

# An Introduction to QFT in curved spacetimes

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# Lecture 3

Qft in curved spacetimes

## Classical theory

**Ingredients:**

- Due to time constrains: restrict to real scalar field  $\phi(\vec{x})$
- Spacetime: assume global hyperbolic  $M_4 = \mathbb{R} \times \Sigma_3$ ,  $g_{\mu\nu}$   
(global hyperbolicity needed for the initial value problem be well posed)
- Action:  $S_{KG} = -\frac{1}{2} \int d^4x \sqrt{-g} (\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2 + \xi R \phi^2)$
- $\nabla_\mu$  covariant derivative defined from  $g_{\mu\nu}$
- $R$  Ricci scalar
- $\xi \in \mathbb{R}$  coupling to the curvature (minimal coupling=0, conformal coupling =1/6)
- E.o.m.:  $(\square - m^2 - \xi R)\phi = 0 \quad \longrightarrow \quad \left( \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - m^2 - \xi R \right) \phi = 0$

- Conjugate momentum  $\Pi(\vec{x})$

Its definition **requires the introduction of a 3+1 foliation** of spacetime in Cauchy hyper surfaces:  $M_4 = \mathbb{R} \times \Sigma_t$

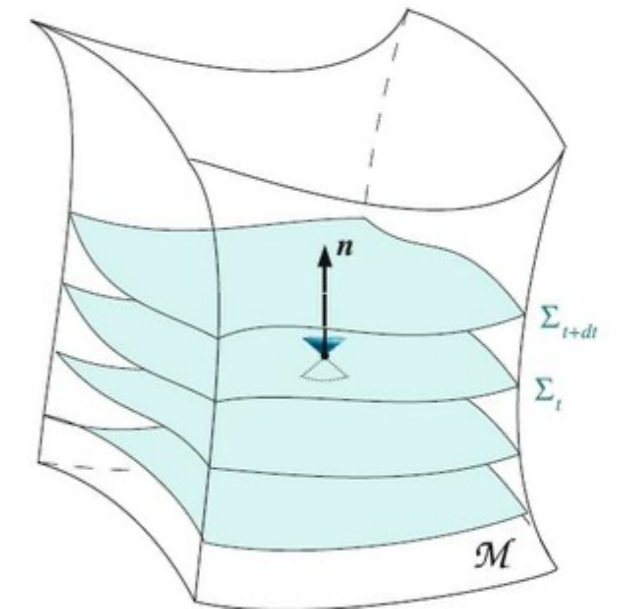
Let  $n^\mu$  be the (time like, future oriented) unit vector field normal to the hypersurfaces  $\Sigma_t$

Let  $h_{\mu\nu} = g_{\mu\nu} + n^\mu n^\nu$  be the metric in  $\Sigma_t$  **induced** by  $g_{\mu\nu}$

With this:  $\Pi(\vec{x}) = \sqrt{h} n^\mu \nabla_\mu \phi(\vec{x})$

- Hamiltonian 
$$H = \int_{\Sigma_t} d^3x \left( \sqrt{h} \mathcal{L} - \Pi(\vec{x}) \phi(\vec{x}) \right)$$

- Phase space



We need to make a choice about the **type of functions** we want to use. Common choice in Minkowski spacetimes and in spatially-flat FLRW: Schwartz space (stable under Fourier transform). But Schwartz space does not generalize to curved spacetimes.

Common choice: **smooth functions of compact support**  $C_0^\infty$

$$\Gamma = \{ (f(\vec{x}), g(\vec{x})) \in C_0^\infty \times C_0^\infty \}$$

Infinite-dimensional, real v.s.

The classical phase space comes equipped with a symplectic structure  $\omega$

$$\omega : \Gamma \times \Gamma \rightarrow \mathbb{R}$$

If  $\gamma_1(\vec{x}) = (f_1(\vec{x}), g_1(\vec{x}))$  and  $\gamma_2(\vec{x}) = (f_2(\vec{x}), g_2(\vec{x}))$

$$\omega(\gamma_1, \gamma_2) = \int d^3x (g_1(\vec{x}) f_2(\vec{x}) - f_1(\vec{x}) g_2(\vec{x}))$$

**Bi-linear, anti-symmetric and weakly degenerate bi-distribution.**

**Weakly degenerate means:**  $\omega(\gamma_1, \gamma_2) = 0 \quad \forall \gamma_2 \iff \gamma_1 = 0$

To quantize the theory, it will be useful to work with the complexities phase space:  $\Gamma_{\mathbb{C}}$

**And the complexities symplectic product:**  $\langle \gamma_1, \gamma_2 \rangle := -i\omega(\gamma_1^*, \gamma_2)$

**(Also known in this theory as Klein-Gordon product)**

# Quantum theory

- **Field and momentum “operators”:**  $\vec{r}(\vec{x}) = (\phi(\vec{x}), \Pi(\vec{x}))$
- **Commutation relations:**  $[r^i(\vec{x}), r^j(\vec{x}')] = i \Omega^{ij} \delta(\vec{x} - \vec{x}')$  where  $\Omega^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- **Recall:**  $\phi(\vec{x}), \Pi(\vec{x})$  are **not** well-defined operators. They need to be “**smearred-out**” with test functions.
- **Linear operators:**  $O_\gamma = \omega(\gamma^*, \vec{r}) = \int d^3x (g^*(\vec{x}) \Pi(\vec{x}) - f^*(\vec{x}) \phi(\vec{x}))$  for all  $\gamma \in \Gamma_{\mathbb{C}}$
- **Algebra of linear operators:**  $[O_{\gamma_1}, O_{\gamma_2}^\dagger] = \langle \gamma_1, \gamma_2 \rangle$  for all  $\gamma_1, \gamma_2 \in \Gamma_{\mathbb{C}}$

We can think of this as an “abstract algebra”, i.e., an algebra with no Hilbert space where to act (yet).

The next step is, following what we did in the previous lecture, construct a **Fock space** and a **representation** of the abstract algebra in it.

## Key ingredient: Kähler structure

Introduce a metric  $\sigma$  in  $\Gamma$  satisfying the following **compatibility conditions**:

1.  $\omega$  is bounded in the norm defined by  $\sigma$

This condition allows one to extend  $\omega$  to  $\Gamma_\sigma$  (the Cauchy-completion of  $\Gamma$ ).

It also allows us to define the inverse of  $\omega : \Omega$

This condition is trivially satisfied when  $\Gamma$  is finite dimensional (explaining why we did not mention it in the previous lectures)

2.  $J^2 = -\mathbb{I}$  in  $\Gamma_\sigma$

where  $J = -\Omega(\cdot, \sigma)$       More explicitly:  $J(\vec{x}, \vec{x}') = - \int d^3 x'' \Omega(\vec{x}, \vec{x}'') \sigma(\vec{x}'', \vec{x}')$

$\implies$  The symplectic eigenvalues of  $J$  are  $i$  and  $-i$

$\implies$  The symplectic eigenvectors of  $J$  are  $\vec{e}_I(\vec{x})$  and  $\vec{e}_I^*(\vec{x})$  with  $I = 1, \dots, \infty$

The triple  $(\omega, \sigma, J)$  defines a Kähler structure in  $\Gamma$

$\implies$  3. The quadratic form:  $\sigma(\gamma^*, \gamma) - i\omega(\gamma^*, \gamma) \geq 0$  is positive semi-definite in  $\Gamma_\sigma^\mathbb{C}$ .

Next, we proceed as explained in the previous lectures. Namely:

The eigenvectors of  $J$  with eigenvalue  $i$ , denoted as  $e_I, I = 1, \dots, \infty$ , define **annihilation operators**  $a_I$

These ops. define a **Fock vacuum**  $|0\rangle$  and, acting with creation operators, the Fock space  $\mathcal{F}$

The following expression provides a representation of the canonical operators in  $\mathcal{F}$

$$\vec{r}(\vec{x}) = \begin{pmatrix} \phi(\vec{x}) \\ \Pi(\vec{x}) \end{pmatrix} = -i \sum_{I=1}^{\infty} \vec{e}_I(\vec{x}) a_I - \vec{e}_I^*(\vec{x}) a_I^\dagger$$

From this, we obtain the representation of any linear operator  $O_\gamma$  via:

$$O_\gamma = \omega(\gamma, \vec{r}) = i\langle \gamma, r \rangle$$

Taking products of these operators we obtain the representation of any other operator. All these operators are well-defined in  $\mathcal{F}$  (i.e. no ultra-violet divergences)

**Remark:** There is a special family of “operators”, such as the energy-momentum tensor, which require further attention due to the appearance of ultra-violet divergences.

## Time evolution

The Heisenberg time evolution of the operators  $O_\gamma$  can be easily obtained as follows.

First, define  $\vec{r}(\vec{x}, t)$  by **evolving the symplectic eigenvectors**  $\vec{e}_I(\vec{x})$  using the **classical equations of motion**

$$\vec{e}_I(\vec{x}) \longrightarrow \vec{e}_I(\vec{x}, t) \quad (\text{classical evolution})$$

$$\implies \vec{r}(\vec{x}, t) = \begin{pmatrix} \phi(\vec{x}, t) \\ \Pi(\vec{x}, t) \end{pmatrix} = -i \sum_{I=1}^{\infty} \vec{e}_I(\vec{x}, t) a_I - \vec{e}_I^*(\vec{x}, t) a_I^\dagger$$

With this:

$$O_\gamma(t) = \omega(\gamma(\vec{x}), \vec{r}(\vec{x}, t)) \quad \text{Heisenberg linear operators}$$

**Remarks:** once again, in field theory  $\phi(\vec{x})$  and  $\Pi(\vec{x})$  must be understood as distributions, and  $\omega(\vec{x}, \vec{x}')$  and  $\sigma(\vec{x}, \vec{x}')$  as bi-distributions.

**Example: Massless scalar field in Minkowski spacetime**

For technical simplicity, let us consider flat spacetime “in a box”, in the sense that we replace the spatial slices with  $\mathbb{R}^3$  topology by a torus  $\mathbb{T}^3$  (this corresponds to a box with periodic boundary conditions).

The **classical phase space** is:  $\Gamma = \{(f(\vec{x}), g(\vec{x})) \in C_0^\infty \times C_0^\infty\}$

Equipped with the **symplectic structure**  $\omega(\vec{x}, \vec{x}') = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta(\vec{x}, \vec{x}')$

The standard, textbook quantization in this spacetime is obtained by introducing the following **metric**:

$$\sigma(\gamma, \gamma') = (f, g) \begin{pmatrix} \Re(\cdot, \cdot)_{-1/2} & 0 \\ 0 & \Re(\cdot, \cdot)_{1/2} \end{pmatrix} \begin{pmatrix} f' \\ g' \end{pmatrix} = \Re(f, f')_{-1/2} + \Re(g, g')_{1/2}.$$

Where  $(\cdot, \cdot)_s$  is a Sobolev product of order  $s$ , defined as:

$$(f, g)_s \equiv \int_{\mathbb{R}^3} \frac{d^3 k}{(2\pi)^3} |\vec{k}|^{2s} \tilde{f}(\vec{k}) \tilde{g}^*(\vec{k}).$$

Note that  $\sigma$  defines the **two-point functions** of field and momentum operators as:

$$\sigma(\vec{x}, \vec{x}') = \begin{pmatrix} \langle 0 | \{ \hat{\Pi}(\vec{x}) \hat{\Pi}(\vec{x}') \} | 0 \rangle & \langle 0 | \{ \hat{\Phi}(\vec{x}) \Pi(\vec{x}') \} | 0 \rangle \\ \langle 0 | \{ \hat{\Phi}(\vec{x}') \Pi(\vec{x}) \} | 0 \rangle & \langle 0 | \{ \hat{\Phi}(\vec{x}) \hat{\Phi}(\vec{x}') \} | 0 \rangle \end{pmatrix}$$

It is easy to check that the expressions for these two-point functions agree with the one obtained using standard textbook quantization.

The linear map: 
$$J(\vec{x}, \vec{x}') = - \int d^3 x'' \Omega(\vec{x}, \vec{x}'') \sigma(\vec{x}'', \vec{x}') = \frac{1}{V_0} \sum_{\vec{k}} e^{i\vec{k}(\vec{x}-\vec{x}')} \begin{pmatrix} 0 & -\frac{1}{k} \\ k & 0 \end{pmatrix}. \quad \text{where } k \equiv |\vec{k}|$$

defines a complex structure in the classical phase space  $\Gamma_\sigma$

Eigenvectors of  $J$  with eigenvalue  $i$ : 
$$\vec{e}_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ -ik \end{pmatrix} e^{i\vec{k}\cdot\vec{x}}$$

The normalization is chosen so  $i\omega(\vec{e}_I^*, \vec{e}_I) = 1$ , since this implies:  $[a_I, a_I^\dagger] = i\omega(\vec{e}_I^*, \vec{e}_I) = 1$

With this:

$$\vec{r}(\vec{x}) = \begin{pmatrix} \phi(\vec{x}) \\ \Pi(\vec{x}) \end{pmatrix} = -i \sum_{\vec{k}} \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ -ik \end{pmatrix} e^{i\vec{k}\cdot\vec{x}} a_{\vec{k}} - \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ ik \end{pmatrix} e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k}}^\dagger$$

Time evolution:

$$\vec{e}_{\vec{k}}(\vec{x}, t) = \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ -ik \end{pmatrix} e^{i(-kt + \vec{k}\cdot\vec{x})}$$

$$\implies \vec{r}(\vec{x}, t) = \begin{pmatrix} \phi(\vec{x}, t) \\ \Pi(\vec{x}, t) \end{pmatrix} = -i \sum_{\vec{k}} \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ -ik \end{pmatrix} e^{i(-kt + \vec{k}\cdot\vec{x})} a_{\vec{k}} - \frac{1}{\sqrt{2k V_0}} \begin{pmatrix} 1 \\ ik \end{pmatrix} e^{-i(-kt + \vec{k}\cdot\vec{x})} a_{\vec{k}}^\dagger$$

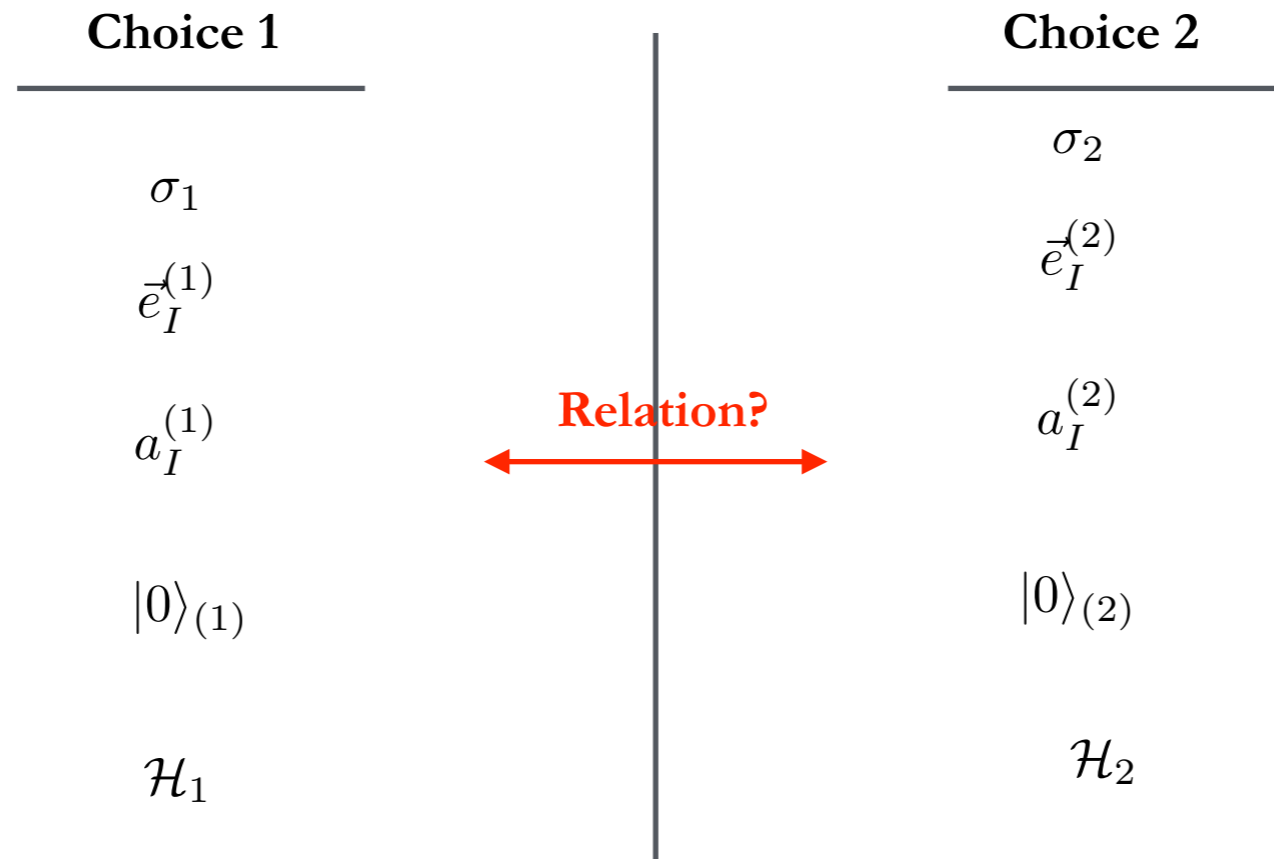
These expressions are equivalent to the ones we learn in introductory courses on qft.

The Fock vacuum defined by this construction is invariant under Lorentz transformations, and time and space translations.

## Bogoliubov Transformations

Formally identical to the discussion in previous lecture

Starting from  $(\Gamma, \omega)$ , introduce **two different**  $\sigma'$ s:  $\sigma_1$  and  $\sigma_2$



**Relation:**

$$\vec{e}_I^{(2)} = \sum_{I'} \alpha_{II'} e_{I'}^{(1)} + \beta_{II'} (e_{I'}^{(1)})^* \quad \text{(relation btw basis of symplectic eigenvectors)}$$

Bogoluibov coefficients

**With this:**

$$a_I^{(2)} = \omega((\vec{e}_I^{(2)})^*, \vec{r}) = \sum_{I'} \alpha_{II'}^* \omega((e_{I'}^{(1)})^*, \vec{r}) + \beta_{II'}^* \omega(e_{I'}^{(1)}, \vec{r}) = \sum_{I'} \alpha_{II'}^* a_{I'}^{(1)} + \beta_{II'}^* a_{I'}^{(1)\dagger}$$

$$a_I^{(2)} = \sum_{I'} \alpha_{II'}^* a_{I'}^{(1)} + \beta_{II'}^* a_{I'}^{(1)\dagger} \quad \text{Bogoluibov relation}$$

From this relation, one can show:  $|0\rangle_{(1)} = N \exp \left[ \frac{1}{2} \lambda_{IJ} a_I^{(2)\dagger} a_J^{(2)\dagger} \right] |0\rangle_{(2)}$

where:  $\lambda_{IJ} = \sum_{I'} \beta_{II'}^* (\alpha^{-1})_{I'J}$

The vacuum  $|0\rangle_{(1)}$  corresponds to a squeezed state in the Fock space of vacuum  $|0\rangle_{(2)}$

- Subtlety

In finite dimensions, there is **guaranteed** that the Fock spaces built from different metrics are **unitarily equivalent** (Stone-Von Neumann Theorem). This is **not true** in field theory.

In field theory, two Fock representations can be **unitarily inequivalent**. They are unitarily equivalent **if and only if** the linear map

$J_1 - J_2$  has finite Hilbert-Schmidt norm:

$$\text{Tr}(J_1 - J_2)^2 < \infty$$

One practical way of checking this

$$\text{Tr}(J_1 - J_2)^2 < \infty \quad \text{iff} \quad \sum_{I,J} |\beta_{IJ}|^2 < \infty$$

In physics language: two Fock representations are unitarily equivalent, **if and only if one Fock-vacuum has finitely many quanta in the other Fock space.**

If two Fock spaces are unitarily inequivalent, they describe different physical theories.

This is considered by many just a “**mathematical**” **problem**, in the sense that, if we **restrict to a finite number of observable** —something we always do in practice— we can always find a map between the two formulations that is unitary.

**Exception**

If  $g_{\mu\nu}$  admits a **globally time-like Killing vector field**  $t^\mu$ , there is a **preferred vacuum**.

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The quanta created by creation operators have a natural **particle interpretation**.  
(Particle = whatever a particle detector detects)

The preferred Fock representations can be constructed as follows:

Identify the symplectic eigenvectors  $e_I(\vec{x}, t)$  with the solutions to the e.o.m. that oscillate **with positive frequency w.r.t. to the Killing time**. Work backwards and construct  $\sigma$  from the symplectic eigenvectors.

The Fock vacuum singled out in this way, is physically preferred because it is the ground state of the (time-independent) Hamiltonian defined by  $t^\mu$

$$\hat{H} = \int_{\Sigma_3} d^3x \sqrt{h} \hat{T}_{\mu\nu} n^\mu t^\nu$$

Message: a **time-like symmetry** is enough to recover a preferred notion of vacuum. A space-like symmetry is not.

Original reference:

A. Ashtekar and A. Magnon. Proc. R. Soc. Lon. A 346, 1975