Loop Quantum Gravity Summer School May 18-20 2026

INTRODUCTION TO QFT IN CST

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Goal: Fock quantization of fields propagating in globally hyperbolic spacetimes.

Only 4 x 50 min lectures, so this is just an "advertisement".

Program:

1. Fock quantization of linear quantum mechanical systems
2. Fock quantization of linear fields
3. Application: the Hawking effect

References:

- QFT and BH thermodynamics, R. Wald
- QFT in curved spacetimes, Birrell and Davies
- Modeling BH evaporation, Fabbri and Navarro Salas

Lectures 1 and 2 :

1. Fock quantization of linear quantum mechanical systems

Motivation:

Many aspects of the theory of linear fields can be reduced to the properties of infinitely many oscillators, possibly coupled with each other via springs. This section provides a reformulation of the theory of one (and multiple) oscillators using a language that will allow generalization to field theory.
Many of the new conceptual and mathematical techniques that appear in quantum field theory in curved spacetimes have a simpler analog in the realm of oscillators.
This chapter introduces these tools in a gentler setting whose physics is surely well understood by all students.

1. Fock quantization of bosonic linear systems

1.1. Non-technical motivation

Simple harmonic oscillator:

$$H = \frac{1}{2}\,\omega\,(p^2 + q^2)\,, \qquad q, p \equiv \text{dimensionless position and momentum}$$

Gral. sol. e.o.m.:

$$q(t) = a\,\frac{e^{-i\omega t}}{\sqrt{2}} + a^{+}\,\frac{e^{i\omega t}}{\sqrt{2}} \qquad (1)$$

Quantum theory: Heisenberg evol.

$$\hat{q}(t) = \hat{a}\,\frac{e^{-i\omega t}}{\sqrt{2}} + \hat{a}^{+}\,\frac{e^{i\omega t}}{\sqrt{2}} \quad (2)\,, \qquad \hat{p}(t) = \frac{1}{\omega}\,\dot{\hat{q}}(t)$$

Inverting $\Rightarrow$ $\hat{a} \equiv \frac{1}{\sqrt{2}}\left(\hat{q}(t_0) + i\,\hat{p}(t_0)\right)$ (3) $\Rightarrow$ $[\hat{a}, \hat{a}^{+}] = 1$

$\uparrow$

$[\hat{q}(t_0), \hat{p}(t_0)] = i$

Hilbert (Fock) space: spanned by the number states basis:

$$\mathcal{F} \approx \overline{\text{span}\,\{|n\rangle\}_{n=0}^{\infty}}\,, \qquad \text{with} \quad \underbrace{\hat{a}^{+}\hat{a}}_{\equiv \hat{N}}|n\rangle = n\,|n\rangle\,, \quad n \in \mathbb{N}$$

Interpretation: $|n\rangle$ is a state with n quanta

Eqn. (2) provides a representation of $\hat{q}(t)$ ...

Questions:

The construction of this quantum representation rests on the (non-Hermit.) operator $\hat{a}$ (Dirac). Its key mathematical property is: $[\hat{a}, \hat{a}^\dagger] = 1$. creation and annihilation opts.

Q: Is $\hat{a}$ unique in any sense?

A: No. $\hat{b} \equiv \cosh\theta \; \hat{a} + \sinh\theta \; \hat{a}^\dagger$, $\theta \in \mathbb{R}$

is such that $[\hat{b}, \hat{b}^\dagger] = \cosh^2\theta - \sinh^2\theta = 1$.

$\Rightarrow$ there are infinitely many possible choices of non-Herm. opt. $\hat{b}$ with the same math. propertis of $\hat{a}$.

Q: Then, why do we use $\hat{a}$ (as defined in (3))

A: Because $\hat{a}$ is "adapted to the Hamiltonian" of the harmonic oscillator, in the following sense:

- Mathematical reason: $[\hat{H}, \hat{a}^\dagger \hat{a}] = 0$
- Physics: the states $|n\rangle$ are eigenstates of $\hat{H}$ (stationary states) $\Rightarrow$ $\hat{a}^\dagger$ creates a quantum of energy.

  $|0\rangle$ is the ground state.

Q: What if $\hat{H}$ is different?

A: If $\hat{H}$ is time-independent and bounded from bel[ow]

one can follow the same approach using the normal modes of the Hamiltonian (see section ...)

$\Rightarrow$ preferred choice of $\hat{a}$.

Otherwise, there is no preferred choice. One can make any choice of $\hat{b}$ / $[\hat{b}, \hat{b}^+] = 1$ and define a basis of number states $|n\rangle$. But $|n\rangle$ will not be eigenstates of $\hat{H}$ (e.g. if $\hat{H}$ is time dependent, $\hat{H}(t)$ does not have eigenstates, since energy is not conserved).

$\Rightarrow$ $\hat{b}^+$ creates quanta, but these "quanta" have only mathematical meaning (no quanta of energy)

$\Rightarrow$ $|0\rangle$ (the state with no quanta) will <u>not</u> be the ground state (so "Fock vacuum" $\neq$ ground state, in general).

Next: How to construct a Fock representation without assuming any property of the Hamiltonian.

1.2. Some aspects of classical linear systems

Def: Linear system defined by Two conditions:

(i) Classical phase space is a vector space

(ii) Hamiltonian is a quadratic polynomial of canonical variables.

- Example: N oscillators (possibly time-dep) coupled via springs define linear systems.
- No example: planar pendulum ; phase space is a cylinder (not a vector space) ; non-polynomial Hamiltonian.

Math. structure of classical theory of linear systs.:

(1) Space of physical states (classical version of Hilbert space)

Phase space : $\Gamma \equiv$ $2N$ - dimesional real vect. space (N degrees of freedom)

(2) Observables : real functions in Γ

Example : Canonical coordinates $q_1, \dots, q_N, p_1 \dots p_N$

Notation : $r \equiv (q_1, \dots, q_N, p_1 \dots p_N)$: vector of canonical observables

components r^i, $i = 1, \dots, 2N$

Other examples: Kinetic energy, potential energy, total energy, angular momentum, etc.

(3) Key mathematical structure in Γ: symplectic structure ω_{ab}

- ω_{ab} is a rank 2 covariant tensor
  - anti-symmetric
  - non-degenerate (if $\omega_{ab}\gamma^b = 0 \iff \gamma^b$ = zero vector)

$\Rightarrow \omega_{ab}$ is invertible: Ω^{ab}

$$\left(\Omega^{ab}\omega_{bc} = \delta^a{}_c \, , \quad \Omega^{ba}\omega_{bc} = -\delta^a{}_c \right)$$

Components of ω in a canonical (or Darboux) coords. r^i:

$$\omega \doteq \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix} = \bigoplus_N \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Omega \doteq \bigoplus_N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The symplectic structure plays two roles:

(a) ω_{ab} and Ω^{ab} can be used to raise and lower indices of tensors in Γ, symilarly to what we do in relativity with the metric. (But be carefull, ω and Ω are anti-symmetric, so watch out for signs).

(b) Ω defines Poisson brackets (= algebraic structure (i.e. a sort of "product") among observables:

f, g functions in Γ: $\quad \{f, g\} \equiv \Omega^{ij} \dfrac{\partial f}{\partial r^i} \dfrac{\partial g}{\partial r^j}$

an operation which assigns a function to any pair of functs

Example: Poisson brackets of canonical observables

$$\{r^i, r^j\} = \Omega^{ij}$$

because $\{r^i, r^j\} = \Omega^{kl} \frac{\partial r^i}{\partial r^k} \frac{\partial r^j}{\partial r^l} = \Omega^{kl} \delta^i{}_k \delta^j{}_l = \Omega^{ij}$.

Time evolution

General quadratic Hamiltonian: $H = \frac{1}{2} h_{ij} r^i r^j + \underbrace{v_i r^i}_{\text{can be eliminated by a linear re-definit of } r^i} + \underbrace{c}_{\text{does not affect dynamics}}$

$\Rightarrow$ without loss of generality, restrict to $H = \frac{1}{2} h_{ij} r^i r^j$, with h_{ij} = symmetric real matrix (possibly time-dep.)

If H quadratic $\Rightarrow$ e.o.m. are linear $\Rightarrow$ $r^i(t)$ and $r^j(0)$ must be linearly related: $r^i(t) = E^i{}_j(t)\, r^j(0)$ with $E^i{}_j(t)$ a real matrix (= Hamiltonian flow in phase space)

Example: If h_{ij} is time-independent,

$$E(t) = \exp(-i J_H t), \quad \text{with } J_H = i\,\Omega \cdot h$$

In components: $(J_H)^j{}_k = i\, \Omega^{jl} h_{lk}$

Proof:

Hamilton's equations: $\dot{r}^i = \{r^i, H\} = \{r^i, \frac{1}{2} h_{kl} r^k r^l\}$

$= \{r^i, r^k\} h_{kl} r^l = \Omega^{ik} h_{kl} r^l = -i (J_H)^i{}_l r^l$

If J_H is time-independent this diff eqn. has the simple soluti[on]

$\dot{r}^i = -i (J_H)^i{}_l r^l$ is $\vec{r}(t) = e^{-i J_H t} \cdot \vec{r}(0)$ $\square$

For a simple harm. oscill., $h = \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\rightarrow \quad J_H = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = i\omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad E = \exp\left[i\omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

Theorem

If $r^i(t) = E^i{}_j(t)\, r^j(0)$ is the solution to Hamilton's eqns. of a classical linear system, then the solution to Heisenberg's eqns. of the correspoding quantum system is

$$\hat{r}^i(t) = E^i{}_j(t)\, \hat{r}^j(0)$$

i.e., the classical solution with hats on r^i.

This implies that dynamics of the classical theory complete determines the quantum dynamics.

Also true in linear field theory.

No generically true for non-linear systems.

Proof:

Hamilton's eqns.:

$$\frac{d}{dt} r^i(t) = \{ r^i, H \} \quad \Rightarrow \quad \frac{d}{dt}\left(E^i{}_j(t)\, r^j \right) = \Omega^{ik} h_{k\ell}\, r^\ell$$

$$\Rightarrow \quad \frac{d}{dt} E^i{}_j(t) = \Omega^{ik} h_{k\ell} \qquad (4)$$

Heisenberg's eqns.:

$$\frac{d}{dt}\hat{r}^i = -i\,[\hat{r}^i, H] \quad (5)$$

Let's check $\hat{r}^i(t) = E^i_{\ j}(t)\,\hat{r}^j(0)$ is a solution:

l.h.s. of (5):

$$\frac{d}{dt}\left(E^i_{\ j}(t)\,\hat{r}^j(0)\right) \overset{(4)}{=} \Omega^{ik} h_{kj}\,\hat{r}^j(0)$$

r.h.s. of (5):

$$[\hat{r}^i, \hat{r}^j] = i\Omega^{ij}$$

$$-i\,[\hat{r}^i, H] = -i\left[\hat{r}^i, \tfrac{1}{2} h_{kl}\,\hat{r}^k\hat{r}^l\right] = \Omega^{ik} h_{kl}\,\hat{r}^l$$

$\square$.

Before describing the quantum theory:

Brief introduction to Gaussian states, since they are closely related to the quantization procedure.

1.4. Fock quantization of linear mechanical systems

Goal: Construct a Fock space and represent the algebra of observables

[Algebra of observables: algebra generated by canonical observables by taking linear combinations as products. See Madhavan's lectures for further details (e.g. Weyl algebra of bounded operators]

We'll do it without assuming any form of the Hamiltonian. The goal is to understand what freedom there exist.

The construction of a Fock representation requires three ingredients, two of which are already present in the classical theory.

Ingredient 1: The classical phase space Γ.

Actually, we'll use the complexification of Γ:

$\Gamma_{\mathbb{C}}$ (= made of all linear combinations of vectors γ in Γ with complex coefficients. If $\Gamma = \mathbb{R}^{2N}$, then $\Gamma_{\mathbb{C}} = \mathbb{C}^{2N}$)

[Equivalently, we could use the space of complex solutions to the e.o.m, $\mathcal{S}_{\mathbb{C}}$. $\mathcal{S}_{\mathbb{C}}$ and $\Gamma_{\mathbb{C}}$ are isomorphic, by identifying solutions and initial data at some time t_0.]

Ingredient 2: Symplectic product in $\Gamma_{\mathbb{C}}$

$$\boxed{\langle \cdot , \cdot \rangle \text{ defined as } \langle \gamma_1 , \gamma_2 \rangle = -i\, \omega(\gamma_1^*, \gamma_2)}, \quad \gamma_1, \gamma_2 \in \Gamma_{\mathbb{C}}$$

The "i" and the conjugate makes $\langle \cdot , \cdot \rangle$ Hermitian:

(a) $\langle \gamma_2 , \gamma_1 \rangle = \langle \gamma_1 , \gamma_2 \rangle^* \quad \forall \; \gamma_1, \gamma_2 \in \Gamma_{\mathbb{C}}$

Other properties:

(b) $\langle \cdot , \cdot \rangle$ is linear in the second entry (and anti-linear in the first entry, bcs of (a))

(c) $\langle \cdot , \cdot \rangle$ is <u>not</u> positive definite

Example: For complex γ, $\langle \gamma , \gamma \rangle = - \langle \gamma^*, \gamma^* \rangle$, so many γ's in Γ have negative symplectic norm

In particular, $\langle \gamma , \gamma \rangle = 0$ for all real γ.

$\Rightarrow$ $\langle \cdot , \cdot \rangle$ does <u>not</u> define an inner product in $\Gamma_{\mathbb{C}}$ (still, will be very useful) ($\Rightarrow$ $\Gamma_{\mathbb{C}}$, $\langle \cdot , \cdot \rangle$ do not define a Hilbert space.)

(d) $\langle \cdot , \cdot \rangle$ is invariant under time evolution:

$\langle \gamma(t), \gamma(t) \rangle$ is time-independent

This is a consequence of the fact that time evolution

This is a consequence of the fact that time evolution leaves invariant ω (canonical transformation)

So far, ingredients 1 and 2 are small extensions of structures borrowed from the classical theory. Now the genuinely new ingredient.

Ingredient 3:

Geometric version: Introduce in the classical phase space Γ a kähler structure:

$$\boxed{\text{metric } \sigma_{ab} \text{ satisfying } \underbrace{J=-\Omega\,\sigma}_{J^a{}_b=-\Omega^{ac}\sigma_{cb}} \text{ is a complex structure}}$$

i.e., $J^2=-\mathbb{I}$

(equivalently, introduce complex structure such that $\sigma_{ab}=\omega_{ac}J^c{}_d$ defines a metric.)

Example: $N=1$ (one degree of freedom) $\rightarrow$ $\Gamma=\mathbb{R}^2$

One possible Kähler structure is defined by $\sigma \doteq \begin{pmatrix}1&0\\0&1\end{pmatrix}$

because $J=-\Omega\cdot\sigma=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $J^2=-\mathbb{I}$ ✓

Another (different) possibility: $\sigma \doteq \begin{pmatrix}e^{2r}&0\\0&e^{-2r}\end{pmatrix}$, $r\in\mathbb{R}$

$$\Rightarrow J=\begin{pmatrix}0&-e^{-2r}\\e^{2r}&0\end{pmatrix} \Rightarrow J^2=-\mathbb{I} \quad ✓$$

Given a Khaler structure, one proceeds as follows:

- Extend the action of σ and J from Γ to $\Gamma_{\mathbb{C}}$ (using linearity).
- J is diagonalizable in $\Gamma_{\mathbb{C}}$ with eigenvalues $\pm i$ (N-degenerate)

Proof:

$\sigma(\cdot^*, \cdot)$ defines an Hermitian inner product in $\Gamma_{\mathbb{C}}$.

Furthermore $\sigma(\overline{(J\gamma_1)}, \gamma_2) \underset{\substack{\uparrow \\ J \text{ is real}}}{=} \sigma(J\bar{\gamma}_1, \gamma_2) \overset{\sigma(\cdot,\cdot) = -\omega(\cdot, J\cdot)}{=} -\omega(J\bar{\gamma}_1, J\gamma_2) =$

$\underset{\substack{\uparrow \\ \sigma(\cdot,\cdot) = \omega(J\cdot,\cdot)}}{=} -\sigma(\bar{\gamma}_1, J\gamma_2) \Rightarrow J$ is anti-Hermitian

$\Rightarrow J$ is diagonalizable

$$\Rightarrow \Gamma_{\mathbb{C}} = \Gamma_+ \oplus \Gamma_-$$

where Γ_+ eigenspace of J with eigenvalue i

Γ_- " " " " " $-i$

Γ_- is merely the complex conjugate of Γ_+)

- The complexified symplectic product $\langle \cdot, \cdot \rangle$ becomes positive definite when restricted to Γ_+ (and negative def. in Γ_-). (And $\Gamma_+ \perp \Gamma_-$: $\langle \gamma_+, \gamma_- \rangle = 0 \ \forall \gamma_+ \in \Gamma_+, \gamma_- \in \Gamma_-$)

Proof

$$0 < \overbrace{\sigma(\bar{\gamma}, \gamma)}^{\sigma \text{ is posit. def}} \underset{\substack{\uparrow \\ \sigma(\cdot,\cdot) = \omega(J\cdot,\cdot)}}{=} \omega(J\bar{\gamma}, \gamma)$$

If $\gamma, \in \Gamma_+$, $\quad J\bar{\gamma} = -i\bar{\gamma}$, so

$$0 < \omega(-i\bar{\gamma}, \gamma) = -i\,\omega(\bar{\gamma}, \gamma) = \langle \gamma_1, \gamma_2 \rangle \quad \square$$

$\Rightarrow h_J \equiv (\Gamma_+, \langle\cdot,\cdot\rangle)$ defines a Hilbert space (N-dimen.)

$\equiv$ one - quanta Hilbert space (not yet the Fock space)

The Fock space is the symmetric Fock space constructed from h_J (see e.g. appendix of Wald's qft book for the definition of symmetric Fock space)

$$\mathcal{F}_s(h_J) \equiv \underbrace{\overset{\mathbb{C}}{\overset{\approx}{h_J^{\circ}}}}_{\substack{\text{vacuum}\\ \text{sector}\\ \text{1 dim.}}} \oplus \underbrace{h_J}_{\substack{\text{1 quanta}\\ \text{sector}\\ \text{N-dim}}} \oplus \underbrace{(h_J \otimes_s h_J)}_{\substack{\text{two quanta}\\ \text{sector (symmetric)}\\ \text{dim } \binom{N+1}{2}}} \oplus \cdots \underbrace{}_{\substack{\text{n- quanta sect.}\\ \text{dim } \binom{N+n-1}{n}}} \quad (7)$$

Annihilation operators: Consider a basis in $\Gamma_+ : \{e_I\}_{I=1}^N$, and choose it orthonormal w.r.t. to the complexified symp. prod. $\langle\cdot,\cdot\rangle$. (All basis vectors are eigenstates of J with eigenvalue i.)

Each e_I defines a annihilation operator (via the association phase space elements $\longrightarrow$ operators

$$e_I \longrightarrow \boxed{\hat{a}_I = -i \langle e_I, \hat{\vec{r}} \rangle, \quad I = 1, \dots, N}$$ Annihilation opts.

It automatically follows that: basis $\{e_I\}_{I=1}^N$ was chosen orthonorm

$$[a_I, a_J] = \langle e_I, \overline{e_J} \rangle = 0$$

$$[a_I, a_J^+] = \langle e_I, e_J \rangle = \circ$$ (follows from orthog. btw Γ_+ and Γ_-)

[There is ambiguity in the choice of basis within Γ_+, but this ambiguity does not change the Fock representation]

With this, the different sectors in (7) are:

- $h_J^0 \approx \mathbb{C}$ is one-dim and spanned by a state we denote as $|0\rangle$: $h_J^0 = \text{span}(|0\rangle)$

  Furthermore: $\hat{a}_I |0\rangle = 0 \quad \forall I$.

  $|0\rangle$ is called the Fock vacuum

- $h_J = \text{span}\{|1_I\rangle\}_{I=1}^N$, where $|1_I\rangle \equiv \hat{a}_I^\dagger |0\rangle$

  One quanta sector

- $h_J \otimes_S h_J = \text{span}\{|1_I 1_J\rangle\}_{I,J=1}^N$, where $|1_I 1_J\rangle \equiv \hat{a}_I^\dagger \hat{a}_J^\dagger |0\rangle$

  Two quanta sector. Note that, because $[a_I^\dagger, a_J^\dagger] = 0$, $|1_I 1_J\rangle$ is equal to $|1_J 1_I\rangle \Rightarrow$ two quanta states are symmetric under the interchange of quanta ($\Rightarrow$ quanta have bosonic character)

  $\Rightarrow \dim h_J \otimes_S h_J = \binom{N+2-1}{2}$

$\vdots$

- $h_J \otimes_S \cdots \otimes_S h_J = \text{Span}\{|1_{I_1} \cdots 1_{I_n}\rangle\}$: n quanta sector

  All states are sym. under quanta interchange.

  Dimension $\binom{N+n-1}{n}$

The Fock space is infinite dimensional, and a generic state is of the form:

$$c\,|0\rangle + \sum_{I=1}^{N} c_I\,|1_I\rangle + \sum_{I,J=1}^{N} c_{IJ}\,|1_I\,1_J\rangle + \sum_{I,J,K=1}^{N} c_{IJK}\,|1_I\,1_J\,1_K\rangle + \cdots$$

with $c_{IJ}, c_{IJK}, \dots$ totally symmetric complex coefficients

and $\quad |c|^2 + \sum_I |c_I|^2 + \sum_{I,J} |c_{IJ}|^2 + \cdots < \infty \quad$ (so norm is finite)

We have not used the Hamiltonian in this construction $\Rightarrow$ the Fock basis $\{|0\rangle, |1_I\rangle, |1_{I,J}\rangle, \dots\}$ are not, in general, eigenstates of the Hamiltonian neither the Fock vacuum $|0\rangle$ is the ground state.

Representation of operators:

In the Fock space the canonical operators $\hat{r} = \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{q}_N \\ \hat{p}_N \end{pmatrix}$ are represented by: $\quad \langle e_I, \hat{r}\rangle = i\, a_I$

$$\hat{r} = -i \sum_{I=1}^{N} \left(e_I\, \hat{a}_I - i\,\bar{e}_I\, \hat{a}_I^{+} \right) \quad (8)$$

Example: Simple harmonic oscillator

$\Gamma = \mathbb{R}^2$. Canonical coords. q and p

Introduce kähler structure defined by $\sigma \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Eigenvectors of J: $\quad e = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ eigenv. i

$\qquad\qquad\qquad\qquad \bar{e} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$ eigenv. $-i$

$\Rightarrow$ the kähler splitting of $\Gamma_{\mathbb{C}} = \mathbb{C}^2$ is

$$\Gamma_{\mathbb{C}} = \Gamma_+ \oplus \Gamma_- \qquad \text{with} \qquad \Gamma_+ = \operatorname{span}(e), \quad \Gamma_- = \overline{\Gamma_+}$$

$$\hat{a} = i \langle e, \hat{r} \rangle = i(-i)\, \omega(e, \hat{r}) =$$

$$\frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 1 & i \end{pmatrix}} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \frac{1}{\sqrt{2}} (\hat{q} + i \hat{p})$$

$$\longrightarrow \quad \hat{a} = \frac{1}{\sqrt{2}} (\hat{q} + i \hat{p}) \quad \checkmark$$

$$\Rightarrow \hat{r} = -i \left(e \hat{a} - \bar{e} \hat{a}^{+} \right) = -i \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \hat{a} + i \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \hat{a}^{+} =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{a} + \hat{a}^{+} \\ -i(\hat{a} - \hat{a}^{+}) \end{pmatrix} \longrightarrow \hat{q} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^{+}) \quad \checkmark$$

Time evolution:

The evolution of the canonical operators is obtained from (8) merely by replacing $e_I \longrightarrow e_I(t)$

$$\boxed{\hat{r}(t) = (-i) \sum_I e_I(t)\, \hat{a}_I - \bar{e}_I(t)\, \hat{a}_I^{+} \qquad (9)}$$

Example: simple h.o. (cont.)

The solution to the e.o.m. with initial data $e = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ at time t_0 is

$$e(t) = \begin{pmatrix} q_e(t) \\ p_e(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-i\omega(t-t_0)}$$

$$\Rightarrow \quad \boxed{\hat{q}(t)} = -i\left(q_e(t)\, \hat{a} - \bar{q}_e(t)\, \hat{a}^+ \right)$$

$$= \boxed{\frac{e^{-i\omega(t-t_0)}}{\sqrt{2}}\, \hat{a} + \frac{e^{+i\omega(t-t_0)}}{\sqrt{2}}\, \hat{a}^+}$$

Take home point: The construction of a Fock representations requires an addional input not present in the classical theory: a kähler structure (consisting in introducing a metric σ in Γ / $J = -\Omega\sigma$ is a complex structure or, equivalently, introducing a complex structure such that $\sigma = \omega J$ is a metric).

This sounds a bit mathy. The connexion to physic is made transparent by this result :

The Fock vacuum is a Gaussian state whose matrix of covariances is precisely the Kähler metric σ :

$$\Delta^2 \hat{r}^i = 2\,\sigma_{ii}$$

In more detail : $\langle 0|\, \hat{r}^i \hat{r}^j \,|0\rangle = \sigma^{ij} + i\,\Omega^{ij}$. (*)

Since a Gaussian state is completely determined by the its second moments (*) [and the first moments, but for vacuum they are all zero : $\langle 0|\,\hat{r}^i\,|0\rangle = 0 \;\forall\, i$], σ completely characterizes the physical properties of the Fock vacuum.

1.v. Bogoliubov transformations

They provide the relation between different Fock representations.

	Representation A		Represent. B	
Kähler struct.	$\sigma_A \overset{\omega}{\longleftrightarrow} J_A$		$\sigma_B \overset{\omega}{\longleftrightarrow} J_B$	
Eigenvectors of J	$e_I \,,\ \bar{e}_I$	Bog. transf. $\longleftrightarrow$	$v_I \,,\ \bar{v}_I$	$I = 1, \dots, N$
Annih. + creat opts	$\hat{a}_I \,,\ \hat{a}_I^{+}$		$\hat{b}_I \,,\ \hat{b}_I^{+}$	
Fock vacuum	$\lvert 0 \rangle_A$		$\lvert 0 \rangle_B$	
Fock space	$\mathcal{F}_A$		$\mathcal{F}_B$	

A Bogoliubov transf. is a unitary map between $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\mathcal{F}_B \underset{\longleftarrow}{\xrightarrow{\hat{U}_{BA}}} \mathcal{F}_A$$

$\hat{U}_{BA}$ permits to translate calculations of physical quantities from one representation to the other.

Interestingly: $\hat{U}_{BA}$ can be obtained from the relation between Kähler structures. Specifically, from the relation between the eigenvectors of J_A and J_B.

Relation between eigenbasis:

Since both sets of eigenvectors form basis, we can write one set in terms of the other:

$$\boxed{v_I = \sum_{J=1}^{N} \alpha_{IJ}\, e_J + \beta_{IJ}\, \bar{e}_J} \qquad (10)$$

(this automatically implies $\bar{v}_I = \sum_{J=1}^{N} \bar{\alpha}_{IJ}\, \bar{e}_J + \bar{\beta}_{IJ}\, e_J$)

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(This automatically implies $v_I = \sum_{J=1} \alpha_{IJ}\, e_J + \beta_{IJ}\, \bar{e}_J$)

Using that basis A is symplectic - orthonormal:

$$\left.\boxed{\begin{aligned} \alpha_{IJ} &= \langle e_J , v_I \rangle \\ \beta_{IJ} &= - \langle \bar{e}_J , v_I \rangle \end{aligned}}\right\} \equiv \text{Bogoluibov coeffs.}$$

In "matrix" form:

N components $\to V$

N components $\to \bar{V}$

N×N matrices

N comp. vector $\begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}$

$$\begin{pmatrix} V \\ \bar{V} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}}_{\equiv B \text{ } 2N\times 2N \text{ matrix}} \begin{pmatrix} e \\ \bar{e} \end{pmatrix}$$

$\begin{pmatrix} \bar{e}_1 \\ \vdots \\ \bar{e}_N \end{pmatrix}$

Since both basis are orthonormal $\Rightarrow$ B must preserve the symplectic product $\Rightarrow$ must be a symplectic transf,

$$B^a{}_c \, \omega_{ab} \, B^b{}_c = \omega_{cb} \quad (11) \quad \big(\Leftrightarrow B \in \underbrace{Sp(\mathbb{R}, 2N)}_{\text{symplectic group in } \mathbb{R}^{2N}} \big)$$

In matrix form: $B^{-1} \cdot \omega \cdot B = \omega$

(11) implies the following relations:

In matrix form: $\alpha \cdot \alpha^{+} - \beta \cdot \beta^{+} = \mathbb{I}_{N\times N}$

$$\alpha \cdot \beta^{T} - \beta \cdot \alpha^{T} = 0$$

Using indices:

$$\sum_K \alpha_{IK}\, \bar{\alpha}_{JK} - \beta_{IK}\, \bar{\beta}_{JK} = \delta_{IJ}$$

$$\sum_K \alpha_{IK}\, \beta_{JK} - \beta_{IK}\, \alpha_{JK} = 0$$

(This is the symplectic analog of an orthogonal transf. in Euclidean space.)

Relation between creation and annihilation opts.

$$\boxed{\hat{b}_I = \sum_J \bar{\alpha}_{IJ}\, \hat{a}_J - \bar{\beta}_{IJ}\, \hat{a}_J^+} \qquad (12)$$

In matrix form: $\begin{pmatrix} \hat{\mathbb{b}} \\ \hat{\mathbb{b}}^+ \end{pmatrix} = (\mathbb{B}^{-1})^T \begin{pmatrix} \hat{\mathbb{a}} \\ \hat{\mathbb{a}}^+ \end{pmatrix}$

Proof

$$\hat{b}_I = i \langle V_I, \hat{r} \rangle = i \langle \sum_J \alpha_{IJ}\, e_I + \beta_{IJ}\, \bar{e}_I, \hat{r} \rangle =$$

$$= i \sum_J \bar{\alpha}_{IJ} \langle e_J, \hat{r} \rangle + \bar{\beta}_{IJ} \langle \bar{e}_J, \hat{r} \rangle =$$

$$= \sum_J \left(\bar{\alpha}_{IJ}\, \hat{a}_J + \bar{\beta}_{IJ}\, \hat{a}_J^+ \right)$$

Relation between representations of $\hat{r}$ (consistency check):

$$\hat{r} = -i \sum_I \left(V_I\, \hat{b}_I - \bar{V}_I\, \hat{b}_I^+ \right) = \underset{(10)+(11)}{\dots} = -i \sum_I \left(e_I\, \hat{a}_I - \bar{e}_I\, \hat{a}_I^+ \right) \checkmark$$

The form of $\hat{U}_B$ is:

$$\hat{U}_B = \underset{\text{normalization}}{N} \exp\left[-\frac{1}{2} \left(\Lambda_{IJ}\, \hat{b}_I^+ \hat{b}_J^+ + \bar{\Lambda}_{IJ}\, \hat{b}_I \hat{b}_J \right) \right]$$

where $\Lambda = \bar{\mathbb{B}} \cdot (\mathbb{\alpha}^{-1})^T$

$\hat{U}_B$ is a squeezing operator.

In particular

$$|0\rangle_A = N \exp\left[-\frac{1}{2} \Lambda_{IJ}\, \hat{b}_I^+ \hat{b}_J^+ \right] |0\rangle_B$$

If $\beta_I = 0 \quad \forall I \Rightarrow \hat{U}_B \propto \hat{\mathbb{I}}$

But if any β_I is $\neq 0$

$\rightarrow |0\rangle_B$ is a squeezed state when written in $\mathcal{F}_A$.

It is straightforward to check that

$${}_B\langle 0| \hat{a}_I^{+} \hat{a}_I |0\rangle_B = \sum_J |\beta_{IJ}|^2 \longrightarrow |0\rangle_B \neq |0\rangle_A \text{ if}$$

$\beta_{IJ} \neq 0$ for some I, J.

1.6. Time independent stable systems

Hamiltonian $\hat{H} = \frac{1}{2} \hat{r}^a h_{ab} \hat{r}^b$

If h_{ab} is time independent and positive definite, there exist a preferred Fock representation, in which the number basis are eigenstates of $\hat{H}$ and the Fock vacuum is the ground state of the Hamiltonian.

The prefferred Kähler structure is obtained by "purifying" h_{ab}:

h_{ab} defines a metric in Γ, but it does not define a Kähler structure becase $J_h \equiv -\Omega \cdot h$ is such that $J_h^2 \neq -\mathbb{I}$. The eigenvalues of iJ_h are $\pm \omega_I$, with ω_I the normal frequencies of the Hamiltonian.

In the <u>basis of normal modes:</u>

$$h \doteq \bigoplus_{I=1}^{N} \omega_I \, \mathbb{I}_2$$

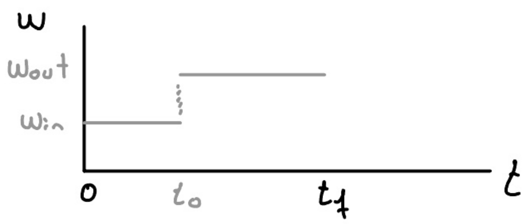
By replacing ω_I by 1 in this equation, we get $\sigma \doteq \bigoplus_I \mathbb{I}_2$, which defines a Kähler structure.

The vacuum state defined from σ is an eigenstate of $\hat{H}$ with the smallest possible eigenvalue. $\equiv$ ground state.

Example: time dependent harmonic oscillator

$$H = \frac{1}{2}\,w(t)\,(p^2+q^2)\ , \quad \text{with}$$

$$w(t) = w_{in}\,\Theta(t_0 - t) + w_{out}\,\Theta(t_0 - t) \ ; \qquad w_{out} \neq w_{in}$$



Two natural Fock representations in this problem:

- In-rep.: based on the time independence of the Hamiltonian at early times $t < t_0$

$$q(t) = (-i)\left(q_e^{in}(t)\,\hat{a}_{in} - \bar{q}_e^{in}(t)\,\hat{a}_{in}^{+} \right)$$

with $q_e^{in}(t)$ solution to e.o.m with initial data $e_{in}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ at $t=0$

The number basis $\{|n\rangle_{in}\}_{n=0}^{\infty}$ has a natural physical interpretation at early times $t < t_0$: $|0\rangle_{in}$ is the ground state and $|n\rangle_{in}$ are eigenstates of the Hamilt. containing n quanta of energy.

- out-rep: based on the time independence of the Hamiltonian at late times $t > t_0$

$$q(t) = (-i)\left(q_e^{out}(t)\,\hat{a}_{out} - \bar{q}_e^{out}(t)\,\hat{a}_{out}^{+} \right)$$

with $\varphi_e^{out}(t)$ solution to e.o.m with initial data $e_{out}(t_f) = \frac{\ }{\sqrt{2}}\begin{pmatrix} \\ 1 \end{pmatrix}$ at $t = t_f$

The number basis $\{|n\rangle_{out}\}_{n=0}^{\infty}$ has a natural physical interpretation at times $t > t_0$.

Bogoliubov coefficients:

$$\alpha = \langle e_{out}(t), e_{in}(t) \rangle , \qquad \beta = -\langle \bar{e}_{out}(t), e_{in}(t) \rangle$$

α and β do not depend on which time we choose to compute them (bcs symplectic products are invariant under time evol.). Let's choose t_0. We need e_{out} and e_{in} at t_0. This is simple to optain because:

$$e_{in}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-i w_{in} t} \quad \text{for} \quad 0 \leq t \leq t_0$$

and

$$e_{out}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-i w_{out}(t - t_f)} \quad \text{for} \quad t_0 \leq t \leq t_f$$

$$\Rightarrow \alpha = \frac{1}{2}\left(\frac{w_{out} + w_{in}}{w_{out}}\right) , \qquad \beta = \frac{1}{2} \frac{w_{out} - w_{in}}{w_{out}}$$

$$\Rightarrow |0\rangle_{in} = \underbrace{\frac{1}{\sqrt{\alpha}} e^{-\frac{1}{2} \beta^*/\alpha \, a_{out}^+ a_{out}^+} |0\rangle_{out}}.$$

This is a <u>squeezed state</u> with squeezing intensity $\operatorname{arcth} |\beta|/|\alpha|$ and sqz. angle $\arg \frac{\beta}{\alpha}$