# Canonical & Path Integral Quantization of Field Theory and Renormalization

## 1. Preparation

QFT: Quantum theory (mechanics) of fields, which is widely accepted as the fundamental theory for describing the strong and electroweak interactions

Dynamical variables of QFT: $\phi(t,\vec{x}) = [\phi(\vec{x})](t)$ (giving up explicit Lorentz covariance for the moment)

A field possesses an uncountably infinite number of degrees of freedom. So, roughly speaking, QFT is just quantum mechanics with infinitely many degrees of freedom.

Space discretization -> countably infinite degrees of freedom

Finite box of space -> finite degrees of freedom

Naively, QFT can be viewed as the proper limit of the above quantum mechanics with finite degrees of freedom when the box size goes to infinity and the cell size goes to zero, where the value of the field at each spatial point (cell) corresponds to one harmonic oscillator (coupled to each other by spatial derivatives, like a continuum). However, for interacting fields, we can only have physically meaningful limits under certain conditions. In particular, the validity of the limit of vanishing cell size is just related to renormalization.

Corresponding to the three pictures of QM, QFT can also have

1) Schrodinger picture: Dynamics is totally due to evolution of the state vector $\Psi[t,\phi(\vec{x})]$, which satisfies the Schrodinger equation
$i\hbar\partial_t\Psi[t,\phi(\vec{x})] = H[\phi(\vec{x}),\hat{\pi}(\vec{x})]\Psi[t,\phi(\vec{x})]$
2) Heisenberg picture: Dynamics is totally due to evolution of the observable operator.
$\left|\psi\right\rangle_H = \left|\psi(0)\right\rangle_S = e^{iHt/\hbar}\left|\psi(t)\right\rangle_S$
$$F_H(t) = e^{iHt/\hbar}F_S e^{-iHt/\hbar} \Rightarrow i\hbar\frac{d}{dt}F_H(t) = i\hbar\frac{\partial}{\partial t}F_H(t) + [F_H(t),H]$$
(the Heisenberg equation)
3) Interaction picture (mostly used): The "free part" of dynamics is due to evolution of the observable operator, while the "non-free part" of dynamics is due to evolution of the state vector, so it is the same as the Heisenberg picture for free fields.
$H = H_0 + H_I, \qquad F_I(t) = e^{iH_0t/\hbar}F_S e^{-iH_0t/\hbar}$
$\left|\psi(t)\right\rangle_I = e^{iH_0t/\hbar}\left|\psi(t)\right\rangle_S = e^{iH_0t/\hbar}e^{-iHt/\hbar}\left|\psi\right\rangle_H$
$i\partial_t\left|\psi(t)\right\rangle_I = H_I(t)\left|\psi(t)\right\rangle_I$ (Exercise)

## Action formalism for fields, Euler-Lagrange equation

An action functional of a field $\phi$:

$$S = \int L dt = \int \hat{L}(\phi, \partial_\mu \phi) d^D x, \qquad L = \int \hat{L}(\phi, \partial_\mu \phi) d^d \vec{x}$$

where the Lagrangian density $\hat{L}(\phi, \partial_\mu \phi)$ could more generally be any local function of $\phi$ and its derivatives (up to a given finite order for local field theories).

Variation of the action:

$$\delta S = \int \delta \hat{L}(\phi, \partial_\mu \phi) d^D x = \int \left[ \frac{\partial \hat{L}}{\partial \phi} \delta\phi + \frac{\partial \hat{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^D x$$

$$= \int \left[ \frac{\partial \hat{L}}{\partial \phi} \delta\phi - \partial_\mu \frac{\partial \hat{L}}{\partial(\partial_\mu \phi)} \delta\phi \right] d^D x + \int_{\partial V} \frac{\partial \hat{L}}{\partial(\partial_\mu \phi)} \delta\phi d^d A_\mu = 0$$

leads to the Euler-Lagrange equation (equation of motion)

$$\frac{\partial \hat{L}}{\partial \phi} = \partial_\mu \left[ \frac{\partial \hat{L}}{\partial(\partial_\mu \phi)} \right]$$ (Exercise: the higher derivative case)

Example: the free real scalar, with the Lagrangian density

$$\hat{L}(\phi, \partial_\mu \phi) = -\frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2$$ ($\hbar = 1 = c$, $(\eta_{\mu\nu}) = \mathrm{diag}(-,+,+,+)$)

The corresponding equation of motion (Klein-Gordon):

$\partial_\mu \partial_\mu \phi(x) = m^2 \phi(x)$ (Exercise)

The Hamiltonian formulation:

Canonical momentum (field) $\pi(\vec{x}) = \dfrac{\delta L}{\delta \dot{\phi}(\vec{x})} = \dfrac{\partial \hat{L}}{\partial \dot{\phi}}$

The Hamiltonian $H[\phi, \pi] = \int \pi(\vec{x}) \dot{\phi}(\vec{x}) d^d \vec{x} - L[\phi, \dot{\phi}]$ (functional Legendre transform)

Hamilton's canonical equations:

$$\dot{\phi}(\vec{x}) = \frac{\delta H}{\delta \pi(\vec{x})}, \qquad \dot{\pi}(\vec{x}) = -\frac{\delta H}{\delta \phi(\vec{x})}$$

(Question: Why equivalent to the Euler-Lagrange equation?)

The Poisson bracket: $\{F, G\} = \int \left( \dfrac{\delta F}{\delta \phi(\vec{x})} \dfrac{\delta G}{\delta \pi(\vec{x})} - \dfrac{\delta F}{\delta \pi(\vec{x})} \dfrac{\delta G}{\delta \phi(\vec{x})} \right) d^d \vec{x}$

Particularly, the bracket between canonical conjugates: $\{\phi(\vec{x}), \pi(\vec{y})\} = \delta^d(\vec{x} - \vec{y})$

Additionally, $\{\phi(\vec{x}), \phi(\vec{y})\} = 0 = \{\pi(\vec{x}), \pi(\vec{y})\}$.

The evolution equation of dynamical quantities: $\dot{F} = \partial_t F + \{F, H\}$ (the Heisenberg equation)

The free real scalar's Hamiltonian: $H = \dfrac{1}{2} \int [\pi^2 + (\nabla \phi)^2 + m^2 \phi^2] d^d \vec{x}$ (Exercise)

## Symmetries and conserved charges of classical fields, Noether's theorem

Noether's theorem (1918):

An *n*-parameter continuous symmetry leads to *n* conserved charges.

Discussion:

1) What is a symmetry transformation?

2) About discrete symmetries.

Concrete forms of symmetry transformations:

$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$ (coordinate transformation of space-time points)

$\phi_A(x) \to \phi'_A(x) = \phi_A(x) + \delta\phi_A(x)$ (field transformation)

The action is invariant under the above transformation:

$$\int_{\Omega'} L(\phi'_A, \partial_\mu\phi'_A, x^\mu) d^D x - \int_\Omega L(\phi_A, \partial_\mu\phi_A, x^\mu) d^D x = 0$$

$$\int_\Omega [L(\phi'_A, \partial_\mu\phi'_A, x^\mu) - L(\phi_A, \partial_\mu\phi_A, x^\mu)] d^D x + \int_{\partial\Omega} L(\phi_A, \partial_\mu\phi_A, x^\mu)\delta x^\nu d^d A_\nu = 0$$

where

$$\begin{aligned} L(\phi'_A, \partial_\mu\phi'_A, x^\mu) - L(\phi_A, \partial_\mu\phi_A, x^\mu) &= \frac{\partial L}{\partial\phi_A}\delta\phi_A + \frac{\partial L}{\partial(\partial_\mu\phi_A)}\partial_\mu\delta\phi_A \\ &= \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi_A)}\delta\phi_A + \frac{\partial L}{\partial(\partial_\mu\phi_A)}\partial_\mu\delta\phi_A \\ &= \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi_A)}\delta\phi_A) \end{aligned}$$

so we have (by virtue of Stokes' theorem)

$$\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\phi_A)}\delta\phi_A + L(\phi_A, \partial_\mu\phi_A, x^\mu)\delta x^\nu] = 0$$

In terms of Lie derivatives:

$$\begin{aligned} \delta x^\mu &= \epsilon\xi^\mu \\ \delta\phi_A &= \epsilon\Phi_A - \epsilon\mathcal{L}_\xi\phi_A \end{aligned}$$

we have the conserved current

$$j^\mu_\xi = \frac{\partial L}{\partial(\partial_\mu\phi_A)}(\mathcal{L}_\xi\phi_A - \Phi_A) - L\xi^\mu, \qquad \partial_\mu j^\mu_\xi = 0$$

Back to the flat space-time:

$\int_t^{t'} \partial_\mu j^\mu_\xi d^D x = \int j^t_\xi d^d\vec{x}\Big|_{t'} - \int j^t_\xi d^d\vec{x}\Big|_t = 0$ (assuming fast decay at spatial infinity)

we have the conserved charge $Q = \int j^t_\xi d^d\vec{x}$.

The case of curved space-times and the theories with higher derivatives.

# 2. Canonical quantization of fields

Quantum mechanics: $\{F, G\} \to \frac{1}{i\hbar}[\hat{F}, \hat{G}], \qquad [x^i, p_j] = i\hbar\delta^i_j$

## Canonical quantization of free real scalar field (Klein-Gordon field)

The Poisson bracket between canonical conjugates gives the canonical commutation relations

$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^d(\vec{x} - \vec{y})$ (...)

while Hamilton's canonical equations give the evolution equations (Heisenberg picture)

$$\dot{\phi}(\vec{x}) = -i[\phi(\vec{x}), H], \qquad H = H[\phi, \hat{\pi}]$$
$$\dot{\pi}(\vec{x}) = -i[\pi(\vec{x}), H]$$

Particularly,

$$H = \frac{1}{2}\int [\pi^2 + (\nabla\phi)^2 + m^2\phi^2] d^d\vec{x} \Rightarrow \ddot{\phi} = \nabla^2\phi - m^2\phi$$ (Exercise)

which is just $(\partial_\mu\partial_\mu - m^2)\phi = 0$, the same form as the classical Klein-Gordon (KG) equation.

The field operator can be expanded in Fourier modes as [inner product]

$$\phi(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d\vec{k}}{\sqrt{2\omega_k}} [a(\vec{k})e^{ik\cdot x} + a^\dagger(\vec{k})e^{-ik\cdot x}], \qquad \omega_k = \sqrt{\vec{k}^2 + m^2}$$
$$(k \cdot x \coloneqq \vec{k} \cdot \vec{x} - \omega_k t)$$

so we have

$$\pi(x) = \dot{\phi}(x) = \frac{-i}{(2\pi)^{d/2}} \int d^d\vec{k} \sqrt{\frac{\omega_k}{2}} [a(\vec{k})e^{ik\cdot x} - a^\dagger(\vec{k})e^{-ik\cdot x}]$$

By virtue of $\delta(\vec{k}) = \frac{1}{(2\pi)^d} \int e^{\pm \vec{k}\cdot\vec{x}} d^d\vec{x}$, we can solve the above relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$ as

$$a(\vec{k}, t) \coloneqq a(\vec{k})e^{-i\omega_k t} = \frac{1}{\sqrt{2\omega_k}} \int d^3x \, e^{ik\cdot x}[\omega_k \phi(x) + i\pi(x)]$$
$$= \frac{i}{\sqrt{2\omega_k}} \int d^3x \, e^{ik\cdot x} \overleftrightarrow{\partial_t} \phi(x)$$

$$a^\dagger(\vec{k}, t) = a^\dagger(\vec{k})e^{i\omega_k t} = \frac{-i}{\sqrt{2\omega_k}} \int d^3x \, e^{-ik\cdot x} \overleftrightarrow{\partial_t} \phi(x), \; f\overleftrightarrow{\partial_t}g \coloneqq f\partial_t g - g\partial_t f$$ (Exercise)

Then by commutation relations of $\phi(x)$ and $\pi(x)$ we can obtain

$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}')$ (Exercise)

The Hamiltonian can be recast as

$$H = \int \frac{\omega_k}{2} [a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})] d^d\vec{k}$$ (Exercise)

Compared with the Hamiltonian of *n*-dim (anisotropic) harmonic oscillator

$$H = \sum_{i=1}^{n} \frac{\omega_i}{2}(a_i^\dagger a_i + a_i a_i^\dagger) = \sum_{i=1}^{n} \omega_i (N_i + \frac{1}{2}), \qquad N_i = a_i^\dagger a_i$$

every momentum mode of the free scalar field corresponds to one degree of freedom of the harmonic oscillator. Thus, defining $N(\vec{k}) = a^\dagger(\vec{k})a(\vec{k})$, we can rewrite the above Hamiltonian as

$$H = \int \omega_k [N(\vec{k}) + \frac{1}{2}] d^d\vec{k}$$

Similar to the ground state $a_{\forall i}|0\rangle = 0$ of the *n*-dim harmonic oscillator, the vacuum state of the free scalar field satisfies $a(\forall \vec{k})|0\rangle = 0$. The creation operator $a^\dagger(\vec{k})$ repeatedly acts on the vacuum state to obtain multi-particle excitations of the momentum mode $\vec{k}$. Note that the creation and annihilation operators for different momentum modes commute, so the particles obey the Bose-Einstein statistics.

Here, an obvious difficulty is that even the vacuum energy is divergent:

$$H|0\rangle = \int \frac{\omega_k}{2} d^d\vec{k}\,|0\rangle$$ (zero-point energy)

So we can define the normal product between operators, which places all annihilation operators to the right of the creation operators. Take the free real scalar as an example:

$$\phi(x) = \phi^{(-)} + \phi^{(+)}, \qquad \phi^{(-)} \sim a^\dagger e^{-ik\cdot x}, \qquad \phi^{(+)} \sim a e^{ik\cdot x}$$

$$:\phi(x)\phi(y): = \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$

Then define all the observables as normal products, for example energy:

$$H = \frac{1}{2}\int :[\pi^2 + (\nabla\phi)^2 + m^2\phi^2]: d^d\vec{x} = \int \omega_k N(\vec{k}) d^d\vec{k}, \qquad H|0\rangle = 0$$

Relativistic covariance of commutation relations (from the mode expansion and the commutation relations of *a* and $a^\dagger$):

$$i\Delta(x-y) = [\phi(x),\phi(y)] = (2\pi)^{-d}\int (e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)})\frac{d^dk}{2\omega_k}$$

$$= (2\pi)^{-d}\int \mathrm{sgn}(k_0)\delta(k^2+m^2)e^{ik\cdot(x-y)} d^Dk$$

with $\Delta(x-y)$ the Pauli–Jordan function. In the equal-time case,

$$i\Delta(x-y) = (2\pi)^{-d}\int_0^\infty d\omega_k \int d^d\vec{k}\delta(k^2+m^2)\left[e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\right] = 0$$

which by its Lorentz invariance leads to its vanishing for any space-like separation of *x* and *y* (microscopic causality).

## Canonical quantization of free spinor field (Dirac field) and electromagnetic field (Maxwell field)

Both not straightforward. Each brings new problems:
Anti-commutation relations and the Fermi-Dirac statistics.
Gauge symmetry and redundant (non-physical) degrees of freedom.

## Interactions of fields and a brief intro to perturbation theory

Generally, possible forms of interactions are determined by symmetries of the theory. Gauge interactions.

We work in the interaction picture (unless otherwise stated):
1) The observable operators are (evolving as) free field operators;
2) If the interaction is weak, perturbation theory can be applied.

Define the time evolution operator *U* by

$$|\Psi(t)\rangle_I = U_I(t,t_0)|\Psi(t_0)\rangle_I, \qquad U_I(t,t) = 1$$

Then from $i\partial_t|\Psi(t)\rangle_I = H_I(t)|\Psi(t)\rangle_I$ we can see that $i\partial_t U_I(t,t_0) = H_I(t)U_I(t,t_0)$.
Discretizing time, we have approximately

$$U_I(t_{i+1},t_0) = [1 - iH_I(t_i)\Delta t]U_I(t_i,t_0), \qquad \Delta t = (t-t_0)/N$$

In the continuum limit, we can obtain

$$U_I(t,t_0) = \lim_{N\to\infty}[1 - iH_I(t_{N-1})\Delta t]\cdots[1 - iH_I(t_1)\Delta t][1 - iH_I(t_0)\Delta t] = \hat{T}\exp - i\int_{t_0}^{t} H_I(t')dt'$$

where $\widehat{T}$ is the time-ordering operator, similar to $(1 + x\Delta t)^N \to e^{x(t-t_0)}$.

Define S-matrix (perturbation expansion) as

$$S = \lim_{\substack{t\to\infty \\ t_0\to-\infty}} U(t,t_0) = \hat{T}\exp - i\int \widehat{H}_I(x)d^Dx = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^Dx_1\cdots\int d^Dx_n \widehat{T}[\widehat{H}_I(x_1)\cdots\widehat{H}_I(x_n)]$$

with its elements $\langle f|S|i\rangle$ directly related to physical observables. Decompose *S* into the non-interacting part and the interacting part:

$$S = 1 + iT, \qquad \langle f|T|i\rangle = (2\pi)^4\delta^4(P_f - P_i)M_{fi}$$

with $M_{fi}$ the Lorentz-invariant amplitude. This form is quite reasonable because the energy-momentum conservation and Lorentz covariance should be obeyed.

The scattering cross section: $d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}|M_{fi}|^2 d\Phi_n$

The decay width: $d\Gamma = \frac{1}{2M}|M_{fi}|^2 d\Phi_n$

Here $d\Phi_n = (2\pi)^4\delta^4(P_f - P_i)\prod_{j=1}^{n}\frac{d^3p_j}{(2\pi)^3 2E_j}$ is the *n*-body final-state Lorentz-invariant differential phase space.

But how to calculate the S-matrix (elements)?

We can introduce the contraction of field operators:

$$\langle \phi_A(x)\phi_B(y)\rangle = \widehat{T}[\phi_A(x)\phi_B(y)] - :\phi_A(x)\phi_B(y):$$

which can be proved to be a c-number, so we have

$$\langle \phi_A(x)\phi_B(y)\rangle = \langle 0|\widehat{T}[\phi_A(x)\phi_B(y)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2 + m^2 - i\epsilon}e^{ik\cdot(x-y)} =: D_F(x-y)$$

which is just the Green's function of the (classical) field equation: $(\partial_\mu\partial_\mu - m^2)D_F(x) = i\delta^4(x)$, also called the Feynman propagator.

Wick's theorem:

$$\widehat{T}[\phi_1(x_1)\cdots\phi_n(x_n)] =: \phi_1(x_1)\cdots\phi_n(x_n): + \text{所有只含一个收缩的正规积} + \cdots$$

The normal product representation of the S-matrix:

$$S = \sum_{\text{图}} \text{对称因子} \times \int \prod d^Dx : \phi\cdots\phi : \times (\text{收缩})^n$$

$$\langle f|S|i\rangle = \langle 0|a\cdots a(\infty)Sa^\dagger\cdots a^\dagger(-\infty)|0\rangle = \sum\int\langle 0|\widehat{T}(\phi\cdots\phi)|0\rangle$$

Since $\langle 0|:\phi\cdots\phi:|0\rangle = 0$, any terms contributing to $\langle f|S|i\rangle$ must only contain products of contractions, which eventually gives Feynman rules (diagrams).

Take the interacting real scalar field $\lambda\phi^4$ theory as an example:

$$\breve{L} = -(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \qquad \breve{H}_I = \frac{\lambda}{4!}\phi^4$$

For the first $\lambda$ order two-to-two scattering,

$$S^{(1)} = -\frac{i\lambda}{4!}\int d^4x\,\widehat{T}[\phi(x)\phi(x)\phi(x)\phi(x)]$$

the initial state $|i\rangle = |p_1, p_2\rangle = a^\dagger(p_1)a^\dagger(p_2)|0\rangle$ and the final state $\langle f| = \langle p_3, p_4| =$

$\langle 0|a(p_3)a(p_4)$. Then the calculation of $\langle f|S^{(1)}|i\rangle$ boils down to

$$\langle 0|\hat{T}[\phi(x_3)\phi(x_4)\phi(x)\phi(x)\phi(x)\phi(x)\phi(x_1)\phi(x_2)]|0\rangle$$

which is just the products of four Feynman propagators (times a symmetry factor). The initial and final operators cannot be contracted in pairs if the energy-momentum conservation cannot hold, as intuitively expected.

[Feynman diagrams]

The complete calculation is kind of straightforward but a bit lengthy. What is the role of the spatial integrals and time derivatives appearing in the expression of $a(\vec{k})$ in terms of $\phi(x)$, as well as that of the space-time integral in $S^{(1)}$? On-shell condition of the final amplitude and amputation of the external legs of the Feynman diagram(s) for $\langle f|S^{(1)}|i\rangle$. Eventually,

$$\langle f|S^{(1)}|i\rangle = -i\lambda(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)$$

At any order of the perturbation theory, one can see that S-matrix elements correspond to amputated Feynman diagrams. Actually, this general structure is encoded in the LSZ (Lehmann–Symanzik–Zimmermann) reduction formula (Heisenberg picture):

$$\boxed{\begin{aligned}&\langle \mathbf{p}_1',\ldots,\mathbf{p}_n',\mathrm{out}|\mathbf{p}_1,\ldots,\mathbf{p}_m,\mathrm{in}\rangle\\ &= i^{n+m}\int\prod_{j=1}^{m}\mathrm{d}^4x_j\,e^{-ip_j\cdot x_j}\prod_{k=1}^{n}\mathrm{d}^4y_k\,e^{+ip_k'\cdot y_k}\\ &\qquad\times\prod_{j=1}^{m}(\Box_{x_j}+m^2)\prod_{k=1}^{n}(\Box_{y_k}+m^2)\ \langle 0|\,\mathrm{T}[\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)]\,|0\rangle\,.\end{aligned}}$$

Here $\langle 0|\hat{T}[\phi(x_1)\cdots\phi(x_m)\phi(y_1)\phi(y_n)]|0\rangle$ is the full (non-perturbative) Green's function.

In the interaction picture, expanding the S-matrix in the full Green's function:

$$\langle 0|\hat{T}[\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)\exp -i\int \breve{H}_I\,d^4x]|0\rangle$$

gives the same result as perturbatively using Wick's theorem.

## 3. Path integral quantization of fields

What is path integral?

It is a formulation of quantum mechanics, equivalent to Schroedinger's wave mechanics and Heisenberg's matrix mechanics.

Actually, path integral should be proposed in the age of wave optics.

Why is path integral useful?

1) A useful tool of quantization;
2) Better for incorporating (space-time and internal) symmetries in quantum theories;
3) Suitable for studying classical-quantum correspondence and semi-classical approx;
4) Statistics, stochastics, polymer physics, and other applications in physics and math.

Path integral in quantum mechanics:

Quantum dynamics is determined by the point-to-point propagator

$$K(t_2,x_2;t_1,x_1) = {}_H\langle t_2,x_2|t_1,x_1\rangle_H = \langle x_2|e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}|x_1\rangle\ ,\quad K(t_2,x_2;t_1,x_1)=\int e^{\frac{i}{\hbar}S[x(t)]}Dx$$

with $|t,x\rangle_H$ the eigenstate of $\hat{x}_H(t)$: $\hat{x}_H(t)|t,x\rangle_H = x|t,x\rangle_H \Longrightarrow |t,x\rangle_H = e^{\frac{i}{\hbar}\hat{H}t}|x\rangle_S$ (Exercise)

For a free particle, $S = \int \frac{m}{2}\dot{x}^2 dt$ and $\hat{H} = \frac{\hat{p}^2}{2m}$.

[The classical limit of the path integral gives the least action principle $\delta S[x] = 0$.]

Huygens' principle:

$$\psi(t_2,x_2) = \int K(t_2,x_2;t_1,x_1)\psi(t_1,x_1)dx_1 \quad (t_2 > t_1)$$

Path integral can be divided/joined with any complete basis of the Hilbert space.

Further, let us consider inserting $x(t)$ ($t_a < t < t_b$) into the path integral (omitting the subscript H because the Heisenberg picture is the default here):

$$\int_{t_a,x_a}^{t_b,x_b} Dx\, e^{iS[x]/\hbar} x(t) = \int dx_1 \cdots dx_j \cdots dx_{N-1} K(t_b,x_b;t_{N-1},x_{N-1}) \cdots K(t_1,x_1;t_a,x_a) x_j$$

$$= \int_{-\infty}^{\infty} dx \langle t_b,x_b|t,x\rangle x \langle t,x|t_a,x_a\rangle = \langle t_b,x_b|\hat{x}(t)|t_a,x_a\rangle$$

Similarly, inserting the product $x(t)x(t')$ gives

$$\int_{t_a,x_a}^{t_b,x_b} Dx\, e^{iS[x]/\hbar} x(t)x(t') = \int dx dx' \begin{cases} \langle t_b,x_b|\hat{x}(t)|t,x\rangle\langle t,x|\hat{x}(t')|t',x'\rangle\langle t',x'|t_a,x_a\rangle & (t > t') \\ \langle t_b,x_b|\hat{x}(t')|t',x'\rangle\langle t',x'|\hat{x}(t)|t,x\rangle\langle t,x|t_a,x_a\rangle & (t' > t) \end{cases}$$

$$= \langle t_b,x_b|T[\hat{x}(t)\hat{x}(t')]|t_a,x_a\rangle$$

with $\hat{T}$ the time-ordering operator.

Alternatively, insertion of dynamical operators in the path integral can be done with the generating functional

$$Z[J(t)] = \int \underset{-\infty}{\overset{\infty}{Dx(t)}} \exp\frac{i}{\hbar}\int_{-\infty}^{\infty}[L + \hbar Jx + \frac{i}{2}\varepsilon x^2]dt \propto \langle 0(\infty)|0(-\infty)\rangle_J$$

which is a functional Fourier transform of $\exp\frac{iS[x]}{\hbar}$, with $J$ a homogeneous external force (source in the case of quantum fields). Then we have

$$\frac{\delta Z[J]}{\delta J(t_1)} = i\int \underset{-\infty}{\overset{\infty}{Dx(t)}}\, x(t_1) \exp\frac{i}{\hbar}\int_{-\infty}^{\infty}[L + \hbar Jx + \frac{i}{2}\varepsilon x^2]dt$$

$$\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)} = i^n \int \underset{-\infty}{\overset{\infty}{Dx(t)}}\, x(t_1)\cdots x(t_n) \exp\frac{i}{\hbar}\int_{-\infty}^{\infty}[L + \hbar Jx + \frac{i}{2}\varepsilon x^2]dt$$

Turning off the source in the end, we obtain

$$\left.\frac{\delta Z[J]}{\delta J(t_1)}\right|_{J=0} = i\int \underset{-\infty}{\overset{\infty}{Dx(t)}}\, x(t_1) \exp\frac{i}{\hbar}\int_{-\infty}^{\infty}[L + \frac{i}{2}\varepsilon x^2]dt$$

$$\propto i\int_{-\infty}^{\infty} dx \langle 0(\infty)|t_1,x\rangle x \langle t_1,x|0(-\infty)\rangle = i\langle 0(\infty)|\hat{x}(t_1)|0(-\infty)\rangle$$

$$\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)} \propto i^n \langle 0(\infty)|T[\hat{x}(t_1)\cdots\hat{x}(t_n)]|0(-\infty)\rangle_J$$

## Path integral and generating functional of the free real scalar field

$$\widehat{L}(\phi,\partial_\mu\phi)=-\frac{1}{2}\partial_\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2\Rightarrow\partial_\mu\partial_\mu\phi(x)=m^2\phi(x)$$

$$Z[J]=\int D\phi(x)\exp i\int\ [\widehat{L}+J(x)\phi(x)+\frac{i}{2}\varepsilon\phi^2]d^Dx\propto\langle 0(\infty)|0(-\infty)\rangle_J$$

$$Z[J]=\int D\phi(x)\exp i\int\ \left(\frac{1}{2}[-\partial_\mu\phi\partial_\mu\phi-(m^2-i\varepsilon)\phi^2]+J\phi\right)d^Dx$$

$$=\int D\phi(x)\exp i\int\ \left(\frac{1}{2}[\phi\partial^2\phi-(m^2-i\varepsilon)\phi^2]+J\phi+\text{边界项}\right)d^Dx$$

$$=\int D\phi(x)\exp i\int\ \left(\frac{1}{2}\phi[\partial^2-m^2+i\varepsilon]\phi+J\phi\right)d^Dx$$

Omitting the computational details, we obtain

$$Z[J]=\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^Dxd^Dy\right)\int D\phi(x)\exp\int\frac{i}{2}\phi[\partial^2-m^2+i\varepsilon]\phi d^Dx$$

$$\propto\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^Dxd^Dy\right)$$

(a functional determinant), where the Feynman propagator

$$\Delta_F(x)=\int\frac{d^Dk}{(2\pi)^D}\frac{e^{-ik\cdot x}}{-k^2-m^2+i\varepsilon},\ D_F(x)=i\Delta_F(x)\ \text{(for } D \text{ even)}$$

In terms of Feynman diagrams,

$$Z[J]\propto 1+\frac{1}{2}J\times\!\longrightarrow\!\times J+\frac{1}{2!}\frac{1}{2^2}\left(J\times\!\longrightarrow\!\times J\right)^2+\frac{1}{3!}\frac{1}{2^3}\left(J\times\!\longrightarrow\!\times J\right)^3\cdots$$

with $\xrightarrow{p}$ $\quad\dfrac{-i}{p^2+m^2-i\varepsilon}$ (momentum-space Feynman rules)

$\xrightarrow{p}\times J\quad iJ(p)$

For the normalization factor,

$$Z[J]=\langle 0(\infty)|0(-\infty)\rangle_J\Rightarrow Z[0]=1\Rightarrow$$

$$Z[J]=\frac{\int D\phi(x)\exp i\int\ \left(\frac{1}{2}\phi[\partial^2-m^2+i\varepsilon]\phi+J\phi\right)d^Dx}{\int D\phi(x)\exp\int\frac{i}{2}\phi[\partial^2-m^2+i\varepsilon]\phi d^Dx}$$

$$=\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^Dxd^Dy\right)$$

Taking functional derivatives of the generating functional,

$$\left.\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}=i^nG(x_1,\cdots,x_n)=i^n\langle 0(\infty)|T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)]|0(-\infty)\rangle$$

Eventually, one can prove Wick's theorem in the canonical quantization using the above expression of $Z[J]$.

## Path integral and generating functional of interacting fields

$$\widehat{L}(\phi,\partial_\mu\phi)=\widehat{L}_0+\widehat{L}_i=-\frac{1}{2}\partial_\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2+\widehat{L}_i(\phi)$$

Perturbatively,

$$Z[J]=\frac{\int D\phi(x)\exp i\int\left(\widehat{L}+J\phi\right)d^Dx}{\int D\phi(x)\exp i\int\widehat{L}d^Dx}=\frac{\int D\phi(x)\exp\left(i\int\widehat{L}_i(\phi)d^Dx\right)\exp i\int\left(\widehat{L}_0+J\phi\right)d^Dx}{\int D\phi(x)\exp i\int\widehat{L}d^Dx}$$

$$=\frac{\int D\phi(x)\exp i\int\widehat{L}_i\left(\frac{\delta}{i\delta J(x)}\right)d^Dx\exp i\int\left(\widehat{L}_0+J\phi\right)d^Dx}{\int D\phi(x)\exp i\int\widehat{L}d^Dx}\propto\exp i\int\widehat{L}_i\left(\frac{\delta}{i\delta J(x)}\right)d^DxZ_0[J]$$

Normalization can be done as

$$Z[J]=\frac{\exp i\int\widehat{L}_i\left(\frac{\delta}{i\delta J(x)}\right)d^DxZ_0[J]}{\text{分子令}J=0}$$

The full Green's function can be obtained by taking functional derivatives of the generating functional $Z[J]$.
The Feynman rules (diagrams) and the corresponding symmetry factors can be automatically generated.

Only connected (Feynman) diagrams contribute to the non-trivial part of physical observables (S-matrix). These diagrams can be generated by the connected generating functional $W[J]$:

$$Z[J]=e^{iW[J]}=e^{i\lambda\int f(\frac{\delta}{\delta J(x)})d^Dx}e^{iW_0[J]}\Longrightarrow\frac{\partial Z[J]}{\partial\lambda}=i\int f(\frac{\delta}{\delta J(x)})d^DxZ[J]$$

$$\Longrightarrow\frac{\partial W[J]}{\partial\lambda}=e^{-iW[J]}\int f(\frac{\delta}{\delta J(x)})d^Dxe^{iW[J]}$$

For the $\lambda\phi^4$ theory,

$$\frac{\delta^2W[J]}{\delta J(x_2)\delta J(x_1)}=\frac{i}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_2)}\frac{\delta Z[J]}{\delta J(x_1)}-\frac{i}{Z[J]}\frac{\delta^2Z[J]}{\delta J(x_2)\delta J(x_1)}$$

$$\left.\frac{\delta^4W[J]}{\delta J(x_4)\cdots\delta J(x_1)}\right|_{J=0}=iG(x_3,x_2)G(x_4,x_1)+iG(x_4,x_2)G(x_3,x_1)$$ (Exercise)

$$+iG(x_4,x_3)G(x_2,x_1)-iG(x_4,x_3,x_2,x_1)$$

$$\left.\frac{\delta^4 W[J]}{\delta J(x_4)\cdots\delta J(x_1)}\right|_{J=0} = i\left(i\;{}^3\!\longleftrightarrow{}^2 - \frac{\lambda}{2}\;{}^3\!\overset{\square}{\longleftrightarrow}{}^2\right)\left(i\;{}^4\!\longleftrightarrow{}^1 - \frac{\lambda}{2}\;{}^4\!\overset{\square}{\longleftrightarrow}{}^1\right)$$

$$+i\left(i\;{}^4\!\longleftrightarrow{}^2 - \frac{\lambda}{2}\;{}^4\!\overset{\square}{\longleftrightarrow}{}^2\right)\left(i\;{}^3\!\longleftrightarrow{}^1 - \frac{\lambda}{2}\;{}^3\!\overset{\square}{\longleftrightarrow}{}^1\right)$$

$$+i\left(i\;{}^4\!\longleftrightarrow{}^3 - \frac{\lambda}{2}\;{}^4\!\overset{\square}{\longleftrightarrow}{}^3\right)\left(i\;{}^2\!\longleftrightarrow{}^1 - \frac{\lambda}{2}\;{}^2\!\overset{\square}{\longleftrightarrow}{}^1\right)$$

$$+i\left({}^1_3\underset{\longleftrightarrow}{\longleftrightarrow}{}^2_4 + {}^1_2\underset{\longleftrightarrow}{\longleftrightarrow}{}^3_4 + {}^1_2\underset{\longleftrightarrow}{\longleftrightarrow}{}^4_3\right) - \frac{\lambda}{2}\left({}^1_3\underset{\longleftrightarrow}{\overset{\square}{\longleftrightarrow}}{}^2_4 + \text{其余5项置换}\right)$$

$$-\frac{\lambda}{4!}\left({}^1_3\times^2_4 + \text{其余23项置换}\right) = -\lambda(\times)$$

One can see that only connected diagrams remain. Going on with the general "proof",

$$f\left(\frac{\delta}{\delta J(x)}\right) = -\frac{1}{4!}\frac{\delta^4}{\delta J(x)^4} \Longrightarrow$$

$$\begin{aligned}
\frac{\partial W[J]}{\partial \lambda} &= -\frac{1}{4!}e^{-iW[J]}\int \frac{\delta^3}{\delta J(x)^3}\frac{i\delta W[J]}{\delta J(x)}e^{iW[J]}d^D x \\
&= -\frac{1}{4!}e^{-iW[J]}\int \frac{\delta^2}{\delta J(x)^2}[i\frac{\delta^2 W[J]}{\delta J(x)^2} - (\frac{\delta W[J]}{\delta J(x)})^2]e^{iW[J]}d^D x \\
&= -\frac{1}{4!}e^{-iW[J]}\int \frac{\delta}{\delta J(x)}[i\frac{\delta^3 W[J]}{\delta J(x)^3} - 3\frac{\delta^2 W[J]}{\delta J(x)^2}\frac{\delta W[J]}{\delta J(x)} - i(\frac{\delta W[J]}{\delta J(x)})^3]e^{iW[J]}d^D x \\
&= -\frac{1}{4!}\int[i\frac{\delta^4 W[J]}{\delta J(x)^4} - 4\frac{\delta^3 W[J]}{\delta J(x)^3}\frac{\delta W[J]}{\delta J(x)} - 3\frac{\delta^2 W[J]}{\delta J(x)^2}\frac{\delta^2 W[J]}{\delta J(x)^2} - 6i\frac{\delta^2 W[J]}{\delta J(x)^2}(\frac{\delta W[J]}{\delta J(x)})^2 + (\frac{\delta W[J]}{\delta J(x)})^4]d^D x
\end{aligned}$$

Actually,

$$\frac{\delta^n}{\delta J(x)^n}e^{iW[J]} = e^{iW[J]}(\frac{\delta}{\delta J(x)} + i\frac{\delta W[J]}{\delta J(x)})^n$$

so we have

$$\frac{\partial W[J]}{\partial \lambda} = -\frac{1}{4!}\int(\frac{\delta}{\delta J(x)} + i\frac{\delta W[J]}{\delta J(x)})^4 d^D x$$

$$\frac{\partial^2 W[J]}{\partial \lambda^2} = (-\frac{1}{4!})^2\int\left([\frac{i\delta}{\delta J(x)}(\frac{\delta}{\delta J(y)} + i\frac{\delta W[J]}{\delta J(y)})^4](\frac{\delta}{\delta J(x)} + i\frac{\delta W[J]}{\delta J(x)})^3 + \cdots\right)d^D x d^D y$$

$W[J] = W_0[J] + \lambda\frac{\partial W[J]}{\partial \lambda}|_0 + \frac{\lambda^2}{2}\frac{\partial^2 W[J]}{\partial \lambda^2}|_0 + \cdots$ (noting that the subscript 0 means $\lambda = 0$)

$\frac{\partial W[J]}{\partial \lambda}|_0 = -\frac{1}{4!}\int[-3\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta^2 W_0[J]}{\delta J(x)^2} - 6i\frac{\delta^2 W_0[J]}{\delta J(x)^2}(\frac{\delta W_0[J]}{\delta J(x)})^2 + (\frac{\delta W_0[J]}{\delta J(x)})^4]d^D x$ (Exercise)

$$\frac{\partial^2 W[J]}{\partial \lambda^2}|_0 = (-\frac{1}{4!})^2\int[\frac{i\delta^4}{\delta J(x)^4}(\frac{\delta W_0[J]}{\delta J(y)})^4 - 4\frac{\delta^3}{\delta J(x)^3}(\frac{\delta W_0[J]}{\delta J(y)})^4\frac{\delta W_0[J]}{\delta J(x)} + 36\frac{i\delta^2}{\delta J(x)^2}(\frac{\delta W_0[J]}{\delta J(y)})^2\frac{\delta^2 W_0[J]}{\delta J(y)^2}\frac{\delta^2 W_0[J]}{\delta J(x)^2} + \cdots]d^D x d^D y$$

Noting the first-order functional derivative

$$\frac{\delta W[J]}{\delta J(x)}=\frac{-i}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}=\frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\equiv\phi(x)$$

we can obtain the quantum effective action $\Gamma[\phi]$ by the functional Legendre transform

$$\Gamma[\phi]=W[J]-\int J(x)\phi(x)d^Dx\Rightarrow$$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)}=-J(x)\Rightarrow W_{xy}\Gamma_{yz}=-\delta^D(x-z)$$ (Exercise)

$$\Rightarrow W_{xyz}=W_{xu}W_{yv}W_{zw}\Gamma_{uvw}$$

which can be shown to only generate one-particle-irreducible (1PI) diagrams (Feynman diagrams that cannot be disconnected by cutting any one internal line).

The classical-quantum correspondence:

| | | |
|---|---|---|
| $\exp\frac{i}{\hbar}S[\phi]$ | $\xleftrightarrow{\text{泛函傅立叶}}$ | Generating functional $Z[J]$ |
| $\ln$ $\downarrow$ | | $\ln$ $\downarrow$ |
| Action $S[\phi]$ | | Connected generating functional $W[J]$ |
| $\uparrow$ $\hbar\to 0$ | Functional Legendre $\longleftrightarrow$ | |
| Quantum effective action $\Gamma[\phi]$ | | |

## Loop expansion of quantum field theory

Restoring Planck's constant, we can expand interacting QFT in terms of the number of loops, instead of the order of $\lambda$.

One can find that the loop expansion is equivalent to semi-classical expansion.

All the tree diagrams give the dynamics of classical fields, while loop diagrams contain momentum integration and may have UV divergences.

Quantum corrections can be conveniently reflected in the deviation of the quantum effective action from the classical action.

## Path integral of Fermionic fields and gauge fields

[Grassmann variables]

[gauge volumes]

# 4. Renormalization (a very brief intro)

Take again the $\lambda\phi^4$ theory as an example. The propagator and the vertex of the theory are

$$\longrightarrow \frac{-i}{p^2+m^2(-i\epsilon)}, \qquad \times \; -i\lambda$$

Concentrate on the 1PI diagrams. All the one-loop divergent 1PI diagrams in the theory are the self-energy graph and the (s, t, u) graphs.

The self-energy graph corresponds to

$$-i\Sigma(-p^2) = -\frac{i\lambda}{2}\int \frac{d^4l}{(2\pi)^4}\frac{-i}{l^2+m^2}$$

with *p* the external momentum and ½ the symmetry factor, which is quadratically divergent.

The (s, t, u) graphs correspond to

$$\Gamma(s) = \Gamma(-p^2) = \frac{(-i\lambda)^2}{2}\int \frac{d^4l}{(2\pi)^4}\frac{-i}{(l-p)^2+m^2}\frac{-i}{l^2+m^2}$$

with $s=-p^2=(p_1+p_2)^2,\ t=-(p_1-p_3)^2,\ u=-(p_1-p_4)^2$ the Mandelstam variables, which are logarithmically divergent (in *p*).

## Coupling constant renormalization

To order $\lambda^2$, the 1PI Green's function (1PI vertex)

$$\Gamma(p_1,p_2,p_3,p_4) = \Gamma(s,t,u) = -i\lambda + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

comprises the tree-graph contribution and one-loop contributions, the latter being (logarithmically) divergent. Taylor series expansion in the Mandelstam variables at the symmetric point (for particles on the shell $p_i^2=-m^2$, ignoring the mass renormalization)

$$s+t+u=4m^2 \Longrightarrow s_0=t_0=u_0=\frac{4m^2}{3}$$

gives

$$\Gamma(s,t,u) = [-i\lambda + 3\Gamma(s_0)] + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

with $\tilde{\Gamma}(s_0)=0$ and $\tilde{\Gamma}(s)$ finite. Ignoring the wavefunction renormalization for the moment, we have

$$\Gamma_{\mathrm{R}}(s,t,u) = \Gamma(s,t,u)$$

Define the renormalized coupling constant by

$$\Gamma_{\mathrm{R}}(s_0,t_0,u_0) = -i\lambda_{\mathrm{R}} \Longrightarrow \lambda_{\mathrm{R}} = \lambda + 3i\Gamma(s_0) =: Z_\lambda^{-1}\lambda$$

with $Z_\lambda$ the vertex renormalization constant. Note that all the renormalization constants are calculated order by order in $\lambda$.

## Mass and wavefunction renormalization and finite observables

Taylor series expansion of the (higher-loop) self-energy graphs in *p* at some value (turning out to be the renormalized mass $m_{\mathrm{R}}$) gives two divergent terms $\Sigma(m_{\mathrm{R}}^2)$ and $\Sigma'(m_{\mathrm{R}}^2)$ (and a finite $\tilde{\Sigma}(-p^2)$), which leads to the mass renormalization

$$m_{\mathrm{R}}^2 = m^2 + \Sigma(m_{\mathrm{R}}^2) =: m^2 + \delta m^2$$

and the wavefunction renormalization

$$\phi_{\mathrm{R}} = \sqrt{1-\Sigma'(m_{\mathrm{R}}^2)}\phi =: Z_\phi^{-1/2}\phi$$

with $Z_\phi$ the wavefunction renormalization constant, respectively.

After the renormalization (redefinition), the propagator

$$i\Delta_{\mathrm{R}}(p) = \frac{-i}{p^2 + m_{\mathrm{R}}^2 + \tilde{\Sigma}(-p^2)} + O(\lambda^2)$$

and the 1PI vertex $\Gamma_{\mathrm{R}}(s,t,u) = -i\lambda_{\mathrm{R}} + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) + O(\lambda^3)$ are completely finite at the corresponding $\lambda$ order, upon which finite theoretical results of observables can be compared with experiments.

## Power counting and renormalizability

For any Feynman diagram in $\lambda\phi^4$, it is easy to see that the number of external lines *E*, internal lines *I* and vertices *n* satisfy

$$4n = 2I + E$$

The number of loop momenta *L*, which will be integrated out, can be shown as

$$L = I - n + 1$$

The superficial degree of divergence is then given by

$$D = 4L - 2I = 4 - E$$

In more general theories, one can prove that (with exceptions)

$$D = 4 - E_{\mathrm{B}} - \frac{3}{2}E_{\mathrm{F}} - \sum_i n_i g_i$$

with $E_{\mathrm{B}}$ the number of external Boson lines, $E_{\mathrm{F}}$ the number of external Fermion lines, $n_i$ the number of the *i*th type vertices, and $g_i$ the dimension of the corresponding coupling constant in units of mass.

1) $g_i > 0$: The *i*th interaction is super-renormalizable, like $\lambda\phi^3$.
2) $g_i = 0$: The *i*th interaction is renormalizable, like $\lambda\phi^4$.
3) $g_i < 0$: The *i*th interaction is non-renormalizable, like the four-Fermion interaction and vertices in perturbative quantum gravity.