Canonical & Path Integral Quantization of Field Theory and Renormalization

Loop 2026

Outline

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- Space discretization $\rightarrow$ countably infinite d.o.f.
- Finite box of space $\rightarrow$ finite d.o.f.

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- For interacting fields, the limit is physically meaningful only under certain conditions.
- Validity of the vanishing cell size limit $\to$ renormalization.

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- Interaction picture (mostly used): "Free part" in Heisenberg picture for operators; "non-free part" in Schrödinger picture for states. $A_I(t) = e^{iH_0 t} A_S e^{-iH_0 t}$, $|\psi_I(t)\rangle = e^{iH_0 t} |\psi_S(t)\rangle$.

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- Euler–Lagrange equation: $\frac{\partial \mathcal{L}}{\partial \phi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} = 0$.

Example: Free Real Scalar Field

- Lagrangian density: $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, $(\mu = 0, 1, \ldots, d$, $\hbar = 1 = c$, $\eta = \mathrm{diag}(-, +, \cdots, +))$.

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- Equation of motion (Klein–Gordon): $(\Box - m^2)\phi = 0$ (Exercise).

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- Poisson bracket: $\{F, G\} = \int d^d\vec{x} \left(\frac{\delta F}{\delta \phi_A} \frac{\delta G}{\delta \pi_A} - \frac{\delta F}{\delta \pi_A} \frac{\delta G}{\delta \phi_A} \right)$.

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- Evolution: $\dot{F} = \partial_t F + \{F, H\}$ (Heisenberg equation).

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- Invariant action: $\int_{\Omega'} \mathcal{L}(\phi'_A, \dots) d^D x' - \int_\Omega \mathcal{L}(\phi_A, \dots) d^D x = 0$.
- Variation of Lagrangian: $\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \delta\phi_A \right)$ (on shell).

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- In flat space: $\partial_\mu j_\xi^\mu = 0 \implies \frac{d}{dt} Q = 0$, $Q = \int j_\xi^t d^d \vec{x}$ (assuming fast decay at spatial infinity).

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- Evolution: $\dot{\phi}(\vec{x}) = -i[\phi(\vec{x}), H]$, $\dot{\pi}(\vec{x}) = -i[\pi(\vec{x}), H]$.
- Gives $(\Box - m^2)\phi = 0$, same form as classical KG equation.

Mode Expansion

- Field expansion: $\phi(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d\vec{k}}{\sqrt{2\omega_k}} [a(\vec{k}) e^{ik\cdot x} + a^\dagger(\vec{k}) e^{-ik\cdot x}]$, $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

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- Similar for $a^\dagger(\vec{k})$.

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- Bosons: creation operators commute → Bose–Einstein statistics.

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- $:\phi(x)\phi(y):= \phi(x)\phi(y) - \langle 0|\phi(x)\phi(y)|0\rangle$.
- Observable energy defined as normal product: $:H:$ finite.

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- $\Delta(x-y)$: Pauli–Jordan function, vanishes for spacelike separations $\to$ microscopic causality.

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- Both require careful handling.

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- Dyson series: $U(t, t_0) = \mathcal{T} \exp\left[-i \int_{t_0}^{t} H_I(\tau) d\tau\right]$.

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- $d\Phi_n = (2\pi)^4 \delta^4(P_f - P_i) \prod \frac{d^3 p_j}{(2\pi)^3 2E_j}$.

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- $\langle f|\, S\, |i\rangle$ reduces to sum of products of contractions $\rightarrow$ Feynman rules/diagrams.

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- Two-point function: $\langle 0 | T\phi(x)\phi(y) | 0 \rangle = -\frac{\delta^2 Z_0}{\delta J(x) \delta J(y)}\Big|_{J=0} = D_F(x-y)$.

Interacting Fields Path Integral

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- Full Green's functions from $\frac{\delta}{\delta J}$ repeatedly acting on $Z[J]$.

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- $Z[J] = e^{iW[J]} = e^{iS_I[-i\delta/\delta J]} e^{iW_0[J]}, \qquad S_I[\phi] = \int \lambda f(\phi) d^D x.$

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- $\frac{\delta^4 W[J]}{\delta J(x_4)\cdots\delta J(x_1)}\Big|_{J=0} =$ $iG(x_3, x_2)G(x_4, x_1) + iG(x_4, x_2)G(x_3, x_1) + iG(x_4, x_3)G(x_2, x_1) - iG(x_4, x_3, x_2, x_1)$ (Exercise).

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  (Exercise).
- To the 1st order in λ, straightforward calculation gives $\frac{\delta^4 W[J]}{\delta J(x_4)\cdots\delta J(x_1)}\Big|_{J=0} = -\lambda(\times).$

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- For $f(\frac{\delta}{\delta J(x)}) = -\frac{1}{4!}\frac{\delta^4}{\delta J(x)^4}$,

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with

$$\frac{\partial W[J]}{\partial \lambda}|_0 = -\frac{1}{4!}\int \left[-3\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta^2 W_0[J]}{\delta J(x)^2} - 6i\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4\right]$$

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Quantum Effective Action $\Gamma[\phi]$

- 1st-order functional derivative: $\frac{\delta W[J]}{\delta J(x)} = \frac{-i}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} =: \phi_c(x).$

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- Key relations:

$$e^{iS[\phi]} \xleftrightarrow{\text{Functional Fourier}} Z[J] = e^{iW[J]},$$

$$\Gamma[\phi] \xleftrightarrow{\text{Functional Legendre}} W[J].$$

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- (s,t,u) graphs (logarithmically divergent): $\Gamma(s) = \frac{(-i\lambda)^2}{2}\int \frac{d^4 l}{(2\pi)^4}\frac{-i}{(l-p)^2+m^2}\frac{-i}{l^2+m^2}$ with $s = -p^2 = -(p_1+p_2)^2$, $t = -(p_1-p_3)^2$, $u = -(p_1-p_4)^2$ the Mandelstam variables.

Coupling Constant Renormalization

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- Self-energy expansion at $-p^2 = m_R^2$: $\Sigma(-p^2) = \Sigma(m_R^2) - \Sigma'(m_R^2)(p^2 + m_R^2) + \tilde{\Sigma}(-p^2)$ with $\tilde{\Sigma}(-p^2)$ finite.

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- Again take $\lambda\phi^4$ as the example: $4n = 2I + E$, $L = I - n + 1$, superficial degree of divergence $D = 4L - 2I = 4 - E$.

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- $g_i = 0$: renormalizable (e.g., $\lambda\phi^4$).

Power Counting and Renormalizability

- Again take $\lambda\phi^4$ as the example: $4n = 2I + E$, $L = I - n + 1$, superficial degree of divergence $D = 4L - 2I = 4 - E$.
- General: $D = 4 - E_B - \frac{3}{2}E_F - \sum n_i g_i$, g_i: dimension of coupling in mass units.
- $g_i > 0$: super-renormalizable (e.g., $\lambda\phi^3$).
- $g_i = 0$: renormalizable (e.g., $\lambda\phi^4$).
- $g_i < 0$: non-renormalizable (e.g., four-Fermion, perturbative quantum gravity).