Quantum mechanism angular momentum & graphical calculus

II

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Loops'26 summer school
Yangzhou University, May 11-12, 2026

$$\mathcal{H}_{SU(2)} = L^2(SU(2), d\mu_H) \xrightarrow{?} \mathcal{H}_{kin} = L^2(\bar{\mathcal{A}}, d\mu_{AL})$$

II. Introduction to:

1. Elementary variables
2. Quantum configuration space
3. A natural measure on the quantum configuration space
3. The Hilbert space
4. Spin network states
5. Elementary operators

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1. Elementary variables: holonomy and flux

The phase space: canonical variables $(A_a^i, \tilde{E}_i^a)$ on Σ

The only non-trivial Poisson bracket:

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \kappa\beta\delta_a^b\delta_j^i\delta^3(x,y), \text{ where } \kappa = 8\pi G$$

The holonomy $h_e(A)$ of connection A_a^i along an edge $e : [0,1] \to \Sigma$

$$A(e(t)) := \dot{e}^a(t)A_a^i(e(t))\tau_i$$

$$h_e(A) := \mathcal{P}\exp\left(\int_e A\right) = \mathbb{I}_2 + \sum_{n=1}^{\infty}\int_0^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \cdots \int_{t_{n-1}}^1 \mathrm{d}t_n\, A(e(t_1))\cdots A(e(t_n)) \in \mathrm{SU}(2)$$

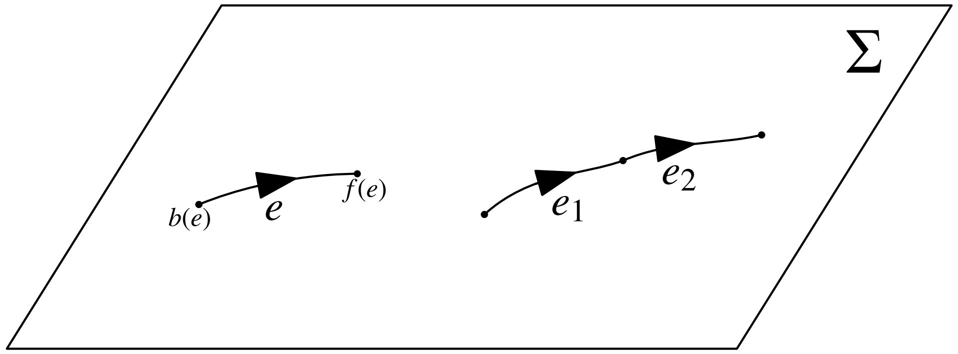
as the unique solution $h_e(A) \equiv h_{e([0,t=1])}(A)$ of

$$\frac{\mathrm{d}h_{e([0,t])}(A)}{\mathrm{d}t} = h_{e([0,t])}(A)\, A(e(t)), \quad h_{e([0,0])}(A) = \mathbb{I}_2$$

Properties of holonomy

$$h_{e_1\circ e_2}(A) = h_{e_1}(A)h_{e_2}(A)$$

$$h_{e^{-1}}(A) = h_e(A)^{-1}$$



The behavior of holonomy under gauge transformation

$$h_e(A^g) = g(b(e))h_e(A)g(f(e))^{-1}$$

$$A \mapsto A^g = -(\mathrm{d}g)g^{-1} + gAg^{-1}$$

The flux $\tilde{E}_i(S)$ of densitized triad $\tilde{E}_i^a$ through a 2-surface $S \subset \Sigma$ is defined by

$$\tilde{E}_i(S) := \int_S \left(*\tilde{E}_i\right)_{ab}$$

The elementary variables are defined in a background-independent way !

1. Elementary variables: holonomy and flux

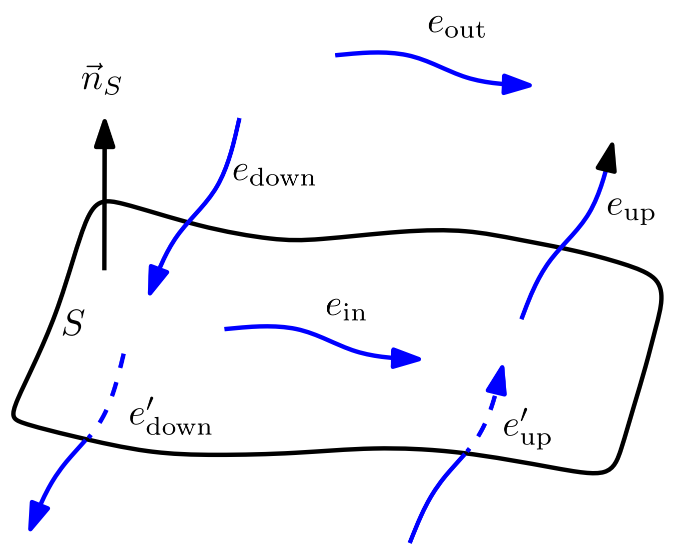
$$L^{(\tau_i)}g^A{}_B = (g\tau_i)^A{}_B \quad \Leftrightarrow \quad L^{(\tau_i)}g = g\tau_i$$
$$R^{(\tau_i)}g^A{}_B = -(\tau_i g)^A{}_B \quad \Leftrightarrow \quad R^{(\tau_i)}g = -\tau_i g$$

$\tau_i = -\frac{\mathrm{i}}{2}\sigma_i$ with σ_i being the Pauli matrices.

$$e \cap S = b(e)/f(e)$$

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \kappa\beta\delta_a^b\delta_j^i\delta^3(x,y)$$

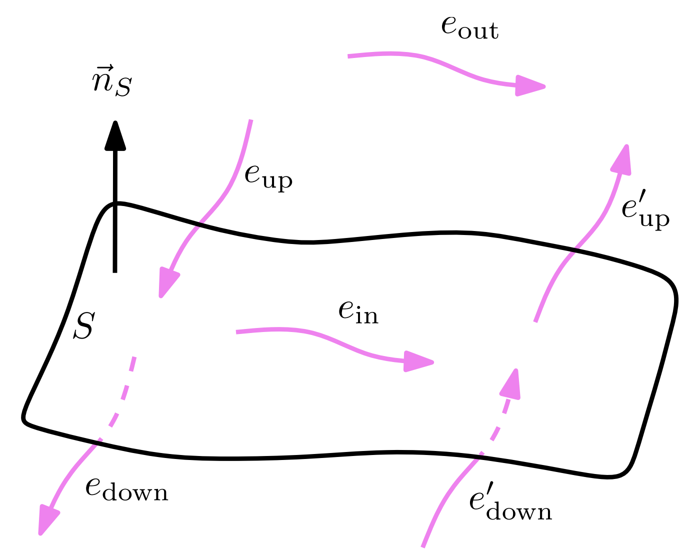
$$\begin{aligned}
\{h_e(A), \tilde{E}_i(S)\} &= \frac{\kappa\beta}{2}\epsilon(e,S)\Big(\delta_{e\cap S,b(e)}\tau_i h_e(A) + \delta_{e\cap S,f(e)}h_e(A)\tau_i\Big)\\
&= \frac{\kappa\beta}{2}\kappa(e,S)\Big(\delta_{e\cap S,b(e)}\tau_i h_e(A) - \delta_{e\cap S,f(e)}h_e(A)\tau_i\Big)\\
&= -\frac{\kappa\beta}{2}\kappa(e,S)\Big(\delta_{e\cap S,b(e)}R_e^{(\tau_i)} + \delta_{e\cap S,f(e)}L_e^{(\tau_i)}\Big)h_e(A)
\end{aligned}$$



$$\left(n_a\dot{e}^a\Big|_{e\cap S} > 0\right)$$

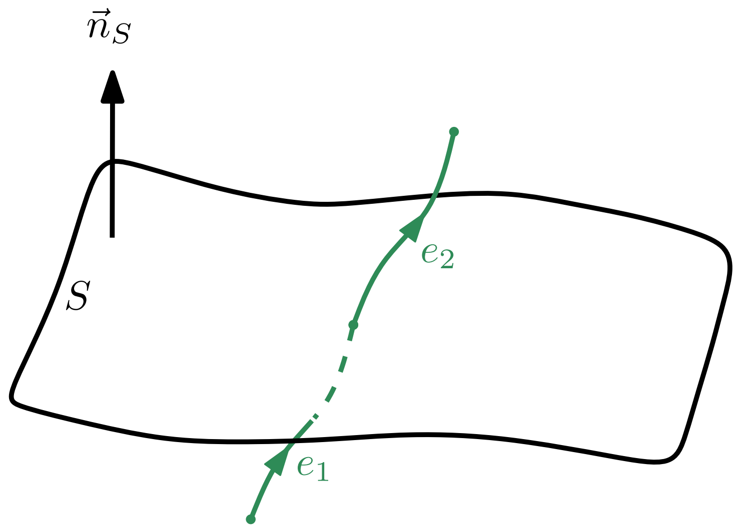
Direction factor: $\epsilon(e,S) := \begin{cases} +1 & e \text{ is the ‘up’ type w.r.t. } S \\ -1 & e \text{ is the ‘down’ type w.r.t. } S \\ 0 & e \text{ is the ‘inside/outside’ type w.r.s. } S \end{cases}$

Steric factor: $\kappa(e,S) := \begin{cases} +1 & e \text{ is the ‘up’ type w.r.t. } S \\ -1 & e \text{ is the ‘down’ type w.r.t. } S \\ 0 & e \text{ is the ‘inside/outside’ type w.r.t. } S \end{cases}$



$$e = e_1 \circ e_2, \quad e_1 \cap e_2 = e \cap S$$

$$\{h_e(A), \tilde{E}_i(S)\} = \kappa\beta\, h_{e_1}(A)\,\tau_i\, h_{e_2}(A)$$



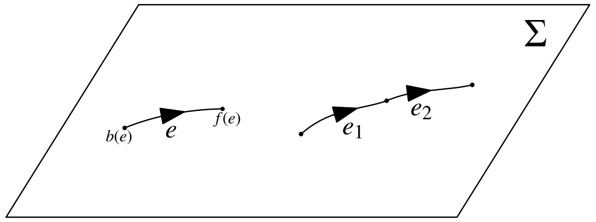
2. Quantum configuration space $\bar{\mathcal{A}}$

The **classical configuration space** $\mathcal{A}$ is the set of smooth connections A on Σ

The holonomy $h_e(A)$ of connection A_a^i along the edge $e : [0,1] \to \Sigma$

$$A(e) \equiv h_e(A) = \mathcal{P}\exp\left(\int_e A\right) \in SU(2)$$

Properties: $A(e_1 \circ e_2) = A(e_1)A(e_2)$, $\quad A(e^{-1}) = A(e)^{-1}$



Def. (Set of paths). *A piecewise analytic path is a composition of edges. The set of paths is denoted by* $\mathcal{P}$.

$\mathcal{A} = \mathrm{Hom}(\mathcal{P}, SU(2))$, be a continue homomorphism map

The **quantum configuration space** $\bar{\mathcal{A}}$ as a specific extension of $\mathcal{A}$

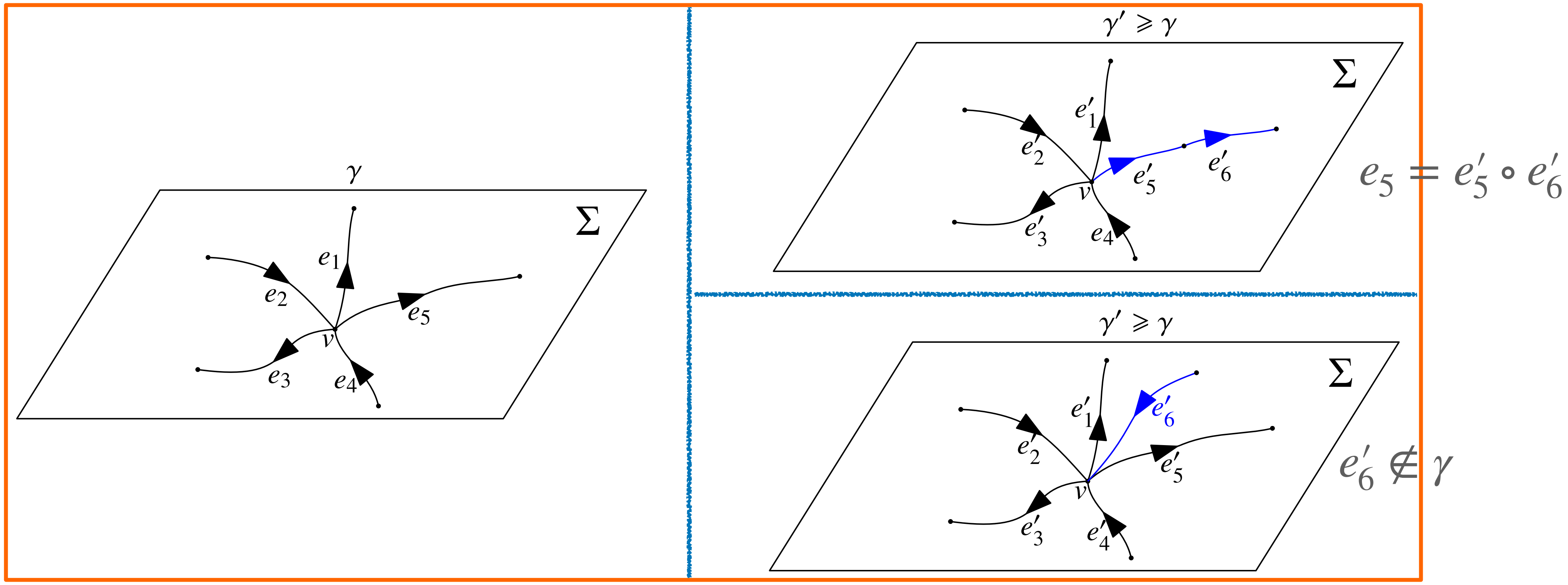
$\bar{\mathcal{A}} := \mathrm{Hom}(\mathcal{P}, SU(2))$, can be an arbitrary discontinue homomorphism map

Relation: in a natural topology (due to Gel'fand), $\mathcal{A}$ is densely embedded in $\bar{\mathcal{A}}$, and $\bar{\mathcal{A}}$ is compact in a natural (Gel'fand) topology.

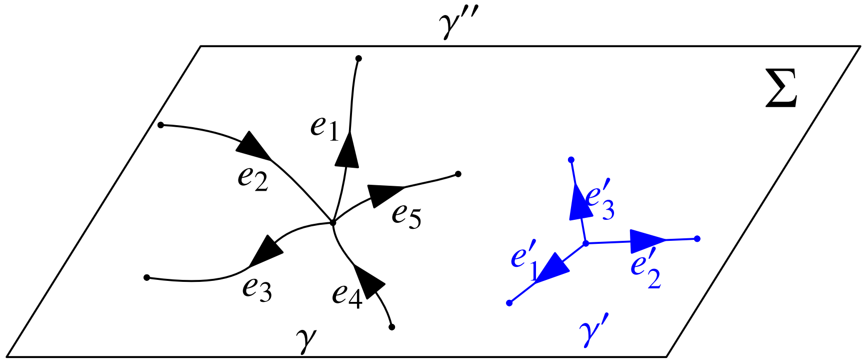
$$\mathcal{H}_{SU(2)} = L^2(SU(2), d\mu_H) \xrightarrow{?} \mathcal{H}_{\mathrm{kin}} = L^2(\bar{\mathcal{A}}, d\mu_{AL})$$

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

Def. (Graph). *A graph γ is a collection of edges and vertices, in which the edges can only intersect each other at their endpoints. A graph γ' is said to be large than another graph γ (or contain γ), $\gamma' \geqslant \gamma$, if every e of γ can be written as $e = e'_1 \circ e'_2 \cdots \circ e'_k$ for some edges $e', \ldots, e'_k$ of γ'.*



Remarks:



(i) Not all two graphs, γ and γ', need to have a containment relation.

(ii) For any two graph γ, γ', there exists γ'' such that $\gamma'' \geqslant \gamma, \gamma'$, where γ'' contain $\gamma \cup \gamma'$.

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

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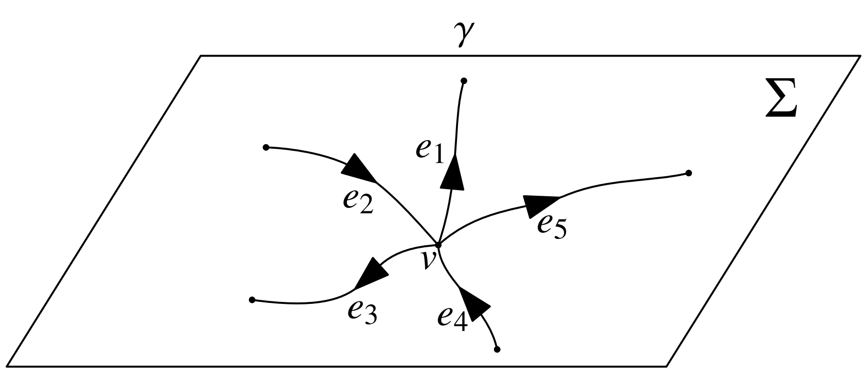
$\bar{\mathcal{A}}_\gamma := \bar{\mathcal{A}}\big|_\gamma$, the restriction of the domain of $\bar{\mathcal{A}}$ on γ with n edges $e_1, \ldots, e_n$



$$\mathcal{H}_{\mathrm{SU}(2)} = L^2(\mathrm{SU}(2), d\mu_H) \xrightarrow{?} \mathcal{H}_{\mathrm{kin}} = L^2(\bar{\mathcal{A}}, d\mu_{AL})$$

glue

$$\left(\bar{\mathcal{A}}_\gamma, d\mu_\gamma\right)$$

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

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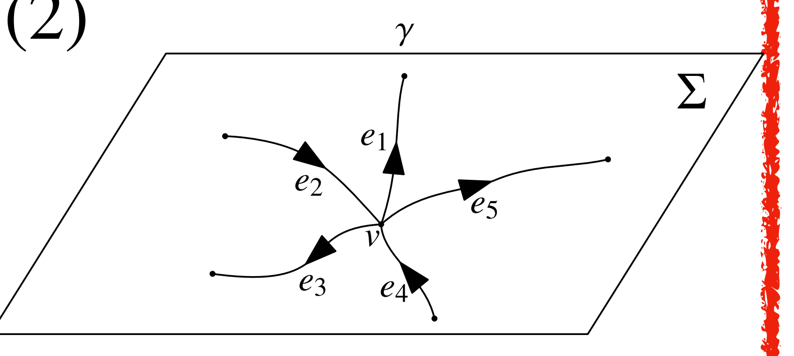
$$\left(\bar{\mathcal{A}}_\gamma, d\mu_\gamma\right)$$

$\bar{\mathcal{A}}_\gamma := \bar{\mathcal{A}}\big|_\gamma$, the restriction of the domain of $\bar{\mathcal{A}}$ on γ with n edges $e_1, \ldots, e_n$

There exists a bijection

$|E(\gamma)|$: the number of edeges of γ

$$\mathcal{I}_E : \bar{\mathcal{A}}_\gamma \to SU(2)^n \equiv SU(2) \times \cdots \times SU(2)$$
$$A \mapsto (A(e_1), \ldots, A(e_n))$$



which helps us to

(i) equip $\bar{\mathcal{A}}_\gamma$ with a topology induced from that of $SU(2)^n$.

(ii) define a nature measure $d\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$ induced from the Haar measure $d\mu_H$ on SU(2).

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

Step I: a natural measure $d\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$

There exists a bijection

$$\mathcal{I}_E : \bar{\mathcal{A}}_\gamma \to SU(2)^n \equiv SU(2) \times \cdots \times SU(2)$$
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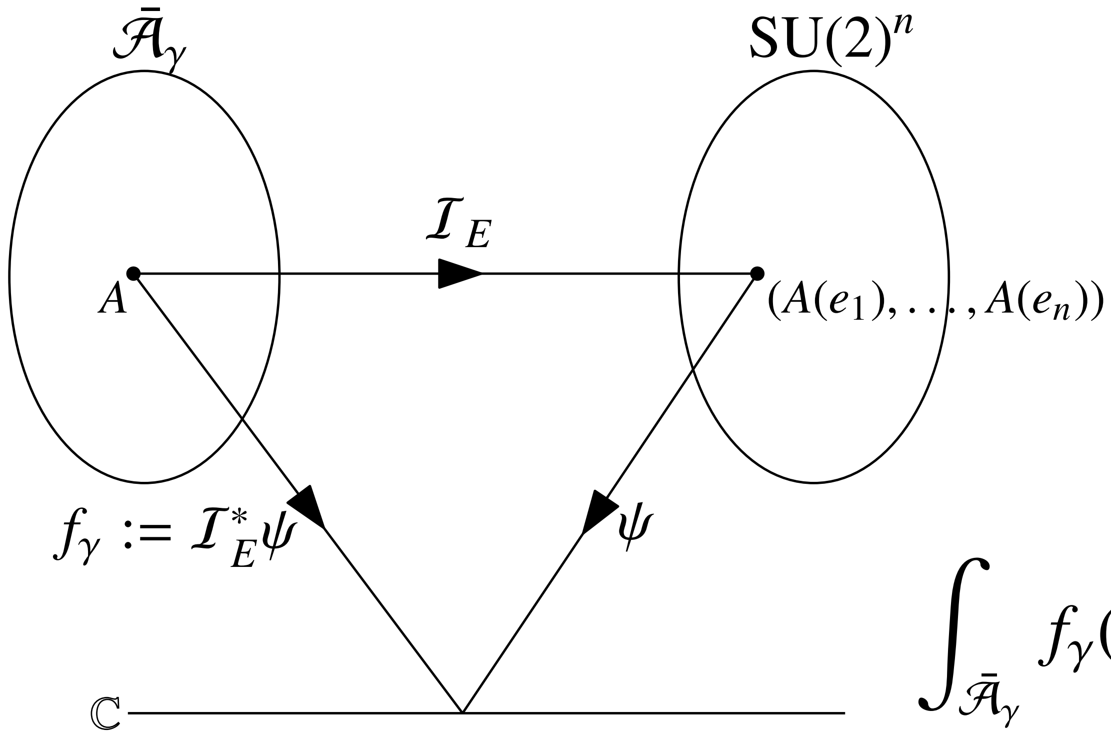

$$\int_{\bar{\mathcal{A}}_\gamma} f_\gamma(A)\, d\mu_\gamma(A) := \int_{SU(2)^n} \psi(A(e_1), \ldots, A(e_n)) \prod_{i=1}^{n} d\mu_H(A(e_i))$$

Remarks:

(i) For any given γ and for each $\bar{A} \in \bar{\mathcal{A}}$, there exists a smooth $A \in \mathcal{A}$ such that

$$\bar{A}(e) = h_e(A), \ \forall e \in \gamma$$

(ii) $f_\gamma(A) = \psi(A(e_1), \ldots, A(e_n)) = \psi(h_{e_1}(A), \ldots, h_{e_n}(A)) \equiv f_\gamma(h_{e_1}(A), \ldots, h_{e_n}(A))$

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

Step I: a natural measure $d\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$

There exists a bijection

$$\mathcal{I}_E : \bar{\mathcal{A}}_\gamma \to SU(2)^n \equiv SU(2) \times \cdots \times SU(2)$$
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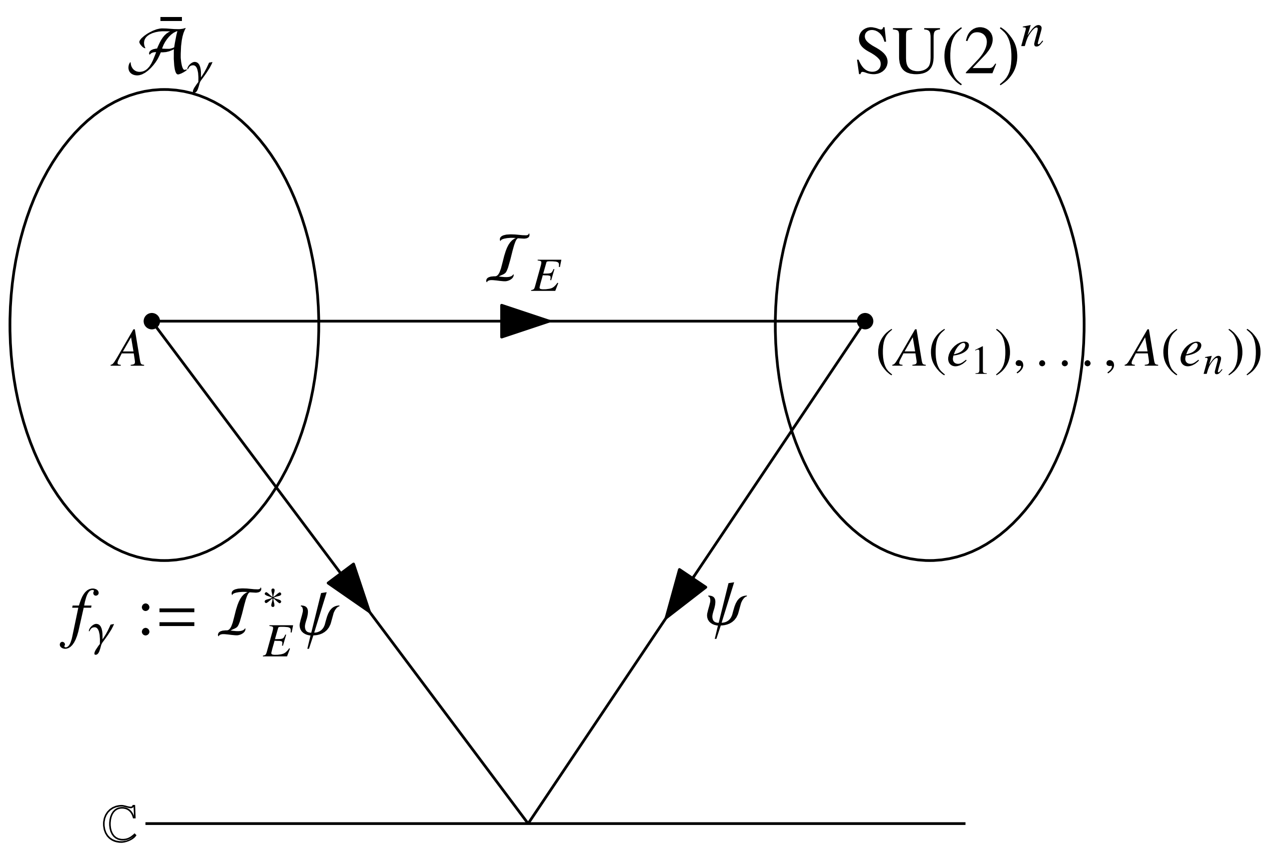


$$\int_{\bar{\mathcal{A}}_\gamma} f_\gamma(A)\, d\mu_\gamma(A) := \int_{SU(2)^n} \psi(A(e_1), \ldots, A(e_n)) \prod_{i=1}^{n} d\mu_H(A(e_i))$$

$\mathcal{H}_{SU(2)} = L^2(SU(2), d\mu_H) \xrightarrow{?} \mathcal{H}_{kin} = L^2(\bar{\mathcal{A}}, d\mu_{AL})$

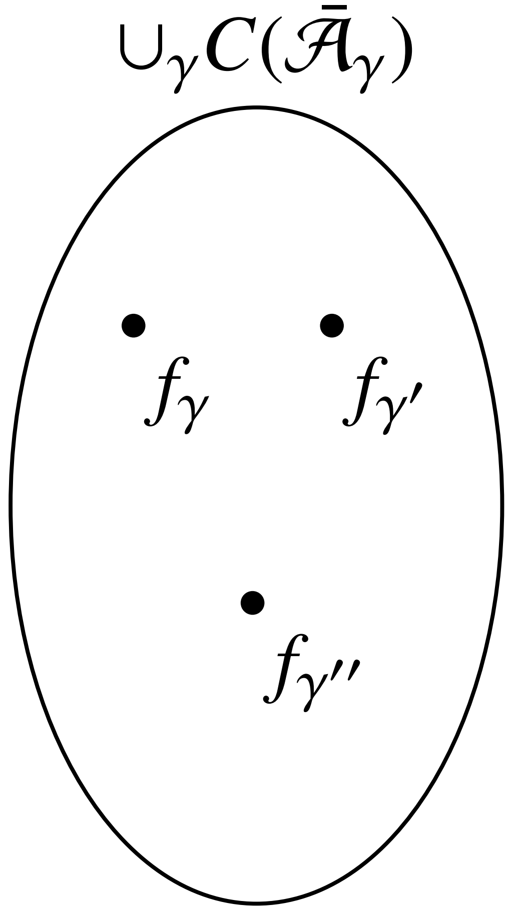
glue

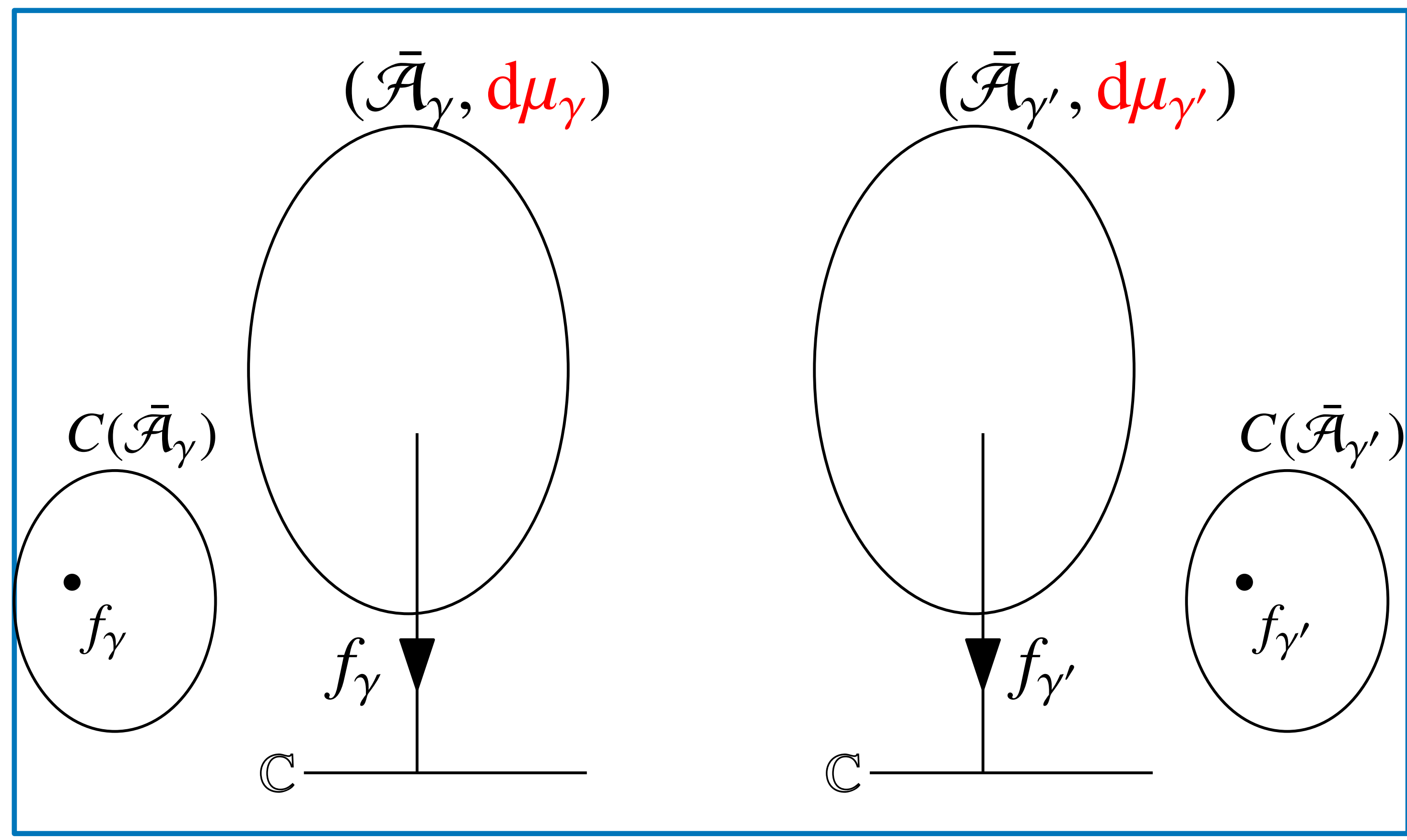
$(\bar{\mathcal{A}}_\gamma, d\mu_\gamma)$

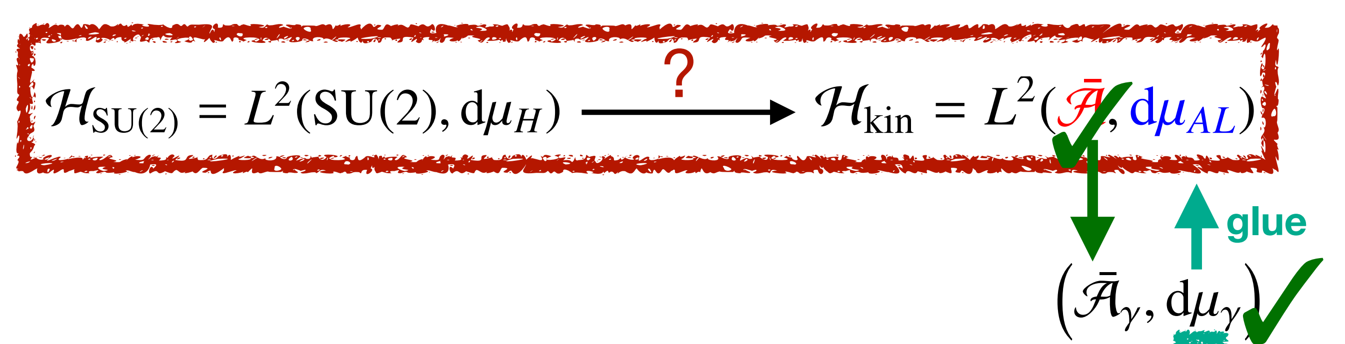
3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

Step II: a natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$ defined by $d\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$

$C(\bar{\mathcal{A}}_\gamma)$: the set of all continuous complex functions on $\bar{\mathcal{A}}_\gamma$







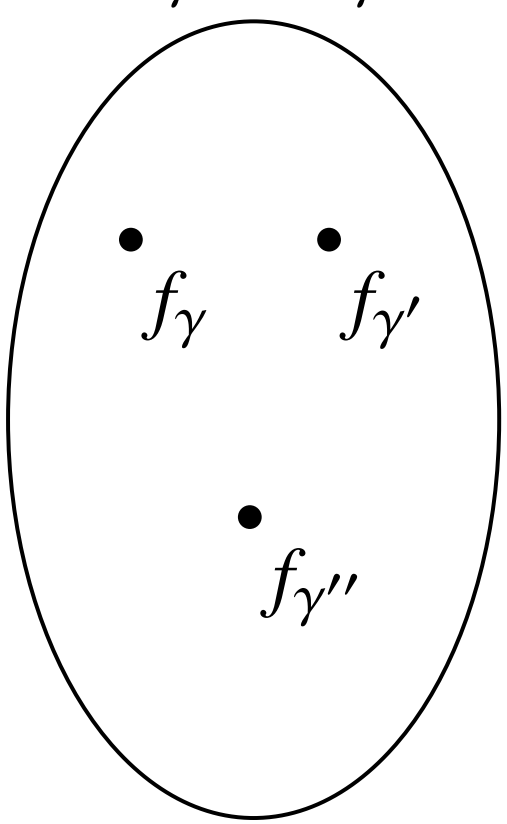
3. A natural measure $\mathrm{d}\mu_{AL}$ on $\bar{\mathcal{A}}$

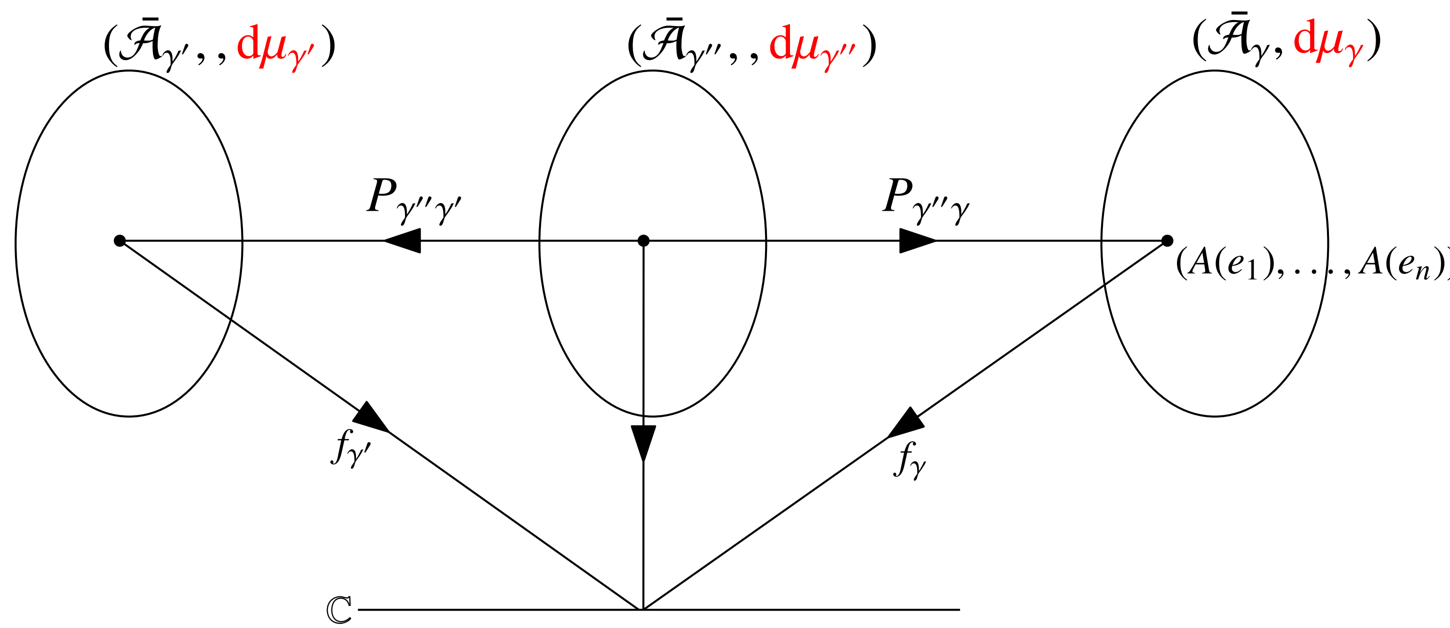
Step II: a natural measure $\mathrm{d}\mu_{AL}$ on $\bar{\mathcal{A}}$ defined by $\mathrm{d}\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$

For $\gamma' \geqslant \gamma$, there exists a surjective map (projective map)

$$P_{\gamma'\gamma} : \bar{\mathcal{A}}_{\gamma'} \to \bar{\mathcal{A}}_\gamma, \quad \text{restricting the domain from } \gamma' \text{ to } \gamma$$

$\cup_\gamma C(\bar{\mathcal{A}}_\gamma)$





$$\forall \gamma'' \geqslant \gamma, \gamma', \quad P^*_{\gamma''\gamma'} f_{\gamma'} = P^*_{\gamma'\gamma} f_\gamma \quad \longrightarrow \quad \int_{\bar{\mathcal{A}}_{\gamma'}} f_{\gamma'}(A)\, \mathrm{d}\mu_{\gamma'}(A) = \int_{\bar{\mathcal{A}}_\gamma} f_\gamma(A)\, \mathrm{d}\mu_\gamma(A)$$

Hence, these two functions are said to be equivalent

$$f_\gamma \sim f_{\gamma'}$$

3. A natural measure $\mathrm{d}\mu_{AL}$ on $\bar{\mathcal{A}}$

Step II: a natural measure $\mathrm{d}\mu_{AL}$ on $\bar{\mathcal{A}}$ defined by $\mathrm{d}\mu_\gamma$ on $\bar{\mathcal{A}}_\gamma$

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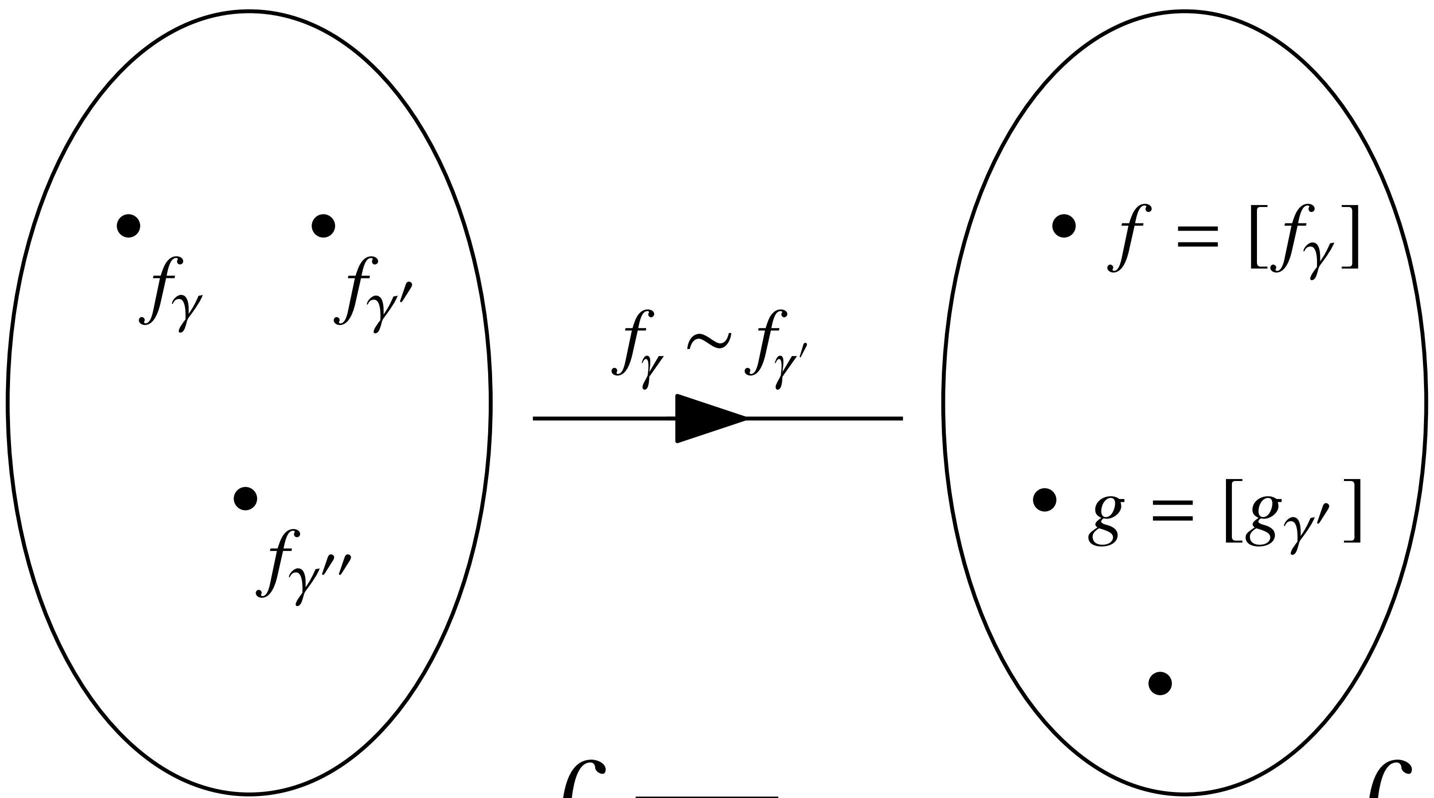
$$P_{\gamma'\gamma} : \bar{\mathcal{A}}_{\gamma'} \to \bar{\mathcal{A}}_\gamma, \quad \text{restricting the domain from } \gamma' \text{ to } \gamma$$

$\cup_\gamma C(\bar{\mathcal{A}}_\gamma)$ $\qquad$ $\mathrm{Cyl}(\bar{\mathcal{A}}) := \cup_\gamma C(\bar{\mathcal{A}}_\gamma)/\sim$



$$\int_{\bar{\mathcal{A}}} f(A)\, \mathrm{d}\mu_{AL}(A) := \int_{\bar{\mathcal{A}}_\gamma} f_\gamma(A)\, \mathrm{d}\mu_\gamma(A)$$

$$\langle g, f\rangle_{\mathrm{Cyl}(\bar{\mathcal{A}})} := \int_{\bar{\mathcal{A}}} \overline{g(A)}\, f(A)\, \mathrm{d}\mu_{AL}(A) := \int_{\bar{\mathcal{A}}_{\gamma''}} \overline{(P^*_{\gamma''\gamma'} g_{\gamma'})(A)}\, (P^*_{\gamma''\gamma} f_\gamma)(A)\, \mathrm{d}\mu_{\gamma''}(A)$$

$\mathcal{H}_{\mathrm{SU(2)}} = L^2(\mathrm{SU(2)}, \mathrm{d}\mu_H) \xrightarrow{?} \mathcal{H}_{\mathrm{kin}} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_{AL})$

glue

$(\bar{\mathcal{A}}_\gamma, \mathrm{d}\mu_\gamma)$

3. A natural measure $d\mu_{AL}$ on $\bar{\mathcal{A}}$

$d\mu_{AL}$: The Ashtekar-Lewandowski measure.

Thm. *The Ashtekar-Lewandowski measure is invariant under internal gauge transformations $g(x)$ and spatial diffeomorphisms φ, i.e.,* $\forall f \in \mathrm{Cyl}(\overline{\mathcal{A}})$,

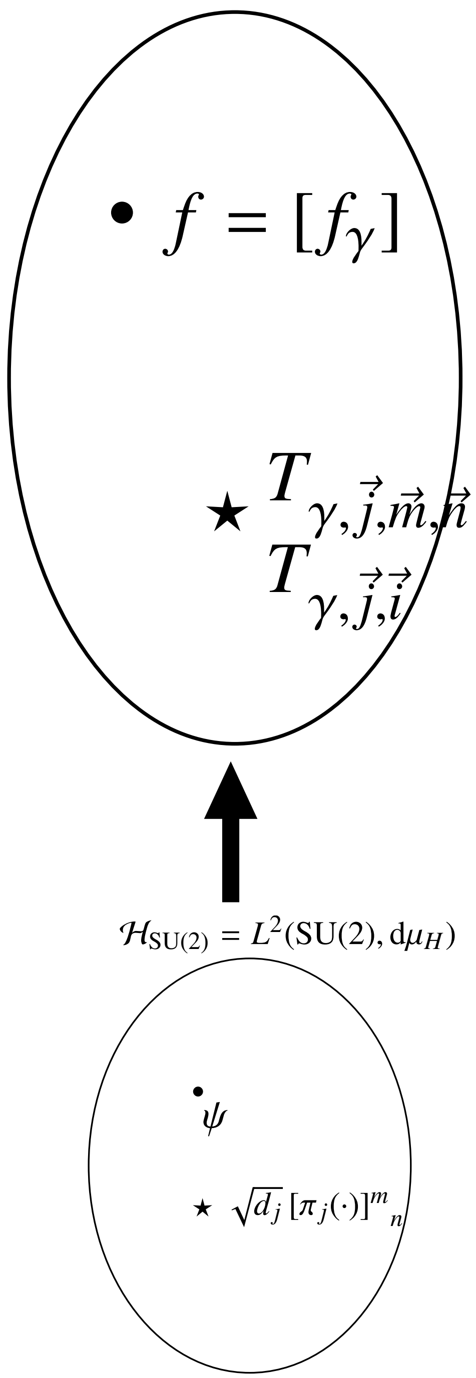
$$\int_{\bar{\mathcal{A}}} g \circ f(A)\, d\mu_{AL}(A) = \int_{\bar{\mathcal{A}}} f(A)\, d\mu_{AL}(A),$$
$$\int_{\bar{\mathcal{A}}} \varphi \circ f(A)\, d\mu_{AL}(A) = \int_{\bar{\mathcal{A}}} f(A)\, d\mu_{AL}(A).$$

4. The kinematical Hilbert space $\mathcal{H}_{\rm kin}$

$$\mathcal{H}_{\rm kin} := \overline{{\rm Cyl}(\bar{\mathcal{A}})} = L^2(\bar{\mathcal{A}}, {\rm d}\mu_{AL})$$

Spin network states: the basis of $\mathcal{H}_{\rm kin}$

$$\mathcal{H}_{\rm kin} := \overline{{\rm Cyl}(\bar{\mathcal{A}})} = L^2(\bar{\mathcal{A}}, {\rm d}\mu_{AL})$$



Spin network states

$$T_{\gamma,\vec{j},\vec{m},\vec{n}}(A) := \prod_{e\in E(\gamma)} \sqrt{d_{j_e}}\,[\pi_{j_e}(A(e))]^{m_e}{}_{n_e}$$

$$\langle T_{\gamma',\vec{j}',\vec{m}',\vec{n}'}, T_{\gamma,\vec{j},\vec{m},\vec{n}}\rangle_{\mathcal{H}_{\rm kin}} = \delta_{\gamma,\gamma'}\delta_{\vec{j}',\vec{j}}\delta_{\vec{m}',\vec{m}}\delta_{\vec{n}',\vec{n}}$$

$$T_{\gamma,\vec{j},\vec{i}}(A) = \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\,\pi_{j_e}(A(e))$$

$$\langle T_{\gamma',\vec{j}',\vec{i}'}, T_{\gamma,\vec{j},\vec{i}}\rangle_{\mathcal{H}_{\rm kin}} = \delta_{\gamma,\gamma'}\delta_{\vec{j}',\vec{j}}\delta_{\vec{i}',\vec{i}}$$

$$i_v \sim \left(i^{J;\vec{a}}_{v;j_1\cdots j_n}\right)_{m_1\cdots m_n}^{\quad M} \equiv (-1)^{j_1-\sum_{i=2}^n j_i - J} \sum_{k_2,\ldots,k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2\rangle \cdots \langle JM | a_{n-1} k_{n-1} j_n m_n\rangle$$

Gauge variant: $i_v \sim \left(i^{J\neq 0;\vec{a}}_{v;j_1\cdots j_n}\right)_{m_1\cdots m_n}^{\quad M\neq 0}$

Gauge invariant: $i_v \sim \left(i^{J=0;\vec{a}}_{v;j_1\cdots j_n}\right)_{m_1\cdots m_n}^{\quad M=0} \equiv \left(i^{J=0;\vec{a}}_{v;j_1\cdots j_n}\right)_{m_1\cdots m_n}$

5. The elementary operators on $\mathcal{H}_{\text{kin}}$

$$\widehat{A(e)} \cdot f_\gamma(A) := A(e) f_\gamma(A)$$

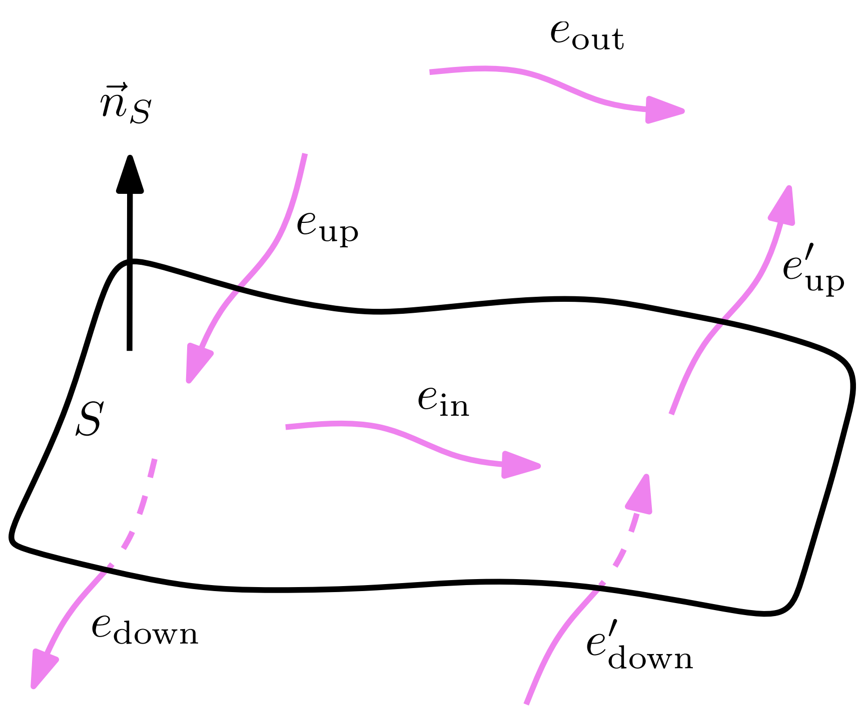
$$\widehat{\tilde{E}_i(S)} \cdot f_\gamma(A) := -\mathrm{i}\hbar \left\{ f_\gamma(A), \tilde{E}_i(S) \right\}$$

$$= \frac{\hbar \kappa \beta}{2} \sum_{v \in V(\gamma)} \left(\sum_{b(e)=v} \kappa(e,S) J^i_{e,R} + \sum_{f(e)=v} \kappa(e,S) J^i_{e,L} \right) f_\gamma(A)$$

$$J^i_{e,R} := \mathrm{i} R_e^{(\tau_i)}, \quad J^i_{e,L} := \mathrm{i} L_e^{(\tau_i)}$$

$$\{h_e(A), \tilde{E}_i(S)\} = -\frac{\kappa\beta}{2} \kappa(e,S) \left(\delta_{e\cap S, b(e)} R_e^{(\tau_i)} + \delta_{e \cap S, f(e)} L_e^{(\tau_i)} \right) h_e(A)$$

Steric factor: $\kappa(e,S) := \begin{cases} +1 & e \text{ is the 'up' type w.r.t. } S \\ -1 & e \text{ is the 'down' type w.r.t. } S \\ 0 & e \text{ is the 'inside/outside' type w.r.t. } S \end{cases}$



Thanks for your attention!