An Introduction to QFT in curved spacetimes

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LQG Summer School 2026

Lecture 4

The Hawking effect

Good References:

Quantum field theory in curved spacetime and black hole thermodynamics, R. M. Wald, The University of Chicago Press, 1994.

Modeling black hole evaporation, A. Fabbri, J. Navarro-Salas, Imperial College Press 2005.

Except where explicitly indicated, natural units are used: $\hbar = G = c = 1$

Stephen W. Hawking, 1942-2018

Commun. math. Phys. 43, 199—220 (1975)

Particle Creation by Black Holes

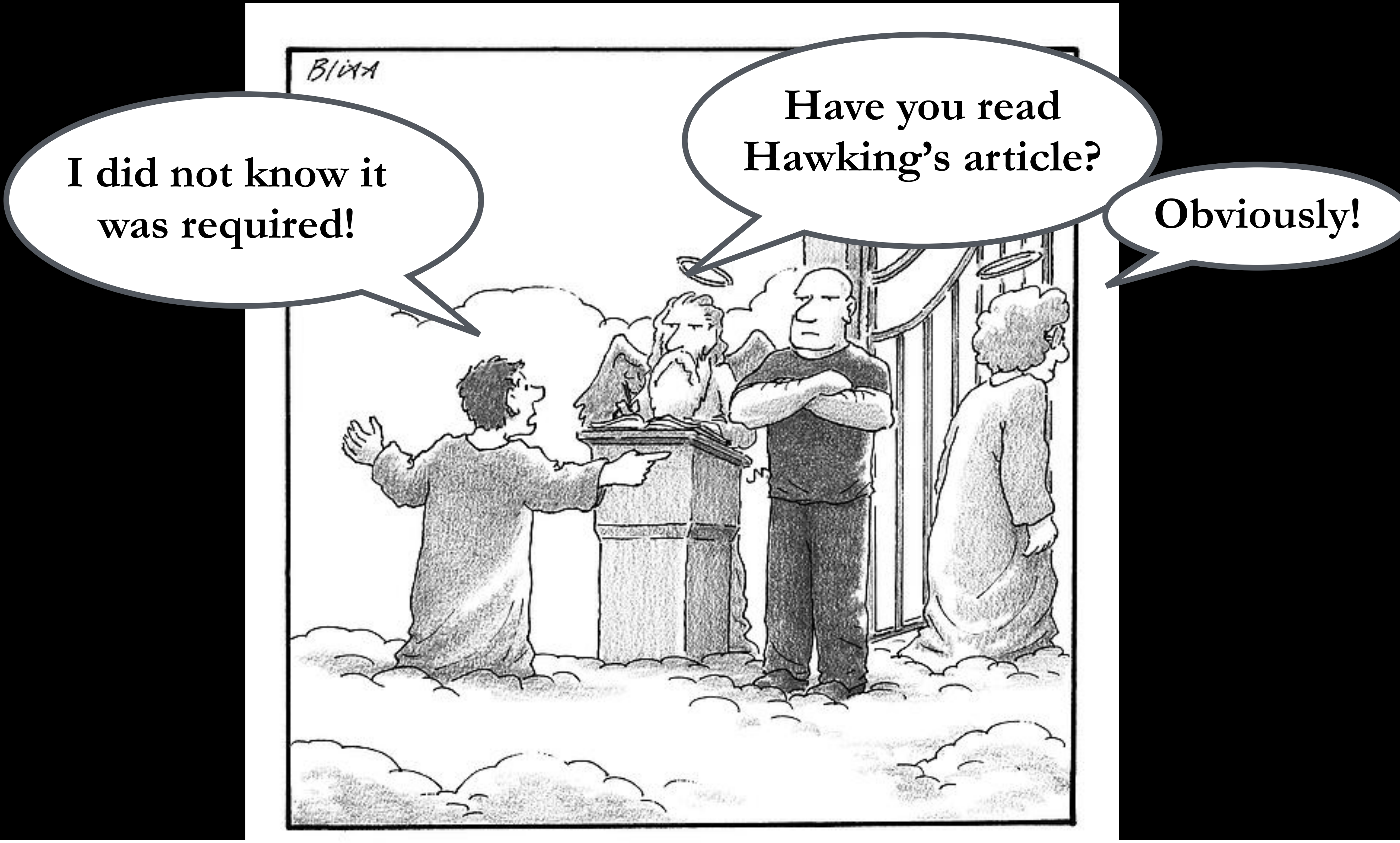
S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

Received April 12, 1975

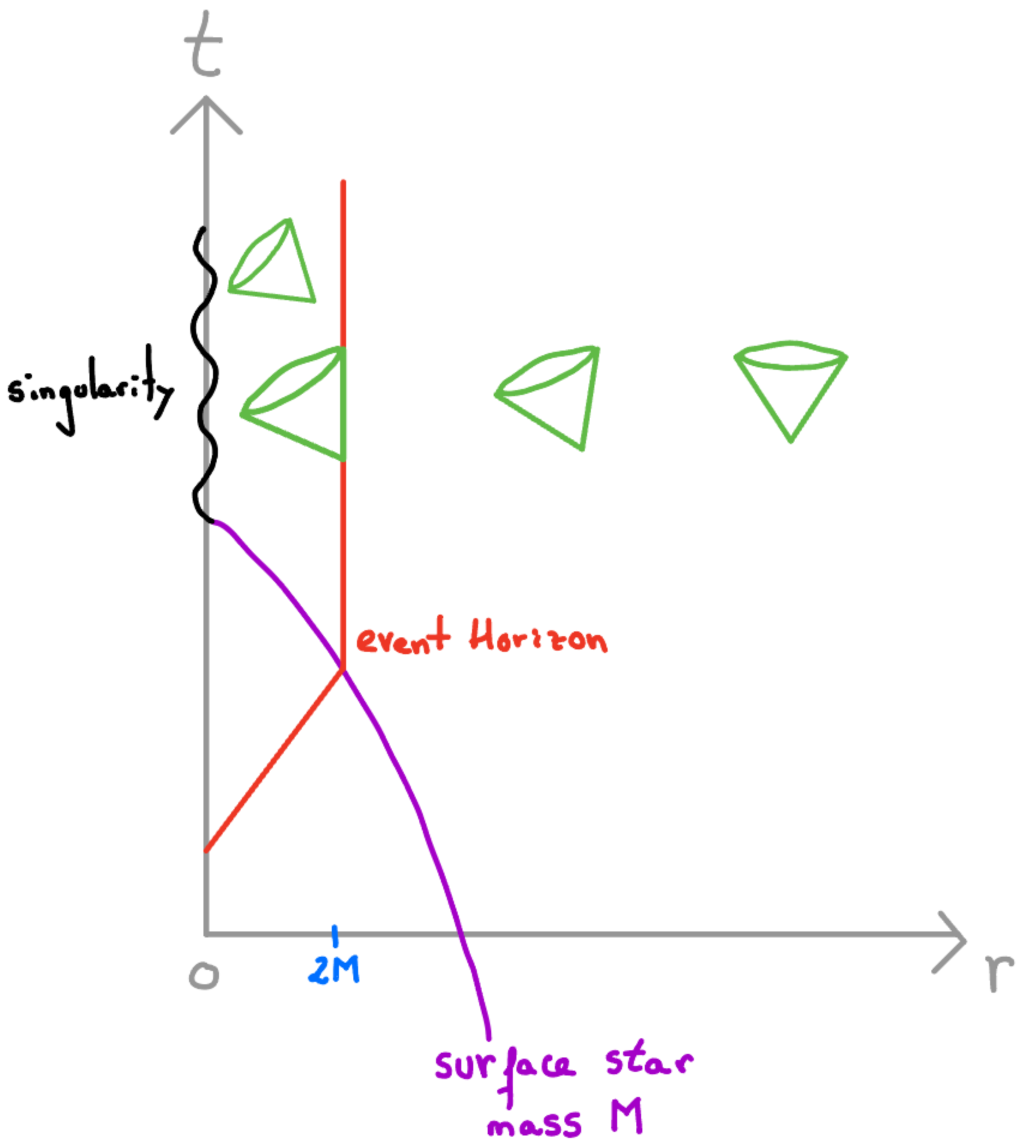
Abstract. In the classical theory black holes can only absorb and not emit particles. However it is shown that quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature $\frac{\hbar\kappa}{2\pi k} \approx 10^{-6}\left(\frac{M_\odot}{M}\right)$ °K where κ is the surface gravity of the black hole. This thermal emission leads to a slow decrease in the mass of the black hole and to its eventual disappearance: any primordial black hole of mass less than about 10^{15} g would have evaporated by now. Although these quantum effects violate the classical law that the area of the event horizon of a black hole cannot decrease, there remains a Generalized Second Law: $S+\frac{1}{4}A$ never decreases where S is the entropy of matter outside black holes and A is the sum of the surface areas of the event horizons. This shows that gravitational collapse converts the baryons and leptons in the collapsing body into entropy. It is tempting to speculate that this might be the reason why the Universe contains so much entropy per baryon.

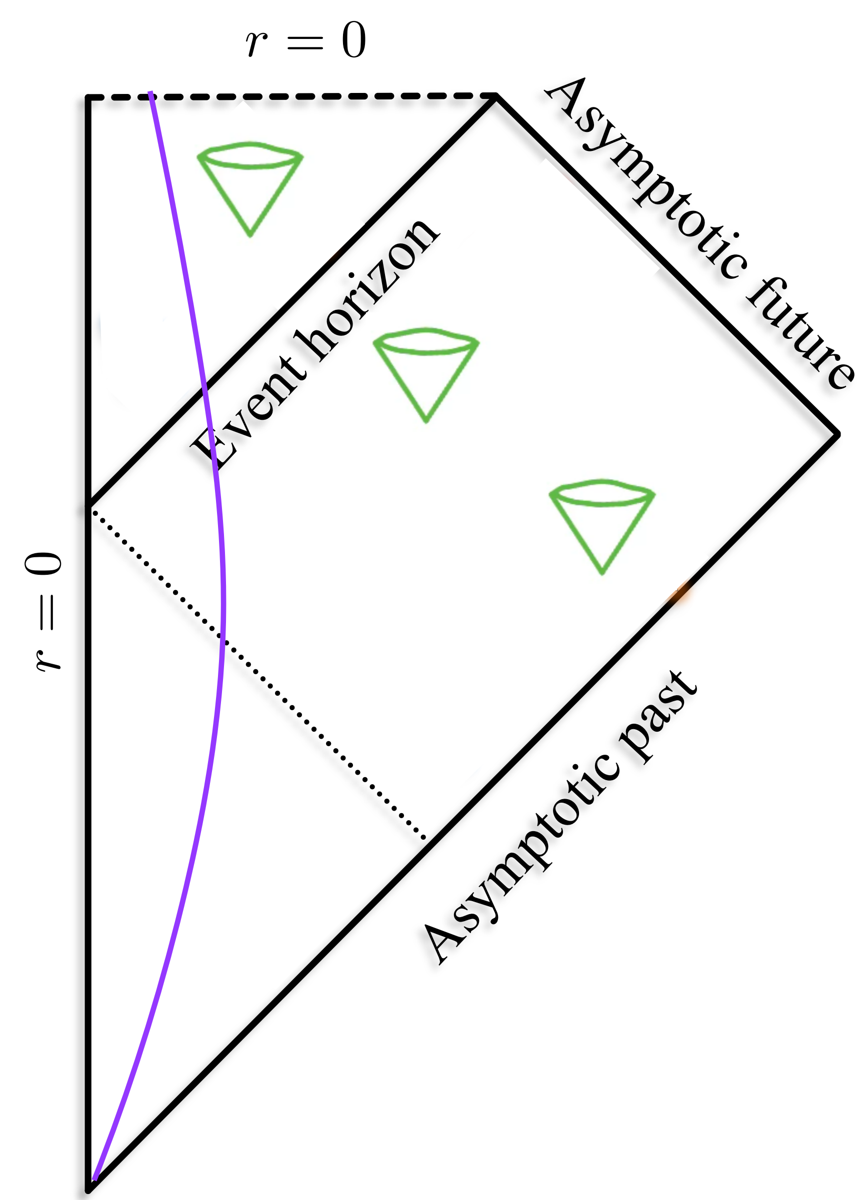
1.


Have you read Hawking’s article?
Obviously!
I did not know it was required!

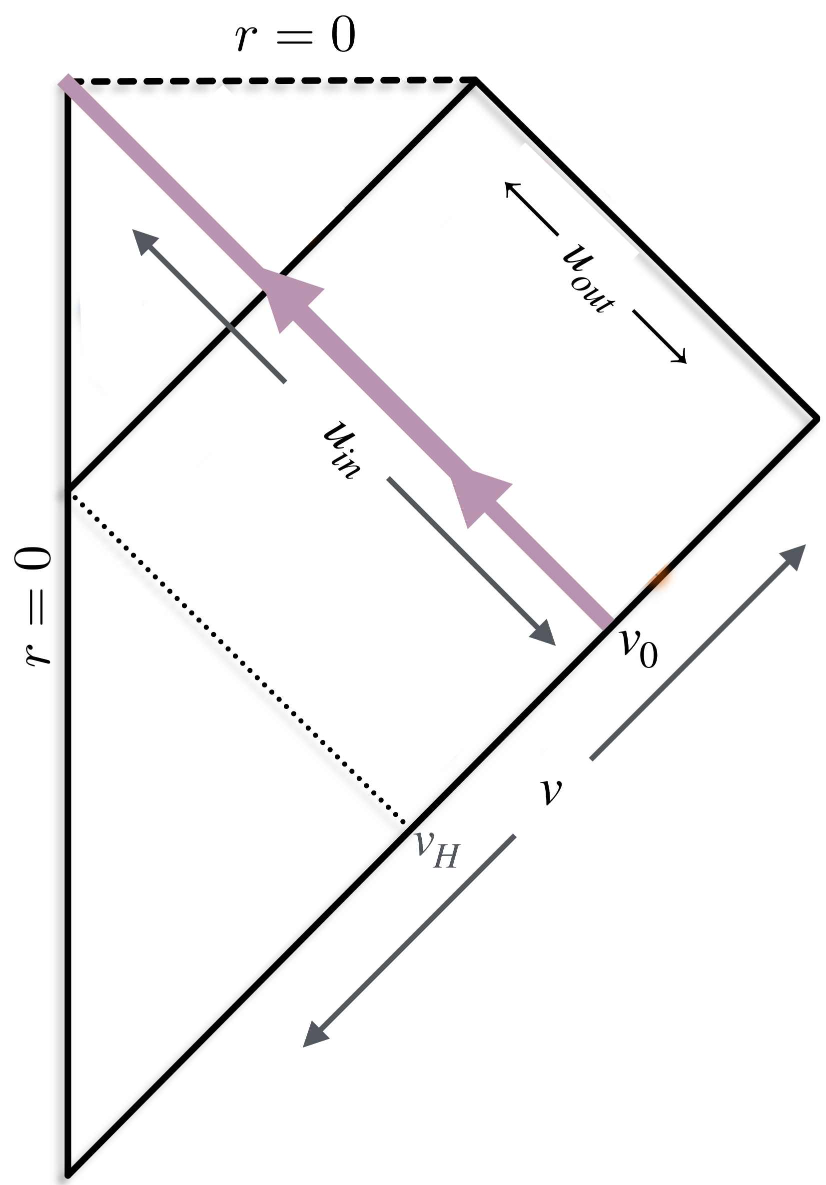
1. Classical Preliminaries

The Hawking effect occurs in the spacetime produced by the formation of a BH by gravitational collapse: time-dependent geometry $\Longrightarrow$ no global time-like Killing vector field





The Hawking effect does not depend on the details of the process of collapse. It only depend on the total mass, angular momentum and electric charge of the collapsing body.
We will focus on J=0, Q=0, and consider a very simple model which permits an analytical treatment: Vaidya spacetime



Vaidya spacetime= BH produced by the collapse of an infinitely thin, spherical symmetric shell of radiation

Double null-coordinates (u,v)

$$u_{in} = t_{in} - r_{in} \qquad v = \begin{cases} t_{in} + r_{in}, & v < v_0 \\ t_{out} + r^*_{out}, & v \geq v_0 \end{cases}$$

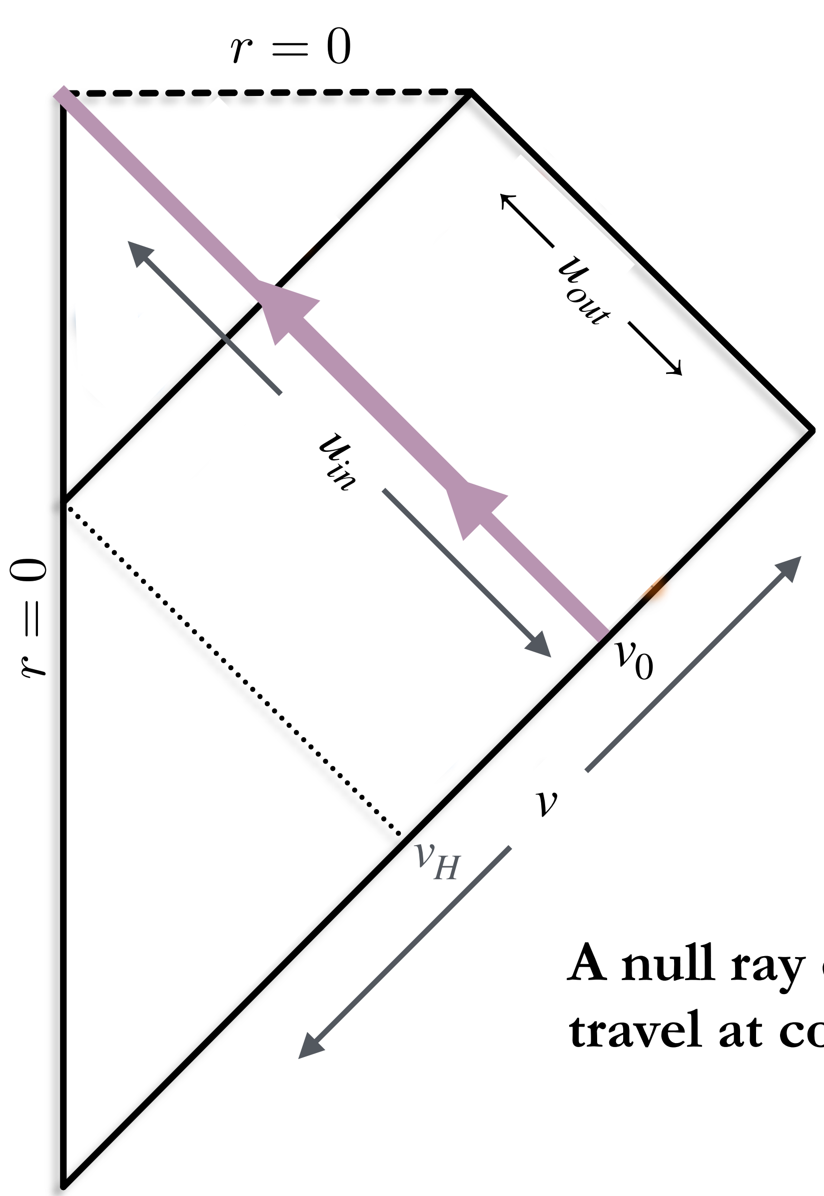
$$u_{out} = t_{out} - r^*_{out}$$

where: $$r^*_{out} := r_{out} + 2M \ln \frac{|r_{out} - 2M|}{2M}$$

Line element:

$$ds^2 = \begin{cases} -dt_{in}^2 + dx^2 + dy^2 + dz^2 = -du_{in}dv + r_{in}^2\, d^2\Omega\,, & \text{for } v \leq v_0 \\ -\left(1 - \frac{2M}{r_{out}}\right) du_{out}dv + r_{out}^2\, d^2\Omega\,, & \text{for } v \geq v_0 \end{cases}$$

A relation between u_{in} and u_{out} can be obtained by demanding continuity of the metric at v_0:



$$ds^2 = \begin{cases} -du_{in}dv + r_{in}^2\, d^2\Omega\,, \text{ for } v \leq v_0 \\ -\left(1-\frac{2M}{r_{out}}\right) du_{out}dv + r_{out}^2\, d^2\Omega\,, \text{ for } v \geq v_0 \end{cases}$$

Continuity of the angular part implies: $r_{out}|_{v_0} = r_{in}|_{v_0}$

$$\Longrightarrow \frac{v_0 - u_{out}}{2} + 2M \ln \frac{|r_{out} - 2M|}{2M} = \frac{v_0 - u_{in}}{2}$$

$$\Longrightarrow u_{out} = u_{in} + \frac{1}{\kappa} \ln(\kappa\,|\,v_0 - u_{in} - 1/\kappa\,|) \qquad \textbf{where } \kappa := \frac{1}{4M}$$

A null ray departing from $\mathscr{I}^-$ at v, will get reflected at $r=0$, and subsequently travel at constant $u_{in} = v$. It will reach $\mathscr{I}^+$ at $u_{out} = u_{out}(u_{in} = v)$:

$$u_{out}(v) = v + \frac{1}{\kappa} \ln(\kappa\,|\,v_H - v\,|)$$

$u_{out} \to -\infty$: $u_{out} \sim v$

$u_{out} \to +\infty$: $u_{out} \sim v_H + \frac{1}{\kappa} \ln(\kappa\,|\,v_H - v\,|)$

I've used $v_H = v_0 - 1/\kappa$

"Late-time" null ray tracing relation

2. Quantum Theory

We want to calculate the quantum evolution from past to future of a massless real scalar field. Because the field is massless, we can use null hyper-surfaces as initial and final Cauchy hyper-surfaces

Initial hyper-surface: $\mathscr{I}^-$ (past null infinity)

Final hyper-surface: $\mathscr{I}^+ \cup H$ (union of future null infinity and the horizon)

In-representation:

The space time is asymptotically flat as we approach ast null infinity $\Longrightarrow$ preferred representation

Physical interpretation: the in-vacuum is the ground state at $\mathscr{I}^-$

Basis of positive-norm modes: $v^{(in)}_{\tilde{\omega},\ell,m}(x)$ such that $v^{(in)}_{\tilde{\omega},\ell,m}|_{\mathscr{I}^-} \sim i\frac{e^{-i\tilde{\omega}v}}{r\sqrt{4\pi\tilde{\omega}}}Y_{\ell,m}(\phi,\theta)$, for $\tilde{\omega} > 0$

$$\Longrightarrow \hat{a}^{(in)}_{\tilde{\omega},\ell,m} = -i\langle v^{(in)}_{\tilde{\omega},\ell,m}, \hat{\phi}\rangle \Longrightarrow |0\rangle_{\text{in}} \text{ in-vacuum}$$

Out-representation:

Basis of positive-norm modes at $\mathscr{I}^+$: $v^{(out)}_{\omega,\ell,m}(x)$ **such that** $v^{(out)}_{\omega,\ell,m}|_{\mathscr{I}^+} \sim i \frac{e^{-i\omega\, u_{out}}}{r\sqrt{4\pi\omega}} Y_{\ell,m}(\phi,\theta)$**, for** $\omega > 0$

$$\Longrightarrow \hat{a}^{(out)}_{\omega,\ell,m} = -\, i\, \langle v^{(out)}_{\omega,\ell,m}, \hat{\phi}\rangle \text{ in-vacuum}$$

Basis of positive-norm modes at H **(horizon):** $v^{(H)}_I(x)$**. No preferred choice. Let's wait a second.**

Evolution: Bogoluibov coefficients:

Question of interest: If the field is prepared in $|0\rangle_{in}$ at $\mathscr{I}^-$ how is the state perceived at $\mathscr{I}^+ \cup H$?

To answer this question, we need to compute Bogoluibov coefficients between $v^{in}_{\tilde\omega,\ell,m}(x)$ and $v^{out}_{\omega,\ell,m}(x)$ and also between $v^{in}_{\tilde\omega,\ell,m}(x)$ and $v^{H}_{I}(x)$.

Calculation of Bogoluibov coefficients between $v^{in}_{\tilde\omega,\ell,m}(x)$ and $v^{out}_{\omega,\ell,m}(x)$:

We need to bring the two modes to the same Cauchy hyper-surface. Let's bring $v^{out}_{\omega,\ell,m}(x)$ to $\mathscr{I}^-$:

To do this calculation accurately, we need to introduce wave-packets $\int_j^{j(j+1)} d\omega\, e^{i\,2\pi\omega n/\epsilon}\, v^{out}_{\omega,\ell,m}(x)$ in order to localize the out-modes around some value of u_{out} . For the sake of time, I'll not write the wave-packet integrals (which nevertheless "factorize" and do not change the calculation) and pretend they are there.

$$v^{(out)}_{\omega,\ell,m}(x)\,|_{\mathscr{I}^+} \sim -\,i\,\frac{e^{-i\omega\, u_{out}(v)}}{r\sqrt{4\pi\omega}}\,Y_{\ell,m}(\phi,\theta)$$

Propagation to $\mathscr{I}^-$ $\longrightarrow$

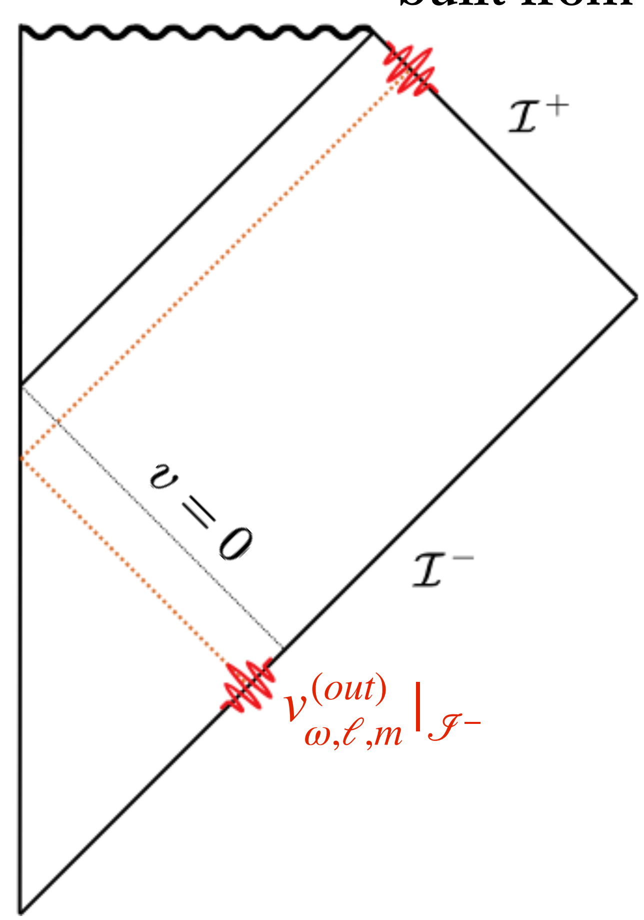
$$-i\,\frac{e^{-i\omega\, v}}{r\sqrt{4\pi\omega}}\,Y_{\ell,m}(\phi,\theta) \qquad \text{for early time modes}$$

$$-i\,\frac{e^{-i\omega\,[v_H-\frac{1}{\kappa}\ln(\kappa(v_H-v))]}}{r\sqrt{4\pi\omega}}\,Y_{\ell,m}(\phi,\theta)\times\Theta(v_H-v) \qquad \text{for late-time modes}$$

Early-time modes ($u_{out} \to -\infty$): By inspection, $\beta_{in,out} = 0 \Longrightarrow$ no particle creation

Late-time modes ($u_{out} \to +\infty$): By inspection, $\beta_{in,out} \neq 0 \Longrightarrow$ particle creation

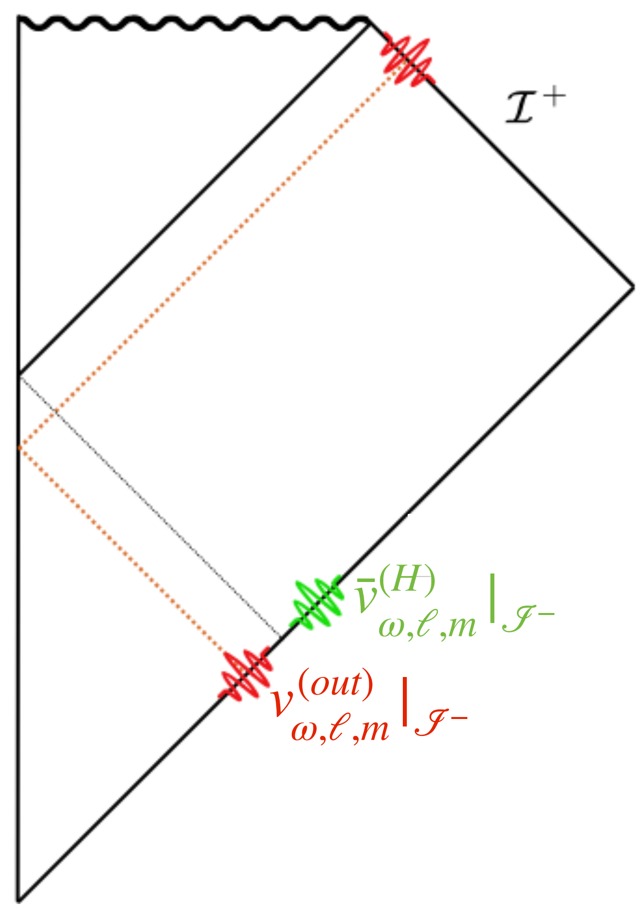
Late-time wavepacket built from $v^{(out)}_{\omega,\ell,m} = i\frac{e^{-i\omega\, u_{out}}}{r\sqrt{4\pi\omega}} Y_{\ell,m}(\phi,\theta)$



The Bogoliubov coefficients can be obtained by using a smart trick proposed in [Wald 1975]. The trick consist in making a smart choice of basis modes at the Horizon, and also at $\mathscr{I}^-$.

- **Step 1:** Define Horizon negative-norm modes by "reflecting" $v^{out}_{\omega,\ell,m}(x)|_{\mathscr{I}^-}$ about v_H:
  (the corresponding positive-norm modes are obtained by complex conjugation)

$$\bar{v}^{(H)}_{\omega,\ell,m}(x)|_{\mathscr{I}^-} \sim -i\frac{e^{-i\omega\,[v_H-\frac{1}{\kappa}\ln(\kappa(v-v_H)]}}{r\sqrt{4\pi\omega}}\,Y_{\ell,m}(\phi,\theta)\times\Theta(v-v_H)$$



- **Step 2:** **Write the following trivial equalities:**

$$v^{(out)}_{\omega,\ell,m}(x) = \alpha_\omega\,[\alpha_\omega\,v^{(out)}_{\omega,\ell,m}(x) + \beta_\omega\,\bar{v}^{(H)}_{\omega,\ell,m}(x)] - \beta_\omega\,[\alpha_\omega\,\bar{v}^{(H)}_{\omega,\ell,m}(x) + \beta_\omega\,v^{(out)}_{\omega,\ell,m}(x)]$$

$$v^{(H)}_{\omega,\ell,m}(x) = \alpha_\omega\,[\alpha_\omega\,v^{(H)}_{\omega,\ell,m}(x) + \beta_\omega\,\bar{v}^{(out)}_{\omega,\ell,m}(x)] - \beta_\omega\,[\alpha_\omega\,\bar{v}^{(out)}_{\omega,\ell,m}(x) + \beta_\omega\,v^{(H)}_{\omega,\ell,m}(x)]$$

With $\alpha_\omega = (1 - e^{-2\pi\omega/\kappa})^{-1/2}$ **and** $\beta_\omega = (e^{2\pi\omega/\kappa} - 1)^{-1/2}$ **. (Notice that** $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$**)**

Let's rename the combinations in the square brackets in the previous equations:

$$e^{(in,I)}_{\omega,\ell,m} := \alpha_\omega\,v^{(out)}_{\omega,\ell,m}(x) + \beta_\omega\,\bar{v}^{(H)}_{\omega,\ell,m}(x)$$

$$e^{(in,II)}_{\omega,\ell,m} := \alpha_\omega\,v^{(H)}_{\omega,\ell,m}(x) + \beta_\omega\,\bar{v}^{(out)}_{\omega,\ell,m}(x)$$

So the previous equations become:

$$v^{(out)}_{\omega,\ell,m}(x) = \alpha_\omega\,e^{(in,I)}_{\omega,\ell,m} + \beta_\omega\,\bar{e}^{(in,II)}_{\omega,\ell,m}$$

$$v^{(H)}_{\omega,\ell,m}(x) = \alpha_\omega\,e^{(in,II)}_{\omega,\ell,m} + \beta_\omega\,\bar{e}^{(in,I)}_{\omega,\ell,m}$$

- **Step 3:** This is the key observation [Wald 1975]:

$e^{(in,I)}_{\omega,\ell,m}$ and $e^{(in,II)}_{\omega,\ell,m}$ are made exclusively if positive frequency modes with respect to time v

(Can be check directly by Fourier transform)

This implies that their associated annihilation operators (call them $\hat{a}^{(in,I)}_{\omega,\ell,m}$ and $\hat{a}^{(in,II)}_{\omega,\ell,m}$) kill the in-vacuum

Using results from our previous lectures, this automatically imply that the in-vacuum is perceived in $\mathscr{I}^+ \cup H$ as

$$|0\rangle_{in} = N\, e^{\left[\frac{1}{2}\sum_{\ell,m}\int_0^\infty d\omega\; e^{-\pi\omega/\kappa}\, \hat{a}^{(out)\dagger}_{\omega,\ell,m}\hat{a}^{(H)\dagger}_{\omega,\ell,m}\right]}\,|0\rangle_{out}\otimes|0\rangle_{in}$$

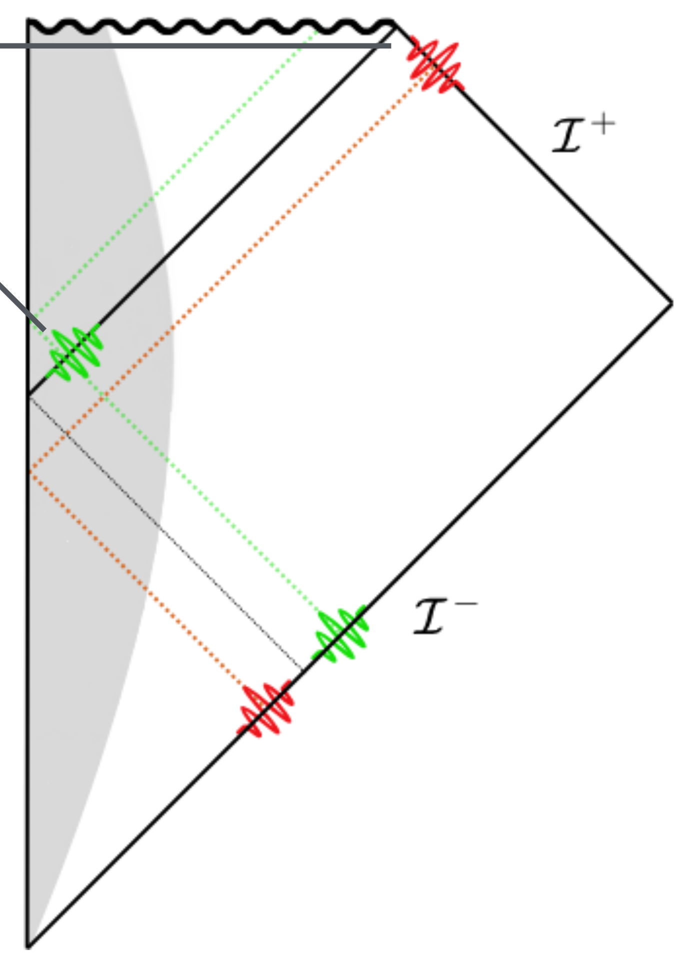
This is a two-mode squeezed vacuum at $\mathscr{I}^+ \cup H$!!!

Two-mode squeezed vacuum in $\mathscr{I}^+ \cup H$

$$|0\rangle_{in} = N\, e^{\left[\frac{1}{2}\sum_{\ell,m}\int_0^\infty d\omega\; e^{-\pi\omega/\kappa}\, \hat{a}^{(out)\dagger}_{\omega,\ell,m}\hat{a}^{(H)\dagger}_{\omega,\ell,m}\right]}\,|0\rangle_{out}\otimes|0\rangle_{H} = N\otimes_{\omega,\ell,m}\sum_{n=0}^{\infty} e^{-n\pi\omega/\kappa}\,|n_{\omega,\ell,m}\rangle_{out}\otimes|n_{\omega,\ell,m}\rangle_{H}$$

Entangled modes!

Entangled modes!



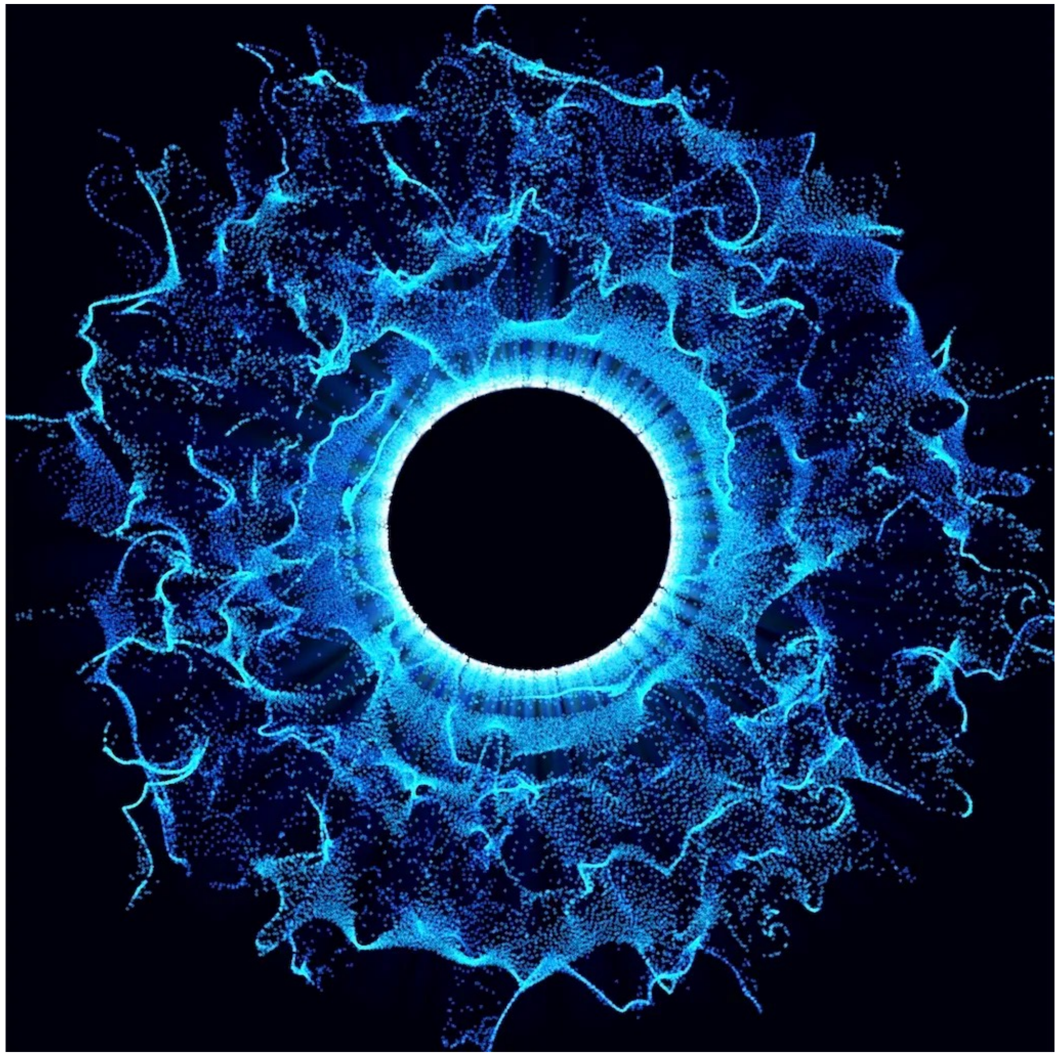
Reduced state at $\mathscr{I}^+$:

$$\hat{\rho}^{out}_{red} = \mathrm{Tr}\,|0\rangle_{in}\langle 0| = N^2 \otimes_{\omega,\ell,m} e^{-2\pi\omega/\kappa}\, |n_{\omega,\ell,m}\rangle_{out}\langle n_{\omega,\ell,m}|$$

This is precisely the density operator of a thermal state at temperature: $T = \dfrac{\kappa}{2\pi} = \dfrac{1}{8\pi M}$

Restoring the constants: $T = \dfrac{\hbar c^3}{8\pi M G k_B}$ **(Hawking temperature). Inverse with *M*!**

(We have neglected back-scattering for reasons of time. If included, the thermal spectrum is modified by some gray-body factors, which can be computed numerically and are small).



Black holes ain't black. They are hot. Their color depends on their Mass!

Observability?

$$T_H \approx 10^{-7}\, K\, \frac{M_\odot}{M}$$

Hawking radiation is over-shined by the Cosmic Microwave Background

(So, astrophysical black holes do not evaporate; they grow)

Smaller black holes are hotter!

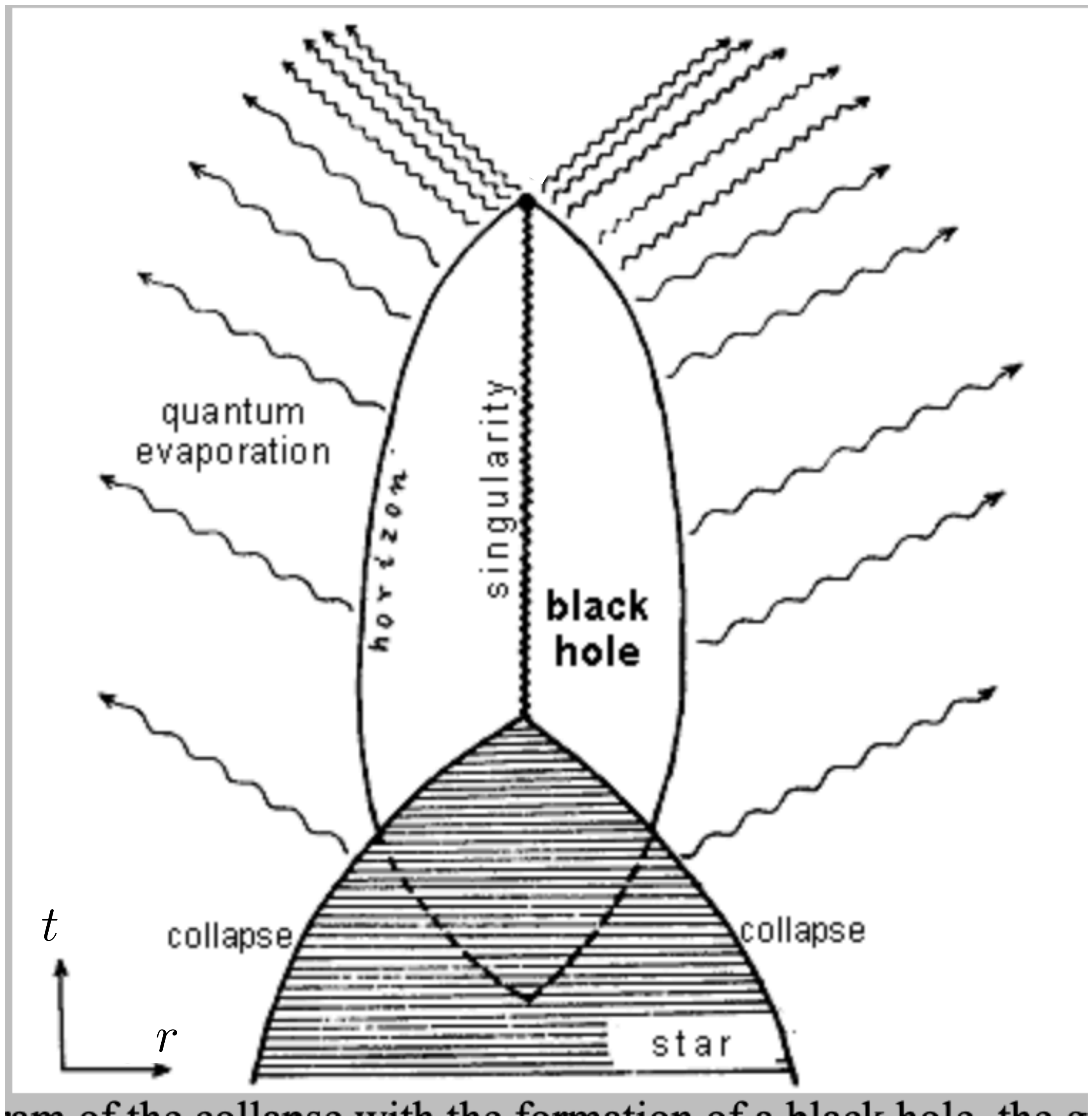
Primordial Black holes, if exist, would have evaporated by now

Breakdown of Predictability in Gravitational Collapse #196

S.W. Hawking (Cambridge U. and Caltech) (1976)

Published in: *Phys.Rev.D* 14 (1976) 2460-2473

DOI cite claim reference search 2,451 citations



Black holes “burn” all info that falls inside. They transforms everything into thermal radiation

Is information lost in Black Hole Evaporation?

You will need to invite me again to hear about this part of the history : -)

Thanks!