\Omega^2 \\ &= -d\tau^2 + r(\tau)^2 d\Omega^2. \end{aligned} \quad (5.10)$$

Thus, the continuity of the reduced metric leads to

$$a(\tau)^2 \tilde{r}_0^2 = r(\tau)^2. \quad (5.11)$$

$r(\tau)$ is a solution to the geodesic equation (5.5). Thus, we need to choose $F(r)$ which results in $r(\tau)$ such that (5.11) is satisfied. Rather than solving (5.5) to get $r(\tau)$, let us take the derivative of (5.11) to get

$$\frac{\dot{a}(\tau) a(\tau) \tilde{r}_0^2}{r(\tau)} = \dot{r}(\tau). \quad (5.12)$$

Substituting this equation into (5.5), we have

$$E^2 - 1 + F(r(\tau)) = \frac{\dot{a}(\tau)^2 a(\tau)^2 \tilde{r}_0^4}{r(\tau)^2} = \dot{a}(\tau)^2 \tilde{r}_0^2 \quad (5.13)$$

where the second equality uses (5.11). Since $d\tilde{t}/d\tau = 1$, $\dot{a}(\tau)^2$ can be solved from the Friedmann equation, i.e., By the Friedmann equation, we have

$$\dot{a}(\tau)^2 = H(\tau)^2 a(\tau)^2 = f(\rho) a(\tau)^2 \quad (5.14)$$

For the APS-LQC case, we have

$$\begin{aligned} \dot{a}(\tau)^2 &= H(\tau)^2 a(\tau)^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right) a(\tau)^2 = \frac{8\pi G}{3} \left(\frac{M}{\frac{4\pi}{3} a(\tau)^3 \tilde{r}_0^3} - \frac{M^2}{\left(\frac{4\pi}{3} a(\tau)^3 \tilde{r}_0^3\right)^2 \rho_c} \right) a(\tau)^2 \\ &= \left(\frac{2GM}{a(\tau) \tilde{r}_0^3} - \frac{2GM^2}{\frac{4\pi}{3} a(\tau)^4 \tilde{r}_0^6 \rho_c} \right). \end{aligned} \quad (5.15)$$

Thus, we have, by (5.13)

$$F(r(\tau)) = 1 - E^2 + \frac{2GM}{a(\tau) \tilde{r}_0} - \frac{2GM^2}{\frac{4\pi}{3} a(\tau)^4 \tilde{r}_0^4 \rho_c} = 1 - E^2 + \frac{2GM}{r(\tau)} - \frac{6GM^2}{4\pi r(\tau)^4 \rho_c} \quad (5.16)$$

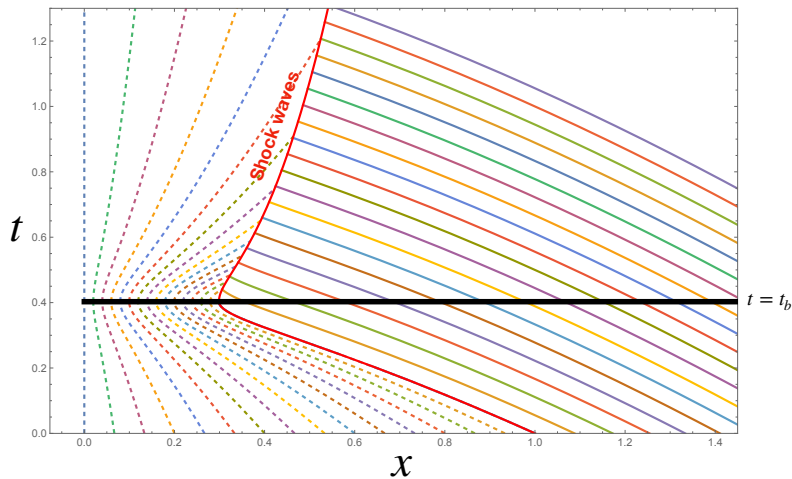


FIG. 6: Shock wave (red curve).

where (5.11) is used again.

Choose the conormal of H_1 to be $\tilde{n}_c = (d\tilde{r})_c / \sqrt{g^{aa}(d\tilde{r})_a(d\tilde{r})_b}$. Considering the extrinsic curvature of H_1 , we have

$$K_{ab}(\tau) \Big|_{H_1} = \tilde{r}_0 a(\tau) (d\theta)_a (d\theta)_b + \tilde{r}_0 a(\tau) \sin^2 \theta (d\phi)_a (d\phi)_b. \quad (5.17)$$

On H_2 , we choose the co-normal vector to be

$$n_a(\tau) = (-\dot{r}(\tau)(dt)_a + \dot{t}(\tau)(dr)_a) / \sqrt{g^{ab}(dt)_a(dt)_b \dot{r}(\tau)^2 + g^{ab}(dr)_a(dr)_b \dot{t}(\tau)^2}. \quad (5.18)$$

We get

$$K_{ab}(\tau) \Big|_{H_2} = Er(\tau) (d\theta)_a (d\theta)_b + Er(\tau) \sin^2 \theta (d\phi)_a (d\phi)_b. \quad (5.19)$$

Thus, the continuity of the extrinsic curvature gives

$$E = 1. \quad (5.20)$$

We therefore obtain from (5.16) that

$$F(r) = \frac{2GM}{r} - \frac{6GM^2}{4\pi\rho_c r^4} \equiv \frac{2GM}{r} - \frac{4\zeta^2 M^2}{r^4} \quad (5.21)$$

B. Inhomogeneous dust profile and shock waves

As shown by Francesco, Viqar and Edward's work [see e.g. *Phys.Rev.D* 109 (2024) 8, 084052], when the dust profile is inhomogeneous, there will be shell crossing singularity appear, as illustrated in Fig. 6.

C. Dust shell in Einstein theory

Let Σ be a 3-D Lorentzian manifold. In Σ , we assume the existence of a dust field described by the energy momentum tensor ${}^{(3)}S_{ab}$. We now embed Σ into the a 4-D Lorentzian manifold M such that Σ is a time-like hypersurface. Let g_{ab} denote the metric on M . We will still use Σ to denote the image of Σ embedded into M . As a hypersurface, Σ can always be given by

$$\xi = 0, \quad (5.22)$$

for certain function ξ on M .

Let \mathbf{n}_a be the normalized conormal vector of Σ , i.e.

$$n_a = (d\xi)_a N^{-1}, \quad (5.23)$$

with

$$N = \sqrt{g^{bc}(d\xi)_b(d\xi)_c}. \quad (5.24)$$

The 4-D correspondence of ${}^{(3)}S_{ab}$ in M is given by

$$\begin{aligned} S_{ab}(x)\mathbf{n}^a(x) &= 0, & S_{ab}(x)v^a w^b &= {}^{(3)}S_{ab}v^a w^b, & \text{for all } v^a, w^a \text{ tangent to } \Sigma, & \forall x \in \Sigma, \\ S_{ab}(y) &= 0, & \forall y &\notin \Sigma. \end{aligned} \quad (5.25)$$

The metric g_{ab} gives a reduced metric ${}^{(3)}\mathfrak{h}_{ab}$ on Σ . The 4-D correspondence to ${}^{(3)}\mathfrak{h}_{ab}$ is given by

$$\mathfrak{h}_{ab} = g_{ab} - \mathbf{n}_a \mathbf{n}_b, \quad (5.26)$$

as it can be verified that

$$\mathfrak{h}_{ab}\mathbf{n}^a = 0, \quad \mathfrak{h}_{ab}v^a w^b = {}^{(3)}\mathfrak{h}_{ab}v^a w^b, \quad \forall v, w \text{ tangent to } \Sigma. \quad (5.27)$$

On M we define the δ_ξ -function as

$$\int_M \sqrt{-\det(g)} f \delta_\xi d^4x = \int_\Sigma \sqrt{-\det({}^{(3)}\mathfrak{h})} f|_\Sigma d^3x, \quad \forall \text{ test function } f. \quad (5.28)$$

Then, the energy moment tensor of the dust field confined on Σ is given by

$$\sigma_{ab} = \delta_\xi S_{ab}. \quad (5.29)$$

The Einstein equation becomes

$$G_{ab} = 8\pi G \sigma_{ab}. \quad (5.30)$$

To see the influence of σ_{ab} , we observed that $\sigma_{ab}\mathbf{n}^a = 0$, leading to

$$G_{ab}\mathbf{n}^a \mathbf{n}^b = 0 = G_{ab}\mathbf{n}^a \mathfrak{h}^b{}_d \quad (5.31)$$

Moreover, according to the Einstein equation, we have

$$G_{ab}g^{ab} = -R = 8\pi G\sigma \quad (5.32)$$

leading to

$$R_{ab} = 8\pi G(\sigma_{ab} - \frac{1}{2}g_{ab}\sigma). \quad (5.33)$$

Therefore, we have

$$R_{ab}\mathfrak{h}^a{}_c \mathfrak{h}^b{}_d = 8\pi G(\sigma_{cd} - \frac{1}{2}\mathfrak{h}_{cd}\sigma). \quad (5.34)$$

By definition of R_{ab} (see (5.35)), we have

$$R_{ab}h^a{}_c h^b{}_d = {}^{(3)}\mathfrak{R}_{cd} - \mathfrak{K}\mathfrak{K}_{cd} - \mathfrak{h}^a{}_c \mathfrak{h}^b{}_d \mathbf{n}^e \nabla_e \mathfrak{K}_{ab} - \frac{1}{N} D_c D_d N \quad (5.35)$$

where ${}^{(3)}\mathfrak{R}_{cd}$ is the Racci curvature of ${}^{(3)}\mathfrak{h}_{ab}$, K_{ab} is the extrinsic curvature of Σ and N is given by (5.24). It is noted that both $\mathfrak{h}^a{}_c \mathfrak{h}^b{}_d \mathbf{n}^e \nabla_e \mathfrak{K}_{ab}$ and $\frac{1}{N} D_c D_d N$ in the right hand side of (5.35) depend on the choice of ξ . For $\frac{1}{N} D_c D_d N$ it is easy to see this because N is defined via ξ . For the term $\mathfrak{h}^a{}_c \mathfrak{h}^b{}_d \mathbf{n}^e \nabla_e \mathfrak{K}_{ab}$, it is ξ -dependence because $\mathbf{n}^e \nabla_e \mathfrak{K}_{ab}$ depends on the extrinsic curvature of the slice in a neighbor of Σ . The slice in a neighbor of Σ is defined by $\xi = \text{constant}$, and thus ξ dependent.

Substituting this result into (5.34), we have

$${}^{(3)}\mathfrak{K}_{cd} - \mathfrak{K}\mathfrak{K}_{cd} - \mathfrak{h}^a{}_c \mathfrak{h}^b{}_d \mathfrak{n}^e \nabla_e \mathfrak{K}_{ab} - \frac{1}{N} D_c D_d N = 8\pi G (\sigma_{cd} - \frac{1}{2} \mathfrak{h}_{cd} \sigma). \quad (5.36)$$

The right-hand-side involve distribution. Therefore, the above equation is interpreted in the sense of distribution, i.e., the following equation is true for all f^{cd} satisfying $f^{cd} n_c = 0 = f^{cd} n_d$:

$$\begin{aligned} & \int_{M_+} \sqrt{-g} \left\{ \left({}^{(3)}\mathfrak{K}_{cd} - \mathfrak{K}\mathfrak{K}_{cd} - \frac{1}{N} D_c D_d N \right) f^{cd} + \nabla_e (n^e f^{ab}) \mathfrak{K}_{ab} \right\} \\ & + \int_{M_-} \sqrt{-g} \left\{ \left({}^{(3)}\mathfrak{K}_{cd} - \mathfrak{K}\mathfrak{K}_{cd} - \frac{1}{N} D_c D_d N \right) f^{cd} + \nabla_e (n^e f^{ab}) \mathfrak{K}_{ab} \right\} \\ & = 8\pi G \int_{\Sigma} \sqrt{-\det({}^{(3)}\mathfrak{h})} f^{cd} (S_{cd} - \frac{1}{2} \mathfrak{h}_{cd} S). \end{aligned} \quad (5.37)$$

Because in M_{\pm} we have

$${}^{(3)}\mathfrak{K}_{cd} - \mathfrak{K}\mathfrak{K}_{cd} - \mathfrak{h}^a{}_c \mathfrak{h}^b{}_d \mathfrak{n}^e \nabla_e \mathfrak{K}_{ab} - \frac{1}{N} D_c D_d N = 0 \quad (5.38)$$

(5.37) gives

$$\int_{M_+} \sqrt{-g} \nabla_e (n^e f^{ab} \mathfrak{K}_{ab}) + \int_{M_-} \sqrt{-g} \nabla_e (n^e f^{ab} \mathfrak{K}_{ab}) = 8\pi G \int_{\Sigma} \sqrt{-\det({}^{(3)}\mathfrak{h})} f^{cd} (S_{cd} - \frac{1}{2} \mathfrak{h}_{cd} S). \quad (5.39)$$

Applying the Gaussian law, we have

$$\int_{\Sigma} \sqrt{-\det({}^{(3)}\mathfrak{h})} f^{ab} [\mathfrak{K}_{ab}] = 8\pi G \int_{\Sigma} \sqrt{-\det({}^{(3)}\mathfrak{h})} f^{cd} (S_{cd} - \frac{1}{2} \mathfrak{h}_{cd} S) \quad (5.40)$$

where $[\mathfrak{K}_{ab}] := \mathfrak{K}_{ab}^+ - \mathfrak{K}_{ab}^-$. We thus get

$$[\mathfrak{K}_{ab}] = -8\pi G (S_{ab} - \frac{1}{2} \mathfrak{h}_{ab} S) \quad (5.41)$$

It is noted that the normal vector \mathfrak{n}^e in (5.37) point in the "future" direction, i.e., from M_- to M_+ , resulting in the $-$ sign in (5.41).

D. Dust shell in spherically symmetric Einstein theory

Now let us consider a dust shell embedded into a spherically symmetric vacuum model. In this case, the metric inside and outside the dust shell read

$$ds_{\pm}^2 = - \left(1 - \frac{2M_{\pm}}{r} \right) dt^2 + \left(1 - \frac{2M_{\pm}}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (5.42)$$

with $M_+ = M$ and $M_- = 0$. Let the world line of each point in the dust shell be given by $\tau \mapsto (t(\tau), x(\tau), \theta, \phi)$ with τ being the proper time. We have

$$- \left(1 - \frac{2M_{\pm}}{r} \right) \dot{t}^2 + \left(1 - \frac{2M_{\pm}}{r} \right)^{-1} \dot{r}^2 = -1. \quad (5.43)$$

A unit conormal vector of the hypersurface is

$$\mathbf{n}_a = \dot{r} (dt)_a - \dot{t} (dr)_a. \quad (5.44)$$

It is conormal vector because $0 = \mathbf{n}_a (\partial/\partial\tau)^a = \mathbf{n}_a (\partial/\partial\theta)^a = \mathbf{n}_a (\partial/\partial\phi)^a$. It is normalized because its norm is

$$\|\mathbf{n}\|^2 = - \left(1 - \frac{2M_{\pm}}{r} \right)^{-1} \dot{x}^2 + \left(1 - \frac{2M_{\pm}}{r} \right) \dot{t}^2 = 1. \quad (5.45)$$

As ds_{\pm}^2 they are solution to the Einstein equation, we get

$$G_{ab}^+ n_+^a n_+^b = 0 = G_{ab}^- n_-^a n_-^b \quad (5.46)$$

Moreover, we have the fomulation

$$\begin{aligned} G_{ab} n^a n^b &= R_{ab} n^a n^b - \frac{1}{2} R = R_{abcd} n^a n^b g^{cd} - \frac{1}{2} R_{abcd} g^{ab} g^{cd} - \frac{1}{2} R_{abcd} n^a n^b n^c n^d \\ &= -\frac{1}{2} R_{abcd} (g^{ab} - n^a n^b) (g^{cd} - n^c n^d) = -\frac{1}{2} R_{abcd} h^{ab} h^{cd} = -\frac{1}{2} \left({}^{(3)}\mathfrak{R} - \mathfrak{R}^2 + \mathfrak{K}_a{}^c \mathfrak{K}_c{}^a \right) \end{aligned} \quad (5.47)$$

According to (5.47), we have

$$\begin{aligned} 0 &= 2 (G_{ab}^+ n_+^a n_+^b - G_{ab}^- n_-^a n_-^b) = \mathfrak{K}_+^2 - \mathfrak{K}_-^2 - [\mathfrak{K}_a{}^c \mathfrak{K}_c{}^a]_+ + [\mathfrak{K}_a{}^c \mathfrak{K}_c{}^a]_- \\ &= (\mathfrak{K}_{ab}^+ + \mathfrak{K}_{ab}^-) [(\mathfrak{K}_+ h^{ab} - \mathfrak{K}_+^{ab}) - (\mathfrak{K}_- h^{ab} - \mathfrak{K}_-^{ab})] \end{aligned} \quad (5.48)$$

According to (5.41), we have

$$[\mathfrak{K} h^{ab} - \mathfrak{K}^{ab}] = -8\pi G \left(-\frac{1}{2} S h^{ab} - S^{ab} + \frac{1}{2} S h^{ab} \right) = 8\pi G S^{ab}. \quad (5.49)$$

We thus get

$$(\mathfrak{K}_{ab}^+ + \mathfrak{K}_{ab}^-) S^{ab} = 0. \quad (5.50)$$

By definition of G_{ab} , we have

$$\begin{aligned} G_{ab} n^a h^b{}_c &= R_{ab} n^a h^b{}_c - \frac{1}{2} R g_{ab} n^a h^b{}_c = R_{ab} n^b h^a{}_c \\ &= h^a{}_c R_{adb}{}^d n^b = -h^a{}_c (\nabla_a \nabla_d n^d - \nabla_d \nabla_a n^d) = -h^a{}_c (\nabla_a (g^{de} \nabla_d n_e) - \nabla_d (g^{de} \nabla_a n_e)) \\ &= -h^a{}_c \left[\nabla_a (h^{de} \nabla_d n_e) - \nabla_d (h^{de} \nabla_a n_e) \right] = -h^a{}_c \nabla_a \mathfrak{K}^d{}_d + h^a{}_c \nabla_d (\mathfrak{K}_a{}^d + h^{de} n_a n^f \nabla_f n_e) \\ &= -h^a{}_c \nabla_a \mathfrak{K}^d{}_d + h^a{}_c \nabla_d \mathfrak{K}_a{}^d + (n^f \nabla_f n_e) \mathfrak{K}^e{}_c = -D_c \mathfrak{K}^a{}_a + D_a \mathfrak{K}^a{}_c. \end{aligned} \quad (5.51)$$

where we used

$$n^e \nabla_a n_e = 0, \quad h^a{}_c \nabla_d \mathfrak{K}_a{}^d = h^a{}_c g^{de} \nabla_d \mathfrak{K}_{ae} = D_a \mathfrak{K}^a{}_c - \mathfrak{K}_{ce} n^d \nabla_d n^e. \quad (5.52)$$

Moreover, we have

$$G_{ab}^+ n_+^a h^b{}_c = 0 = G_{ab}^- n_-^a h^b{}_c \quad (5.53)$$

According to (5.51), we have

$$G_{ab}^+ n_+^a h^b{}_c - G_{ab}^- n_-^a h^b{}_c = D_a [\mathfrak{K}^a{}_c] - D_c [\mathfrak{K}^a{}_a] = -8\pi G \left[D_a (S^a{}_c - \frac{1}{2} h^a{}_c S) + \frac{1}{2} D_c S \right] = -8\pi G D_a S^a{}_c \quad (5.54)$$

We thus get

$$D_a S^a{}_c = 0 \quad (5.55)$$

To get the junction surface, we need to compute the acceleration

$$(\partial_\tau)^a \nabla_a (\partial_\tau)^c = (\partial_\tau)^a D_a (\partial_\tau)^c + n^c n_d (\partial_\tau)^a \nabla_a (\partial_\tau)^d = (\partial_\tau)^a D_a (\partial_\tau)^c - n^c K_{ab} (\partial_\tau)^a (\partial_\tau)^d. \quad (5.56)$$

Let us consider the presureless dust, i.e.,

$$S^{ab} = \rho (\partial_\tau)^a (\partial_\tau)^b. \quad (5.57)$$

Then, (5.55) leads to

$$0 = D_a (\rho (\partial_\tau)^a) (\partial_\tau)^b + \rho (\partial_\tau)^a D_a (\partial_\tau)^b \Rightarrow D_a (\rho (\partial_\tau)^a) = 0 \text{ and } (\partial_\tau)^a D_a (\partial_\tau)^b = 0 \quad (5.58)$$

where we reach the conclusion by using $(d\tau)_a$ to contract both sides of the equation. With this conclusion, we have

$$\mathbf{n}_c(\partial_\tau)^a \nabla_a (\partial_\tau)^c = -\mathfrak{K}_{ab}(\partial_\tau)^a (\partial_\tau)^b = -\frac{1}{\rho} \mathfrak{K}_{ab} S^{ab}. \quad (5.59)$$

According to (5.50), we have

$$\mathbf{n}_c^+(\partial_\tau)^a \nabla_a^+ (\partial_\tau)^c + \mathbf{n}_c^-(\partial_\tau)^a \nabla_a^- (\partial_\tau)^c = 0. \quad (5.60)$$

In the Schwarzschild metric, we have

$$(\partial_\tau)^a \nabla_a (\partial_\tau)^c = \left(\partial_\tau^2 t + \frac{2M \partial_\tau r \partial_\tau t}{r(r-2M)} \right) (\partial_t)^c + \left(\partial_\tau^2 r + \frac{M(r-2M) \partial_\tau t}{r^3} - \frac{M(\partial_\tau r)^2}{r(r-2M)} \right) (\partial_r)^c. \quad (5.61)$$

Since $\|\partial_\tau\|^2 = -1$, we get

$$\begin{aligned} - \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 &= -1 \\ \frac{d}{d\tau} \left(- \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 \right) &= 0 \end{aligned} \quad (5.62)$$

from which we can solve $\partial_\tau^2 t$ and $\partial_\tau t > 0$. Substituting the results into (5.61), we have

$$\mathbf{n}_c(\partial_\tau)^a \nabla_a (\partial_\tau)^c = -\frac{M + r^2 \partial_\tau^2 r}{r^2 \sqrt{1 - \frac{2M}{r} + \partial_\tau r^2}} \quad (5.63)$$

Therefore, (5.60) gives us

$$\begin{aligned} \frac{M + r^2 \partial_\tau^2 r}{r^2 \sqrt{1 - \frac{2M}{r} + \partial_\tau r^2}} + \frac{r^2 \partial_\tau^2 r}{r^2 \sqrt{1 + \partial_\tau r^2}} &= 0 \\ \Rightarrow \partial_\tau^2 r \left(\sqrt{1 + \partial_\tau r^2} + r \sqrt{1 - \frac{2M}{r} + \partial_\tau r^2} \right) &= -M \sqrt{1 + \partial_\tau r^2} r^{-2} \\ \Rightarrow \sqrt{1 + (\partial_\tau r)^2} &= a + \frac{M}{2ar}, \quad \text{for some constant } a. \end{aligned} \quad (5.64)$$

According to (5.41), we have

$$\begin{aligned} -4\pi G\rho &= [\mathfrak{K}_{ab}](\partial_\tau)^a (\partial_\tau)^b = -\mathbf{n}_c^+(\partial_\tau)^a \nabla_a^+ (\partial_\tau)^c + \mathbf{n}_c^-(\partial_\tau)^a \nabla_a^- (\partial_\tau)^c \\ &= \frac{M + r^2 \partial_\tau^2 r}{r^2 \sqrt{1 - \frac{2M}{r} + \partial_\tau r^2}} - \frac{r^2 \partial_\tau^2 r}{r^2 \sqrt{1 + \partial_\tau r^2}} = -\frac{M}{ar^2}, \end{aligned} \quad (5.65)$$

namely

$$4\pi Gr^2 \rho = M/a. \quad (5.66)$$

VI. SOLVING THE TRAJECTORY OF DUST SHELL IN CANONICAL GRAVITY

A. Einstein tensor from the action

In this section, let us investigate how to get the Einstein tensor from the action. By definition of $g_{\mu\nu}$ and $g^{\mu\nu}$, we have

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2}, & \frac{N^a}{N^2} \\ \frac{N^a}{N^2}, & q^{ab} - \frac{N^a N^b}{N^2} \end{pmatrix}, \quad (6.1)$$

i.e.,

$$g^{\mu\nu}(\partial_\mu)^{b_1}(\partial_\nu)^{b_2} = -\frac{1}{N^2} \left((\partial_t)^{b_1} - N^{b_1} \right) \left((\partial_t)^{b_2} - N^{b_2} \right) + q^{b_1 b_2}. \quad (6.2)$$

Therefore, we have

$$\begin{aligned} & \delta g^{\mu\nu}(\partial_\mu)^{b_1}(\partial_\nu)^{b_2} \\ &= \frac{2}{N^3} \left((\partial_t)^{b_1} - N^{b_1} \right) \left((\partial_t)^{b_2} - N^{b_2} \right) \delta N \\ & \quad + \frac{1}{N^2} \left((\partial_t)^{b_2} - N^{b_2} \right) \delta N^{b_1} + \frac{1}{N^2} \left((\partial_t)^{b_1} - N^{b_1} \right) \delta N^{b_2} \\ & \quad + \delta q^{b_1 b_2} \\ &= \frac{2}{N} n^{b_1} n^{b_2} \delta N + \frac{2}{N} n^{(b_1} \delta N^{b_2)} + \delta q^{b_1 b_2}. \end{aligned} \quad (6.3)$$

Thus, we get

$$\delta S = \int \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} = \int \sqrt{-g} \frac{2}{N} \left(n^{b_1} n^{b_2} G_{b_1 b_2} \delta N + G_{b_1 b_2} n^{b_1} \delta N^{b_2} \right) + \sqrt{-g} G_{b_1 b_2} \delta q^{b_1 b_2}, \quad (6.4)$$

implying that

$$\frac{\delta S}{\delta N} = \frac{2}{N} \sqrt{-g} G_{b_1 b_2} n^{b_1} n^{b_2}, \quad \frac{\delta S}{\delta N^\mu} = \frac{2}{N} \sqrt{-g} G_{\mu\nu} n^\nu, \quad \frac{\delta S}{\delta q^{ab}} = \sqrt{-g} G_{ab}. \quad (6.5)$$

On the other hand, we know that

$$S = \int d^4x \left(-\frac{1}{2G} K_1 \dot{E}^1 - \frac{1}{G} K_2 \dot{E}^2 - NH - N^x H_x \right). \quad (6.6)$$

Thus, we get

$$G_{b_1 b_2} n^{b_1} n^{b_2} = -\frac{H}{2\sqrt{q}}, \quad G_{\mu a} n^\mu = -\frac{H_a}{2\sqrt{q}}. \quad (6.7)$$

To get what is $\delta S/\delta q^{ab}$, let us introduce P_{ab} such that

$$\{q^{cd}(x), P_{ab}(y)\} = 2\delta_a^{(c} \delta_b^{d)} \delta(x, y). \quad (6.8)$$

Then, we have

$$S = \int d^4x (P_{ab} \partial_t q^{ab} - HN - H_x N^x). \quad (6.9)$$

Thus, we have

$$\frac{\delta S}{\delta q^{ab}} = -\partial_t P_{ab} + \{P_{ab}, H[N] + H_x[N^x]\} = \sqrt{-g} G_{ab}, \quad (6.10)$$

where we need to relate P_{ab} with $\partial_t q_{ab}$ by the equations

$$\partial_t q_{ab} = \{q_{ab}, H[N] + H_x[N^x]\}. \quad (6.11)$$

It is noted that by calculating $\frac{\delta S}{\delta q^{ab}}$, we need to write S as an function of q_{ab} and its derivatives. By doing this, $\delta S/\delta q^{ab}$ gives the Euler-Lagrangian equation with respect to q^{ab} . However, we did not do this in (6.10). The reason why the correct results can be obtained without rewriting S as an function of q_{ab} can be seen as follows.

Consider the mechanical system with $L(q, \dot{q})$. Then, we have the Euler-Lagrangian equation

$$\frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}. \quad (6.12)$$

As we know, after the Legendre transformation, we have

$$\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q} \Big|_{p=\frac{\partial L}{\partial \dot{q}}}. \quad (6.13)$$

We thus have that the Euler-Lagrangian equation (6.12) can be rewritten as

$$\frac{\delta S}{\delta q} = \left(-\frac{\partial H}{\partial q} - \frac{d}{dt}p \right) \Big|_{p=\frac{\partial L}{\partial \dot{q}}} = \left(-\frac{d}{dt}p + \{p, H\} \right) \Big|_{p=\frac{\partial L}{\partial \dot{q}}}. \quad (6.14)$$

The above results in the spherically symmetric model by using MMA codes. Moreover, in the spherically symmetric model, we have

$$q^{ab}\partial_a \otimes \partial_b = \frac{E^1}{(E^2)^2}\partial_x \otimes \partial_x + \frac{1}{E^1} \left(\partial_\theta \otimes \partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\phi \otimes \partial_\phi \right). \quad (6.15)$$

The symplectic potential is

$$\begin{aligned} \vartheta &= -\frac{1}{2G}K_1\delta E^1 - \frac{1}{G}K_2\delta E^2 = -\frac{1}{2G}K_1\delta \left(\frac{1}{q^{\theta\theta}} \right) - \frac{1}{G}K_2\delta \left(\frac{1}{\sqrt{q^{\theta\theta}q^{xx}}} \right) \\ &= \frac{1}{2G}K_1 \frac{\delta q^{\theta\theta}}{(q^{\theta\theta})^2} + \frac{1}{2G}K_2 \frac{1}{\sqrt{q^{\theta\theta}q^{xx}}^3} (q^{\theta\theta}\delta q^{xx} + q^{xx}\delta q^{\theta\theta}) \\ &= \frac{1}{2G} \left(K_1 \frac{1}{(q^{\theta\theta})^2} + K_2 \frac{q^{xx}}{\sqrt{q^{\theta\theta}q^{xx}}^3} \right) \delta q^{\theta\theta} + \frac{1}{2G}K_2 \frac{q^{\theta\theta}}{\sqrt{q^{\theta\theta}q^{xx}}^3} \delta q^{xx} \\ &= \frac{1}{2G} \frac{K_2(E^2)^3}{E^1} \delta q^{xx} + \frac{E^1(K_1E^1 + K_2E^2)}{2G} \delta q^{\theta\theta} \\ &= \frac{1}{2G} \frac{K_2(E^2)^3}{E^1} \delta q^{xx} + \frac{E^1(K_1E^1 + K_2E^2)}{4G} \delta q^{\theta\theta} + \frac{E^1(K_1E^1 + K_2E^2)\sin^2(\theta)}{4G} \delta q^{\phi\phi} \end{aligned} \quad (6.16)$$

Thus, we have

$$P_{ab}dx^a dx^b = \frac{1}{2G} \frac{K_2(E^2)^3}{E^1} dx^2 + \frac{1}{4G} E^1 (K_1E^1 + K_2E^2) (d\theta^2 + \sin^2(\theta)d\phi^2). \quad (6.17)$$

Now, we can reconstruct the Einstein tensor as follows

$$\begin{aligned} G_{b_1b_2} &= G_{b'_1b'_2} \left(h_{b_1}^{b'_1} - n_{b_1}n^{b'_1} \right) \left(h_{b_2}^{b'_2} - n_{b_2}n^{b'_2} \right) \\ &= G_{b'_1b'_2} h_{b_1}^{b'_1} h_{b_2}^{b'_2} - G_{b'_1b'_2} h_{b_1}^{b'_1} n_{b_2}n^{b'_2} - G_{b'_1b'_2} h_{b_2}^{b'_2} n_{b_1}n^{b'_1} + G_{b'_1b'_2} n^{b'_1} n^{b'_2} n_{b_1} n_{b_2} \\ &= -\frac{H}{2\sqrt{q}} n_{b_1} n_{b_2} + \frac{H_a}{\sqrt{q}} h_{(b_1}^a n_{b_2)} + \frac{1}{\sqrt{-g}} \left(-\partial_t P_{ab} + \{P_{ab}, H[N] + H_x[N^x]\} \right) h_{b_1}^a h_{b_2}^b. \end{aligned} \quad (6.18)$$

B. Dust shell in classical model: dust shell as discontinuity

From now on, we will consider the geometrized unit with $G = 1 = c$. For the gravitational part, the phase space contains two canonical pairs (K_I, E^I) with $I = 1, 2$. The satisfies the Poisson-bracket relation

$$\{K_1(x), E^1(y)\} = 2\delta(x, y), \quad \{K_2(x), E^2(y)\} = \delta(x, y). \quad (6.19)$$

In the model, we have the constraints H_x^g and H^g , which read

$$\begin{aligned} H_x^g &= -\frac{1}{2} (K_1\partial_x E^1 - 2E^2\partial_x K_2), \\ H^g &= \frac{\sqrt{E^1}\partial_x^2 E^1}{2E^2} - \frac{\sqrt{E^1}\partial_x E^1\partial_x E^2}{2(E^2)^2} + \frac{(\partial_x E^1)^2}{8\sqrt{E^1}E^2} - \frac{E^2(K_2)^2}{2\sqrt{E^1}} - \frac{E^2}{2\sqrt{E^1}} - \sqrt{E^1}K_1K_2. \end{aligned} \quad (6.20)$$

1. Constraint of the dust shell part

Once the dust shell is considered, the dust-shell part of the Hamiltonian and diffeomorphism constraint of the model is

$$H^{\text{sh}} = \delta(x - \mathfrak{r}) \sqrt{\mathfrak{p}^2 \frac{E^1}{(E^2)^2} + m^2}, \quad H_x^{\text{sh}} = -\mathfrak{p} \delta(x - \mathfrak{r}). \quad (6.21)$$

Then, we have the EOM

$$\begin{aligned} \dot{E}^1 &= N^x \partial_x E^1 + 2N \sqrt{E^1} K_2, \\ \dot{E}^2 &= \frac{N(E^1 K_1 + E^2 K_2)}{\sqrt{E^1}} + \partial_x(N^x E^2), \\ \dot{K}_1 &= N \left(-\frac{\partial_x E^1 \partial_x E^2}{2\sqrt{E^1}(E^2)^2} - \frac{(\partial_x E^1)^2 - 4E^1 \partial_x^2 E^1}{8\sqrt{E^1}^3 E^2} + \frac{E^2(1 + (K_2)^2)}{2\sqrt{E^1}^3} - \frac{K_1 K_2}{\sqrt{E^1}} + \frac{G\mathfrak{p}^2 \delta(x - \mathfrak{r})}{(E^2)^2 \sqrt{\mathfrak{p}^2 \frac{E^1}{(E^2)^2} + m^2}} \right) \\ &\quad + \frac{\partial_x E^1 \partial_x N + 2E^1 \partial_x^2 N}{2\sqrt{E^1} E^2} - \frac{\sqrt{E^1} \partial_x E^2 \partial_x N}{(E^2)^2} + \partial_x(N^x K_1), \\ \dot{K}_2 &= N \left(\frac{(\partial_x E^1)^2}{8\sqrt{E^1}(E^2)^2} - \frac{(K_2)^2 + 1}{2\sqrt{E^1}} - \frac{G\mathfrak{p}^2 E^1 \delta(x - \mathfrak{r})}{(E^2)^3 \sqrt{\mathfrak{p}^2 \frac{E^1}{(E^2)^2} + m^2}} \right) + \frac{\sqrt{E^1} \partial_x E^1 \partial_x N}{2(E^2)^2} + N^x \partial_x K_2 \end{aligned} \quad (6.22)$$

It is equivalent to the Einstein's equation:

$$G_{\mu\nu} = 8\pi G \sigma \delta(x - \mathfrak{r}) u_\mu u_\nu \quad (6.23)$$

where \mathfrak{r} denotes the radial of the dust shell, u_μ is the 4-velocity of the shell and σ is the area density of the shell. We have

$$u^\mu \partial_\mu \propto \partial_t + \mathfrak{t} \partial_x. \quad (6.24)$$

Then, we define

$$n_\nu dx^\mu = \mathfrak{t} dt - dx, \quad \text{and} \quad m^\mu \partial_\mu = \partial_\theta. \quad (6.25)$$

We will have

$$G_{\mu\nu} n^\nu = 0 = G_{\mu\nu} m^\nu. \quad (6.26)$$

These are the equations do not contain δ -function explicitly.

Now let us consider these two equations.

2. The Einstein tensor from Hamiltonian formulation

According to (6.18), we have

$$\begin{aligned} G_{\mu\nu} dx^\mu dx^\nu &= -\frac{H}{2\sqrt{q}} N^2 dt^2 - N \frac{H_x}{\sqrt{q}} (dx + N^x dt) dt \\ &\quad + \frac{1}{N\sqrt{q}} (-\partial_t P_{xx} + \{P_{xx}, H[N] + H_x[N^x]\}) (dx + N^x dt)^2 \\ &\quad + \frac{1}{N\sqrt{q}} (-\partial_t P_{\theta\theta} + \{P_{\theta\theta}, H[N] + H_x[N^x]\}) (d\theta^2 + \sin^2 \theta d\phi^2) \end{aligned} \quad (6.27)$$

where P_{xx} and $P_{\theta\theta}$ are defined in (6.17). Then, we have

$$\begin{aligned} dx^\mu G_{\mu\nu} n^\nu &= \left(-\frac{\partial_x E^1 (N^x + \mathfrak{t})}{(E^2)^2} - \frac{2N\sqrt{E^1} \partial_x K_2}{(E^2)^2} - \frac{2N^x (\mathfrak{t} \partial_x K_2 + \partial_t K_2)}{N\sqrt{E^1}} + \dots \right) dt \\ &\quad - \left(\frac{2 (\mathfrak{t} \partial_x K_2 + \partial_t K_2)}{N\sqrt{E^1}} + \dots \right) dx \end{aligned} \quad (6.28)$$

where \dots denote the terms involving N , N^x , E^I , and their first order derivatives so that they will not produce δ functions if we use the assumption that the metric is continuous. It will be useful to prove the following relation:

- (1) Given any function $f(x)$ which is discontinuous at $x = 0$, usually we have $\partial_x f = (f(\epsilon) - f(-\epsilon))\delta(x) + \dots$ where \dots denotes some regular function. Mathematically, it actually means

$$\int \partial_x \chi f + \int_{x < 0} \chi \partial_x f + \int_{x > 0} \chi \partial_x f = -\chi(0)(f(\epsilon) - f(-\epsilon)) \quad \forall \text{test functions } \chi. \quad (6.29)$$

- (2) Now let us consider a function $f(t, x)$ which is discontinuous along $t \rightarrow \mathfrak{r}(t)$. We have, for any test function,

$$\begin{aligned} & \iint dt dx \partial_x \chi f + \iint_{x < \mathfrak{r}(t)} dt dx \chi \partial_x f + \iint_{x < \mathfrak{r}(t)} dt dx \chi \partial_x f \\ &= \iint_{x < \mathfrak{r}(t)} dt dx \vec{\partial} \times (-\chi f, 0) + \iint_{x < \mathfrak{r}(t)} dt dx \vec{\partial} \times (-\chi f, 0) \\ &= \int_{\text{along } \mathfrak{r}(t)} \chi(\mathfrak{r}(t)) [f(t, \mathfrak{r}(t) - \epsilon) - f(t, \mathfrak{r}(t) + \epsilon)] dt \end{aligned} \quad (6.30)$$

where $\vec{\partial} = (\partial_t, \partial_x)$, we used the Stokes' theorem, and choose the orientation $dt \wedge dx$ so that $\vec{dl} = -(dt, \dot{\mathfrak{r}}dt)$ for $x < \mathfrak{r}$ and $\vec{dl} = (dt, \dot{\mathfrak{r}}dt)$ for $x > \mathfrak{r}$, and

$$\begin{aligned} & \iint dt dx \partial_t \chi f + \iint_{x < \mathfrak{r}(t)} dt dx \chi \partial_t f + \iint_{x < \mathfrak{r}(t)} dt dx \chi \partial_t f \\ &= \iint_{x < \mathfrak{r}(t)} dt dx \vec{\partial} \times (0, \chi f) + \iint_{x < \mathfrak{r}(t)} dt dx \vec{\partial} \times (0, \chi f) \\ &= - \int_{\text{along } \mathfrak{r}(t)} \chi(\mathfrak{r}(t)) [f(t, \mathfrak{r}(t) - \epsilon) - f(t, \mathfrak{r}(t) + \epsilon)] \dot{\mathfrak{r}} dt. \end{aligned} \quad (6.31)$$

We thus get

$$\partial_x f = [f]\delta(x - \mathfrak{r}) + \dots, \quad \partial_t f = -[f]\dot{\mathfrak{r}}\delta(x - \mathfrak{r}) + \dots \quad (6.32)$$

According to (6.32), we have

$$\dot{\mathfrak{r}}\partial_x K_2 + \partial_t K_2 = 0, \quad \text{in the sense of distribution} \quad (6.33)$$

Then, $dx^\mu G_{\mu\nu} n^\nu = 0$ leads to

$$[\partial_x E^1](\hat{N}^x + \dot{\mathfrak{r}}) + 2\hat{N}\sqrt{\hat{E}^1}[K_2] = 0 \quad (6.34)$$

where we used the convention $[f] = f(t, \mathfrak{r}(t) + \epsilon) - f(t, \mathfrak{r}(t) - \epsilon)$, and $\hat{g} = g(t, \mathfrak{r}(t))$. Similarly, from $G_{\mu\nu} m^\nu = 0$, we get

$$-\left(\dot{\mathfrak{r}} + \hat{N}^x\right)[K_1] - \frac{\sqrt{\hat{E}^1}[\partial_x N]}{\hat{E}^2} = \frac{(\hat{N}^x + \dot{\mathfrak{r}})\hat{E}^2[K_2]}{\hat{E}^1} + \frac{\hat{N}[\partial_x E^1]}{2\sqrt{\hat{E}^1}\hat{E}^2} \quad (6.35)$$

3. How jump condition determines the trajectory of the dust shell

Now let us consider the how the equations (6.34) and (6.35) determine the dynamics.

The key point in the dynamics is that E^I , N and N^x , as components of the metric, are continuous. N^x and N , as Lagrangian multiplier, are chosen by hand in the dynamics. Thus, we keep the continuity of these variable, we must require that E^I are continuous in the evolution.

By the Hamilton's equation (6.22), we have

$$\begin{aligned} E^1(t + \delta t, \mathfrak{r}(t + \delta t)) - E^1(t, \mathfrak{r}(t + \delta t)) &= (N^x \partial_x E^1 + 2N\sqrt{E^1}K_2 + \partial_x E^1 \dot{\mathfrak{r}})\delta t + O(\delta t^2) \\ &= ((N^x + \dot{\mathfrak{r}})\partial_x E^1 + 2N\sqrt{E^1}K_2)\delta t + O(\delta t^2) \end{aligned} \quad (6.36)$$

Thus, in order to keep the continuity of E^1 in the evolution, we must have

$$(\hat{N}^x + \dot{\mathfrak{t}})[\partial_x E^1] + 2\hat{N}\sqrt{\hat{E}^1}[K_2] = 0, \quad (6.37)$$

which is actually just (6.34). Now let us consider the evolution equation for E^2 in (6.22). Similarly, to keep the continuity for E^2 , we must have

$$-(\hat{N}^x + \dot{\mathfrak{t}})[\partial_x E^2] = \frac{\hat{N}(\hat{E}^1[K_1] + \hat{E}^2[K_2])}{\sqrt{\hat{E}^1}} + [\partial_x N^x]\hat{E}^2 \quad (6.38)$$

Now, we have the equations (6.34), (6.35) and (6.38). They comes from $G_{\mu\nu}n^\mu = 0 = G_{\mu\nu}m^\nu$ and requiring that E^I keep their continuity in the evolution. We should regard these equations as equation for \hat{N} , $\hat{N}^x + \dot{\mathfrak{t}}$, $[\partial_x N]$ and $\partial_x N^x$ as other quantities involved in these equations are determined by the dynamics. Solving these equations, we have

$$\begin{aligned} \hat{N}^x + \dot{\mathfrak{t}} &= -\frac{2\sqrt{\hat{E}^1}\hat{N}[K_2]}{[\partial_x E^1]}, \\ [\partial_x N^x] &= \frac{2\sqrt{\hat{E}^1}\hat{N}[\partial_x E^2][K_2]}{\hat{E}^2[\partial_x E^1]} - \frac{\sqrt{\hat{E}^1}\hat{N}[K_1]}{\hat{E}^2} - \frac{\hat{N}[K_2]}{\sqrt{\hat{E}^1}} \\ [\partial_x N] &= \frac{2\hat{E}^2\hat{N}[K_1][K_2]}{[\partial_x E^1]} + \frac{2(\hat{E}^2)^2\hat{N}[K_2]^2}{\hat{E}^1[\partial_x E^1]} - \frac{\hat{N}[\partial_x E^1]}{2\hat{E}^1} \end{aligned} \quad (6.39)$$

Dynamics under the gauge $\dot{\mathfrak{t}} = 0$

Now let us consider that the gauge is chosen such that $\dot{\mathfrak{t}} = 0$ leading to $\mathfrak{r}(t) = \mathfrak{r}_0$. In this case, what we need to do is to solve the following vacuum EOM for $x < \mathfrak{r}_0$ and $x > \mathfrak{r}_0$ respectively,

$$\begin{aligned} \dot{E}^1 &= N^x \partial_x E^1 + 2N\sqrt{E^1}K_2, \\ \dot{E}^2 &= \frac{N(E^1 K_1 + E^2 K_2)}{\sqrt{E^1}} + \partial_x(N^x E^2), \\ \dot{K}_1 &= N \left(-\frac{\partial_x E^1 \partial_x E^2}{2\sqrt{E^1}(E^2)^2} - \frac{(\partial_x E^1)^2 - 4E^1 \partial_x^2 E^1}{8\sqrt{E^1}^3 E^2} + \frac{E^2(1 + (K_2)^2)}{2\sqrt{E^1}^3} - \frac{K_1 K_2}{\sqrt{E^1}} \right) \\ &\quad + \frac{\partial_x E^1 \partial_x N + 2E^1 \partial_x^2 N}{2\sqrt{E^1} E^2} - \frac{\sqrt{E^1} \partial_x E^2 \partial_x N}{(E^2)^2} + \partial_x(N^x K_1), \\ \dot{K}_2 &= N \left(\frac{(\partial_x E^1)^2}{8\sqrt{E^1}(E^2)^2} - \frac{(K_2)^2 + 1}{2\sqrt{E^1}} \right) + \frac{\sqrt{E^1} \partial_x E^1 \partial_x N}{2(E^2)^2} + N^x \partial_x K_2, \end{aligned} \quad (6.40)$$

where the lapse function N and the shift vector N^x must be chosen such that the jump condition (6.39) holds. In this case, (6.39) can be simplified as

$$\begin{aligned} \hat{N}^x &= -\frac{2\hat{N}\sqrt{\hat{E}^1}[K_2]}{[\partial_x E^1]}, \\ [\partial_x N^x] &= \frac{2\hat{N}\sqrt{\hat{E}^1}[\partial_x E^2][K_2]}{\hat{E}^2[\partial_x E^1]} - \frac{\hat{N}\sqrt{\hat{E}^1}[K_1]}{\hat{E}^2} - \frac{\hat{N}[K_2]}{\sqrt{\hat{E}^1}} \\ [\partial_x N] &= \frac{2\hat{E}^2\hat{N}[K_1][K_2]}{[\partial_x E^1]} + \frac{2\hat{N}(\hat{E}^2)^2[K_2]^2}{\hat{E}^1[\partial_x E^1]} - \frac{\hat{N}[\partial_x E^1]}{2\hat{E}^1} \\ &= -\frac{\hat{N}^x \hat{E}^2}{\sqrt{\hat{E}^1}}[K_1] - \frac{\hat{N}^x (\hat{E}^2)^2}{\sqrt{\hat{E}^1}^3}[K_2] - \frac{\hat{N}[\partial_x E^1]}{2\hat{E}^1}. \end{aligned} \quad (6.41)$$

Due to these equations, we can choose the lapse function N and shift vector N^x obeying

$$\begin{aligned} \partial_x N^x &= -N^x \frac{\partial_x E^2}{E^2} - \frac{K_2}{\sqrt{E^1}} - \sqrt{E^1} \frac{K_1}{E^2}, \\ \partial_x N &= N \left(-\frac{N^x E^2}{\sqrt{E^1}^3} (K_1 E^1 + K_2 E^2) - \frac{\partial_x E^1}{2E^1} \right) \end{aligned} \quad (6.42)$$

with the boundary condition

$$\hat{N}^x = N^x(t, \mathbf{r}) = -\frac{2\sqrt{E^1(t, \mathbf{r})}[K_2]}{[\partial_x E^1]}, \quad \hat{N} = 1. \quad (6.43)$$

Here, we add an overall factor N in the RHS of $\partial_x N$ to guarantee that N is positive definite.

It is worth noting that the gauge choice (6.42) is not required to hold for all x . In fact, ensuring the jump condition (6.41) only demands that this gauge be imposed in a neighborhood of the dust shell. We require the gauge to become a familiar one away from the shell. To this end, let us consider the gauge choice such that $\dot{E}^1 = 0 = \dot{E}^2$, leading to

$$\begin{aligned} 0 &= N^x \partial_x E^1 + 2N\sqrt{E^1}K_2, \\ 0 &= \frac{N(E^1 K_1 + E^2 K_2)}{\sqrt{E^1}} + \partial_x(N^x E^2) \\ &= -\frac{N^x \partial_x E^1 (E^1 K_1 + E^2 K_2)}{2K_2 E^1} + \partial_x(N^x E^2) \\ &= -N^x E^2 \left(\frac{\partial_x K_2}{K_2} + \frac{\partial_x E^1}{2E^1} \right) + \partial_x(N^x E^2) \\ &= -N^x E^2 \partial_x \ln \left(|K_2 \sqrt{E^1}| \right) + \partial_x(N^x E^2) \end{aligned} \quad (6.44)$$

where we used $H_x = 0$, leading to

$$K_1 \partial_x E^1 = 2E^2 \partial_x K_2. \quad (6.45)$$

According to (6.44), we have

$$N^x = -\frac{K_2 \sqrt{E^1}}{E^2}, \quad N = \frac{1}{2} \frac{\partial_x E^1}{E^2}, \quad (6.46)$$

We therefore choose the gauge

$$N = g_s \dot{N} + \frac{g_b}{2} \frac{\partial_x E^1}{E^2}, \quad N^x = g_s \dot{N}^x - g_b \frac{K_2 \sqrt{E^1}}{E^2} \quad (6.47)$$

where \dot{N} and \dot{N}^x are solutions to (6.42), i.e.,

$$\begin{aligned} \partial_x \dot{N}^x &= -\dot{N}^x \frac{\partial_x E^2}{E^2} - \frac{K_2}{\sqrt{E^1}} - \sqrt{E^1} \frac{K_1}{E^2}, \\ \partial_x \dot{N} &= \dot{N} \left(-\frac{\dot{N}^x E^2}{\sqrt{E^1}^3} (K_1 E^1 + K_2 E^2) - \frac{\partial_x E^1}{2E^1} \right), \end{aligned} \quad (6.48)$$

and g_s is such a function that $g_s(x) = 1$ for x around \mathbf{r} and $g_s(x) = 0$ for $|x - \mathbf{r}| \gg 1$, and

$$g_b = 1 - g_s.$$

The full EOMs

Let us again define

$$s_1 = E^1, \quad s_2 = K_2, \quad s_3 = \frac{K_1}{E^2}, \quad s_4 = \frac{\partial_x s_1}{E^2}, \quad s_5 = \frac{\partial_x s_4}{E^2}, \quad s_6 = N^x E^2, \quad s_7 = \frac{\partial_x N}{E^2} \sqrt{s_1} \quad (6.49)$$

Considering the equations of motion (6.40), we have

$$\begin{aligned} \dot{E}^2 &= \partial_x s_6 + \frac{NE^2}{\sqrt{s_1}} (s_1 s_3 + s_2), \\ \partial_t(E^2 s_1) &= \partial_x (s_1 s_6) + 2NE^2 \sqrt{s_1} s_2 + NE^2 \sqrt{s_1} (s_1 s_3 + s_2) \\ &= \partial_x (s_1 s_6) + NE^2 \sqrt{s_1} (3s_2 + s_1 s_3), \\ \partial_t(E^2 s_4) &= \partial_x (\partial_t s_1) = \partial_x (s_4 s_6 + 2N \sqrt{s_1} s_2), \end{aligned} \quad (6.50)$$

and

$$\begin{aligned}
\partial_t(E^2 s_2) &= \partial_x(s_6 s_2) - \frac{NE^2}{2\sqrt{s_1}} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4}\right) + \frac{s_4 \partial_x N \sqrt{s_1}}{2} + NE^2 s_2 \left(\sqrt{s_1} s_3 + \frac{s_2}{\sqrt{s_1}}\right) \\
&= \partial_x \left(s_6 s_2 + \frac{Ns_4 \sqrt{s_1}}{2}\right) - \frac{NE^2}{2\sqrt{s_1}} \left(1 - (s_2)^2 - \frac{(s_4)^2}{4}\right) - \frac{\partial_x s_4 N \sqrt{s_1}}{2} - \frac{s_4 N \partial_x s_1}{4\sqrt{s_1}} + NE^2 s_2 \sqrt{s_1} s_3 \\
&= \partial_x \left(s_6 s_2 + \frac{Ns_4 \sqrt{s_1}}{2}\right) - \frac{NE^2}{2\sqrt{s_1}} \left(1 - (s_2)^2 - \frac{(s_4)^2}{4}\right) - \frac{E^2 (s_4)^2 N}{4\sqrt{s_1}} - \frac{E^2 N \sqrt{s_1}}{2} (s_5 - 2s_2 s_3) \\
&= \partial_x \left(s_6 s_2 + \frac{Ns_4 \sqrt{s_1}}{2}\right) - \frac{NE^2}{2\sqrt{s_1}} \left(1 - (s_2)^2 - \frac{(s_4)^2}{4}\right) - \frac{E^2 (s_4)^2 N}{4\sqrt{s_1}} - \frac{E^2 N}{2\sqrt{s_1}} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4}\right) \\
&= \partial_x \left(s_6 s_2 + \frac{Ns_4 \sqrt{s_1}}{2}\right) - \frac{NE^2}{\sqrt{s_1}}
\end{aligned} \tag{6.51}$$

where we used the Hamiltonian constraint leading to

$$s_5 - 2s_2 s_3 = \frac{(s_2)^2}{s_1} - \frac{(s_4)^2}{4s_1} + \frac{1}{s_1} \tag{6.52}$$

and

$$\begin{aligned}
\partial_t(E^2 s_3) &= \partial_x(s_6 s_3 + s_7) - \frac{2NE^2 s_2 s_3}{\sqrt{s_1}} - NE^2 \sqrt{s_1} (s_3)^2 + \frac{NE^2 (s_2)^2}{2(s_1)^{3/2}} - \frac{NE^2 (s_4)^2}{8(s_1)^{3/2}} + \frac{NE^2 s_5}{2\sqrt{s_1}} + \frac{NE^2}{2(s_1)^{3/2}} \\
&\quad + NE^2 s_3 \left(\sqrt{s_1} s_3 + \frac{s_2}{\sqrt{s_1}}\right) \\
&= \partial_x(s_6 s_3 + s_7) + \frac{NE^2}{2(s_1)^{3/2}} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4}\right) + \frac{NE^2}{2\sqrt{s_1}} (s_5 - 2s_2 s_3) \\
&= \partial_x(s_6 s_3 + s_7) + \frac{NE^2}{(s_1)^{3/2}} \left(1 + (s_2)^2 - \frac{(s_4)^2}{4}\right) \\
&= \partial_x(s_6 s_3 + s_7) + \frac{2mNE^2}{(s_1)^2}
\end{aligned} \tag{6.53}$$

where in the last step we used (6.52), and

$$m = \frac{\sqrt{s_1}}{2} \left[1 + (s_2)^2 - \frac{(s_4)^2}{4}\right], \tag{6.54}$$

Thus, define \vec{u} to be

$$\vec{u} = (E^2, E^2 s_1, E^2 s_4, E^2 s_2, E^4 s_3) \tag{6.55}$$

the equation of motions reads

$$\partial_t \vec{u} = \partial_x F + J \tag{6.56}$$

with

$$F = \begin{pmatrix} s_6 \\ s_1 s_6 \\ s_4 s_6 + 2N\sqrt{s_1} s_2 \\ s_6 s_2 + \frac{Ns_4 \sqrt{s_1}}{2} \\ s_6 s_3 + s_7 \end{pmatrix}, \quad J = \begin{pmatrix} \frac{NE^2}{\sqrt{s_1}} (s_1 s_3 + s_2) \\ NE^2 \sqrt{s_1} (3s_2 + s_1 s_3) \\ 0 \\ -\frac{NE^2}{\sqrt{s_1}} \\ \frac{2mNE^2}{(s_1)^2} \end{pmatrix}, \tag{6.57}$$

By definition, we have

$$N = g_s \dot{N} + \frac{g_b}{2} s_4, \quad s_6 = g_s \dot{s}_6 - g_b s_2 \sqrt{s_1} \quad (6.58)$$

where we introduce

$$\dot{s}_6 = E^2 \dot{N}^x, \quad (6.59)$$

and \dot{N} and \dot{s}_6 satisfy according to (6.48)

$$\begin{aligned} \frac{\partial_x \dot{s}_6}{E^2} &= -\frac{s_2}{\sqrt{s_1}} - \sqrt{s_1} s_3, \\ \frac{\partial_x \dot{N}}{E^2} &= \dot{N} \left(-\frac{\dot{s}_6}{\sqrt{s_1}^3} (s_3 s_1 + s_2) - \frac{s_4}{2s_1} \right), \end{aligned} \quad (6.60)$$

For the scalar s_7 , we have

$$\begin{aligned} s_7 &= \sqrt{s_1} \frac{\partial_x N}{E^2} \\ &= \sqrt{s_1} \frac{(\partial_x g_s) \dot{N}}{E^2} + \sqrt{s_1} g_s \dot{N} \left(-\frac{\dot{s}_6}{\sqrt{s_1}^3} (s_3 s_1 + s_2) - \frac{s_4}{2s_1} \right) + \sqrt{s_1} \frac{(\partial_x g_b) s_4}{2E^2} + \sqrt{s_1} \frac{g_b}{2} s_5 \\ &= \sqrt{s_1} \frac{\partial_x g_s}{E^2} \left(\dot{N} - \frac{s_4}{2} \right) + g_s \dot{N} \left(-\frac{\dot{s}_6}{s_1} (s_3 s_1 + s_2) - \frac{s_4}{2\sqrt{s_1}} \right) + \frac{g_b}{2} \left(\frac{2m}{s_1} + 2\sqrt{s_1} s_2 s_3 \right) \end{aligned} \quad (6.61)$$

with applying (6.52) leading to

$$s_5 = \frac{2m}{\sqrt{s_1}^3} + 2s_2 s_3. \quad (6.62)$$

The above EOMs now can be solved numerically. We get the result as shown in (7).

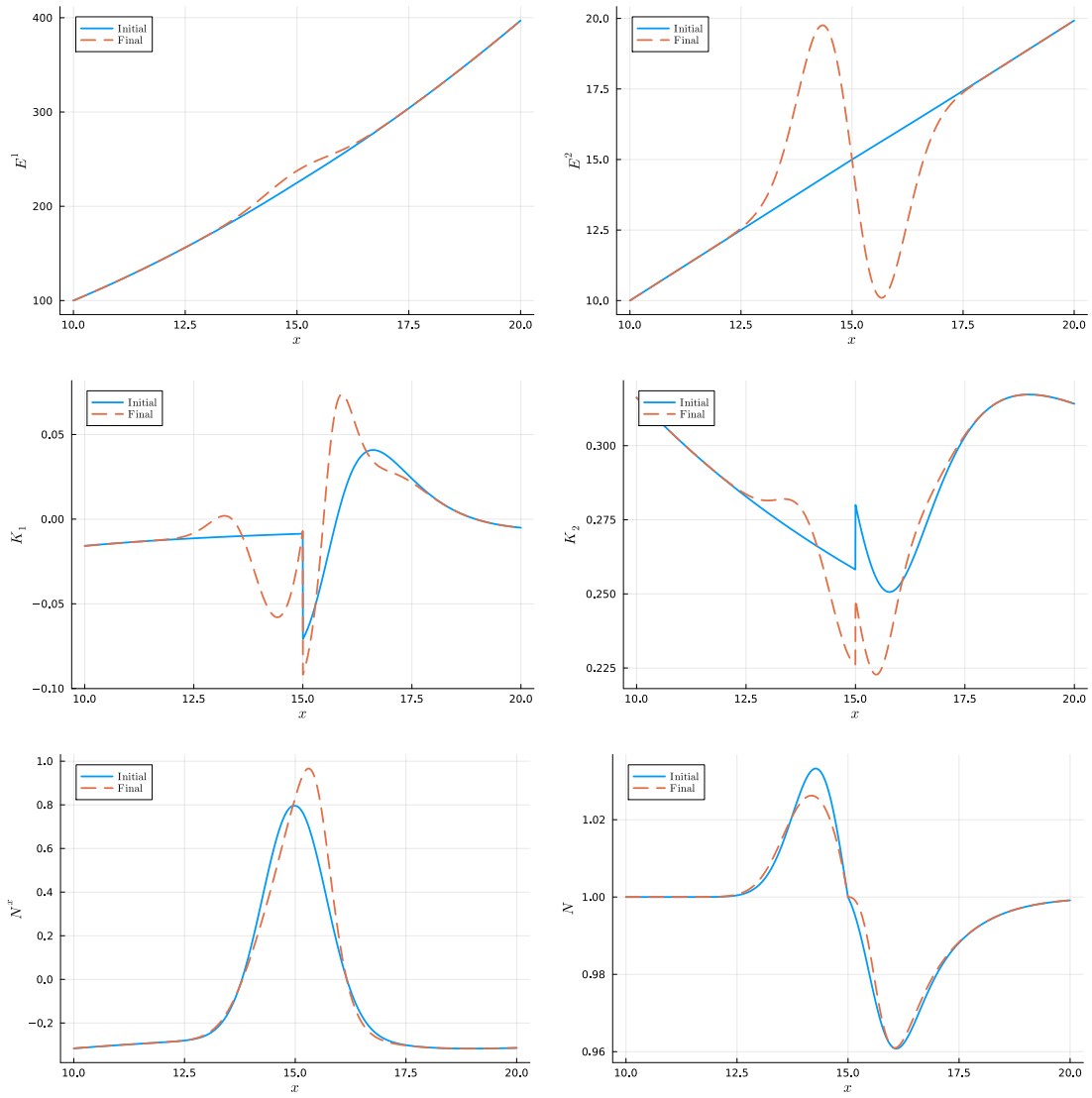


FIG. 7: Initial (solid blue) and evolved (dashed red) profiles of E^I , K_I , N , and N^x for $m_i = 1/2$ and $m_e = 1$, shown at $t = 0.4$.