Loop Quantum Gravity Summer School 2026

Quanutum Information and Quantum Entanglement

Yi Ling（凌意）
Institute of High Energy Physics, CAS
（中国科学院高能物理研究所）

05/12/2026, Yangzhou University

Outlines

- **Lecture I: Background：quantum gravity meets quantum information**
- **Lecture II: Basic concepts in quantum information**
- **Lecture III: Quantum entanglement in tensor networks**
- **Lecture IV: Quantum entanglement and the structure of spacetime**

Acknowledgement to collaborators：

Peng Liu, Yu-Xuan Liu, Chao Niu, Jianpin Wu, Meng-he Wu, Zhuoyu Xian, Yikang Xiao

lecture I

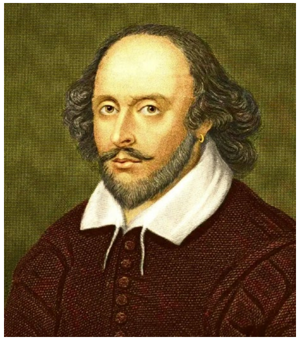
Background:

quantum gravity meets quantum information

- It or Bit
- Black hole information loss paradox

To be, or not to be, that is the question

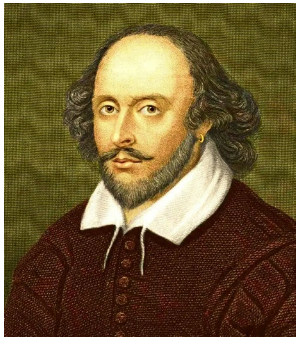
—— 《Hamlet》



William Shakespearer

To be, or not to be, that is the question

—— 《Hamlet》



William Shakespearer

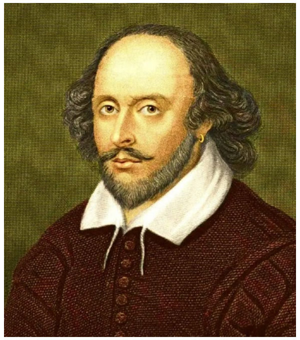
It, or Bit, that is the question

- Matter
- Energy

- Information
- Entropy

To be, or not to be, that is the question

—— 《Hamlet》



William Shakespearer

It, or Bit, that is the question

- Matter
- Energy

- Information
- Entropy

Life feeds on negative entropy

—— 《What is life?》

Erwin Schrödinger

The Sun as a low-entropy energy source

—— 《Road to reality》

Roger Penrose

It from bit

"It from bit. Otherwise put, every 'it' – every particle, every field of force, even the space-time continuum itself – derives its function, its meaning, its very existence entirely — even if in some contexts indirectly — from the apparatus-elicited answers to yes-or-no questions, binary choices, bits."

John Wheeler

It or Bit

- Implications to the structure of spacetime

Classical gravity: General Relativity

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Geometry ⟷ Matter

↓

What is the relationship between matter and spacetime at the quantum mechanical level?

It or Bit

- Implications to the structure of spacetime

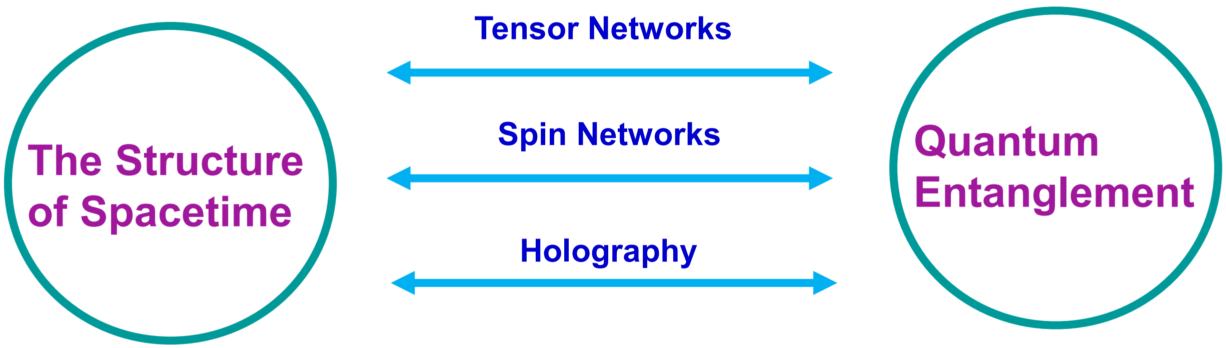
Classical gravity： General Relativity

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Geometry ⟷ Matter

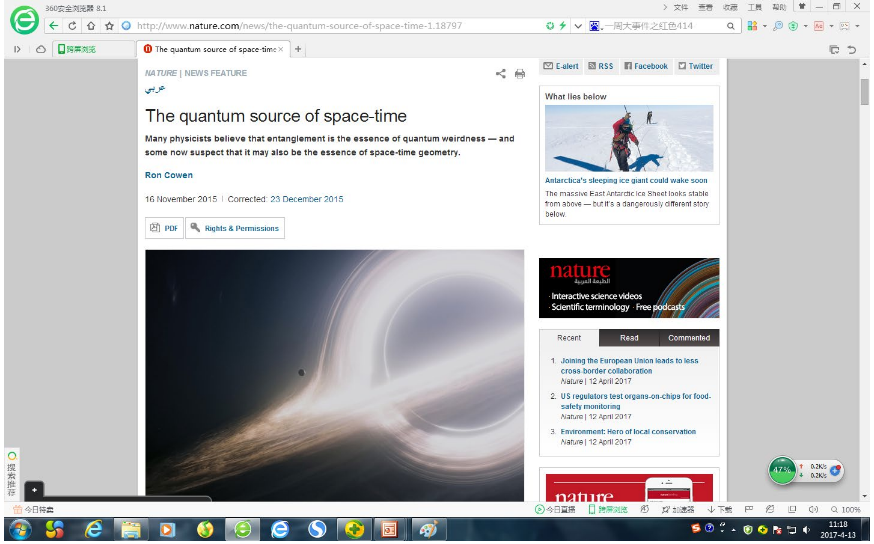
↓

What is the relationship between matter and spacetime at the quantum mechanical level?



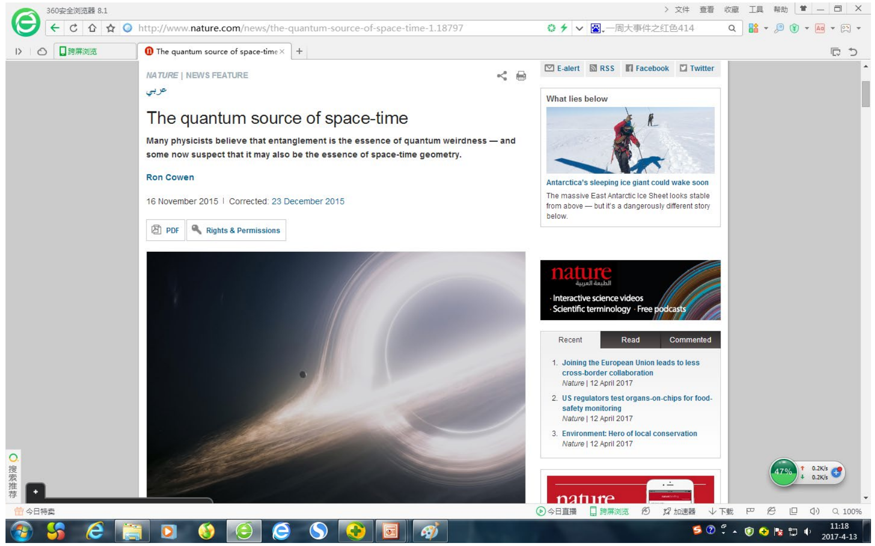
It or Bit

- The entanglement may be the essence of spacetime



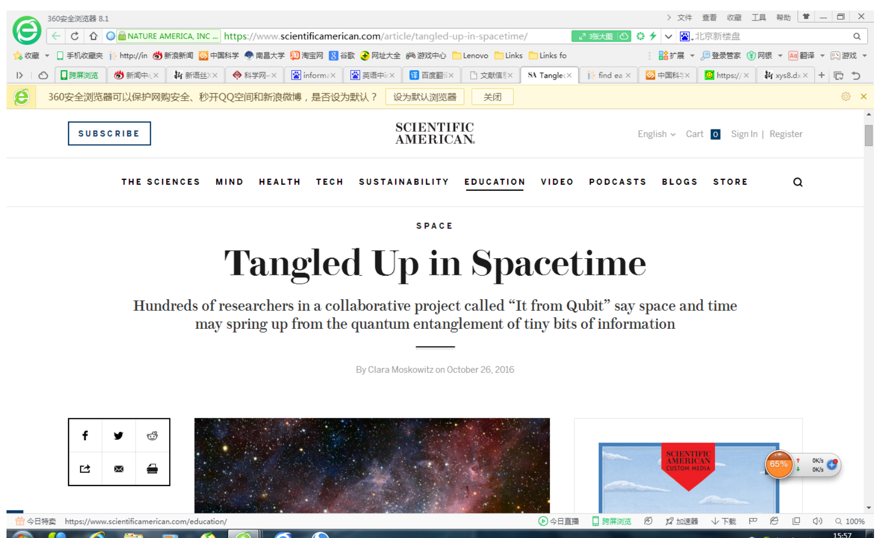
http://www.nature.com/news/the-quantum-source-of-space-time-1.18797

- The entanglement may be the essence of spacetime



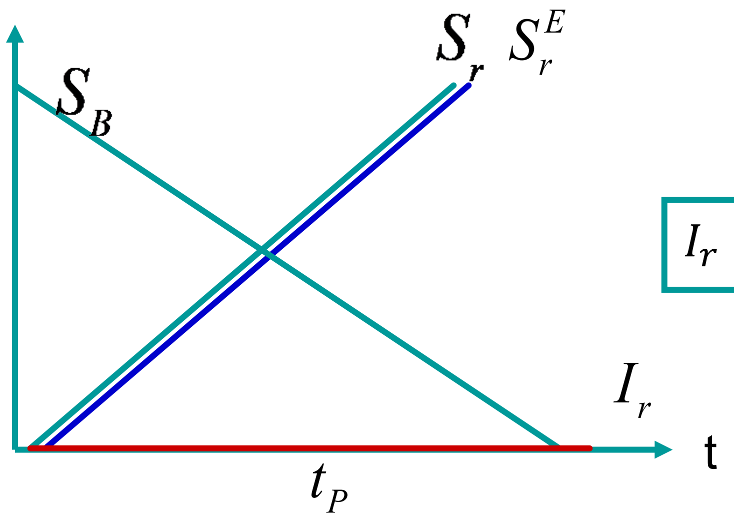
http://www.nature.com/news/the-quantum-source-of-space-time-1.18797

- “It from Qubit” project

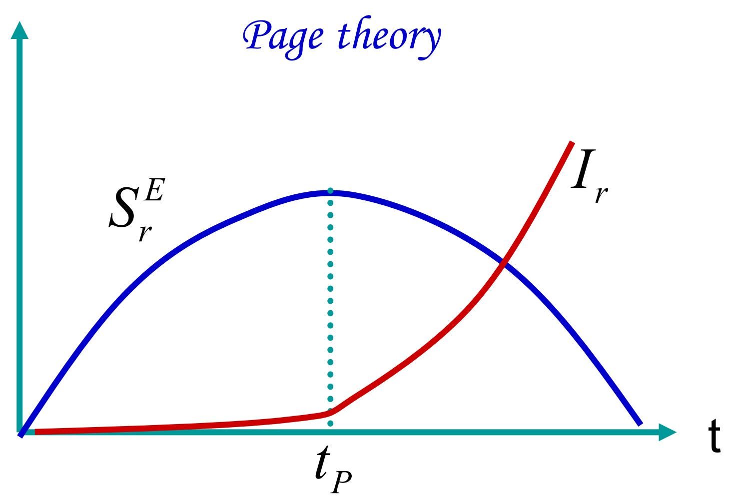


The information loss paradox

The information loss paradox manifests itself in the middle stage of evaporation

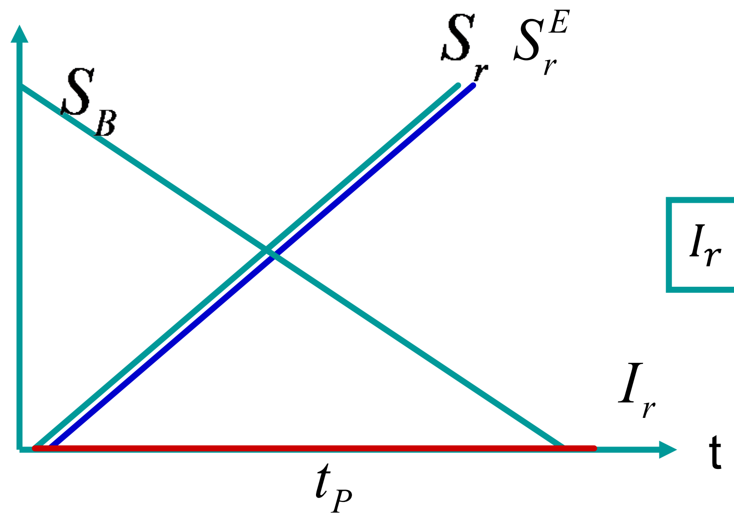


$$I_r = S_r - S_r^E$$

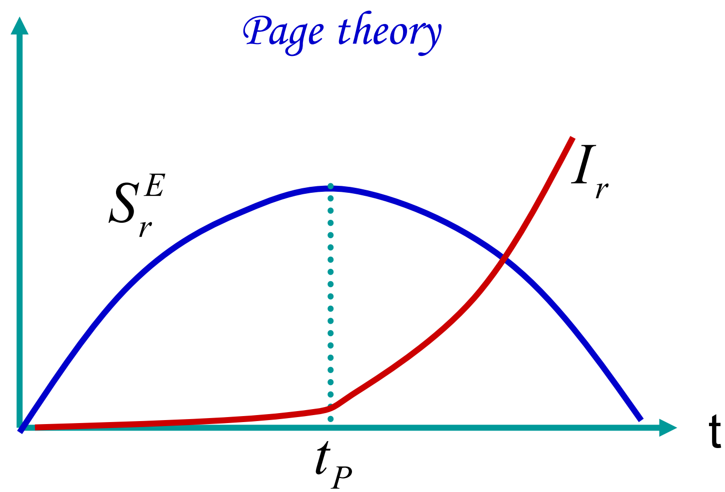


The information loss paradox

The information loss paradox manifests itself in the middle stage of evaporation



$$I_r = S_r - S_r^E$$



Key issue:
How to compute the entanglement entropy between the Hawking radiation and the black hole in a quantum field theory?

Lecture II
Basic concepts about Quantum Information and entanglement

- Pure State
- Bipartite system
- Entanglement

Quantum information and entanglement

Pure state

A pure state $|\psi\rangle$ is a quantum state that can be completely described by a single wave function or state vector in Hilbert space $\mathcal{H}_A$.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad\qquad |\psi\rangle \in \mathcal{H}_A$$

- Normalization: the state satisfies $\langle\psi|\psi\rangle = 1$

Quantum information and entanglement

Pure state

A pure state $|\psi\rangle$ is a quantum state that can be completely described by a single wave function or state vector in Hilbert space $\mathcal{H}_A$.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad |\psi\rangle \in \mathcal{H}_A$$

- Normalization: the state satisfies $\langle\psi|\psi\rangle = 1$

- the density matrix ρ given by $|\psi\rangle\langle\psi|$

$$\rho = \begin{pmatrix} \langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle & \langle\uparrow|\psi\rangle\langle\psi|\downarrow\rangle \\ \langle\downarrow|\psi\rangle\langle\psi|\uparrow\rangle & \langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- Purity criterion: the trace of the square of the density matrix is $\mathrm{Tr}\rho^2=1$.

Quantum information and entanglement

Pure state

A pure state $|\psi\rangle$ is a quantum state that can be completely described by a single wave function or state vector in Hilbert space $\mathcal{H}_A$.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad |\psi\rangle \in \mathcal{H}_A$$

- Normalization: the state satisfies $\langle\psi|\psi\rangle = 1$

- the density matrix ρ given by $|\psi\rangle\langle\psi|$

$$\rho = \begin{pmatrix} \langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle & \langle\uparrow|\psi\rangle\langle\psi|\downarrow\rangle \\ \langle\downarrow|\psi\rangle\langle\psi|\uparrow\rangle & \langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- Purity criterion: the trace of the square of the density matrix is $\mathrm{Tr}\rho^2=1$.

- Physical meaning: the system is in a definite quantum superposition, exhibiting full coherence and capable of interference.

Quantum information and entanglement

Pure state of 2-qubit system

- The Hilbert space :

$$\mathcal{H}_{AB} = \mathcal{H}_A \bigotimes \mathcal{H}_B \qquad Dim(\mathcal{H}_{AB}) = 2 \times 2 = 4$$

Quantum information and entanglement

Pure state of 2-qubit system

- The Hilbert space :

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad\qquad Dim(\mathcal{H}_{AB}) = 2 \times 2 = 4$$

$$|\psi_I\rangle = |\psi\rangle_A \otimes |\psi\rangle_B = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_B$$

$$= \frac{1}{2}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B)$$

$$\equiv \frac{1}{2}(|\uparrow\rangle_A|\uparrow\rangle_B + |\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B)$$

Quantum information and entanglement

Pure state of 2-qubit system

- The Hilbert space :

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad Dim(\mathcal{H}_{AB}) = 2 \times 2 = 4$$

$$|\psi_I\rangle = |\psi\rangle_A \otimes |\psi\rangle_B = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_B$$

$$= \frac{1}{2}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B)$$

$$\equiv \frac{1}{2}(|\uparrow\rangle_A|\uparrow\rangle_B + |\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B)$$

- The density matrix ρ_{AB} given by $|\psi\rangle\langle\psi|$ is a 4×4 matrix

Convention for bases in $\mathcal{H}_{AB}$:

$$|\uparrow\rangle_A|\uparrow\rangle_B: \quad \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |\uparrow\rangle_A|\downarrow\rangle_B: \quad \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|\downarrow\rangle_A|\uparrow\rangle_B: \quad \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |\downarrow\rangle_A|\downarrow\rangle_B: \quad \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|\psi\rangle = \alpha_1|\uparrow\rangle_A|\uparrow\rangle_B + \alpha_2|\uparrow\rangle_A|\downarrow\rangle_B + \alpha_3|\downarrow\rangle_A|\uparrow\rangle_B + \alpha_4|\downarrow\rangle_A|\downarrow\rangle_B \qquad \forall\, \psi \in \mathcal{H}_{AB}$$

Quantum information and entanglement

- **Bipartition of 2-qubit system**

$$|\psi_I\rangle\text{: } \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$$

$$\rho_{AB} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$Tr\rho_{AB}=1,\ Tr\rho_{AB}{}^2=1$$

Quantum information and entanglement

Bipartition of 2-qubit system

$$|\psi_I\rangle\text{: } \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$$

$$\rho_{AB} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$Tr\rho_{AB}=1,\ Tr\rho_{AB}{}^2=1$$

- The reduced density matrix ρ_A:

$$\rho_A = Tr_B\rho_{AB} = \langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle_B + \langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle_B$$

$$\rho_A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- Tr$\rho_A{}^2$=1, Tr$\rho_B{}^2$=1, no correlations between A and B.

$$|\psi_I\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$$

Quantum information and entanglement

- **Pure state of 2-qubit system** $|\psi_{II}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

EPR state $$|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B \pm |\downarrow\rangle_A|\downarrow\rangle_B)$$

$|\psi_{II}\rangle$: $\alpha_1 = \alpha_3 = 0, \alpha_2 = \alpha_4 = \frac{1}{\sqrt{2}}$

Quantum information and entanglement

- **Pure state of 2-qubit system** $|\psi_{II}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

EPR state $$|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B \pm |\downarrow\rangle_A|\downarrow\rangle_B)$$

$|\psi_{II}\rangle$: $\alpha_1 = \alpha_3 = 0, \alpha_2 = \alpha_4 = \frac{1}{\sqrt{2}}$

- The density matrix ρ_{AB} given by $|\psi_{II}\rangle\langle\psi_{II}|$ is a 4×4 matrix

$$\rho_{AB} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$Tr\rho_{AB}=1$, $Tr\rho_{AB}{}^2=1$

Quantum information and entanglement

- **Pure state of 2-qubit system** $|\psi_{II}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

EPR state $|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B \pm |\downarrow\rangle_A|\downarrow\rangle_B)$ $|\psi_{II}\rangle$: $\alpha_1 = \alpha_3 = 0, \alpha_2 = \alpha_4 = \frac{1}{\sqrt{2}}$

- The density matrix ρ_{AB} given by $|\psi_{II}\rangle\langle\psi_{II}|$ is a 4×4 matrix

$$\rho_{AB} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$Tr\rho_{AB}=1$, $Tr\rho_{AB}{}^2=1$

- The reduced density matrix ρ_A:

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$\mathrm{Tr}\rho_A=1$, $\mathrm{Tr}\rho_A{}^2<1$

- $\mathrm{Tr}\rho_A{}^2 < 1$, the subsystem A is correlated with B.

$$|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B) \neq |\psi\rangle_A \otimes |\varphi\rangle_B$$

- $\mathrm{Tr}\rho_A{}^2 < 1$, the subsystem A can not be described by a pure state in $\mathcal{H}_A$, dubbed as a mixed "state".

Quantum information and entanglement

- **Pure state of 2-qubit system** $|\psi_{II}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B) \neq |\psi\rangle_A \otimes |\varphi\rangle_B$$

How to distinguish a pure state and a mixed state?

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \quad \Longleftrightarrow \quad \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Quantum information and entanglement

- **Pure state of 2-qubit system** $|\psi_{II}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi_{II}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B) \neq |\psi\rangle_A \otimes |\varphi\rangle_B$$

How to distinguish a pure state and a mixed state?

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \quad \Longleftrightarrow \quad \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

- pure states possess maximized information in the sense that it is unitary.
- while mixed states, due to their classical uncertainty, contain less information.

How to evaluate the correlation between subsystem A and B?

How to evaluate the information lost during the course of tracing subsystem B?

Quantum information and entanglement

The entanglement between A and B

- Entanglement entropy

For a pure state of two subsystems, the von Neumann entropy of the subsystem is defined as the entanglement entropy.

$$\boxed{S_A = -Tr\rho_A \ln \rho_A}$$

$$0 \leq S_A \leq S_{max} = ln2 \qquad 0 \leq S_A \leq log_2 2 = 1$$

For a pure state $S_A = S_B = -Tr\rho_B \ln \rho_B$

Quantum information and entanglement

The entanglement between A and B

- Entanglement entropy

For a pure state of two subsystems, the von Neumann entropy of the subsystem is defined as the entanglement entropy.

$$S_A = -Tr\rho_A \ln \rho_A$$

$0 \le S_A \le S_{max} = ln2$ $\quad 0 \le S_A \le log_2 2=1$

For a pure state $\quad S_A = S_B = -Tr\rho_B \ln \rho_B$

For $|\psi_I\rangle$ $\quad S_A = 0$ $\quad \rho_A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

For EPR state $|\psi_{II}\rangle$ $\quad S_A = -Tr\frac{I}{2}\ln\left(\frac{I}{2}\right) = \ln 2$

EPR state is the maximally entangled state.

Quantum information and entanglement

The entanglement between A and B

- Entanglement entropy

For a pure state of two subsystems, the von Neumann entropy of the subsystem is defined as the entanglement entropy.

$$S_A = -Tr\rho_A \ln \rho_A$$

$0 \le S_A \le S_{max} = ln2$ $\quad 0 \le S_A \le log_2 2=1$

For a pure state $\quad S_A = S_B = -Tr\rho_B \ln \rho_B$

For $|\psi_I\rangle$ $\quad S_A = 0$ $\quad \rho_A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

For EPR state $|\psi_{II}\rangle$ $\quad S_A = -Tr\frac{I}{2}\ln\left(\frac{I}{2}\right) = \ln 2$

EPR state is the maximally entangled state.

- Information contained in subsystem A:

$$I_A \equiv S_{max} - S_A = \ln d - S_A$$

$$lnd \ge I_A \ge 0$$

Quantum information and entanglement

The entanglement between A and B

- Entanglement spectrum (Renyi entropy):

$$S_n = \frac{1}{1-n} \ln \frac{Tr\rho_A{}^n}{(Tr\rho_A)^n}$$

$$S = -Tr\rho_A \ln \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} Tr\rho_A{}^n = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

Quantum information and entanglement

- **Pure state of 3-qubit system** $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

  - GHZ state: $|\psi_I\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C + |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C)$ ρ_{ABC}: 9 ×9 matrix

Quantum information and entanglement

- **Pure state of 3-qubit system** $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

  - GHZ state: $|\psi_I\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C + |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C)$ ρ_{ABC}: 9 ×9 matrix

$$\rho_{AB} = Tr_C\rho_{ABC} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_{AB} = -Tr\rho_{AB}\ln\rho_{AB} = \ln 2$$

Quantum information and entanglement

- **Pure state of 3-qubit system** $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

  - GHZ state: $|\psi_I\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C + |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C)$ ρ_{ABC}: 9 ×9 matrix

$$\rho_{AB} = Tr_C\rho_{ABC} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_{AB} = -Tr\rho_{AB}\ln\rho_{AB} = \ln 2$$

  - S_{AB} measures the entanglement between subsystem (AB) and C.

    $\mathrm{Tr}\rho_{AB}=1$, $\mathrm{Tr}\rho_{AB}{}^2<1$

  - ρ_{AB} is a mixed state

Is the subsystem A entangled with B in ρ_{AB}?

Quantum information and entanglement

- Separability criterion: For a bipartite density matrix ρ_{AB} , it is separable if there exists a probability distribution $\{p_i\}$ and local states ρ_i^A ,ρ_i^B such that :

$$\rho_{AB} = \sum_i p_i \ \rho_i^A \otimes \rho_i^B$$

Clearly, it can be written as a probabilistic mixture of product states:

$$\rho_{AB} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|$$

no bipartite entanglement between particles A and B!

Quantum information and entanglement

- Separability criterion: For a bipartite density matrix ρ_{AB} , it is separable if there exists a probability distribution $\{p_i\}$ and local states ρ_i^A ,ρ_i^B such that :

$$\rho_{AB} = \sum_i p_i \ \rho_i^A \otimes \rho_i^B$$

Clearly, it can be written as a probabilistic mixture of product states:

$$\rho_{AB} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|$$

no bipartite entanglement between particles A and B!

- a necessary and sufficient condition for separability: if the partial transpose of ρ_{AB} with respect to one subsystem (e.g., $\rho_{AB}^{T_B}$ （has only non-negative eigenvalues, then ρ_{AB} is separable.

Quantum information and entanglement

- Separability criterion: For a bipartite density matrix ρ_{AB} , it is separable if there exists a probability distribution $\{p_i\}$ and local states ρ_i^A ,ρ_i^B such that :

$$\rho_{AB} = \sum_i p_i \ \rho_i^A \otimes \rho_i^B$$

Clearly, it can be written as a probabilistic mixture of product states:

$$\rho_{AB} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|$$

no bipartite entanglement between particles A and B!

- a necessary and sufficient condition for separability: if the partial transpose of ρ_{AB} with respect to one subsystem (e.g., $\rho_{AB}^{T_B}$ (has only non-negative eigenvalues, then ρ_{AB} is separable.

Physical meaning: In the GHZ state, genuine three-partite entanglement exists, but any two particles are not entangled – their correlations are purely classical!

Quantum information and entanglement

Pure state of 3-qubit system

- Mutual Information of GHZ state: $|\psi_I\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C + |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C)$

$$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = \ln 2$$

In the GHZ state, the mutual information between any two particles equals **1 bit**, which comes entirely from classical correlations (since ρ_{AB} is separable).

no bipartite entanglement between particles A and B.

Quantum information and entanglement

- **Pure state of 3-qubit system** $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

  - W state: $|\psi_{II}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$

$$\rho_{23} = Tr_1\rho_{123} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S = -Tr\rho_{23}\ln\rho_{23} = -\left(\frac{2}{3}log_2\frac{2}{3} + \frac{1}{3}log_2\frac{1}{3}\right) = log_2 3 - \frac{2}{3} \cong 0.918$$

Quantum information and entanglement

- **Pure state of 3-qubit system** $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

  - W state: $|\psi_{II}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$

$$\rho_{23} = Tr_1\rho_{123} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S = -Tr\rho_{23}\ln\rho_{23} = -\left(\frac{2}{3}log_2\frac{2}{3} + \frac{1}{3}log_2\frac{1}{3}\right) = log_2 3 - \frac{2}{3} \cong 0.918$$

  - Now partial transpose on qubit 2: map (ij, kl) -> (il, kj) : (12, 21) -> (11, 22)

$$\rho^{T_2}{}_{23} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

  - $\rho_{AB}^{T_B}$ has a negative eigenvalue: $\lambda = \frac{1 \pm \sqrt{5}}{2}$ the reduced two-qubit state of the W state is entangled (non-separable).

Quantum information and entanglement

The extension to N-qubit system $\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots\cdots \otimes \mathcal{H}_N$

- GHZ state: $$|\psi_I\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\cdots\downarrow\rangle)$$

  "All or nothing" entanglement. The state is extremely non-local but fragile.

- W state: $$|\psi_{II}\rangle = \frac{1}{\sqrt{N}}(|100\cdots 0\rangle + |010\cdots 0\rangle + |00\cdots 01\rangle)$$

  "One-excitation" entanglement. The state remains entangled even after losing some particles, making it more suitable for noisy environments or quantum networks.

- Both states serve as essential resources in quantum information:

  GHZ for secret sharing and extreme non-locality tests,

  W for robustness in lossy channels and distributed quantum computing.

Quantum information and entanglement

- **The typical states in an N-qubit system** $\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots\cdots \otimes \mathcal{H}_N$

  - Page theorem:

Let $d_A = 2^k$, $d_B = 2^{N-k}$, and $d_A \leq d_B$ $(k \leq N/2)$

The average entanglement entropy is

$$\bar{S}(\rho_A) = \sum_{j=d_B+1}^{d_A d_B} \frac{1}{j} - \frac{d_A - 1}{2d_B}$$

For large dimensions

$$\bar{S}(\rho_A) = ln d_A - \frac{d_A}{2d_B} + 0(\frac{1}{d_B})$$

Quantum information and entanglement

- **The typical states in an N-qubit system** $\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots\cdots \otimes \mathcal{H}_N$

  - Page theorem:

    Let $d_A = 2^k$, $d_B = 2^{N-k}$, and $d_A \leq d_B$ $(k \leq N/2)$

    The average entanglement entropy is

$$\bar{S}(\rho_A) = \sum_{j=d_B+1}^{d_A d_B} \frac{1}{j} - \frac{d_A - 1}{2d_B}$$

    For large dimensions

$$\boxed{\bar{S}(\rho_A) = ln d_A - \frac{d_A}{2d_B} + 0(\frac{1}{d_B})}$$

  - Key implications for N-qubit systems

    1. Typical states are highly entangled.

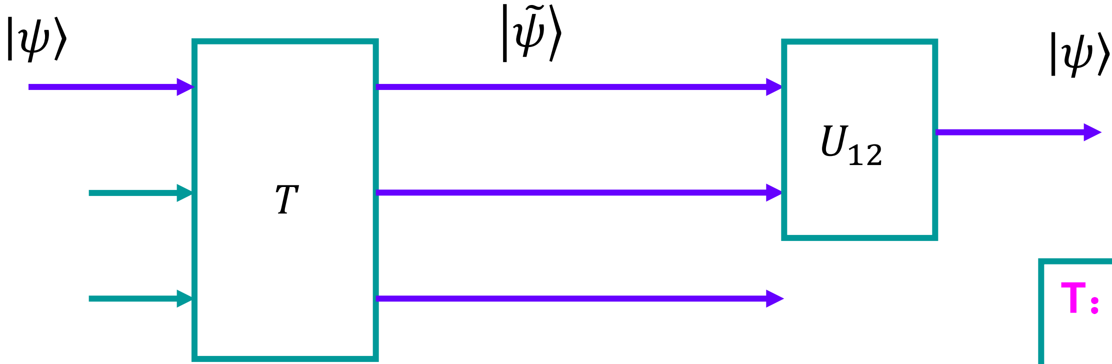
    2. Contrast with GHZ and W states.

       GHZ S =ln2 for 1<k<N

    3. The theorem inspired the Page curve in black hole physics.

Quantum information and entanglement

Entanglement and Qunatum Error Correction (QEC)



T：More outgoing channels than incoming ones

$$|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle \xrightarrow{\text{Entanglement}} |\tilde{\psi}\rangle = \sum_{i=0}^{2} a_i |\tilde{i}\rangle$$

$$(U_{12} \otimes I_3)|\tilde{i}\rangle = |i\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$(U_{12} \otimes I_3)|\tilde{\psi}\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

U_{12}

$$|00\rangle \to |00\rangle \quad |11\rangle \to |01\rangle \quad |22\rangle \to |02\rangle$$
$$|01\rangle \to |12\rangle \quad |12\rangle \to |10\rangle \quad |20\rangle \to |11\rangle$$
$$|02\rangle \to |21\rangle \quad |10\rangle \to |22\rangle \quad |21\rangle \to |20\rangle$$

$$|\tilde{0}\rangle = (|000\rangle + |111\rangle + |222\rangle)/\sqrt{3}$$
$$|\tilde{1}\rangle = (|012\rangle + |120\rangle + |201\rangle)/\sqrt{3}$$
$$|\tilde{2}\rangle = (|021\rangle + |102\rangle + |210\rangle)/\sqrt{3}$$

Lecture III
Quantum entanglement in tensor networks

- Singular Value Decomposition (SVD) of matrix
- Tensor and tensor chain decomposition
- Matrix Product State (MPS)
- Entanglement entropy of MPS
- Operators and expectation values

Quantum entanglement in tensor networks

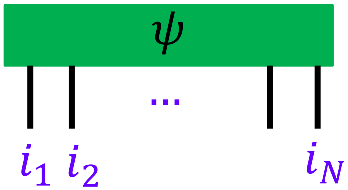
- **From states to tensors**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \ldots, d$$

Quantum entanglement in tensor networks

- **From states to tensors**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \ldots, d$$

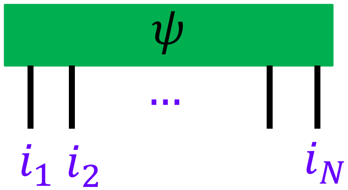


$$\dim(\mathcal{H}_L) = d^N$$

Quantum entanglement in tensor networks

From states to tensors

$$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} \psi_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \dots, d$$

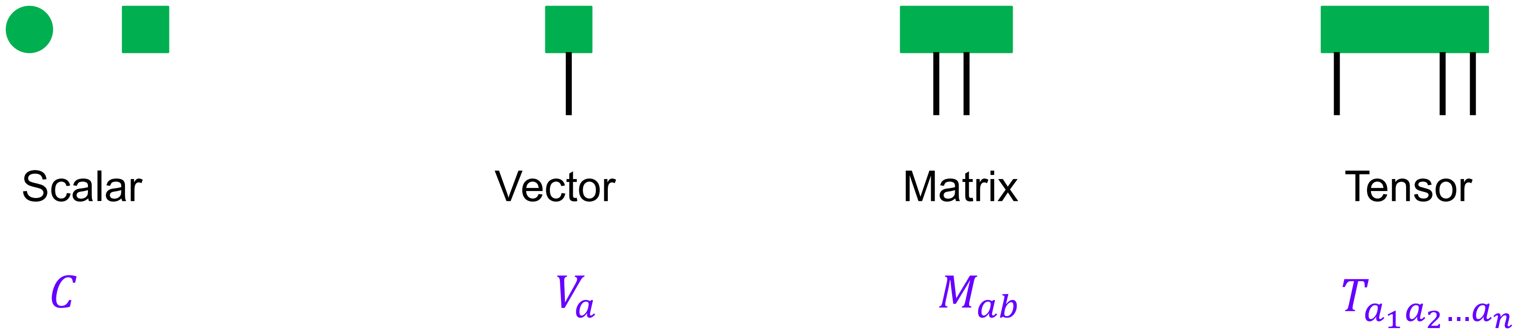


$$\dim(\mathcal{H}_L) = d^N$$

- For $d = 2$, $N>20$, the Hilbert space is too huge to compute.
- For a many-body system, it is hard to compute the energy spectrum and the ground state and excitations.
- Let alone the entanglement structure of the state.

Quantum entanglement in tensor networks

- **Diagram for tensors**

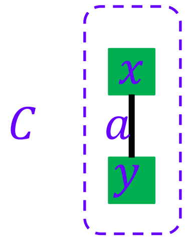


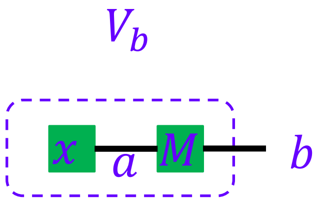
Quantum entanglement in tensor networks

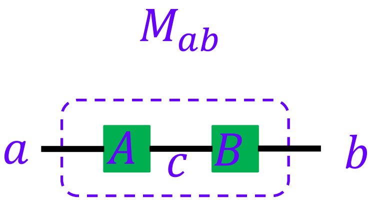
Diagram for tensors

Scalar	Vector	Matrix	Tensor
C	V_a	M_{ab}	$T_{a_1 a_2 \ldots a_n}$

- Contractions:







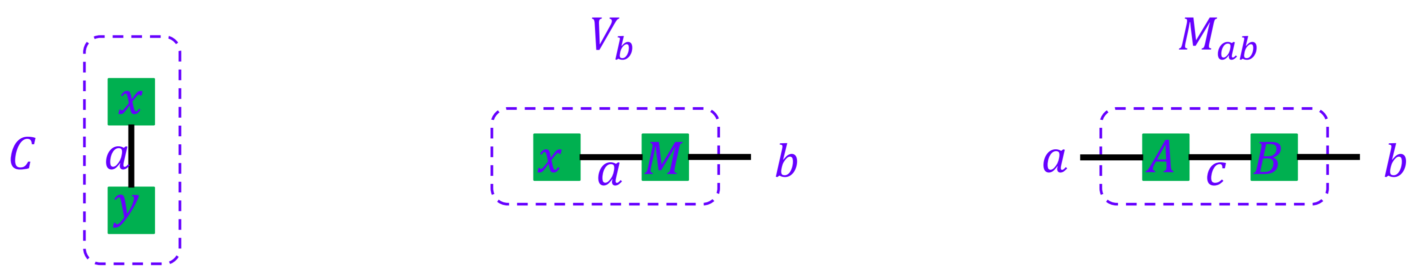
$C = x_a\ y_a$ $\qquad$ $V_b = x_a\ M_{ab}$ $\qquad$ $M_{ab} = A_{ac}\ B_{cb}$

Quantum entanglement in tensor networks

Diagram for tensors

Scalar	Vector	Matrix	Tensor
C	V_a	M_{ab}	$T_{a_1 a_2 \dots a_n}$

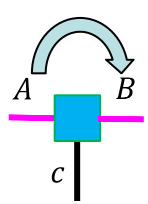
- Contractions:



$C = x_a\, y_a$ $\quad$ $V_b = x_a\, M_{ab}$ $\quad$ $M_{ab} = A_{ac}\, B_{cb}$

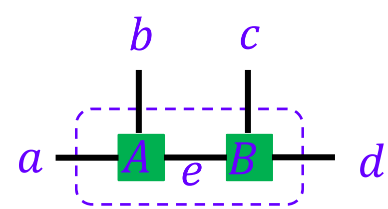
- The order of indices:

clockwise:



T_{ABc}

T_{abcd}



$T_{abcd} = A_{abe}\, B_{ecd}$

Quantum entanglement in tensor networks

The key ideas in tensor network approach

1. Applying SVD to decompose tensors into the product of tensors with lower orders.

2. Applying the approximate method by truncating the diagonal matrix.

Quantum entanglement in tensor networks

The key ideas in tensor network approach

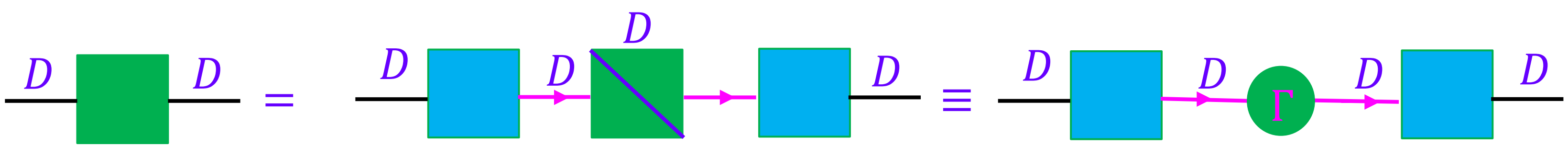
1. Applying SVD to decompose tensors into the product of tensors with lower orders.

2. Applying the approximate method by truncating the diagonal matrix.

Eigenvalue decomposition of the matrix

Hermitian matrix: $M = M^{\dagger}$ ➡ $M = U \Gamma U^{\dagger}$

Γ is a diagonal matrix, referred to as the eigenvalue spectrum



Quantum entanglement in tensor networks

The key ideas in tensor network approach

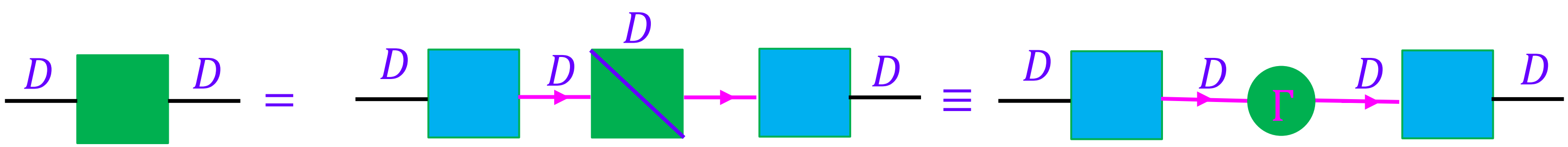
1. Applying SVD to decompose tensors into the product of tensors with lower orders.

2. Applying the approximate method by truncating the diagonal matrix.

Eigenvalue decomposition of the matrix

Hermitian matrix: $M = M^{\dagger}$ → $M = U\Gamma U^{\dagger}$

Γ is a diagonal matrix, referred to as the eigenvalue spectrum



The maximum eigenvalue problem

$$\lim_{K\to\infty} M^{K} = \Gamma_0^{K} u^{(0)} u^{(0)T}$$

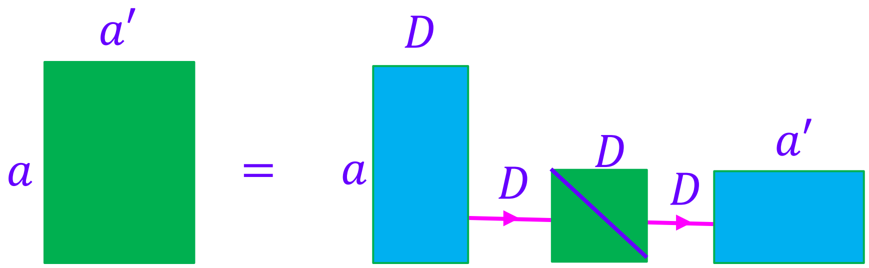
Truncation of the diagonal matrix: $(\Gamma_{00} \geq \Gamma_{11} \dots \geq \Gamma_{(D-1)(D-1)})$

Quantum entanglement in tensor networks

- **Singular value decomposition**

M: $a \times a'$ matrix

$$\boxed{M = U\Lambda V^{\dagger}} \qquad UU^{\dagger} = I, \qquad VV^{\dagger} = I$$



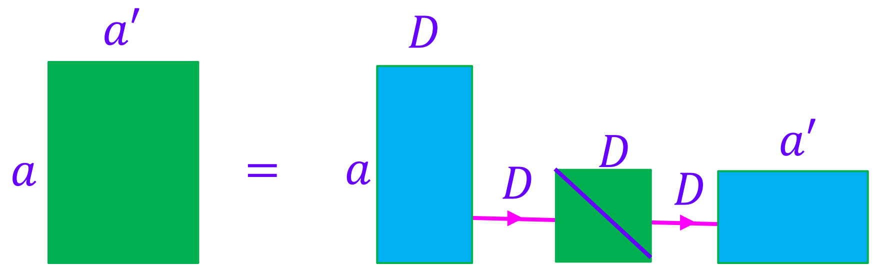
Rank: the number of non-zero eigenvalues

Quantum entanglement in tensor networks

Singular value decomposition

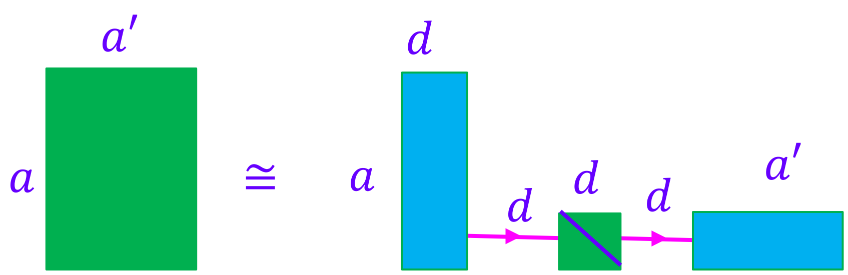
M: $a \times a'$ matrix

$$\boxed{M = U\Lambda V^\dagger} \qquad UU^\dagger = I, \qquad VV^\dagger = I$$



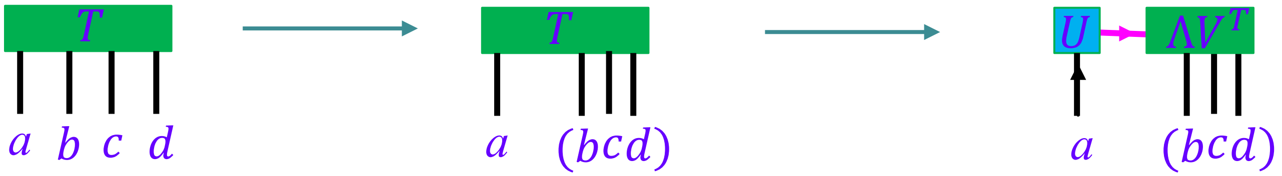
Rank: the number of non-zero eigenvalues

Truncation of the diagonal matrix Λ



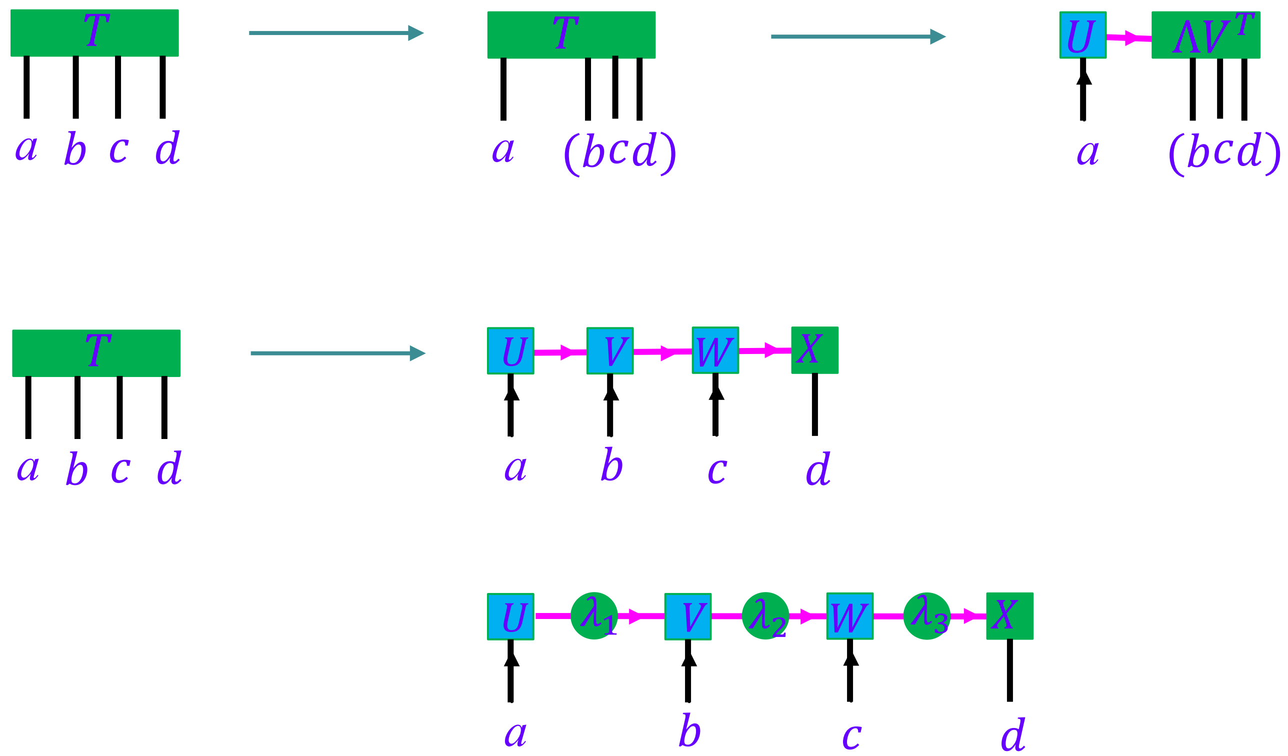
Quantum entanglement in tensor networks

- **Tensor train decomposition:**



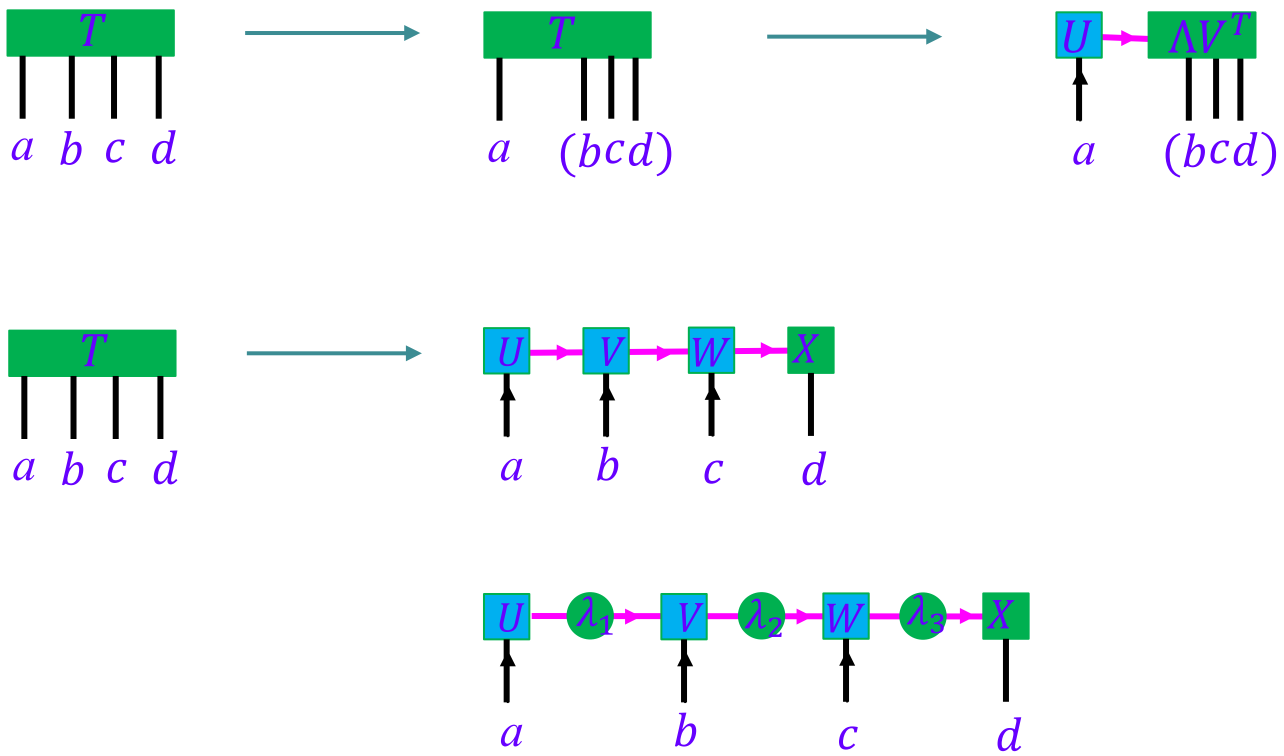
Quantum entanglement in tensor networks

- **Tensor train decomposition:**



Quantum entanglement in tensor networks

- **Tensor train decomposition:**



1. Without any approximate TT decomposition, the "exponential wall" problem cannot be solved without rank loss.

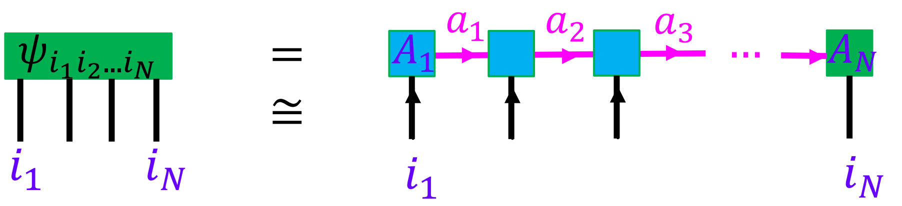
2. Orthogonality is typically indicated by an arrow: it stipulates that contracting the inner indices of a tensor with its conjugate tensor yields the identity matrix.

Quantum entanglement in tensor networks

- **Matrix Product State (MPS):**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \ldots, d$$

$$\psi_{i_1 i_2 \ldots i_N} = A^{(1)}_{i_1 a_1} A^{(2)}_{i_2 a_1 a_2} \ldots A^{(N-1)}_{i_{N-1} a_{N-2} a_{N-1}} A^{(N)}_{i_N a_{N-1}}$$

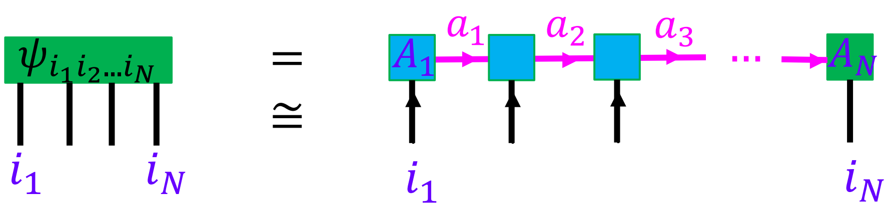


$a_1 = 1, \ldots, D$

- **Matrix Product State (MPS):**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \ldots, d$$

$$\psi_{i_1 i_2 \ldots i_N} = A^{(1)}_{i_1 a_1} A^{(2)}_{i_2 a_1 a_2} \ldots A^{(N-1)}_{i_{N-1} a_{N-2} a_{N-1}} A^{(N)}_{i_N a_{N-1}}$$



$a_1 = 1, \ldots, D$

$$\dim(\mathcal{H}_L) = d^N \quad \gg \quad \dim(\mathcal{H}_R) = NdD^2$$

MPS reduces the parameter complexity of quantum multibody from exponential to linear levels!

Quantum entanglement in tensor networks

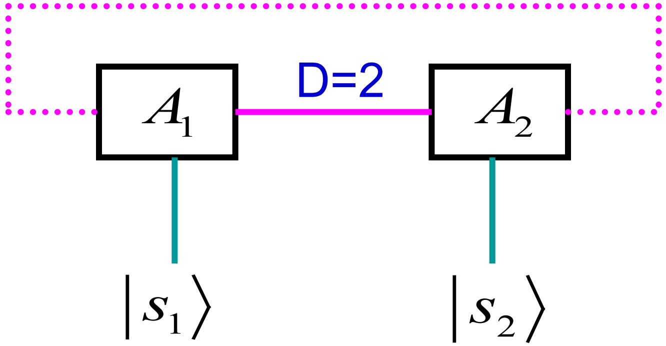
- **EPR state as the simplest Matrix Product State (MPS):**

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|\psi\rangle = \sum_{s_1 s_2=\uparrow}^{\downarrow} Tr\left(A_{[1]}^{s_1} A_{[2]}^{s_2}\right)|s_1 s_2\rangle$$

$$A_{[1]}^{\uparrow} = A_{[2]}^{\downarrow} = \frac{1}{2^{1/4}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$A_{[1]}^{\downarrow} = A_{[2]}^{\uparrow} = \frac{1}{2^{1/4}}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$



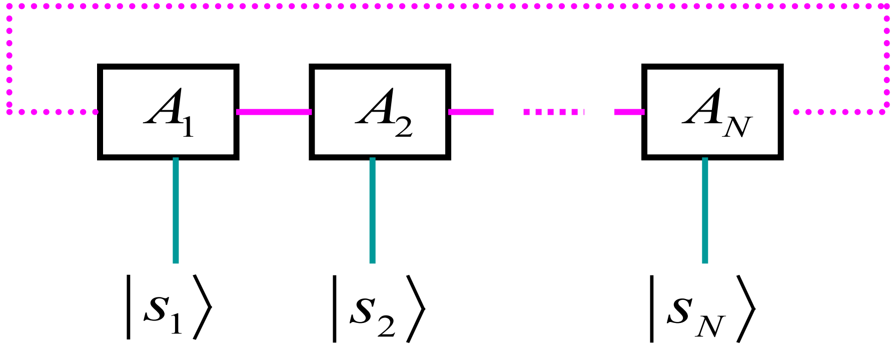
- **GHZ state**

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$$

$$|\psi\rangle = \sum_{s_1 s_2=\uparrow}^{\downarrow} Tr\left(A_{[1]}^{s_1} A_{[2]}^{s_2} \dots A_{[N]}^{s_N}\right)|s_1 s_2 \dots s_N\rangle$$

$$A_{[i]}^{0} = \frac{1}{2^{1/2N}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

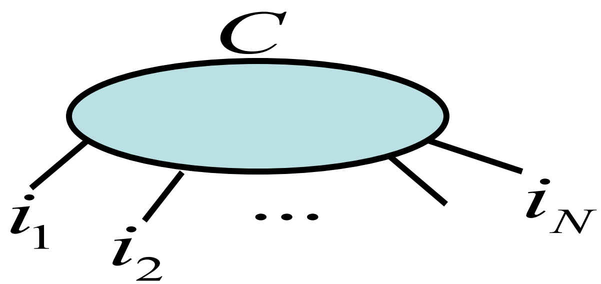
$$A_{[i]}^{1} = \frac{1}{2^{1/2N}}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$



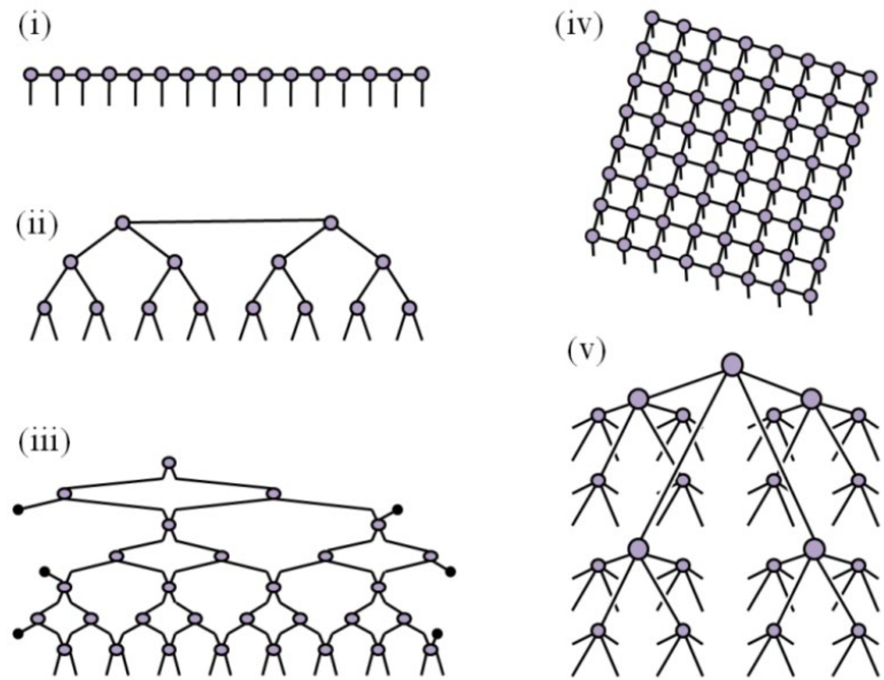
Quantum entanglement in tensor networks

- **Various tensor network states**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} C_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$$



1. **Matrix Product State（MPS）**
2. **Tree Tensor Networks（TTN）**
3. **Multiscale Entanglement Renormalization Ansatz（MERA）**

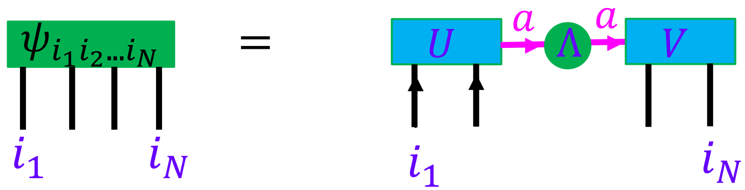


Quantum entanglement in tensor networks

- **Schmidt decomposition and entanglement spectrum:**

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle \qquad i_1 = 1, \ldots, d$$

$$\psi_{i_1 i_2 \ldots i_N} = \sum_{a=0}^{D-1} U_{i_1 i_2 \ldots i_k, a} \Lambda_a V^*_{i_{k+1} i_{k+2} \ldots i_N, a}$$



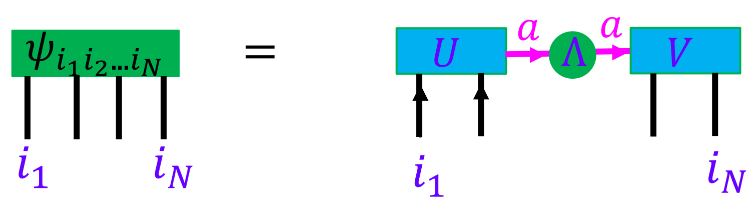
$$|\psi\rangle = \sum_{a=0}^{D-1} \Lambda_a |U^a\rangle |V^a\rangle$$

Quantum entanglement in tensor networks

Schmidt decomposition and entanglement spectrum:

$$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} \psi_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle \qquad i_1 = 1, \dots, d$$

$$\psi_{i_1 i_2 \dots i_N} = \sum_{a=0}^{D-1} U_{i_1 i_2 \dots i_{k,a}} \Lambda_a V^*_{i_{k+1} i_{k+2} \dots i_{N,a}}$$



$$|\psi\rangle = \sum_{a=0}^{D-1} \Lambda_a |U^a\rangle |V^a\rangle$$

1. The Schmidt decomposition of quantum states corresponds to the singular value decomposition of their coefficient matrices

2. Λ_a is the entanglement spectrum:

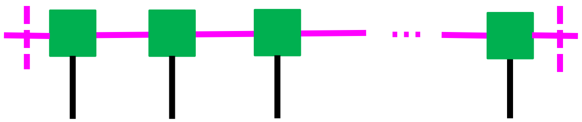
$$\sum_a {\Lambda_a}^2 = 1 \qquad S = -Tr\rho_A \ln \rho_A = -\sum_a {\Lambda_a}^2 ln\, {\Lambda_a}^2 \leq lnD$$

Quantum entanglement in tensor networks

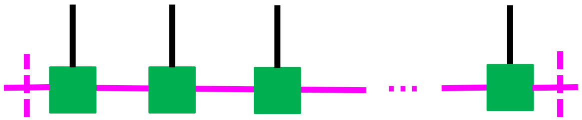
- **The reduced density matrix and entanglement entropy of MPS:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_{2\ldots}=1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

$|\psi_{MPS}\rangle$



$\langle\psi_{MPS}|$

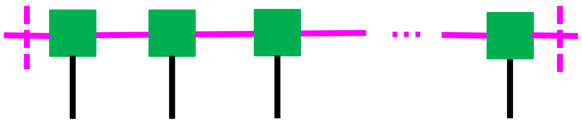


Quantum entanglement in tensor networks

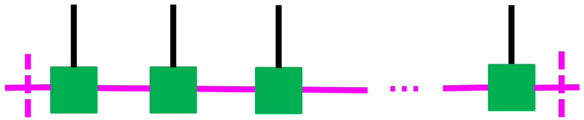
- **The reduced density matrix and entanglement entropy of MPS:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_2 \ldots =1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

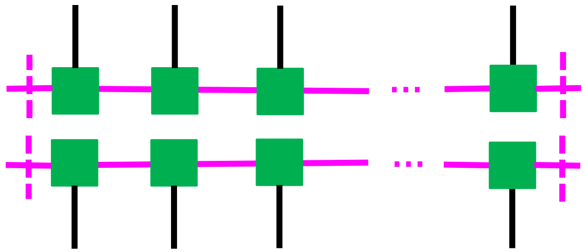
$|\psi_{MPS}\rangle$



$\langle\psi_{MPS}|$



$\rho = |\psi_{MPS}\rangle \langle\psi_{MPS}|$

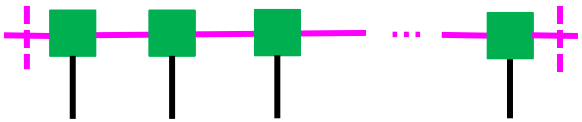


Quantum entanglement in tensor networks

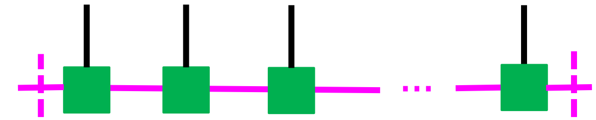
- **The reduced density matrix and entanglement entropy of MPS:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_2 \ldots =1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

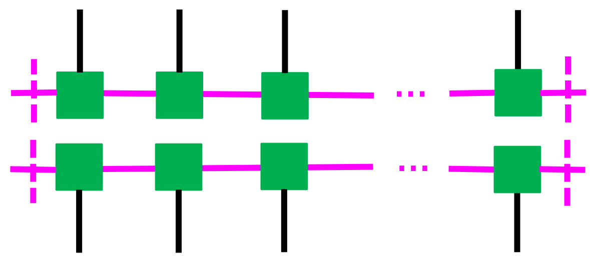
$|\psi_{MPS}\rangle$



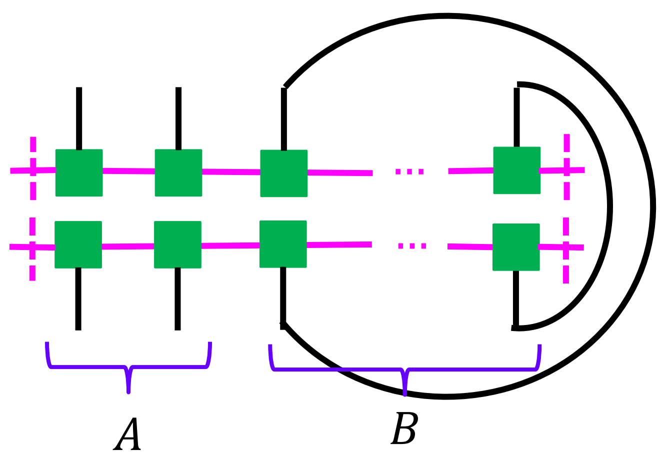
$\langle\psi_{MPS}|$



$\rho = |\psi_{MPS}\rangle \langle\psi_{MPS}|$

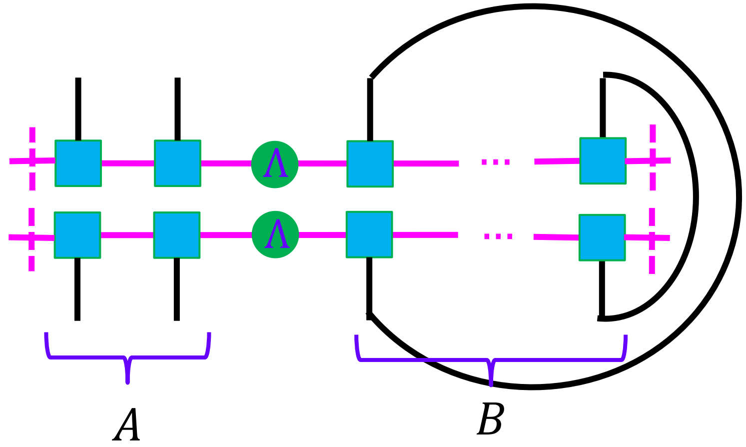


$\rho_A = Tr_B \rho$



Quantum entanglement in tensor networks

For SVD:



$$\rho_A = Tr_B \rho = \left(M^{\dagger}\right)_{d^2 \times D} \Lambda^2{}_{D \times D} \, (M)_{D \times d^2} = -\sum_a {\Lambda_a}^2 ln\, {\Lambda_a}^2$$

Quantum entanglement in tensor networks

- **Higher-order singular value decomposition (HOSVD) :**

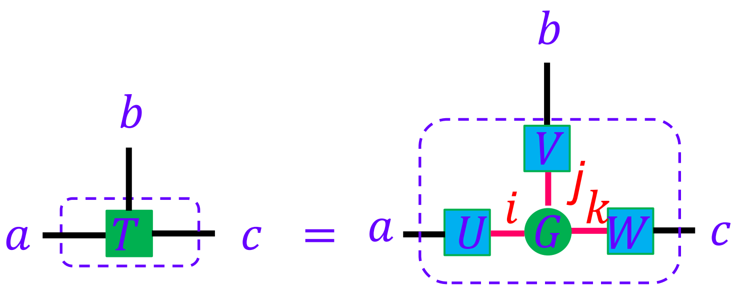
$$T_{abc} = G_{ijk} U_{ai} V_{bj} W_{ck}$$

G_{ijk} is the core tensor of T_{abc}

Bond reduced matrix

$$J_{ii'} = G_{ijk}\, G_{i'jk}$$

It is a non negative definite diagonal matrix, and the elements are arranged in non ascending order $(J_{00} \geq J_{11} \dots \geq 0)$



Quantum entanglement in tensor networks

Operators on states:

the coefficients of an operator are $d^N \times d^N$ dimensional tensors

$$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} \psi_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle$$

$$|\psi'\rangle = \hat{O}\,|\psi\rangle \qquad \hat{O} = \hat{O}^{(1)} \otimes \hat{O}^{(2)} \otimes \dots \otimes \hat{O}^{(N)}$$

Quantum entanglement in tensor networks

Operators on states:

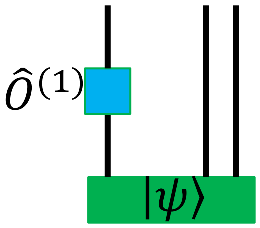
the coefficients of an operator are $d^N \times d^N$ dimensional tensors

$$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} \psi_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle$$

$$|\psi'\rangle = \hat{O}\, |\psi\rangle \qquad \hat{O} = \hat{O}^{(1)} \otimes \hat{O}^{(2)} \otimes \dots \otimes \hat{O}^{(N)}$$

$$\hat{O} = \hat{\sigma}^{(1)} \otimes I^{(2)} \otimes \dots \otimes I^{(N)}$$

$$\psi'_{i_1 i_2 \dots i_N} = \sigma^{(1)}_{i_1 j_1}\, \psi_{j_1 i_2 \dots i_N}$$



Quantum entanglement in tensor networks

- **Operators on states:**

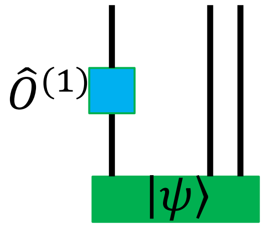
the coefficients of an operator are $d^N \times d^N$ dimensional tensors

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} \psi_{i_1 i_2 \ldots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$$

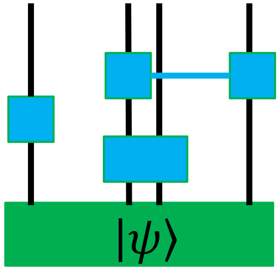
$$|\psi'\rangle = \hat{O}\, |\psi\rangle \qquad \hat{O} = \hat{O}^{(1)} \otimes \hat{O}^{(2)} \otimes \ldots \otimes \hat{O}^{(N)}$$

$$\hat{O} = \hat{\sigma}^{(1)} \otimes I^{(2)} \otimes \ldots \otimes I^{(N)}$$

$$\psi'_{i_1 i_2 \ldots i_N} = \sigma^{(1)}_{i_1 j_1}\, \psi_{j_1 i_2 \ldots i_N}$$



General operators on states:



Quantum entanglement in tensor networks

- **Expectation values:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_2 \ldots = 1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

$$\langle\psi_{MPS}|\hat{O}\,|\psi_{MPS}\rangle = \sum_{\substack{i_1 i_2 \ldots = 1 \\ i'_1 i'_2 \ldots = 1}}^{d} Tr\left(\prod_{m=1}^{N} A_{[m]}^{i_m}\right) Tr\left(\prod_{m=1}^{N} \overline{A}_{[m]}^{i'_m}\right) \prod_{m=1}^{N} \langle i'_m\,|\hat{O}_m|i_m\rangle$$

$$Tr(A \otimes B) = TrATrB$$

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$$

Quantum entanglement in tensor networks

- **Expectation values:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_2 \ldots =1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

$$\langle\psi_{MPS}|\hat{O}\,|\psi_{MPS}\rangle = \sum_{\substack{i_1 i_2 \ldots =1 \\ i_1' i_2' \ldots =1}}^{d} Tr\left(\prod_{m=1}^{N} A_{[m]}^{i_m}\right) Tr\left(\prod_{m=1}^{N} \overline{A}_{[m]}^{i_m'}\right) \prod_{m=1}^{N} \langle i_m'\,|\hat{O}_m|i_m\rangle$$

$$Tr(A \otimes B) = TrATrB$$

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$$

$$= Tr\left[\prod_{m=1}^{N} \sum_{\substack{i_1 i_2 \ldots =1 \\ i_1' i_2' \ldots =1}}^{d} \langle i_m'|\hat{O}_m|i_m\rangle\, A_{[m]}^{i_m} \otimes \overline{A}_{[m]}^{i_m'}\right]$$

Quantum entanglement in tensor networks

- **Expectation values:**

$$|\psi_{MPS}\rangle = \sum_{i_1 i_2 \ldots =1}^{d} Tr\left(A_{[1]}^{i_1} A_{[2]}^{i_2} \ldots A_{[N]}^{i_N}\right) |i_1 i_2 \ldots i_N\rangle$$

$$\langle\psi_{MPS}|\hat{O}\,|\psi_{MPS}\rangle = \sum_{\substack{i_1 i_2 \ldots =1 \\ i'_1 i'_2 \ldots =1}}^{d} Tr\left(\prod_{m=1}^{N} A_{[m]}^{i_m}\right) Tr\left(\prod_{m=1}^{N} \overline{A}_{[m]}^{i'_m}\right) \prod_{m=1}^{N} \langle i'_m\,|\hat{O}_m|i_m\rangle$$

$$Tr(A \otimes B) = TrATrB$$

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$$

$$= Tr\left[\prod_{m=1}^{N} \sum_{\substack{i_1 i_2 \ldots =1 \\ i'_1 i'_2 \ldots =1}}^{d} \langle i'_m|\hat{O}_m|i_m\rangle\, A_{[m]}^{i_m} \otimes \overline{A}_{[m]}^{i'_m}\right]$$

$$E_{O_m} := \sum_{\substack{i_1 i_2 \ldots =1 \\ i'_1 i'_2 \ldots =1}}^{d} \langle i'_m|\hat{O}_m|i_m\rangle\, A_{[m]}^{i_m} \otimes \overline{A}_{[m]}^{i'_m}$$

$$\langle\psi_{MPS}|\hat{O}\,|\psi_{MPS}\rangle = Tr(E_{O_1} \ldots\ldots E_{O_m})$$

Quantum entanglement in tensor networks

- Correlations:

$$\langle\psi_{MPS}|I\otimes\cdots I\otimes\hat{O}^{(i)}\otimes I\ldots\ I\otimes\hat{O}^{(i+\Delta)}..\otimes I|\psi_{MPS}\rangle=\langle\psi_{MPS}|\hat{O}^{(i)}\hat{O}^{(i+\Delta)}|\psi_{MPS}\rangle$$

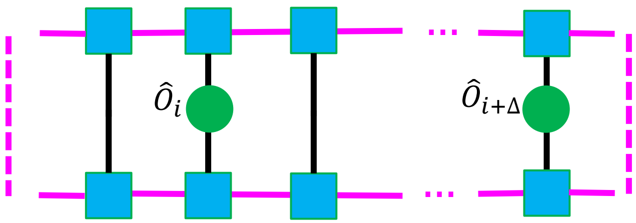
$$\boxed{\langle\psi_{MPS}|\hat{O}^{(i)}\hat{O}^{(i+\Delta)}|\psi_{MPS}\rangle-\langle\psi_{MPS}|\hat{O}^{(i)}|\psi_{MPS}\rangle\,\langle\psi_{MPS}|\hat{O}^{(i+\Delta)}|\psi_{MPS}\rangle}$$

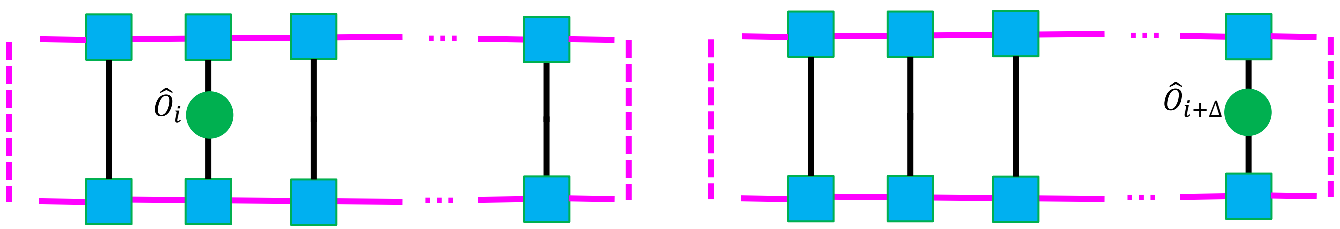
Quantum entanglement in tensor networks

- Correlations:

$$\langle \psi_{MPS} | I \otimes \cdots I \otimes \hat{O}^{(i)} \otimes I \ldots\ I \otimes \hat{O}^{(i+\Delta)} .. \otimes I | \psi_{MPS} \rangle = \langle \psi_{MPS} | \hat{O}^{(i)} \hat{O}^{(i+\Delta)} | \psi_{MPS} \rangle$$

$$\langle \psi_{MPS} | \hat{O}^{(i)} \hat{O}^{(i+\Delta)} | \psi_{MPS} \rangle - \langle \psi_{MPS} | \hat{O}^{(i)} | \psi_{MPS} \rangle \langle \psi_{MPS} | \hat{O}^{(i+\Delta)} | \psi_{MPS} \rangle$$



$-$ 

$\sim e^{-\Delta}$

Quantum entanglement in tensor networks

Basic procedures to find the ground state:

1. Given the Hamiltonian, starting from a random MPS state.

2. Evaluate the expectation value of the Hamiltonian at this state by SVD and tensor contractions.

3. Leave the most left tensor uncontracted, diagonalize the matrix, find E_{min}.

4. Reshape the eigenvector to form a new tensor for the most left tensor.

5. Repeat the above steps for the next tensor neighboring to the most left one.

6. Repeat the above steps until E_{min} unchanged.

Quantum entanglement in tensor networks

- **Basic procedures to find the ground state**:

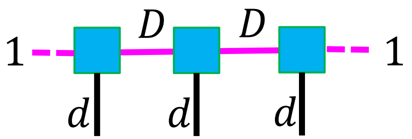
1. Given the Hamiltonian, starting from a random MPS state.
2. Evaluate the expectation value of the Hamiltonian at this state by SVD and tensor contractions.
3. Leave the most left tensor uncontracted, diagonalize the matrix, find E_{min}.
4. Reshape the eigenvector to form a new tensor for the most left tensor.
5. Repeat the above steps for the next tensor neighboring to the most left one.
6. Repeat the above steps until E_{min} unchanged.

$$\hat{H} = \sum_{i=1}^{N} \hat{O}_i \, \hat{O}_{i+1}$$

$N = 3$ $$\langle \hat{H} \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle + \langle \hat{O}_2 \hat{O}_3 \rangle$$

Random MPS state

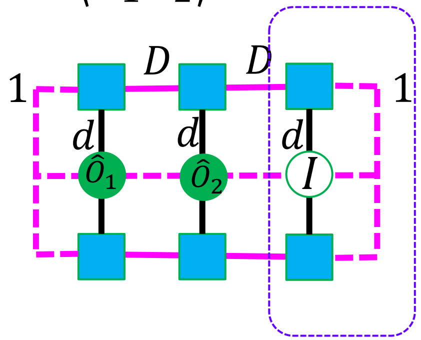
SVD:



Quantum entanglement in tensor networks

$$\langle \hat{H} \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle + \langle \hat{O}_2 \hat{O}_3 \rangle$$

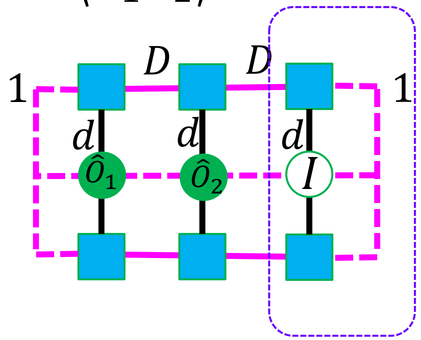
$\langle \hat{O}_1 \hat{O}_2 \rangle$

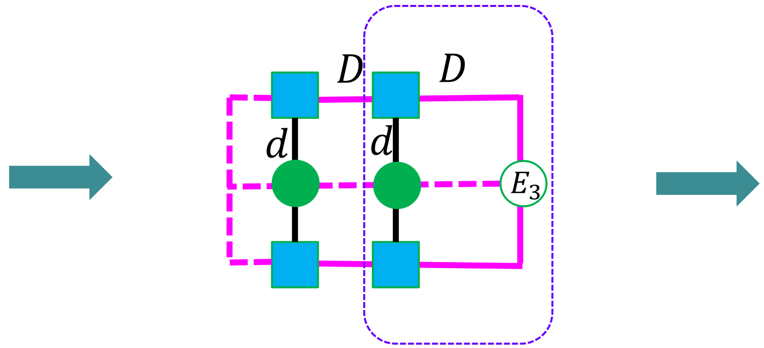


Quantum entanglement in tensor networks

$$\langle \hat{H} \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle + \langle \hat{O}_2 \hat{O}_3 \rangle$$

$\langle \hat{O}_1 \hat{O}_2 \rangle$

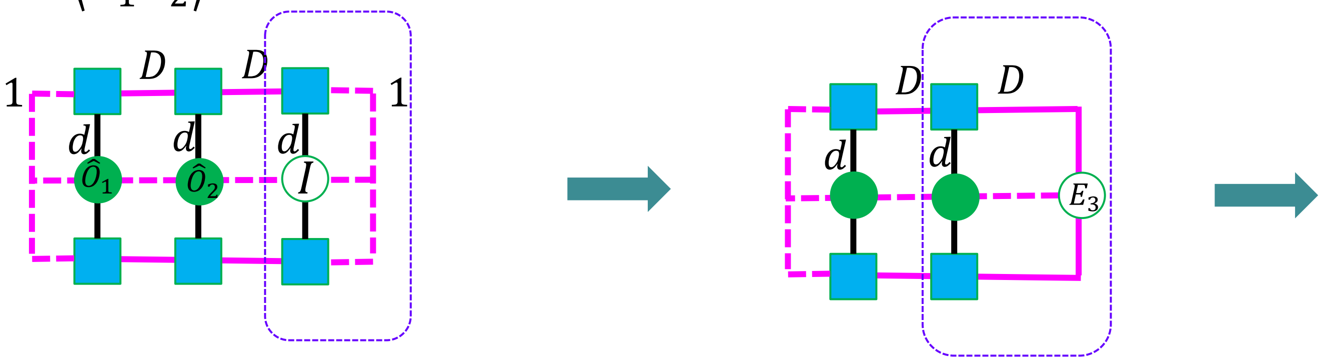


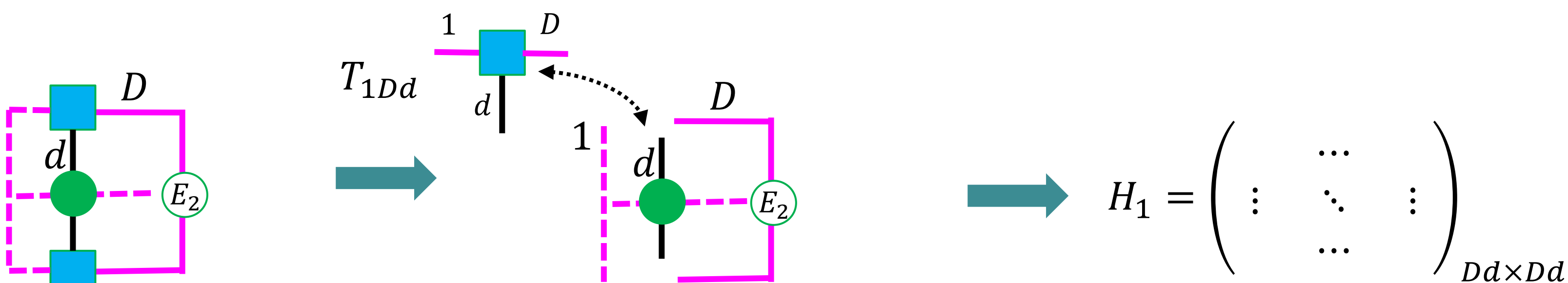


Quantum entanglement in tensor networks

$$\langle \hat{H} \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle + \langle \hat{O}_2 \hat{O}_3 \rangle$$

$\langle \hat{O}_1 \hat{O}_2 \rangle$

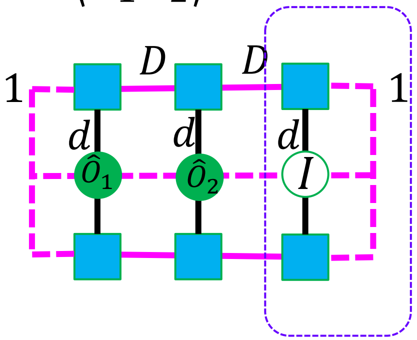


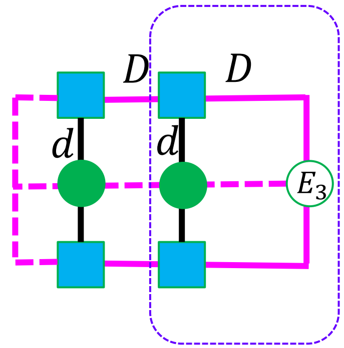


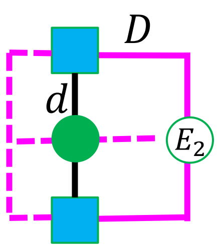
Quantum entanglement in tensor networks

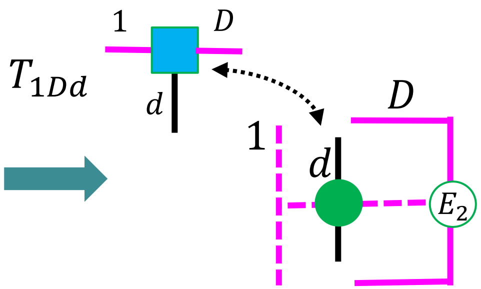
$$\langle \hat{H} \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle + \langle \hat{O}_2 \hat{O}_3 \rangle$$

$\langle \hat{O}_1 \hat{O}_2 \rangle$









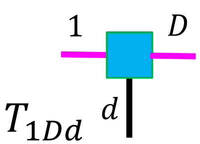
$$H_1 = \begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}_{Dd \times Dd}$$

$\langle \hat{O}_2 \hat{O}_3 \rangle$

$$H_2 = \begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}_{Dd \times Dd}$$

$$H = H_1 + H_2$$

diagonalize H and find E_{min}
and φ_g which is dD dim.



Quantum entanglement in tensor networks

- **Imaginary time evolution to find the ground state**:

When the system temperature is extremely low ($\beta \to \infty$), the system density operator is given by the eigenstate with the lowest Hamiltonian (denoted as $|g\rangle$), which is called the ground state of the system. The corresponding eigenvalue E_g is called the ground state energy.

$$\lim_{\beta\to\infty} e^{-\beta\hat{H}}/Z = |g\rangle\langle g|$$

$$\hat{H}|g\rangle = E_g|g\rangle$$

Ground state solution refers to finding the lowest eigenstate and eigenvalues of the matrix corresponding to the Hamiltonian, corresponding to the following optimization problem

$$E_g = \langle g|\hat{H}|g\rangle$$

$$\boxed{\lim_{\beta\to\infty} e^{-\beta\hat{H}}|\varphi_{MPS}\rangle \to |g\rangle}$$

$\beta = N\tau, \quad N \to \infty, \tau$ *is* small

$$\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{34}$$

$$e^{\tau(\hat{H}_{12}+\hat{H}_{23}+\hat{H}_{34})} \approx e^{\tau(\hat{H}_{12}+\hat{H}_{34})}\, e^{\tau(\hat{H}_{23})} = e^{\tau\hat{H}_{12}}\, e^{\tau\hat{H}_{34}}\, e^{\tau\hat{H}_{23}}$$