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# Advanced General Relativity, Holonomy

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Xiangdong Zhang(张向东)

South China University of Technology (华南理工大学)

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# Outline

- Basic elements of  $SU(2)$  group
- Connection dynamics from canonical transformation
- Connection dynamics from Lagrangian
- Connection dynamics beyond GR
- Holonomy

# Historical developments

- There are four fundamental interactions in the nature, three of which (EM, Weak, Strong interactions) are described by gauge theories.
- What about the remaining force, gravity? Can it also be reformulated as a gauge theory, thereby allowing us to draw on quantization techniques from gauge field theory?
- The core feature of a gauge theory lies in its dynamics being rooted in the concept of connection.
- The quantum theory is highly relied on the corresponding classical theory.
- The geometric dynamics: From ADM (Arnowitt, Deser, Misner) formalism to Palatini formalism.
- Yet neither framework constitutes a true connection dynamics.
- In 1986, the Ashtekar formalism—a complex connection-based dynamical framework—was established, laying the foundational groundwork for loop quantum gravity.

- However, quantization based on the complex connection cannot be rigorously implemented due to the non-compact nature of the internal gauge group.
- In 1995, Barbero revised Ashtekar's complex variables into real canonical variables.
- Holst subsequently constructed a generalized Palatini action to support this real connection dynamics.
- Despite containing a free parameter (the Barbero-Immirzi parameter  $\beta$ ) and featuring a more complex Hamiltonian constraint than the Ashtekar framework, the generalized Palatini Hamiltonian with real connections is now broadly embraced by loop theorists for quantization efforts.
- Applicable scope of connection dynamics: Connection dynamics beyond GR, coupled with matter fields....

# Basic elements of SU(2) group

**Def. 1.** A **group**  $G$  is a set equipped with a map (group-product, composition)  $G \times G \rightarrow G$  such that

(i)  $(g_1 g_2) g_3 = g_1 (g_2 g_3), \quad \forall g_1, g_2, g_3 \in G.$

(ii)  $\exists$  identity element  $e$  such that  $ge = eg = g, \quad \forall g \in G.$

(iii)  $\forall g \in G, \exists$  its inverse  $g^{-1}$  such that  $gg^{-1} = g^{-1}g = e.$

**Def. 2.** Let  $G$  and  $G'$  be two groups. A map  $\varphi : G \rightarrow G'$  is called a **homomorphism**, if  $\varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2), \quad \forall g_1, g_2 \in G.$

**Def. 3.** A homomorphism  $\varphi : G \rightarrow G'$  is called an **isomorphism** if the map  $\varphi$  is 1-1 as well as onto. An isomorphism  $\varphi : G \rightarrow G$  is called a **automorphism** on  $G$ .

**Def. 4.** A **Lie group** of dimension  $n$  is a group  $G$  such that  $G$  is also a smooth manifold of (real) dimension  $n$ .

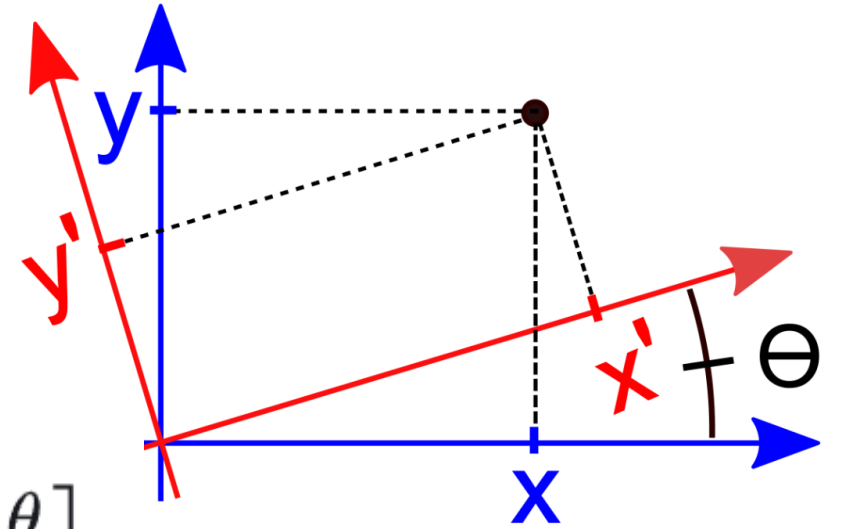
- A group is called abelian such that  $gh=hg$ , for any elements in  $G$

# SO (2) group

- For a rotation along Z-axis, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



- So, we only need to consider the matrix
- This is the SO(2) rotational group, which is an abelian group

$$\begin{aligned} \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos(\beta + \alpha) & -\sin(\beta + \alpha) \\ \sin(\beta + \alpha) & \cos(\beta + \alpha) \end{bmatrix} \end{aligned}$$

# SO (2) group

- For  $\theta = 0$  case, we have the identity element 
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$
- The inverse element 
$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1}$$
- Now, we are going to demonstrate SO(2) is isomorphic to  $\mathbb{R}$   $\rho : \mathbb{R} \rightarrow SO(2)$
- This map send identity to identity as  $0 \mapsto \rho(0) = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = I$   $\theta \mapsto \rho(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
- Inverse element to inverse element  $-\theta \mapsto \rho(-\theta) = - \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
- Keep the additivity  $(\alpha + \beta) \mapsto \rho(\alpha + \beta) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$

# SO (2) group

- SO(2) is also isomorphic to U(1) group, which reads  $\{e^{i\theta} | \theta \in \mathbb{R}\}(\cdot, 1)$

- This map from  $\mathbb{R}$  to U(1) group  $\tau : \mathbb{R} \rightarrow \{e^{i\theta} | \theta \in \mathbb{R}\}$

$$\theta \mapsto \tau(\theta) = e^{i\theta}$$

- This map keep the operation  $(\alpha + \beta) \mapsto \tau(\alpha + \beta) = e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$

- Moreover,  $\rightarrow \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$

- Is also realized as  $\theta \mapsto \prod_{i=1}^n e^{i\theta} = e^{in\theta}$

# SO(3) group

- Now we come the rotation matrix along x-axis  $R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\theta}{N} & \sin \frac{\theta}{N} \\ 0 & -\sin \frac{\theta}{N} & \cos \frac{\theta}{N} \end{pmatrix}^N$

- Note that  $N \rightarrow \infty$ ,  $\cos \frac{\theta}{N} \rightarrow 1$ ,  $\sin \frac{\theta}{N} \approx \frac{\theta}{N}$ .

- We have  $R_x = \lim_{N \rightarrow \infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\theta}{N} \\ 0 & -\frac{\theta}{N} & 1 \end{pmatrix}^N = \lim_{N \rightarrow \infty} g^N = \lim_{N \rightarrow \infty} (I + \frac{\theta}{N} X)^N$

- with X being :  $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ ,

- Now the group element  $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\theta}{N} \\ 0 & -\frac{\theta}{N} & 1 \end{pmatrix} = I + \frac{\theta}{N} X$

# SO(3) group

- Similarly, we have  $X_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ ,  $X_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $X_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- The definition of SO(3) group  $SO(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det R = 1\}$

- Let an element of SO(3) group as  $O: O = e^{\theta J}$ ,

$$O^T O = (e^{\theta J})^T e^{\theta J} = I$$

- We have  $\implies J^T + J = 0$

- Moreover, since  $\det(e^A) = e^{\text{tr}(A)}$ ,

- We arrival  $\det(O) = 1 \implies \det(e^{\theta J}) = e^{\theta \text{tr}(J)} = 1$   
 $\implies \text{tr}(J) = 0$

- The complete basis of 3X3 matrix read

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Introduce  $J_k = i\mathcal{J}_k$  we have  $\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ ,  $\mathcal{J}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$ ,  $\mathcal{J}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- These basis satisfy  $[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k$

- With this basis, all group element can be represented as

$$R = e^{i(\theta^x \mathcal{J}_x + \theta^y \mathcal{J}_y + \theta^z \mathcal{J}_z)} = e^{i\theta^k \mathcal{J}_k}$$

# SU(2) group

- The SU(2) group is a 2X2 complex matrix, such that

$$U^\dagger U = UU^\dagger = 1 \quad , \text{and} \quad \det(U) = 1$$

- Let a group element  $U = e^{iJ_i}$ , we have
 
$$U^\dagger U = (e^{iJ_i})^\dagger e^{iJ_i} = 1$$

$$\Rightarrow (e^{iJ_i})^\dagger e^{iJ_i} = e^{-iJ_i^\dagger} e^{iJ_i} = 1$$

$$\Rightarrow e^{-iJ_i^\dagger + iJ_i + \frac{1}{2}[J_i^\dagger, J_i] + \dots} = 1$$

$$\Rightarrow J_i^\dagger = J_i$$
- Moreover  $\det(U) = \det(e^{iJ_i}) = 1$ 

$$\Rightarrow \det(e^{iJ_i}) = e^{i \operatorname{tr}(J_i)} = 1$$

$$\Rightarrow \operatorname{tr}(J_i) = 0$$

- The complete basis of trace zero 2X2 complex matrix is the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# SU(2) group

- The commutation relation between Pauli matrix read  $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
- Introduce  $J_i \equiv \frac{1}{2}\sigma_i$
- We have  $[J_i, J_j] = i\epsilon_{ijk}J_k$
- SU(2) group is the covering of SO(3) group
- Lie algebra is the tangent space of the identity element, so if we write a group element as  $e^{i\vec{\theta}\cdot\vec{T}}$
- Then T forms an Lie algebra

# The ADM formalism of GR

- Consider the Einstein-Hilbert action on an 4-manifold  $M$ :

$$S_H[g_{\alpha\beta}] = \frac{1}{2\kappa} \int_M d^4x(e)R[g].$$

- To carry out the Hamiltonian analysis, suppose  $M$  is topologically  $\Sigma \times \mathbf{R}$  for some 3-dimensional compact manifold  $\Sigma$  without boundary. For noncompact  $\Sigma$ , the dynamical fields have to satisfy certain boundary condition such that the boundary terms could be removed.
- We introduce a foliation parameterized by a smooth function  $t$  and a time-evolution vector field  $t^\alpha$  such that  $t^\alpha(dt)_\alpha = 1$  in  $M$ , where  $t^\alpha$  can be decomposed with respect to the unit normal vector  $n^\alpha$  of  $\Sigma$  as:

$$t^\alpha = Nn^\alpha + N^\alpha,$$

here  $N$  is called the *lapse function* and  $N^\alpha$  the *shift vector*.

- After the 3+1 decomposition and the Legendre transformation, the Hilbert action can be expressed as

$$S_H = \int_{\mathbf{R}} dt \int_{\Sigma} d^3x [\tilde{\pi}^{ab} \mathcal{L}_t h_{ab} - \mathcal{H}(h_{ab}, \tilde{\pi}^{ab}, N, N^c)],$$

where  $h_{ab}$  is the induced spatial metric on  $\Sigma$ , and from which the symplectic structure on the classical phase space is obtained as

$$\{h_{ab}(x), \tilde{\pi}^{cd}(y)\} := \delta_{(a}^c \delta_{b)}^d \delta^3(x - y),$$

and the Hamiltonian reads

$$\mathcal{H} = N^a \mathcal{V}_a(h, \tilde{\pi}) + NS(h, \tilde{\pi}).$$

- The smeared diffeomorphism constraint  $\mathcal{V}(\vec{N})$  generates the infinitesimal spatial diffeomorphism by the vector field  $N^a$  on  $\Sigma$ ;
- The smeared Hamiltonian constraint  $\mathcal{S}(N)$  generates the infinitesimal bubble time evolution off  $\Sigma$ .
- They form a first-class constraint system as

$$\begin{aligned} \{\mathcal{V}(\vec{N}), \mathcal{V}(\vec{N}')\} &= \mathcal{V}([\vec{N}, \vec{N}']), \\ \{\mathcal{V}(\vec{N}), \mathcal{S}(M)\} &= -\mathcal{S}(\mathcal{L}_{\vec{N}}M), \\ \{\mathcal{S}(N), \mathcal{S}(M)\} &= -\mathcal{V}((N\partial_b M - M\partial_b N)h^{ab}). \end{aligned}$$

Introduce momentum  $p^{ab} = \frac{\sqrt{h}}{2}(K^{ab} - Kh^{ab})$  conjugate to spatial metric  $h_{ab}$

- Diffeomorphism and Hamiltonian constraints

$$V_a = -2D^b p_{ab},$$

$$H_{gr} = \frac{2}{\sqrt{h}} \left( p^{ab} p_{ab} - \frac{1}{3} p^2 \right) - \frac{1}{2} \sqrt{h} {}^3R$$

- Wheeler-DeWitt equation.

The 3-metric is expressed in terms of *triad* as  $q_{ab} := \delta_{jk} e_a^j e_b^k$

Notice that this relation is invariant under local SO(3) rotations  $e_a^i \mapsto O_j^i e_a^j$

Introduce  $K_a^i$  through extrinsic curvature as  $K_a^i = K_{ab} e_j^b \delta^{ij}$

Due to the symmetry property of  $K_{\{ab\}}$ , we thus have the following constraint :

$$G_{ab} := K_{[a}^j e_{b]}^j = 0$$

Consider the quantity  $E_i^a = \sqrt{h} e_i^a$

With the help of above equation, one can equivalently write  $G_{\{ab\}}$  in the form

$$G_{jk} := K_{a[j} E_{k]}^a = 0$$

We can also write the diffeomorphism and Hamiltonian constraint as functions on the extended phase space, which one can check to be explicitly given by

$$H_a := 2s D_b [K_a^j E_j^b - \delta_a^b K_c^j E_j^c]$$

$$H := -\frac{s}{\sqrt{\det(q)}} (K_a^l K_b^j - K_a^j K_b^l) E_j^a E_l^b - \sqrt{\det(q)} R$$

- Let us equip the extended phase space coordinatised by  $(K^i_a, E^a_i)$  with the symplectic structure

$$\{E_j^a(x), E_k^b(y)\} = \{K_a^j(x), K_b^k(y)\} = 0, \quad \{E_i^a(x), K_b^j(y)\} = \frac{\kappa}{2} \delta_b^a \delta_i^j \delta(x, y)$$

Note that the compatibility condition  $D_a q_{bc} = 0$ , implies that

$$D_a e_b^j = 0 \Rightarrow \Gamma_{ajk} = -e_k^b [\partial_a e_b^j - \Gamma_{ab}^c e_c^j]$$

By using the explicit formula to introduce  $\Gamma^i_a$  as

$$\begin{aligned} \Gamma_a^i &= \frac{1}{2} \epsilon^{ijk} e_k^b [e_{a,b}^j - e_{b,a}^j + e_j^c e_a^l e_{c,b}^l] \\ &= \frac{1}{2} \epsilon^{ijk} E_k^b [E_{a,b}^j - E_{b,a}^j + E_j^c E_a^l E_{c,b}^l] \\ &\quad + \frac{1}{4} \epsilon^{ijk} E_k^b \left[ 2E_a^j \frac{(\det(E))_{,b}}{\det(E)} - E_b^j \frac{(\det(E))_{,a}}{\det(E)} \right] \end{aligned}$$

- We can then write the Gauss constraint in the form

$$G_j = 0 + \epsilon_{jkl} ({}^{(\beta)}K_a^k) ({}^{(\beta)}E_l^a) = \partial_a ({}^{(\beta)}E_j^a) + \epsilon_{jkl} [\Gamma_a^k + ({}^{(\beta)}K_a^k)] ({}^{(\beta)}E_l^a)$$

$$=: ({}^{(\beta)}\mathcal{D}_a ({}^{(\beta)}E_j^a)$$

- This equation suggests introducing the new connection

$$A_a^i := \Gamma_a^i + \beta K_a^i,$$

- This is the exact definition of Ashtekar-Barbero connection.
- Take Ashtekar connection A and triad E as the conjugate pair, they satisfy the following commutation relation:

$$\{A_a^i(\vec{x}), E_j^b(\vec{x}')\} = 8\pi G\beta \delta_j^i \delta_a^b \delta^3(\vec{x} - \vec{x}')$$

- It remains to write the constraints in terms of these new variables. To that end we introduce the curvatures

$$R_{ab}^j := 2\partial_{[a}\Gamma_{b]}^j + \epsilon_{jkl}\Gamma_a^k\Gamma_b^l$$

$$^{(\beta)}F_{ab}^j := 2\partial_{[a}^{(\beta)}A_{b]}^j + \epsilon_{jkl}^{(\beta)}A_a^k^{(\beta)}A_b^l$$

- Let us expand  $F$  in terms of  $\Gamma$  and  $K$

$$^{(\beta)}F_{ab}^j = R_{ab}^j + 2\beta D_{[a}K_{b]}^j + \beta^2\epsilon_{jkl}K_a^kK_b^l$$

- Contracting with  $E$  yields  $^{(\beta)}F_{ab}^j^{(\beta)}E_j^b = \frac{R_{ab}^jE_j^b}{\beta} + 2D_{[a}(K_{b]}^jE_j^b) + \beta K_a^jG_j$

- Bianchi identity means

$$\epsilon_{ijk}\epsilon^{efc}R_{ef}^je_c^k = 0 \Rightarrow \frac{1}{2}\epsilon_{ijk}\epsilon^{efc}R_{ef}^je_c^ke_a^i = \frac{1}{2}E_j^b\epsilon_{cab}\epsilon^{efc}R_{ae}^j = R_{ab}^jE_j^b = 0$$

- Now we compare it with the expression of Diffeomorphism constraint and thus arrive at the conclusion

$${}^{(\beta)}F_{ab}^j {}^{(\beta)}E_j^b = -sH_a + {}^{(\beta)}K_a^j G_j$$

- Next, we note that

$$\begin{aligned} {}^{(\beta)}F_{ab}^j \epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b = & -\det(q) \frac{R_{abkl} e_k^a e_l^b}{\beta^2} - 2 \frac{E_j^a D_a G_j}{\beta} \\ & + (K_a^j E_j^a)^2 - (K_b^j E_j^a)(K_a^k E_k^b) \end{aligned}$$

- With this in hand, the Hamiltonian constraint can be written as

$$\begin{aligned}
& {}^{(\beta)}F_{ab}^j \epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b + 2 {}^{(\beta)}E_j^a D_a G_j \\
&= \sqrt{\det(q)} \left[ -\sqrt{\det(q)} \frac{R}{\beta^2} - \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
&= \frac{\sqrt{\det(q)}}{\beta^2} \left[ -\sqrt{\det(q)} R - \beta^2 \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
&= \frac{\sqrt{\det(q)}}{\beta^2} \left[ H + (s - \beta^2) \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
&= s\sqrt{\det(q)} \left[ -\frac{s}{\sqrt{\det(q)}} [(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2] - \frac{s}{\beta^2} \sqrt{\det(q)} R \right] \\
&= s\sqrt{\det(q)} \left[ H + \left(1 - \frac{s}{\beta^2}\right) \sqrt{\det(q)} R \right]
\end{aligned}$$

- Now we can solve the Hamiltonian constraint as

$$H = \frac{\beta^2}{\sqrt{|\det({}^{(\beta)}E\beta)|}} \left[ {}^{(\beta)}F_{ab}^j \epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b + 2 {}^{(\beta)}E_j^a D_a G_j \right]$$

$$+ (\beta^2 - s) \frac{({}^{(\beta)}K_b^j {}^{(\beta)}E_j^a) ({}^{(\beta)}K_a^j {}^{(\beta)}E_j^b) - ({}^{(\beta)}K_c^k {}^{(\beta)}E_k^c)^2}{\sqrt{|\det({}^{(\beta)}E\beta)|}}$$

- At this end, the Gauss, Diffeo, Hamiltonian constraints read respectively

$$G_i = D_a E_i^a$$

$$V_a = \frac{1}{\kappa\beta} F_{ab}^i E_i^b$$

$$H_{gr} = \frac{\epsilon^{ijk} E_i^a E_j^b}{2\kappa\sqrt{h}} F_{ab}^k - 2(\beta^2 + 1) \frac{E_{[i}^a E_{j]}^b}{2\kappa\sqrt{h}} K_a^i K_b^j$$

# The Palatini formalism

- The Lagrangian formulation
  - Consider an 4-manifold,  $M$ , on which the basic dynamical variables in the Palatini framework are tetrad  $e_I^\alpha$  and  $so(1, 3)$ -valued connection  $\omega_\alpha^{IJ}$  (not necessarily torsion-free), where the capital Latin indices  $I, J, \dots$  denote the internal Minkowski space indices and the Greek indices  $\alpha, \beta, \dots$  denote spacetime indices.
  - A tensor with both spacetime indices and internal indices is named as a generalized tensor.
  - The internal space is equipped with a Minkowskian metric  $\eta_{IJ}$  (of signature  $-, +, +, +$ ) fixed once for all, such that the spacetime metric reads:

$$g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J.$$

# The Palatini formalism

- The Palatini action is given by:

$$S_P[e_K^\beta, \omega_\alpha^{IJ}] = \frac{1}{2\kappa} \int_M d^4x(e) e_I^\alpha e_J^\beta \Omega_{\alpha\beta}^{IJ},$$

where the  $so(1, 3)$ -valued curvature 2-form  $\Omega_{\alpha\beta}^{IJ}$  of the connection  $\omega_\alpha^{IJ}$  reads:

$$\Omega_{\alpha\beta}^{IJ} := 2\mathcal{D}_{[\alpha}\omega_{\beta]}^{IJ} = \partial_\alpha\omega_\beta^{IJ} - \partial_\beta\omega_\alpha^{IJ} + \omega_\alpha^{IK} \wedge \omega_{\beta K}^J,$$

here  $\mathcal{D}_\alpha$  denotes the  $so(1, 3)$  covariant derivative with respect to  $\omega_\alpha^{IJ}$  acting on internal indices.

- Besides spacetime diffeomorphism transformations, the action is also invariant under internal  $SO(1, 3)$  rotations:

$$(e, \omega) \mapsto (e', \omega') = (b^{-1}e, b^{-1}\omega b + b^{-1}db),$$

for any  $SO(1, 3)$ -valued function  $b$  on  $M$ .

# The Palatini formalism

- The gravitational field equations are obtained by varying the Palatini action with respect to  $e_I^\alpha$  and  $\omega_\alpha^{IJ}$  respectively. We first study the variation with respect to the connection  $\omega_\alpha^{IJ}$ .
- Using the Leibnitz rule and integral by parts, one obtains

$$\delta S_P = -\frac{1}{\kappa} \int_M \delta \omega_\beta^{IJ} \mathcal{D}_\alpha [(e) e_I^\alpha e_J^\beta],$$

where we have extended  $\mathcal{D}_\alpha$  to a generalized  $so(1,3)$  covariant derivative acting on both spacetime and internal indices, and used the fact that  $\mathcal{D}_\alpha \tilde{\lambda}^\alpha = \partial_\alpha \tilde{\lambda}^\alpha$  for all vector density  $\tilde{\lambda}^\alpha$  of weight  $+1$  and neglected the surface term.

# The Palatini formalism

- Thus it gives the equation of motion:

$$\mathcal{D}_\alpha[(e)e_{[I}^\alpha e_{J]}^\beta] = -\frac{1}{4}\mathcal{D}_\alpha[\tilde{\eta}^{\alpha\beta\gamma\delta}\epsilon_{IJKL}e_\gamma^K e_\delta^L] = 0,$$

where  $\tilde{\eta}^{\alpha\beta\gamma\delta}$  is the spacetime Levi-Civita symbol and  $\epsilon_{IJKL}$  the internal Levi-Civita symbol. This equation leads to the torsion-free Cartan's first equation:

$$\mathcal{D}_{[\alpha}e_{\beta]}^I = 0, \quad (1)$$

which means that the connection  $\omega_\alpha^{IJ}$  is the unique torsion-free Levi-Civita spin connection compatible with the tetrad  $e_I^\alpha$ .

- The variation of  $S_P$  wrt.  $e_J^\beta$  gives

$$e_I^\alpha \Omega_{\alpha\beta}^{IJ} - \frac{1}{2}\Omega_{\alpha\gamma}^{IK} e_I^\alpha e_K^\gamma e_\beta^J = 0,$$

which is equivalent to the vacuum Einstein equation

# The Palatini formalism

- After the 3+1 decomposition and the Legendre transformation, one obtains from the Palatini action the configuration and conjugate momentum respectively as:

$$\begin{aligned}\omega_a^{IJ} &:= \omega_\alpha^{IJ} h_a^\alpha, \\ \tilde{P}^a_{IJ} &:= \tilde{E}^a_{[I} n_{J]},\end{aligned}$$

- where  $\tilde{E}^a_I \equiv \sqrt{\det h} e_I^\alpha h_a^\alpha$ , and the internal normal vector is defined as  $n_I \equiv n_\alpha e_I^\alpha$
- One also has the Lagrangian multipliers:  $N, N^\alpha, \omega_\alpha^{IJ} t^\alpha$
- There exist second-class constraints. Solving them, one results in dynamical variables  $(\tilde{E}^a_i, K_a^i)$ ,  $i = 1, 2, 3$ , where  $K_a^i := q_i^j h_a^\alpha \omega_\alpha^{IJ} n_J$ ,
- here  $q_i^j$  denotes the internal projection map with respect to the internal normal  $n^I$ .

# The Palatini formalism

- Note that the "boost part" of the Gauss constraint wrt  $n^I$  is solved, and  $K_a^i$  will be related to the extrinsic curvature of  $\Sigma$  on shell.
- The symplectic structure on the classical phase space is obtained as

$$\{\tilde{E}_i^a(x), K_b^j(y)\} := \delta_i^j \delta_b^a \delta^3(x - y).$$

The remained first-class constraints read

$$G'_i \equiv \epsilon_{ijk} K_a^j \tilde{E}^{ak} \approx 0,$$

$$\mathcal{V}'_a \equiv \nabla_b (K_a^j \tilde{E}_j^b - K_c^j \tilde{E}_j^c \delta_a^b) \approx 0,$$

$$\mathcal{S}' \equiv 2\sqrt{\det h}^{-1} \tilde{E}_i^a \tilde{E}_j^b K_{[a}^i K_{b]}^j + \sqrt{\det h} R \approx 0,$$

where  $\nabla_a$  is the  $SO(3)$  generalized derivative operator compatible with triad  $e_i^a$ , and  $R$  is its scalar curvature.

- Note that  $K_a^i$  is not a  $SO(3)$  connection. Hence we are still in geometrical dynamics.

# The generalized Palatini formalism

- The generalized Palatini action in which we are interested is given by [Holst 1996]:

$$S_G[e_K^\beta, \omega_\alpha^{IJ}] = \frac{1}{2\kappa} \int_M d^4x (e) e_I^\alpha e_J^\beta (\Omega_{\alpha\beta}^{IJ} + \frac{1}{2\beta} \epsilon^{IJ}{}_{KL} \Omega_{\alpha\beta}{}^{KL}),$$

where  $\beta$  is the Barbero-Immirzi parameter.

- Note that the generalized Palatini action returns to the Palatini action when  $\frac{1}{\beta} = 0$  and gives the (anti)self-dual Ashtekar formalism when one sets  $\frac{1}{\beta} = \pm i$ .
- Moreover, besides spacetime diffeomorphism transformations, the action is also invariant under internal  $SO(1, 3)$  rotations.

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- Moreover, besides spacetime diffeomorphism transformations, the action is also invariant under internal  $SO(1, 3)$  rotations.

- The variation of  $S_G$  with respect to the connection  $\omega_\alpha^{IJ}$  gives

$$\begin{aligned}\delta S_G &= \frac{1}{2\kappa} \int_M d^4x(e) e_I^\alpha e_J^\beta (\delta\Omega_{\alpha\beta}^{IJ} + \frac{1}{2\beta} \epsilon^{IJ}{}_{KL} \delta\Omega_{\alpha\beta}{}^{KL}) \\ &= -\frac{1}{\kappa} \int_M (\delta\omega_\beta{}^{IJ} + \frac{1}{2\beta} \epsilon^{IJ}{}_{KL} \delta\omega_\beta{}^{KL}) \mathcal{D}_\alpha [(e) e_I^\alpha e_J^\beta],\end{aligned}$$

which leads also to the conclusion that the connection  $\omega_\alpha^{IJ}$  is the unique torsion-free Levi-Civita spin connection compatible with the tetrad  $e_I^\alpha$ .

- As a result, the second term in the action  $S_G$  can be calculated as:

$$(e) e_I^\alpha e_J^\beta \epsilon^{IJKL} \Omega_{\alpha\beta KL} = \eta^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta},$$

which is exactly vanishing due to the symmetric properties of Riemann tensor. [\[Nie-Yan term\]](#)

- So the generalized Palatini action returns to the Palatini action, which will certainly give the Einstein field equation.

- We consider only the case where the Barbero-Immirzi parameter  $\beta$  is real. Then the Hamiltonian analysis of the generalized Palatini action will involve second-class constraints.
- However, in the 3+1 decomposition it is convenient to carry out a partial gauge fixing, i.e., fix an internal constant vector field  $n^I$  with  $\eta_{IJ}n^I n^J = -1$ . Then the internal vector space  $V$  is 3+1 decomposed with a 3-dimensional subspace  $W$  orthogonal to  $n^I$ , which will be the internal space on  $\Sigma$ .
- An internal reduced metric  $\delta_{ij}$  is also obtained by the projection map  $q_I^i$ .
- The spatial metric and the internal reduced metric are related by:

$$h_{ab} = \delta_{ij} e_a^i e_b^j,$$

where the orthonormal co-triad on  $\Sigma$  is defined by

$$e_a^i := e_\alpha^I q_I^i h_a^\alpha.$$

- Note that the internal gauge group  $SO(1, 3)$  is reduced to its subgroup  $SO(3)$  which leaves  $n^I$  invariant.
- Two Levi-Civita symbols are obtained respectively as

$$\begin{aligned}\epsilon_{ijk} &:= q_i^I q_j^J q_k^K n^L \epsilon_{LIJK}, \\ \underline{\eta}_{abc} &:= h_a^\alpha h_b^\beta h_c^\gamma t^\delta \underline{\eta}_{\delta\alpha\beta\gamma}.\end{aligned}$$

- Using the connection 1-form  $\omega_\alpha^{IJ}$ , one can obtain a spin connection on  $\Sigma$  as:

$$\Gamma_a^i := \frac{1}{2} h_a^\alpha q_I^i \epsilon^{IJ}_{KL} n_J \omega_\alpha^{KL}.$$

- After the 3+1 decomposition and the Legendre transformation, the generalized Palatini action becomes:

$$S_G = \int_{\mathbf{R}} dt \int_{\Sigma} d^3x [\tilde{P}_i^a \mathcal{L}_t A_a^i - \mathcal{H}_{tot}(A_a^i, \tilde{P}_j^b, \Lambda^i, N, N^c)],$$

where the configuration and conjugate momentum are defined respectively by:

$$\begin{aligned} A_a^i &:= \Gamma_a^i + \beta K_a^i, \\ \tilde{P}_i^a &:= \frac{1}{2\kappa\beta} \tilde{\eta}^{abc} \epsilon_{ijk} e_b^j e_c^k = \frac{1}{\kappa\beta} \sqrt{\det h} e_i^a, \end{aligned}$$

here the determinant of the 3-metric  $h_{ab}$  on  $\Sigma$  satisfies  $\det h = (\kappa\beta)^3 \det \tilde{P}$ .

- In the definition of the configuration variable  $A_a^i$ , we should emphasize that  $\Gamma_a^i$  is restricted to be the unique torsion free  $so(3)$ -valued spin connection compatible with the triad  $e_i^a$ . This conclusion is obtained by solving a second class constraint in the Hamiltonian analysis.

- Thus the symplectic structure on the classical phase space is obtained as

$$\{A_a^i(x), \tilde{P}_j^b(y)\} := \delta_j^i \delta_b^a \delta^3(x - y).$$

- In the Hamiltonian formalism, one starts with the fields  $(A_a^i, \tilde{P}_i^a)$ .  
Then neither the basic dynamical variables nor their Poisson brackets depend on the Barbero-Immirzi parameter  $\beta$ .
- Hence, for the case of pure gravitational field, the dynamical theories with different  $\beta$  are symplectic equivalent.
- The Hamiltonian density  $\mathcal{H}_{tot}$  is a linear combination of constraints:

$$\mathcal{H}_{tot} = \Lambda^i G_i + N^a V_a + NS,$$

where  $\Lambda^i \equiv -\frac{1}{2} \epsilon^i{}_{jk} \omega_t{}^{jk}$ .

- The three constraints are expressed as:

$$G_i = D_a \tilde{P}_i^a := \partial_a \tilde{P}_i^a + \epsilon_{ij}{}^k A_a^j \tilde{P}_k^a,$$

$$V_a = \tilde{P}_i^b F_{ab}^i - \frac{1 + \beta^2}{\beta} K_a^i G_i,$$

$$S = \frac{\kappa \beta^2}{2\sqrt{\det q}} \tilde{P}_i^a \tilde{P}_j^b [\epsilon^{ij}{}^k F_{ab}^k - 2(1 + \beta^2) K_{[a}^i K_{b]}^j] \\ + \kappa(1 + \beta^2) \partial_a \left( \frac{\tilde{P}_i^a}{\sqrt{\det q}} \right) G^i,$$

here the configuration variable  $A_a^i$  performs as a  $su(2)$ -valued connection on  $\Sigma$ , and  $F_{ab}^i$  is the  $su(2)$ -valued curvature 2-form of  $A_a^i$ :

$$F_{ab}^i := 2D_{[a} A_{b]}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} A_a^j A_b^k.$$

- The Gaussian constraint  $G_i$  has crucial importance in formulating the general relativity into a dynamical theory of connections.
- The corresponding smeared constraint function,  $\mathcal{G}(\Lambda) := \int_{\Sigma} d^3x \Lambda^i(x) G_i(x)$ , generates a transformation on the phase space as:

$$\begin{aligned} \{A_a^i(x), \mathcal{G}(\Lambda)\} &= -D_a \Lambda^i(x) \\ \{\tilde{P}_i^a(x), \mathcal{G}(\Lambda)\} &= \epsilon_{ij}{}^k \Lambda^j(x) \tilde{P}_k^a(x), \end{aligned}$$

which are just the infinitesimal versions of the following gauge transformations for the  $so(3)$ -valued connection 1-form  $\mathbf{A}$  and internal rotation for the  $so(3)$ -valued densitized vector field  $\tilde{\mathbf{P}}$  respectively:

$$(\mathbf{A}_a, \tilde{\mathbf{P}}^b) \mapsto (g^{-1} \mathbf{A}_a g + g^{-1} (dg)_a, g^{-1} \tilde{\mathbf{P}}^b g).$$

- To display the meaning of the vector constraint  $V_a$ , one may consider the smeared constraint function:

$$\mathcal{V}(\vec{N}) := \int_{\Sigma} d^3x (N^a \tilde{P}_i^b F_{ab}^i - (N^a A_a^i) G_i).$$

It generates the infinitesimal spatial diffeomorphism by the vector field  $N^a$  on  $\Sigma$  as:

$$\{A_a^i(x), \mathcal{V}(\vec{N})\} = \mathcal{L}_{\vec{N}} A_a^i(x), \quad \{\tilde{P}_i^a(x), \mathcal{V}(\vec{N})\} = \mathcal{L}_{\vec{N}} \tilde{P}_i^a(x).$$

- The smeared scalar constraint is weakly equivalent to the following function

$$\begin{aligned} \mathcal{S}(N) &:= \int_{\Sigma} d^3x N(x) S(x) \\ &= \frac{\kappa \beta^2}{2} \int_{\Sigma} d^3x N \frac{\tilde{P}_i^a \tilde{P}_j^b}{\sqrt{|\det q|}} [\epsilon^{ij}{}_k F_{ab}^k - 2(1 + \beta^2) K_{[a}^i K_{b]}^j], \end{aligned}$$

which also generates the infinitesimal bubble time evolution

- The constraints algebra has the following form:

$$\begin{aligned}
\{\mathcal{G}(\Lambda), \mathcal{G}(\Lambda')\} &= \mathcal{G}([\Lambda, \Lambda']), \\
\{\mathcal{G}(\Lambda), \mathcal{V}(\vec{N})\} &= -\mathcal{G}(\mathcal{L}_{\vec{N}}\Lambda), \\
\{\mathcal{G}(\Lambda), \mathcal{H}(N)\} &= 0, \\
\{\mathcal{V}(\vec{N}), \mathcal{V}(\vec{N}')\} &= \mathcal{V}([\vec{N}, \vec{N}']), \\
\{\mathcal{V}(\vec{N}), \mathcal{S}(M)\} &= -\mathcal{S}(\mathcal{L}_{\vec{N}}M), \\
\{\mathcal{S}(N), \mathcal{S}(M)\} &= -\mathcal{V}((N\partial_b M - M\partial_b N)h^{ab}) \\
&\quad -\mathcal{G}((N\partial_b M - M\partial_b N)h^{ab}A_a) \\
&\quad - (1 + \beta^2)\mathcal{G}\left(\frac{[\tilde{P}^a\partial_a N, \tilde{P}^b\partial_b M]}{\det h}\right).
\end{aligned}$$

- Hence the constraints are all of first class.

- Note that the evolution of constraints is consistent since the Hamiltonian  $H = \int_{\Sigma} d^3x \mathcal{H}_{tot}$  is the linear combination of the constraint functions.
- The evolution equations of the basic canonical pair read

$$\mathcal{L}_t A_a^i = \{A_a^i, H\}, \quad \mathcal{L}_t \tilde{P}_i^a = \{\tilde{P}_i^a, H\}.$$

- Together with the three constraint equations, they are completely equivalent to the Einstein field equations.
- Thus general relativity is cast as a dynamical theory of connections with a compact structure group.

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# Connection dynamics beyond GR

- The advantage of a dynamical theory of connections is that it is convenient to be quantized background independently.
- Modified gravity theories have received increased attention recently, since it can explain the "dark universe" without recourse to dark energy and dark matter.
- As one of the representative modified gravity theories,  $f(R)$  theories has been successfully cast into a connection dynamics formalism and then been quantized through extending LQG scheme [\[XZ and Ma, 2011, PRL\]](#).
- The non-perturbative loop quantization procedure is not only valid for GR but also valid for a general class of metric theories of gravity.

- It is also straightforward to construct the connection dynamics for more general scalar-tensor theories of gravity [XZ and Ma, 2011, PRD ] with the action reads

$$S(g) = \frac{1}{16\pi G} \int_{\Sigma} d^4x \sqrt{-g} \left[ \phi \mathcal{R} - \frac{\omega(\phi)}{\phi} (\partial_{\mu}\phi) \partial^{\mu}\phi - 2V(\phi) \right]$$

- Now the Ashtekar variable becomes  $A_a^i = \Gamma_a^i + \beta \tilde{K}_a^i$ ,
- where  $\tilde{K}_a^i \equiv \tilde{K}_{ab} e_i^b$  with

$$\tilde{K}^{ab} = \phi K^{ab} + \frac{q^{ab}}{2N} (\dot{\phi} - N^c \partial_c \phi)$$

- The Gauss constraint reads  $\mathcal{G}_i = \mathcal{D}_a E_i^a \equiv \partial_a E_i^a + \epsilon_{ijk} A_a^j E^{ak} = 0$ ,
- And the diffeomorphism constraint reads

$$V_a = \frac{1}{8\pi G \beta} F_{ab}^i E_i^b + \pi \partial_a \phi,$$

- The Hamiltonian constraint

$$\begin{aligned}
 H = & \frac{\phi}{16\pi G} \left[ F_{ab}^j - (\beta^2 + \frac{1}{\phi^2}) \varepsilon_{jmn} \tilde{K}_a^m \tilde{K}_b^n \right] \frac{\varepsilon_{jkl} E_k^a E_l^b}{\sqrt{q}} \\
 & + \frac{1}{(3 + 2\omega(\phi))} \left( \frac{(\tilde{K}_a^i E_i^a)^2}{8\pi G \phi \sqrt{q}} + 2 \frac{(\tilde{K}_a^i E_i^a) \pi}{\sqrt{q}} + \frac{8\pi G \pi^2 \phi}{\sqrt{q}} \right) \\
 & + \frac{1}{8\pi G} \left[ \frac{\omega(\phi)}{2\phi} \sqrt{q} (D_a \phi) D^a \phi + \sqrt{q} D_a D^a \phi + \sqrt{q} V(\phi) \right],
 \end{aligned}$$

- Loop quantum gravity in its Hamiltonian form relies on a connection formulation of the gravitational phase space with three key properties: 1.) a compact gauge group, 2.) real variables, and 3.) canonical Poisson brackets.
- Besides the modified gravity, Weyl gravity, supersymmetry gravity, higher dimensions, are also have corresponding connection dynamical formalism [Bodendorfer, Eder and XZ, Handbook of Quantum Gravity. Springer, Singapore (2023)].

# Coupled with the matter fields

- Matter and the gravity are coupled to each other: matter tells the spacetime how to curve, while the spacetime tell the matter how to move.
- In quantum mechanics,  $i \hbar \partial \Psi / \partial t = \hat{H} \Psi$ , Hamiltonian generates time evolution
- However, the notion of time is absent in pure GR since we only have Hamiltonian constraints rather than a true Hamiltonian.
- Deparametrized framework: Using the scalar field as an internal time.
- For massless scalar field with the action

$$S_\varphi = -\frac{1}{2} \int_M d^4 X \sqrt{g} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$$

- The diffeomorphism constraint reads

$$C_a(x) = C_a^{\text{gr}}(x) + \pi(x) \phi_{,a}(x),$$

# Coupled with the matter fields

- The Hamiltonian constraint

$$C(x) = C^{\text{gr}}(x) + \frac{1}{2} \frac{\pi^2(x)}{\sqrt{q(x)}} + \frac{1}{2} q^{ab}(x) \phi_{,a}(x) \phi_{,b}(x) \sqrt{q(x)}$$

- We can solve the vector constraint  $C_a(x) = 0$ , obtaining  $\phi_{,a} = -\frac{C_a^{\text{gr}}}{\pi}$ .
- Then the scalar constraint becomes

$$C'(x) = \pi(x) - h(x),$$
$$h := \sqrt{-\sqrt{q} C^{\text{gr}} + \sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}.$$

- For Dirac observable  $O$  commute with  $C = \pi - h$  lead to  $\frac{\partial}{\partial \phi} O = \{O, h\}$ .
- The equation of motion (EOM) of the operator  $O$  describes its evolution with respect to the scalar field  $\phi$

# Ashtekar formalism of the non-rotational dust

- The deparametrization formalism of LQG were realized in a few system including massless scalar field [[Lewandowski et.al, 2010](#)] as well as dust field [[Husain and Pawlowski,2012](#); [Thiemann et.al, 2015](#)].
- These coupled system can be achieved both at the classical level as well as the quantum level. The combination of LQG with the deparametrization framework makes it possible to solve the quantum Hamiltonian.
- Currently, there exist two different strategies to solve this coupled Hamiltonian constraint.
- The first strategy is to deparametrize the system first and then quantize the coupled system [[Lewandowski et.al, 2010](#); [Husain and Pawlowski,2012](#); [Thiemann et.al, 2015](#)].

# Ashtekar formalism of the non-rotational dust

- The second strategy is proposed by Lewandowski and Sahlmann [[PRD \(2016\)](#)]. They quantize the coupled system of gravity and a massless scalar field. However, the commutator of two Hamiltonian operators does not vanish and hence no nontrivial solutions could be obtained.
- Instead, we consider the non-rotational dust model coupled with GR which was also widely investigated in LQG literatures [[Lewandowski et.al, 2015](#); [Thiemann et.al, 2010, 2015](#); [XZ et.al, 2020](#)].
- Our purpose is to quantize the coupled system without imposing any gauge fixing. As a consequence, the Dirac quantization procedure can be realized in loop quantum gravity by this system [[XZ and Ma, PLB 2023](#)].
- Our starting point is the action which reads

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \frac{1}{2} \int d^4x \sqrt{-g} M \left( g^{ab} \partial_a T \partial_b T + 1 \right).$$

# Ashtekar formalism of the non-rotational dust

- The diffeomorphism and Hamiltonian constraints read respectively

$$C_a(x) = C_a^{gr}(x) + \pi(x) T_{,a}(x) = 0,$$

$$C_{total} = C^{gr} + \frac{1}{2} \left( \frac{\pi^2}{M\sqrt{q}} + M\sqrt{q} \left( 1 + q^{ab} T_{,a} T_{,b} \right) \right) = 0,$$

- where  $T(x)$  is the configuration variable of the non-rotational dust,  $\pi(x)$  is the conjugate momentum of  $T(x)$  satisfying

$$\pi(x) = \pm M\sqrt{q} \sqrt{1 + q^{ab} T_{,a}(x) T_{,b}(x)}.$$

- Substituted this condition into scalar constraint, we obtain

$$|C_{total}| = |C^{gr}(x)| + \pi(x) \sqrt{1 + q^{ab} T_{,a}(x) T_{,b}(x)} = 0$$

# Ashtekar formalism of the non-rotational dust

- Now the question is how to directly implement quantization by the following scalar constraints

$$C^{total} = \pi(x) + \frac{|C^{gr}|}{\sqrt{1 + q^{ab} T_{,a} T_{,b}}} = 0.$$

- The appearance of term  $\mathcal{I} \equiv \frac{1}{\sqrt{1+q^{ab} T_{,a} T_{,b}}}$  makes it hard to tame at the quantum level.
- Our key observation is that

$$\mathcal{I} = \frac{\sqrt{q}}{\sqrt{q + E_i^a E_i^b T_{,a} T_{,b}}} = \frac{2}{\kappa\beta\sqrt{q}} E_i^a \{A_a^i, S[1]\} - \frac{2S(x)}{\sqrt{q}},$$

where  $q = \frac{1}{3!} |\varepsilon_{abc} \varepsilon^{ijk} E_i^a E_j^b E_k^c|$  and

$$S[f] \equiv \int d^3x f(x) S(x) := \int d^3x f(x) \sqrt{q(x) + E_i^a E_i^b T_{,a} T_{,b}}.$$

# Ashtekar formalism of the non-rotational dust

- Therefore, the smeared version of Hamiltonian constraint reads

$$C(N) = \int_{\Sigma} d^3x N(\pi(x) + h(x)) = 0$$

- where

$$\begin{aligned} h(x) &\equiv |C^{gr}(x)| \left( \frac{2}{\kappa\beta\sqrt{q}} E_i^a \{A_a^i, S[1]\} - \frac{2S(x)}{\sqrt{q}} \right) \\ &=: h_1(x) + h_2(x). \end{aligned}$$

- The above Eq implies that one can define a physical Hamiltonian  $h(x)$  which generates the evolution of the system with respect to the dynamical dust "time"  $T(x)$ .
- The the whole coupled system can be quantized successfully by LQG method, and the Dirac quantization procedure can be realized [XZ and Ma, PLB 2023].

# Holonomy

**Holonomy group** is the group of linear transformations induced by parallel transport around closed loops based at a point  $p$  in a manifold  $M$ , with respect to a given connection  $\nabla$  on a vector bundle  $E$ . Specifically, for each piecewise-smooth loop  $\gamma$  at  $p$ , the connection defines a parallel transport map  $P_\gamma : E_p \rightarrow E_p$ , which is invertible and linear. The set of all such maps forms a subgroup of  $GL(E_p)$ , called the **holonomy group** at  $p$ , denoted  $\text{Hol}_p(\nabla)$ .

The equation of holonomy reads 
$$\dot{h}_{e(t)}(A) = h_{e(t)}(A) A_a(e(t)) \dot{e}^a(t), \quad h_{e(0)} = 1.$$

The solution to this equation would be 
$$h(c) = P \exp \left( \int_0^1 [A_a^i \dot{c}^a \tau_i] dt \right)$$

which could also be expanded as

$$h_e(A) = \mathcal{P} \exp \int_e A = 1 + \sum_{n=1}^{\infty} \int_0^1 dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 A(t_1) \cdots A(t_n)$$

# Properties of Holonomy

- The beginning point, final point and range of a curve  $c \in C$  is defined, respectively, by

$$b(c) := c(0), \quad f(c) := c(1), \quad r(c) := c([0, 1])$$

- Composition  $\circ : C \times C \rightarrow C$  of composable curves  $c_1, c_2 \in C$  (those with  $f(c_1) = b(c_2)$ ) and inversion  $-1 : C \rightarrow C$  of  $c \in C$  are defined by

$$(c_1 \circ c_2)(t) \begin{cases} := c_1(2t) & t \in [0, \frac{1}{2}] \\ c_2(2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}, \quad c^{-1}(t) := c(1 - t)$$

- Then we have
  1.  $h_{c_1 \circ c_2}(A) = h_{c_1}(A)h_{c_2}(A)$
  2.  $h_{c^{-1}}(A) = h_c(A)^{-1}$

# Properties of Holonomy

- For gauge transformation  $A^g := -dg g^{-1} + \text{Ad}_g(A)$

- The holonomy transform as

$$h_c^g(A) := h_c(A^g) = g(b(c))h_c(A)g(f(c))^{-1}$$

- Some reference use different notion (Ashtekar and Lewandowski, CQG 2004), the  $\mathfrak{c}$   $A_\alpha \mapsto g_\alpha^{-1}A_\alpha g_\alpha + g_\alpha^{-1}d_\alpha g_\alpha$ ,

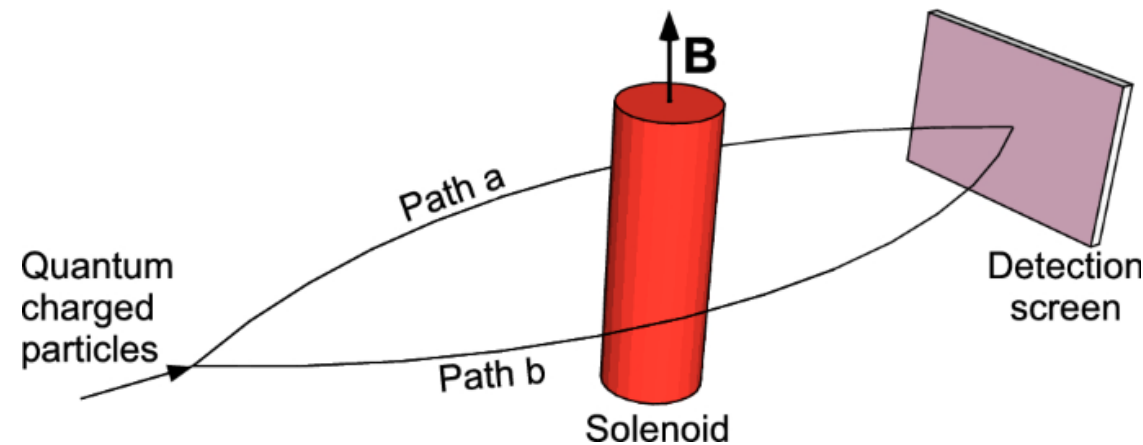
- Then the resulted equation for holonomy is

$$\frac{d}{dt}U_e(t, t_1; A) = -A_a(e(t))\dot{e}^a(t)U(t, t_1; A), \quad \text{and} \quad U(t_1, t_1; A) = I.$$

# Aharonov-Bohm effect

- Covariant derivative for EM field reads  $D_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu$
- Then the parallel transport for electron should be  $D_\mu\psi = 0$

- This means  $D_\mu\psi = 0$ 
  - $\Rightarrow \left(\partial_\mu + i\frac{e}{\hbar}A_\mu\right)\psi = 0$
  - $\Rightarrow \partial_\mu\psi = -i\frac{e}{\hbar}A_\mu\psi$
  - $\Rightarrow \psi(x) = \psi(x_0) \exp\left(-i\frac{e}{\hbar} \int A_\mu dx^\mu\right)$



- Even for interior which absence of B field, the phase crosses a shift as

$$\Delta\theta = \frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{e}{\hbar} \int \mathbf{B} \cdot d\mathbf{S} = \frac{e}{\hbar} \Phi$$

[C. N Yang and T T Wu, 1975]

# Holonomy for LQC

- The densitized triad and spin connection can be simplified as

$$E_i^a = pV_0^{-\frac{2}{3}} \sqrt{\det({}^0q)} e_i^a, \quad A_b^j = cV_0^{-\frac{1}{3}} \omega_b^j.$$

- Commutation relation between  $c$  and  $p$  reads

$$\{c, p\} = \frac{\kappa\beta}{3}$$

- The  $SU(2)$  holonomy along an edge  $e$  is given by

$$A(e) = \cos \frac{\ell c}{2} + 2 \left[ \sin \frac{\ell c}{2} \right] (\dot{e}^{a0} \omega_a^i) \tau^i$$

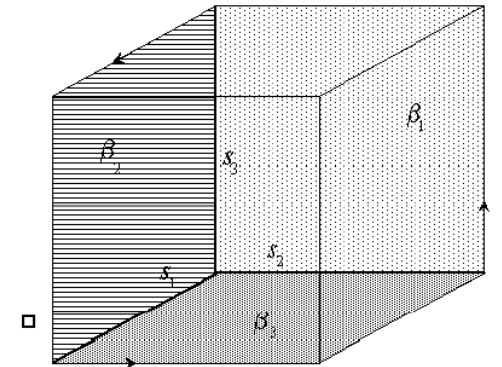


FIG. 3, An elementary cell 2 in a cubic partition.  $s_1, s_2, s_3$  are the edges of the cell and 1, 2, 3 the three oriented loops which are boundaries of faces orthogonal to these edges.

- Given a functional

$$E[f] = \frac{1}{\kappa\beta} \int E_i^a(x) f_a^i(x),$$

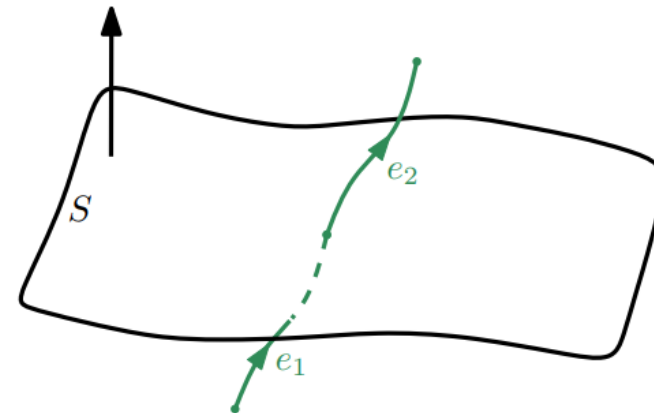
- The Holonomy-Flux algebra

$$\{h_e(A), \tilde{E}_i(S)\} = \frac{\kappa\beta}{2} \epsilon(e, S) \left( \delta_{e \cap S, b(e)} \tau_i h_e(A) + \delta_{e \cap S, f(e)} h_e(A) \tau_i \right)$$

- Direction factor:  $\epsilon(e, S) := \begin{cases} +1 & e \text{ is the 'up' type w.r.t. } S \quad \left( n_a \dot{e}^a \Big|_{e \cap S} > 0 \right) \\ -1 & e \text{ is the 'down' type w.r.t. } S \\ 0 & e \text{ is the 'inside/outside' type w.r.s. } S \end{cases}$

- Example:  $e = e_1 \circ e_2, \quad e_1 \cap e_2 = e \cap S$

$$\{h_e(A), \tilde{E}_i(S)\} = \kappa\beta h_{e_1}(A) \tau_i h_{e_2}(A)$$



# Thiemann trick

- The final complication is the presence of the volume element  $\sqrt{\det P}$  in the denominator.

$$e_a^i := \frac{\sqrt{k\beta}}{2} \eta_{abc} \epsilon^{ijk} \frac{P_j^b P_k^c}{\sqrt{\det P}}.$$

- This can be expressed as a manageable Poisson bracket

$$e_a^i(x) = \frac{2}{k\beta} \{A_a^i(x), V\}.$$

- Using this fact, the Euclidean scalar constraint part  $\mathcal{C}^{\text{Eucl}}(N)$  is written as

$$\mathcal{C}^{\text{Eucl}}(N) = -\frac{2}{k^2 \beta^{\frac{3}{2}}} \int_M d^3x N(x) \eta^{abc} \text{Tr} \left( F_{ab}(x) \{A_c(x), V\} \right)$$

- Moreover, we also note that  $K^i_a$  can be expressed as a Poisson bracket

$$K^i_a = \frac{1}{k\beta} \{A^i_a, \bar{K}\}$$

- where  $\bar{K}$  is the integral of the trace of the extrinsic curvature

$$\bar{K} = k\beta \int_M d^3x K^i_a P^a_i.$$

- Now  $\bar{K}$  itself can be expressed as a Poisson bracket

$$\bar{K} = \beta^{-\frac{3}{2}} \{C^{\text{Eucl}}(1), V\}.$$

- Hence  $\mathcal{T}(N)$ , can be expressed as:

$$\mathcal{T}(N) = -\frac{2}{k^4\beta^3} \int_M d^3x N(x) \eta^{abc} \text{Tr} \left( \{A_a(x), \bar{K}\} \{A_b(x), \bar{K}\} \{A_c(x), V\} \right).$$

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